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"SCALAR SELF-FORCE REGULARIZATION FOR CIRCULAR GEODESICS IN SCHWARZSCHILD SPACETIME"

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CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Resumo

por Alexandre Sampaio da Cruz

Um dos principais desafios experimentais associados à detecção de ondas gravitacionais está na necessidade de se possuir um profundo conhecimento prévio sobre a forma dessas ondas para sua detecção. Tal comenhecimento depende, é claro, do entendimento da dinânica dos sistemas físicos que as emitem. Dentre tais sistemas, uma classe de particular interesse atual é composta dos sistemas chamados de *Inspirais de Razão de Massa Extrema* (*Extreme Mass Ratio Inspirals*, EMRIs), isto é, sistemas binários nos quais um dos corpos é muito mais massivo que o outro.

A princpial abordagem utilizada para o estudo da física de EMRIs é a teoria de auto-força gravitacional. Nessa abordagem a geometria do espaço-tempo é tida, em primeira aproximação, como determinada apenas pela influência gravitacional do corpo de maior massa. Isto é, o corpo menos massivo se comporta como uma particula teste que se move em uma geodésica neste espaço-tempo dito “de fundo”. Para entender o inspiralamento, é necessário ir além dessa aproximação. É preciso considerar a perturbação que o corpo de menor massa causa nesse fundo. Sob a perspetiva da teoria de auto-força, o desvio do movimento desse corpo de uma geodésica do espaço-tempo de fundo é visto como uma força, cuja origem é própria perturbação gravitacional gerada pelo mesmo. Esse tipo de problema pertence a uma classe mais ampla de problemas de auto-forca, que requerem o mesmo tipo de tratamento matemático associado ao seu caráter singular.

Nesse trabalho, várias técnicas associadas ao cáculo da auto-força escalar são estudadas. Em particular, o método de regularização por decomposição em multipolos [1] é adotado. Para a obtenção dos modos do campo gerado pela carga, o método MST [2] é revisitado e utilizado. Tais modos são regularizados de duas maneiras: (i) utilizando a técnica de regularização analítica por expansão pós-Newtoniana [3]; (ii) pela subtração de parâmetros de regularização já conhecidos na literatura, porém derivados nesse trabalho de uma maneira alternativa, através da utilização do método WKB para resolver as equações de campo no espaço de frequências.

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Abstract

by Alexandre Sampaio da Cruz

One of the main experimental challenges regarding the detection of gravitational waves lies on the fact that one is required to have a profound knowledge of these waves prior to their detection. Such knowledge depends, of course, on the understating of the dynamics of the physical system that emit these waves. One class of such systems that draws particular interest at the moment are the so-called *Extreme Mass Ratio Inspirals* (EMRIs), binary systems in which one of the bodies is much more massive than the other.

The main approach for studying the physics of EMRIs is gravitational self-force theory. In this approach, one considers the geometry of the spacetime to be, in first approximation, determined only by the influence of the more massive body. Meaning that the less massive body behaves like a test particle that moves on a geodesic of this background spacetime. To understand the inspiralling, one needs to go beyond this first approximation and consider the perturbation that the less massive body causes in the background spacetime. Under the perspective of self-force theory, the deviation of the motion of this body from a geodesic of the background spacetime is seen as a force, whose origin is the gravitational perturbation sourced by the body itself. This type of problem belongs to a broader class of problems called self-force problems, which require similar mathematical treatment due to their singular character.

In this work, various techniques associated with the calculation of the scalar self-force are studied. In particular, the mode-sum regularization method [1] is adopted. For the obtention of the field modes, the MST method [2] is revisited and applied. Such modes are regularized in two ways: (i) with the use of the post-Newtonian regularization technique [3] (ii) by the subtraction of regularization parameters that are known in the literature, but are obtained in an alternative way, from WKB solutions to the field equations in the frequency domain.

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Chapter 1

INTRODUCTION

1.1 THE TWO-BODY PROBLEM IN GENERAL RELATIVITY

The two-body problem is undoubtedly one of the most recurring and important problems in physics. Its description is rather simple: two isolated point-like bodies interact with each other by a given force and one is interested in the prediction of the motion of such bodies. Despite the name, two-body problems are actually a class of problems with varying complexity depending on the nature of the prescribed interaction. The equations describing the dynamics of this type of system are in principle coupled differential equations for the positions of the two particles. In the context of Newtonian physics, the decoupling of these equations can be achieved by reformulating the problem in terms of the motion of the center of mass of the system and the relative motion of the particles [7]. With this reformulation, the two-body problem can be solved analytically in closed-form for a large variety of interaction potentials between the particles.

The most remarkable two-body problem in classical physics is the Kepler problem, in which the interaction force is given by Newton's Law of Universal Gravitation. Solutions to the Kepler problem, called Keplerian orbits, offer a good description of the movement of the majority of the bodies in the solar system around the Sun[1]. One famous exception to that is the precession of the orbit of Mercury, which can only be described by considering corrections from the theory of General Relativity (GR). In fact, the accurate description of this motion was one of the first experimental results that paved the way for the establishment of GR.

[1] Though a more accurate description of the movement of any given body should take into consideration corrections coming from the gravitational interaction between that body and the remaining bodies of the solar system other than the Sun.

In comparison to its Newtonian counterpart, the description of a system of two massive bodies in General Relativity poses a much greater challenge in both its physical and mathematical aspects. First of all, the non-linearity of Einstein's Field Equations makes it unfeasible to find closed-form solutions for the majority of systems in GR, with two-body systems being no different except for some very specific scenarios. Second is the fact that a two-body system produces a time-dependent gravitational field, which implies the irradiation of gravitational waves (GW). As consequence of the loss of energy and angular momentum due to the emission of GW, periodic orbits do not exist in GR and the bound orbits of binary systems eventually inspiral into each other[2].

A variety of tools have been developed for the the study of binary systems in General Relativity. Their efficiency is determined by the characteristic parameters of the system, such as the ratio between the masses of the two bodies and their spatial separation. Because the bodies in such systems eventually inspiral into each other, the most suitable method for describing the dynamics of one particular system varies along its evolution. The four most prominent existing tools for the study of binary systems in GR are: post-Newtonian (pN) expansions, numerical relativity, self-force/black perturbation theory and effective one-body theory. The regimes in which each one of these methods excel are depicted in Figure 1.1. A brief overview of each one this methods shall now be given.

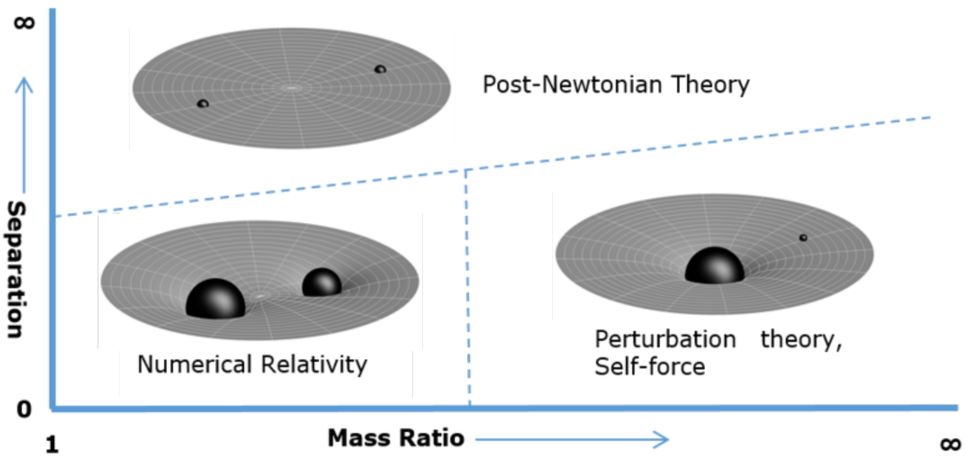

FIGURE 1.1: Schematic representation of the regimes of applicability of the different approaches to the two-body problem in terms of the spatial separation between the two the bodies and their mass ratio. As commented in the main text, the effective-one body approach draws information from the other three approach and hence is in principle useful for the entire parameter space. Source: Wikipedia [4]

[2]Special configurations of the two body-problem can generate stationary gravitational fields and, consequently, no inspiralling. However, that is not the general picture.

The post-Newtonian method relies on perturbatively solving Einstein's field equations by expanding relevant quantities (e.g. the metric and energy-momentum tensor) about Newtonian physics, that is for small velocities of the bodies and weak gravitational fields. This method excels at describing the dynamics of binary systems when the separation between the bodies is large, so that the spacetime geometry at their positions is efficiently described in terms of perturbations to the Minkowski geometry and the typical velocities of the bodies are small. It was via the application of this method that the aforementioned relativistic corrections to the motion of Mercury were first calculated. As the two bodies in a binary system inspiral into each other, the convergence of pN series becomes slower as the separation between the bodies shrinks and the spacetime geometry around them starts to deviate greatly from Minkowski. At this stage, one could resort to the tools of numerical relativity to solve Einstein's equations. Despite being computationally expensive, the numerical approach has proven to be the most reliable method for studying the small separation and comparable mass region of the parameter space. However, when the two bodies possess very disparate masses, the weak-field approximation becomes inaccurate very early in the inspiral, meaning that the numerical calculation would need to be carried out for a very large amount of orbital cycles. Fortunately, it is exactly in this regime that another perturbative method becomes valuable tool.

For two particles having disparate values of mass, the ratio $\mu \equiv m/M$ between the masses of the less massive body m and of the more massive body M, can be used as a small parameter for the perturbatively solving Einstein's field equations. At order-zero in this perturbation theory, solutions to Einstein's field equations are given by a metric tensor that is associated with presence of only the larger mass M. At this order, the less massive body of mass m travels along a geodesic on this background spacetime. First-order deviations from this geodesic motion are seen as a force, which causes the inspiralling and whose origin is the presence of the mass m itself. This force is called a self-force and this approach shall be the one studied throughout this work.

Another tool that exists for the study of binary systems in General Relativity is the Effective One-Body approach [8]. This approach draws information from all the other methods discussed in order to map the two-body problem onto an effective one-body problem. Its applicability is in principle the entirety of the parameter space, as it uses information from all the other methods.

1.2 EXTREME MASS RATIO INSPIRALS AND THE SELF-FORCE PROBLEM

The direct detection of Gravitational Waves (GW) by the *Laser Interferometer Gravitational-Wave Observatory* (LIGO-Virgo) [9] in 2015 was one of the most celebrated results in XXI century physics. In addition to confirming one of the major predictions of General Relativity that had remained (directly) undetected for almost a century, the LIGO-Virgo detection is seen as a mark of the dawn of a new upcoming era in multi-messenger astronomy which promises to put to test our understanding of gravity and cosmology.

While the LIGO experiment is an Earth-based interferometer capable of detecting high frequency gravitational waves (10Hz to 10kHz). The upcoming *Laser Interferometer Space Antenna* (LISA) mission [10] is a space-based experiment which will be capable of detecting frequencies in the range of the milihertz. One of the sources of GW in LISA's frequency range are the so called *Extreme Mass Ratio Inspirals.* These consist of binary systems in which one of the bodies is much more massive than the other ($\mu \sim 10^{-5}$ or less). From an astrophysical perspective, this type of system is typically composed of a small neutron star or black hole orbiting a supermassive black-hole, like the ones that are present in the center of galaxies.

One challenging aspect of GW detection is the necessity to produce great models of the signals to be detected in order to separate them from a noisy background. The data analysis done in the LIGO experiment relies heavily on wave-form templates obtained from numerical relativity [11]. In the case of the upcoming LISA experiment, the picture remains the same, one must be equipped with precise templates of the GW signals in order to tell them apart from the background, except for the fact that the more effective method of studying the dynamics of EMRIs is black hole perturbation theory.

Huge progress in the study of EMRIs using the tools of black hole peturbation and self-force theory has been made in the last few years, much of this progress is greatly overviewed in L. Barack & A. Pound (2019) [12] and A. Pound & B. Wardell (2021) [13]. In black hole perturbation theory, as it was first developed, one attempts to solve Einstein's equations by expanding the exact metric tensor $\mathbf{g}_{\mu\nu}$ of the spacetime in a binary system as

$$\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + O(\varepsilon^2), \tag{1.1}$$

where $g_{\mu\nu}$ is the background metric associated with the more massive black hole, $h^{(1)}_{\mu\nu}$ is the first-order correction due to the presence of the small mass m and ε is the typical size of the components of $h^{(1)}_{\mu\nu}$ in some coordinate system. Even though point-particle approximations in General relativity are problematic, they hold in the context of linearized gravity [12], so that in the context of first-order black hole perturbation theory, one can make the point-particle approximation to the smaller mass m that sources the perturbation $h^{(1)}_{\mu\nu}$. A discussion about the generalization of this approach to a finite sized mass is found in [5].

One of the most challenging aspects of the self-force approach is its fundamentally singular nature. In the self-force picture, the trajectory of a particle of mass m deviates from a geodesic of the background spacetime due to the effect of $h^{(1)}_{\mu\nu}$. However, since this perturbation is sourced by the point-mass m itself, it is formally divergent at the position of the particle. Thus, this approach requires the development of regularization methods capable of extracting a finite result for the effect of the perturbation field $h^{(1)}_{\mu\nu}$ on the motion of the point particle.

The above described problem of extracting a physical result for the gravitational self-force experienced by the point mass m due to the linearized gravity field $h^{(1)}_{\mu\nu}$ belongs to a broader class of general self-force problems in curved spacetime that also includes scalar and electromagnetic counterparts. In this class of problems, one is interested in extracting the physical self-force experienced by the interaction of a charge travelling through curved spacetime with its own field, be it a scalar, electromagnetic or linearized gravity field. In all of those cases, the self-force is a formally divergent quantity that requires the adoption of regularization procedures. The main aspects of such procedures do not depend on the particular type of field considered. Therefore, it is common that they are first proposed and studied in the context of the more simple scalar self-force and later generalized to be applied to the other fields. For this reason, this work will focus on the study of the regularization of the scalar self-force. At some point, the specialization of the background spacetime geometry to the Schwarzschild geometry will also be made. Throughout this work the system of geometrized units $G = c = 1$ — where c is the speed of light in vacuum and G is the universal gravitational constant — is adopted.

Chapter 2

DYNAMICS OF SCALAR CHARGES IN BLACK HOLE SPACETIMES

The notion of a self-force first arises in the context of classical electrodynamics. It is a consequence of Maxwell's equations that accelerated charges emit electromagnetic (EM) radiation. These waves carry energy away from the particle and therefore decelerate it, in a phenomenon called radiation reaction. In the absence of any other external field, it is clear that this deceleration can only be explained as an interaction of the particle with its own field. The force involved in this interaction is called the electromagnetic self-force. The non-relativistic expression for the EM self-force was first derived by Lorentz. This result was later generalized to the context of special relativity by the efforts of Abraham and Dirac [14], becoming what is now called the Abraham-Lorentz-Dirac force. Further generalization to the curved spacetimes of General Relativity (GR) was achieved by DeWitt & Brehme [15] e Hobbs [16].

Motivated by the necessity to model the evolution of EMRIs, Mino, Sasaki and Tanaka [17] and Quinn & Wald [18] extended the results concerning the EM self-force in curved spacetime to the description of a gravitational self-force experienced by a point-mass. Equivalent results for the case a scalar charge in curved spacetime were also obtained by Quinn [19]. In all cases, one of the main challenges for the computation of the respective self-force lies on the fact that the field (be it scalar, EM or a metric perturbation field) diverges at the position of the particle. Thus, the obtention of physically meaningful results rely on the development of regularization methods capable of curing the self-force from its intrinsically singular nature.

In the following sections, the foundations of the so-called Green's function method for the calculation of the scalar self-force are introduced based on the review by E. Poisson et al [5], where a deep discussion of the electromagnetic and gravitational counterparts can also be found. Even though this method shall not be directly employed for the self-force calculations in this work, valuable notions that will subsidize the discussions in the upcoming chapters will be gained by introducing this framework. At the end of the chapter, the mode-sum method that will be applied for the actual self-force calculations performed in this work is introduced.

2.1 THE EQUATIONS OF MOTION FOR A SCALAR POINT-CHARGE IN SCHWARZSCHILD SPACETIME

Let a particle of scalar charge q travel along a worldline γ in a curved spacetime in which the metric tensor is $g_{\mu\nu}$. The particle creates a massless scalar field, $\Phi(x)$, which satisfies the Klein-Gordon (KG) equation:

$$\Box\Phi(x) = -4\pi\mu(x), \tag{2.1}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the d'Alambertian operator. The source for the field equation (2.1) is given by the charge density $\mu(x)$ of the point particle travelling along γ,

$$\mu(x) = q\int_\gamma \delta^4(x, z(\tau))d\tau. \tag{2.2}$$

Here $z(\tau)$ are the spacetime points along γ parameterized by the proper time τ and $\delta^4(x, x')$ stands for the invariant Dirac's distribution in four-dimensional spacetime,

$$\delta^4(x, x') = \frac{\delta^4(x - x')}{\sqrt{-g}}, \tag{2.3}$$

where $\delta^4(x - x') = \delta(x^0 - x'^0)\delta(x^1 - x'^1)\delta(x^2 - x'^2)\delta(x^3 - x'^3)$ is the "coordinate" four-dimensional Dirac distribution and g is the metric determinant at point x.

A solution to the Klein-Gordon equation can be expressed in terms of a Green's function $G(x, x')$ as

$$\Phi(x) = \int d^4x' G(x, x')\mu(x')\sqrt{-g'} = q\int_\gamma d\tau G(x, z(\tau)), \tag{2.4}$$

where the first integral in the above equation is taken over the entire spacetime and $G(x, x')$ is a solution to Green's equation,

$$\Box G(x, x') = -4\pi\delta^4(x, x'). \tag{2.5}$$

Being a second order linear differential equation, this equation admits a set of two linearly independent solutions, which can be given in the form of retarded and advanced solutions. The retarded Green's function, $G^{ret}(x, x')$, is defined as a solution to Green's equation (2.5) that vanishes outside the causal past of the field point x. Similarly, the advanced solution, $G^{adv}(x, x')$, is defined as a solution that vanishes outside the causal future of x. The physical field is, of course, one calculated from equation (2.4) with choice of the retarded Green's function.

In flat spacetime, the retarded and advanced solutions to Green's equation are non-vanishing only on the past and future light cones of x, respectively. In curved spacetime, however, the support of $G^{ret}(x, x')$ and $G^{adv}(x, x')$ may also include the interior of these light cones. This is due to the fact that waves travelling through curved spacetime may develop *tails* [20], that propagate with velocities less than c. For spacetimes in which wave propagation exhibits this characteristic, the retarded field at a point x generated by a source particle depends on knowledge about the entire past history of the source, as illustrated in Figure (2.1).

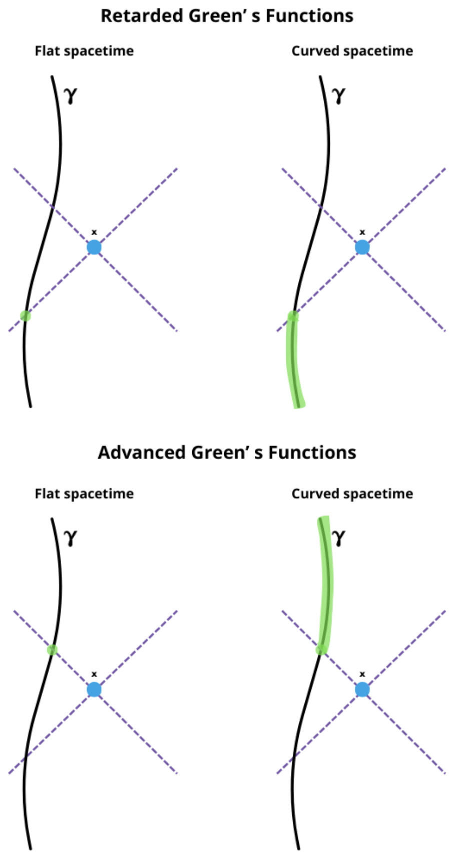

FIGURE 2.1: Schematic representation of the support of the retarded and advanced Green's functions in flat and curved spacetime. The support of the curved spacetime Green's functions includes the interior the light cones of the field point x.

Let the points $z(\tau)$ along the worldline be represented by coordinates $z^\mu(\tau)$ and let $u^\mu(\tau) \equiv dz^\mu(\tau)/d\tau$ be the four-velocity of the point source, the equations governing its motion can be written as

$$m(\tau)\frac{Du^\mu}{d\tau} = q(g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu \Phi(z), \tag{2.6}$$

where $D/d\tau$ stands for the derivative with respect to τ along γ, $m(\tau)$ is the mass of the particle and $\nabla_\nu \Phi(z)$ is the gradient of the field generated by the particle evaluated at its own current position z. Since the particle irradiates monopole waves, it experiences a loss of mass over time, which is governed by $dm/d\tau = -qu^\mu \nabla_\mu \Phi(z)$. Preceding any form of regularization, the scalar self-force on the particle at a point z on the worldline is formally defined from the right-hand side of equation (2.6) as

$$F_\alpha(z) \equiv \lim_{x\to z} q(\delta_\alpha^\beta + u_\alpha u^\beta)\nabla_\beta \Phi(x). \tag{2.7}$$

It is clear though, that this definition needs to be supplemented with some form of regularization procedure, since $\Phi(z)$ is a divergent quantity. Throughout this work, the self-force prior to any regularization in equation (2.7) shall be referred to as the full self-force[1]. In terms of the retarded Green's function, this full self-force (2.7) can be expressed as

$$F_\alpha = q^2(\delta_\alpha^\beta + u_\alpha u^\beta) \lim_{\substack{x \to z(0) \\ \varepsilon \to 0_+}} \nabla_\beta \int_{-\infty}^{\varepsilon} d\tau G(x, z(\tau)), \tag{2.8}$$

where the evaluation point was taken to be $x = z(\tau = 0)$ and the upper limit of this integral is chosen so that the current position of the particle lies inside the region in which the Green's function is being integrated.

2.2 THE DETWEILER-WHITING DECOMPOSITION

From now on, the point-particle fields generated at a point x with the choice of the Green's functions $G^{ret}(x, x')$ and $G^{adv}(x, x')$ shall be denoted $\Phi^{ret}(x)$ and $\Phi^{adv}(x)$, respectively. The term "self-field" and the notations $\Phi^{ret}(z)$ and $\Phi^{adv}(z)$ shall refer to the evaluation of these retarded and advanced fields at the position of the source. From the properties of the Green's functions introduced, one can see that the radiation zone behaviour of $\Phi^{ret}(x)$ and $\Phi^{adv}(x)$ is, respectively, that of *outgoing* and *incoming* waves at infinity. Other solutions to Green's equation can be proposed as linear combinations of $G^{ret}(x, x')$ and $G^{adv}(x, x')$. In particular, a solution

$$G^s(x, x') \equiv \frac{1}{2}\Big(G^{ret}(x, x') + G^{adv}(x, x')\Big), \tag{2.9}$$

is defined to contain equal amounts of *incoming* and *outgoing* radiation at infinity. This Green's function does not distinguish between past and future and the field $\Phi^s(x)$ calculated from it does not affect the motion of the particle [14]. Thus, no self-force. While not producing any effect on the particle, the self-field $\Phi^s(z)$ is just as singular as $\Phi^{ret}(z)$ and $\Phi^{adv}(z)$, as it is a solution to the same field equation with the point charge source (2.1). Therefore, subtracting it from the retarded field has the effect of removing the singular behaviour with no effect on the actual motion of the particle. Mathematically, a regularized self-field, $\Phi^r(z)$, could be defined as

$$\Phi^r(z) \equiv \Phi^{ret}(z) - \Phi^s(z). \tag{2.10}$$

[1] This name is taken from Hikida et al [3].

Or, at the level of the Green's function, one could define

$$G^r(x,x') \equiv G^{ret}(x,x') - G^s(x,x'). \tag{2.11}$$

and calculate the field associated with $G^r(x,x')$. It follows from this definition that the two-point function $G^r(x,x')$ is a solution to homogeneous version of Green's equation (2.5) which ensures that the self-field $\Phi^r(z)$ is regular.

While the above definitions allow for the introduction of a regular self-field, cured from the singular character of $\Phi^{ret}(z)$, they may introduce another problem. The symmetric Green's function $G^s(x,x')$, as defined in equation (2.9), is non-vanishing on both the causal past and future of point x. This non-causal support is inherited by $G^r(x,x')$, as defined in (2.11). This is not a problem for the evaluation of $\Phi^r(z)$ if the supports of $G^{ret}(x,x')$ and $G^{adv}(x,x')$ are restricted to the surface of the light cones, since both retarded and advanced points (i.e. the intersections of the null cones with the worldline) are mapped to the position z of the particle in the $x \to z$ limit (See Figure 2.2). Thus, in this case, the self-fields $\Phi^s(z)$ and $\Phi^r(z)$ are indeed causal. If, however, $G^{ret}(x,x')$ and $G^{adv}(x,x')$ are non-vanishing inside the light cones, the fields $\Phi^s(x)$ and, consequently, $\Phi^r(x)$ possess a dependence on the chronological future of the particle that persists at the self-field limit (i.e. when evaluating $\Phi^s(z)$ and $\Phi^r(z)$). Hence, the self-field $\Phi^r(z)$, if defined as in the equations above, is not generally causal in curved spacetime.

A solution to this problem was proposed by Detweiler & Whiting [21] by introducing a new field $\Phi^S(x)$ which shares the same desirable properties of $\Phi^s(x)$ but is devoid of its non-causal nature when evaluated at the position of the particle. Their construction for the Green's function $G^S(x,x')$, associated with a field $\Phi^S(x)$, relies on adding to $G^s(x,x')$ in equation (2.9), a function $H(x,x')$ that has property of canceling out its support inside the future light cone of point x. Namely, the Detweiler-Whiting Singular Green's function, $G^S(x,x')$, is defined as

$$G^S(x,x') \equiv \frac{1}{2}\Big(G^{ret}(x,x') + G^{adv}(x,x') - H(x,x')\Big). \tag{2.12}$$

The requirement that $G^S(x,x')$ is still a solution to Green's equation (2.5) implies that the function $H(x,x')$ must be a solution to the homogeneous version of that equation. Furthermore, to keep "no self-force" property of $\Phi^s(x)$, one must require that $H(x,x')$ is symmetric, meaning that it should also cancel the support of $G^s(x,x')$ inside the past light cone. Assuming that such a function exists, a regular two-point function can be defined as:

$$G^R(x,x') \equiv G^{ret}(x,x') - G^S(x,x') = \frac{1}{2}\Big(G^{ret}(x,x') - G^{adv}(x,x') + H(x,x')\Big). \tag{2.13}$$

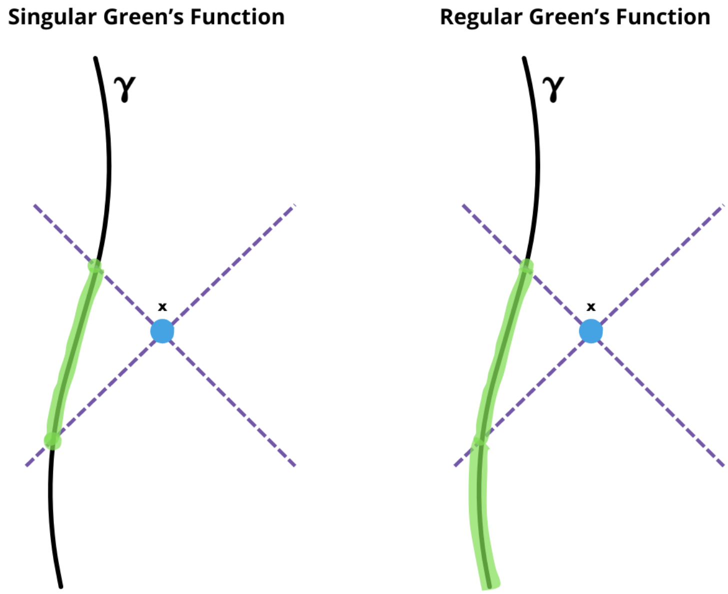

FIGURE 2.2: Scheme of the support of the Singular and Regular Green's functions

From this point on, the labels singular (S) and regular (R) shall refer to the Detweiler-Whiting quantities $G^S(x,x')$ and $G^R(x,x')$ and their associated fields. The supports of these Green's functions are illustrated in Figure 2.2. It follows from this construction that, in the limit $x \to z$, $\Phi^R(z)$ is both finite and causal. Unfortunately, direct calculation of $G^R(x,x')$ is not generally possible and the regularization procedure usually involves the obtention of the retarded and singular fields through different means.

2.3 THE LOCAL PICTURE AND THE SINGULAR FIELD

In this section, the foundations for the construction of "local expansions" for small spacetime separation between the arguments of the Green's functions defined in the last section is presented. The obtention of such expansions consists in a effective method for computing the singular self-field $\Phi^S(z)$, since this self-field is, by construction, devoid of any dependence on the chronological past and future of the particle. A complete derivation of such expansions requires the introduction of many aspects of the theory of bitensors. This topic is greatly reviewed in work by E. Poisson et al [5], where one can also find a detailed derivation of the expansions for the Green's functions and field quantities. These calculations shall not be repeated in this work. Instead, the important notions that will be necessary to subsidize the discussion in the upcoming chapters shall be outlined.

2.3.1 Foundations of a local expansion and the Hadamard Form

The construction of expansions for "small separation" in the context of General Relativity must be consistent with the curved nature of the spacetime. One could consider the length along a geodesic segment that connects two points x and x' as a natural measure of the separation between these points. This, however, may involve some ambiguity, since two points in curved spacetime are not generally linked by only one geodesic segment (in fact, such geodesic segment may not even exist). A normal convex neighborhood of a point x, denoted $\mathcal{N}(x)$, is defined as a region around this point in which every point $x' \in \mathcal{N}(x)$ is linked to x by a unique geodesic that lies inside this region. Then, for points satisfying $x \in \mathcal{N}(x')$ an expansion based on these unique geodesics can be constructed.

Let the unique geodesic β that passes through two points x and x' be described by coordinates $y^\mu(\lambda)$, where λ is a parameter along the curve. The Synge world function is defined as

$$\sigma(x, x') \equiv \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} g_{\mu\nu} t^\mu t^\nu d\lambda, \tag{2.14}$$

where $t^\mu \equiv dy^\mu/d\lambda$ is the vector tangent to β and $x' = y^\mu(\lambda_0)$ and $x = y^\mu(\lambda_1)$. If the parameter λ is taken to be the proper time τ for timelike t^μ or the proper distance s for spacelike t^μ, this function evaluates to half the squared geodesic length from x' to x along β. If t^μ is a null vector, then $\sigma(x, x') = 0$. Furthermore, the first derivatives of $\sigma(x, x')$ can be used to construct a curved spacetime analogue to the flat spacetime separation vector, $(x^\alpha - x^{\alpha'})$. These derivatives are bivectors, meaning that they behave as a vector with respect to operations in the tangent space of the point where the derivative was taken and as a scalar with respect to operations relative to the other point. Through variation of equation (2.14), one can obtain the following expressions for the first derivatives of $\sigma(x, x')$:

$$\sigma_\alpha \equiv \partial_\alpha \sigma = (\lambda_1 - \lambda_0) g_{\alpha\beta} t^\beta; \qquad \sigma_{\alpha'} \equiv \partial_{\alpha'} \sigma = -(\lambda_1 - \lambda_0) g_{\alpha'\beta'} t^{\beta'}. \tag{2.15}$$

where the primed and unprimed indices of σ_α and $\sigma_{\alpha'}$ indicate that they are elements of the tangent space of x and x', respectively. Closer inspection of equation (2.15) reveals that the object σ_α is a vector tangent to β at point x with norm $g_{\alpha\beta}\sigma^\alpha\sigma^\beta = 2\sigma$. Similarly, $\sigma_{\alpha'}$ is proportional to the reflected tangent to β at x' with norm 2σ. Hence, both of these bivectors carry information about the direction and magnitude of the separation between points x and x'. Expressions for tensors near the worldline can then be constructed by expanding these quantities in powers of $\sigma^{\alpha'}$.

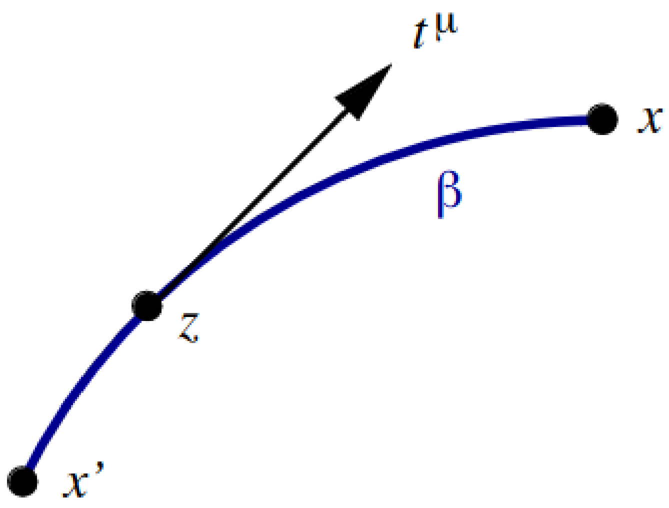

FIGURE 2.3: Schematic representation of the Synge's world function. Source: Poisson et al (2011) [5]

The support of the retarded Green's function introduced in the previous section together with the properties of $\sigma(x, x')$ motivates the introduction of the following ansatz for $G^{ret}(x, x')$ for points $x \in \mathcal{N}(x')$:

$$G^{ret}(x, x') \overset{\mathcal{N}}{=} U(x, x')\delta_-(\sigma) + V(x, x')\Theta_-(-\sigma). \tag{2.16}$$

The functions $U(x, x')$ and $V(x, x')$ are smooth biscalars and the symbol $\overset{\mathcal{N}}{=}$ indicates that the equality is only valid for $x \in \mathcal{N}(x')$. The quantities $\delta_-(\sigma)$ and $\Theta_-(-\sigma)$ are the past light-cone distributions in curved spacetime [5]. These are defined as following:

$$\delta_-(\sigma) \equiv \delta(\sigma)\theta_-(x, x'), \tag{2.17}$$

$$\Theta_-(-\sigma) \equiv \Theta(-\sigma)\theta_-(x, x'), \tag{2.18}$$

where $\theta_-(x, x')$ is a step-like function defined to equal 1 if x' is in the causal past of x and vanish otherwise and $\delta(x)$ and $\Theta(x)$ are the standard Dirac's Delta and Heaviside's Theta. The ansatz in equation (2.16) is called the Hadamard form [20] of the retarded Green's function. Similarly, the the advanced Green's function can be expressed in Hadamard form as

$$G^{adv}(x, x') \overset{\mathcal{N}}{=} U(x, x')\delta_+(\sigma) + V(x, x')\Theta_+(-\sigma). \tag{2.19}$$

The future light cone distributions, $\delta_+(\sigma)$ and $\Theta_+(-\sigma)$, are given by

$$\delta_+(\sigma) \equiv \delta(\sigma)\theta_+(x,x'), \tag{2.20}$$

$$\Theta_+(-\sigma) \equiv \Theta(-\sigma)\theta_+(x,x'), \tag{2.21}$$

with $\theta_+(x,x') = 1$ if x' is in the causal future of x and zero otherwise. Retarded and advanced solutions to Green's equation are shown to satisfy a reciprocity relation $G^{ret}(x,x') = G^{adv}(x',x)$ [5], which implies that $V(x,x')$ should be symmetric in its arguments.

Together with the definition of the singular Green's function (2.12), the ansatzes (2.16) and (2.19) imply the following Hadamard form for the Singular Green's Function:

$$G^S(x,x') \overset{\mathcal{N}}{=} \frac{1}{2}\big(U(x,x')\delta(\sigma) + V(x,x')\Theta(-\sigma) - H(x,x')\big) \tag{2.22}$$

Here the identities $\delta(\sigma) = \delta_+(\sigma) + \delta_-(\sigma)$ and $\Theta(-\sigma) = \Theta_+(-\sigma) + \Theta_-(-\sigma)$ have been used. Recalling the discussion prior to the definition of the Detweiler-Whiting singular Green's function, the quantity $H(x,x')$ was included to cancel the support of $G_S(x,x')$ on points inside the future and past light cones of x. One can see that, for points inside these light cones, the Hadamard form in the above equation reduces to $G_S(x,x') \overset{\mathcal{N}}{=} (V(x,x') - H(x,x'))/2$ (for $\sigma \neq 0$). This, along with the condition that $H(x,x')$ should be a symmetric homogeneous solution to the field equation implies that

$$H(x,x') \overset{\mathcal{N}}{=} V(x,x'), \tag{2.23}$$

provided that $V(x,x')$ is itself a homogeneous solution of the field equation. The Hadamard form of the singular Green's function then takes the form:

$$G^S(x,x') \overset{\mathcal{N}}{=} \frac{1}{2}\big(U(x,x')\delta(\sigma) - V(x,x')\Theta(\sigma)\big). \tag{2.24}$$

The support of the singular Green's function on the outside of the past and future light cones, i.e. for spacelike intervals, is made explicit by the presence of the $\Theta(\sigma)$ term.

What then remains to this construction of the Green's functions is the determination of the biscalars $U(x,x')$ and $V(x,x')$. Differential equations that determine these quantities can be obtained by substitution of either one of the ansatzes for the Green's functions into Green's equation (2.5). Using the distributional identities for the light cone distributions (see section 13.2 in [5]), the field equation for $G^{(ret/adv)}(x,x')$ in the region of validity of the Hadamard expressions becomes:

$$\Box G^{(ret/adv)}(x,x') \overset{\mathcal{N}}{=} -4\pi\delta_4\left(x,x'\right)U + \delta'_{\pm}(\sigma)\left\{2U_{,\alpha}\sigma^{\alpha} + \left(\sigma^{\alpha}{}_{\alpha} - 4\right)U\right\} + \\ + \delta_{\pm}(\sigma)\left\{-2V_{,\alpha}\sigma^{\alpha} + \left(2 - \sigma^{\alpha}_{\alpha}\right)V + \Box U\right\} + \Theta_{\pm}(-\sigma)\Box V = -4\pi\delta_4(x,x') \tag{2.25}$$

Where the arguments of $U(x,x')$ and $V(x,x')$ were omitted for clarity and $\sigma^{\alpha}{}_{\alpha} = \nabla^{\alpha}\sigma_{\alpha}$. By comparing the left and right-hand sides of this equation, one can see that it is satisfied if the terms containing $\delta_{\pm}, \delta'_{\pm}$ and $\theta_{\pm}$ vanish and if, at the limit $x \to x'$, the pre-factors to the $\delta^4(x,x')$ terms match. This last restriction implies that

$$\lim_{x\to x'} U(x,x') = 1. \tag{2.26}$$

The requirement that the pre-factor to the $\delta'_{\pm}$ term vanishes implies the following differential equation:

$$2U_{,\alpha}\sigma^{\alpha} + \left(\sigma^{\alpha}_{\alpha} - 4\right)U = 0, \qquad \text{for } x \text{ and } x' \text{ such that } \sigma(x,x') = 0. \tag{2.27}$$

This two restrictions are proven [5; 15] to be enough for determining that $U(x,x') = \sqrt{\Delta(x,x')}$, where $\Delta(x,x')$ is the Van Vleck biscalar [22], which is related to the focusing or divergence of the geodesics in the background spacetime (See [5; 15]). Next, by requiring that the remaining terms vanish, one can obtain the restrictions for the other biscalar, $V(x,x')$. Namely, the condition that the term containing $\Theta_{\pm}(-\sigma)$ vanishes implies that it satisfies the following differential equation:

$$\Box V(x,x') = 0, \tag{2.28}$$

while the remaining term, containing $\delta_{\pm}(\sigma)$, gives a restriction for $V(x,x')$ for $\sigma = 0$ (since the $\delta_{\pm}(\sigma)$ guarantees that this term always vanishes for timelike and spacelike curves) in the form of

$$-2V_{,\alpha}\sigma^{\alpha} + \left(2 - \sigma^{\alpha}_{\alpha}\right)V + \Box U = 0 \qquad \text{for } x \text{ and } x' \text{ such that } \sigma(x,x') = 0. \tag{2.29}$$

These equations, together with the requirement that $V(x,x')$ is smooth at $x \to x'$, determine this biscalar. Expansions for these biscalar functions can be obtained by expanding quantities in powers of derivatives of Synge's world function (2.15) [5].

2.3.2 Expressions for the Retarded and Singular Fields

Based on the Hadamard construction, formal expressions for the fields $\Phi^{ret}(x)$ and $\Phi^{S}(x)$ at a point x near the worldine shall now be written. To do so, let $z(\tau_{-})$ and

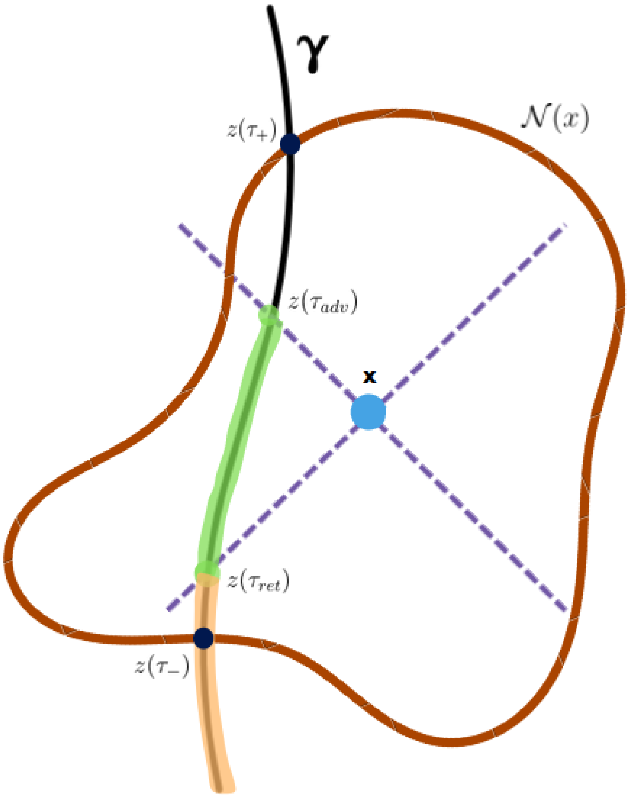

FIGURE 2.4: Depiction of the different intervals of integration under the assumption that x is "close enough" to the worldline.

$z(\tau_+)$ be defined as the intersection points between worldline and the boundaries of the maximum $\mathcal{N}(x)$ for a nearby field point x. A generic field $\Phi^{(X)}$ ($X = ret,\ adv,\ S$ or R) is obtained by the integration of equation (2.4) with the choice of the respective Green's function $G^{(X)}(x, x')$. Considering the region of validity of the Hadamard expressions, the integration along the worldline can be broken into three separate intervals (see Figure 2.4),

$$\Phi^{(X)}(x) = q\int_{-\infty}^{\tau_-} d\tau G^{(X)}(x, z(\tau)) + q\int_{\tau_-}^{\tau_+} d\tau G^{(X)}(x, z(\tau)) + q\int_{\tau_+}^{\infty} d\tau G^{(X)}(x, z(\tau)). \tag{2.30}$$

The middle integral is restricted to $x \in \mathcal{N}(x')$ and it can be performed by expressing the Green's function in Hadamard form. Next, the above expression shall be specialized for the retarded and singular fields. It will be convenient to define the retarded and advanced points, $z(\tau_{ret})$ and $z(\tau_{adv})$, as the intersection points of the worldline with the past and future light cones of x, respectively. It shall be assumed that the point x is close enough to γ, so that both $z(\tau_{ret})$ and $z(\tau_{adv})$ lie inside $\mathcal{N}(x)$, as depicted in Figure 2.4.

For the purpose of calculating retarded field, one can immediately see that the last integral in equation (2.30) vanishes, since both limits of integration lie on the causal future of x. Furthermore, the integrand of the middle integral is non-vanishing only in the causal past, meaning that its upper limit can be set to τ_{ret}. Hence, the specialization of (2.30) to the retarded field can be written as

$$\Phi^{ret}(x) = q\int_{-\infty}^{\tau_-} d\tau G^{ret}(x, z(\tau)) + q\int_{\tau_-}^{\tau_{ret}} d\tau G^{ret}(x, z(\tau)) \tag{2.31}$$

or, by expressing the second integral in terms of the Hadamard form, as

$$\Phi^{ret}(x) = q\int_{\tau_-}^{\tau_{ret}} d\tau U(x, z(\tau))\delta_-(\sigma) + q\int_{\tau_-}^{\tau_{ret}} d\tau V(x, z(\tau))\Theta_-(-\sigma) + \\ + \int_{-\infty}^{\tau_-} d\tau G^{ret}(x, z(\tau)). \tag{2.32}$$

Evaluation of the last term in this equation requires knowledge of the Green's function for points outside the $\mathcal{N}(x)$, which cannot be obtained from the Hadamard construction discussed. The first term in (2.32) can be integrated by changing variables to σ. As one passes through the point $z(\tau_{ret})$ traveling along γ, the geodesic β that links x to $z(\tau)$ goes from being null to being spacelike and σ increases, one can then write $d\tau = u^\mu\sigma_\mu d\sigma$. The integration of this term over σ yields:

$$\Phi^{ret}(x) = q\frac{U(x, z(\tau_{ret}))}{r_{ret}} + q\int_{\tau_-}^{\tau_{ret}} d\tau V(x, z(\tau)) + \int_{-\infty}^{\tau_-} d\tau G^{ret}(x, z(\tau)), \tag{2.33}$$

where $r_{ret} \equiv u^{\alpha'}\sigma_{\alpha'}|_{\tau=\tau_{ret}}$ has the interpretation of being the retarded spatial distance between x and the retarded point $z(\tau_{ret})$ in a frame that is co-moving with the particle [23]. Though the expansions for the biscalars $U(x, x')$ and $V(x, x')$ were not shown here, from the fact that $\lim_{x\to x'} U(x, x') = 1$, one can already see that the first term in equation is singular in the self-field limit — that is, as $r_{ret} \to 0$ $(x \to z)$.

Next, a similar expression for the singular field shall be written. Following from the fact that $G^S(x, x')$ has no support on the chronological future or past of x, the first and last integrals in (2.30) can be dropped and the limits of the middle integral can be set to τ_{ret} and τ_{adv},

$$\Phi^S(x) = \frac{1}{2}\bigg(\int_{\tau_{ret}}^{\tau_{adv}} d\tau U(x, z(\tau))\delta_-(\sigma) + \int_{\tau_{ret}}^{\tau_{adv}} d\tau U(x, z(\tau))\delta_+(\sigma) - \int_{\tau_{ret}}^{\tau_{adv}} d\tau V(x, z(\tau))\Theta(\sigma)\bigg), \tag{2.34}$$

where $\delta(\sigma)$ was re-expressed as $\delta_-(\sigma) + \delta_+(\sigma)$. The first term in the expression above contains exactly the same integral as in equation (2.32) and integration of the second term can be carried out in a similar manner. Except that the $\delta_+(\sigma)$ now yields quantities evaluated at the advanced point. The singular field can then be expressed as:

$$\Phi^S(x) = q\frac{U(x,z(\tau_{ret}))}{2r_{ret}} + q\frac{U(x,z(\tau_{adv}))}{2r_{adv}} - \frac{1}{2}q\int_{\tau_{ret}}^{\tau_{adv}} d\tau V(x,z(\tau)), \tag{2.35}$$

where $r_{adv} \equiv -u^{\alpha'}\sigma_{\alpha'}|_{\tau=\tau_{adv}}$ is the advanced distance between x and the advanced point $z(\tau_{adv})$ in the particle's frame [23]. As expected from its construction, the singular field is completely determined by the quantities $U(x,x')$ and $V(x,x')$. Thus, by supplying the equations with expansions for these biscalars, for a nearby point field point, $\Phi^S(x)$ can be completely determined as a series expansion for small spacetime separation.

Lastly, the expressions for the gradients of the retarded and singular fields shall be written. To calculate the gradient of biscalar functions evaluated at the retarded and advanced points, one must take into account the fact that a variation of the field point x induces a variation of $z(\tau_{ret})$ and $z(\tau_{adv})$ so that they are still connected to x by a null geodesic. Therefore, the gradient of a generic biscalar evaluated at the retarded point, $A(x,z(\tau_{ret}))$, is written as:

$$\partial_\alpha A(x,z(\tau_{ret})) = \partial_\alpha A(x,z(\tau_{ret})) + u^{\alpha'}\partial_{\alpha'}A(x,z(\tau_{ret}))\partial_\alpha\tau_{ret}, \tag{2.36}$$

where the x_α-derivative of $z(\tau_{ret})$ was re-expressed in terms of the four-velocity and of the derivative of the proper time parameter at the retarded point and the notation $\partial_{\alpha'}$ refers to the derivative with respect to the coordinates $z^{\alpha'}_{ret}$ of the retarded point. With this is mind, the gradient of the retarded field is found to be

$$\begin{aligned}\nabla_\alpha\Phi^{ret}(x) = &-\frac{q}{r^2_{ret}}U(x,z(\tau_{ret}))\partial_\alpha r_{ret} + \frac{q}{r_{ret}}\partial_\alpha U(x,z(\tau_{ret})) + \frac{q}{r_{ret}}\partial_{\alpha'}U(x,x')u^{\alpha'}\partial_\alpha\tau_{ret}\\ &+ qV(x,x')\partial_\alpha\tau_{ret} + \nabla_\alpha\Phi^{tail}(x),\end{aligned} \tag{2.37}$$

$$\Phi^{tail}(x) = q\int_{\tau_-}^{\tau_{ret}} d\tau V(x,z(\tau)) + \int_{-\infty}^{\tau_-} d\tau G^{ret}(x,z(\tau)). \tag{2.38}$$

One can see that the tail term given above, $\Phi^{tail}(x)$, contains contributions from both the particle *recent* and *distant* past. The full self-force (i.e. prior to any regularization) is formally given by substitution of this expression into equation (2.7). Similarly, the gradient of the singular field (2.35) is found to be:

$$\begin{aligned}\nabla_\alpha\Phi^S(x) = &-\frac{q}{2r^2_{ret}}U(x,z_{ret})\partial_\alpha r_{ret} - \frac{q}{2r^2_{adv}}U(x,z_{adv})\partial_\alpha r_{adv} + \frac{q}{2r}\partial_\alpha U(x,z_{ret}) +\\ &+ \frac{q}{2r_{ret}}\partial_{\alpha'}U(x,z_{ret})u^{\alpha'}\partial_\alpha\tau_{ret} + \frac{q}{2r_{adv}}\partial_\alpha U(x,z_{adv}) + \frac{q}{2r_{adv}}\partial_{\alpha''}U(x,z_{adv})u^{\alpha''}\partial_\alpha\tau_{adv}\\ &+ \frac{1}{2}qV(x,z_{ret})\partial_\alpha\tau_{ret} - \frac{1}{2}qV(x,z_{adv})\partial_\alpha\tau_{adv} - \frac{1}{2}\int_{\tau_{ret}}^{\tau_{adv}} \nabla_\alpha V(x,z(\tau)).\end{aligned} \tag{2.39}$$

Here $\partial_{\alpha''}$ stands for the derivative with respect to the coordinates of the advanced point and $u^{\alpha''}$ is the four-velocity evaluated at this point. To obtain this expression, quantities referring to advanced point were treated in analogous fashion to (2.36). The singular part of the self-force F^S_α (i.e. the self-force due to Φ^S), which is formally given by the substitution of this gradient into equation (2.7), of course has no tail term. The regularized self-force at some point z along the particle's trajectory is given by:

$$F^R_\alpha \equiv F_\alpha - F^S_\alpha = q(\delta^\beta_\alpha + u_\alpha u^\beta) \lim_{x\to z} \big(\nabla_\alpha \Phi^{ret}(x) - \nabla_\alpha \Phi^S(x)\big), \tag{2.40}$$

Note that the definitions of F_α and F^S_α only hold in the formal sense, since both field gradients diverge at the position of the particle. To obtain a finite result for F^R_α, one must first perform the subtraction between the two field gradients at some point a x and then take the $x \to z$ limit. This can be achieved by introduce expansions for the advanced and retarded quantities in equation (2.37) and (2.39), subtracting them and taking the $x \to z$ limit at the end. All the singular terms, of course, vanish. However, a complete determination of F^R_α still relies on being able to evaluate the tail term in equation (2.38).

2.4 THE MODE-SUM METHOD

When supplemented with small distance expansions, the Detweiler-Whitting [21] decomposition presented consists in a powerful scheme for tackling the regularization of the self-force. However, the obtention of the regularized self-force using this framework still relies on the integration of the tail term in equation (2.38), for which one needs information about the distant past support of the retarded Green's function. Since an exact analytical calculation of the retarded Green's function is not generally possible, additional methods must be introduced to evaluate the tail contribution to the self-force. One method for computing this contribution is the method of matched expansions. In this method, one computes the "quasilocal" contribution to self-force, coming from the first term in equation (2.38), by integrating the expansions of $V(x, x')$. The difficult part then becomes the evaluation of the "distant past" integral in the second term of that same equation. Obtention of expressions for the Green's function in the "distant past" regime was proven possible by Casals et al [24] for the case of the Nairai spacetime by adopting of a decomposition into quasi-normal modes. Both "quasilocal" and "distant past" expansions for the Green's function are shown to match in a common region of validity. Another method, which shall be the one adopted in this work, is the mode-sum method, introduced by Barack and Ori [1]. This method relies on the decomposition of both retarded and singular self-fields into a basis of spherical harmonics to perform the regularization in a mode-by-mode fashion. Each ℓ-mode of these self-fields is proven to be finite and their singular nature is only expressed as the divergence of the sum over all the ℓ-modes. Therefore, by regularizing the-ℓ modes before performing their summation, a finite result is obtained.

2.4.1 Mode-sum Regularization

From this point on, it will be convenient to restrict the discussion to the Schwarzschild spacetime on the exterior of a black hole of mass M, described by the line element

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2.41}$$

Where $f(r) = 1 - 2M/r$ and $\{t, r, \theta, \phi\}$ are the standard Schwarzschild coordinates. The spherically symmetric nature of this spacetime induces the decomposition of the retarded and singular fields into a basis of spherical harmonics,

$$\Phi^{(ret/S)}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Phi^{(ret/S)}_{\ell m}(r,t) Y_{\ell m}(\theta,\phi). \tag{2.42}$$

Following the discussion in Barack and Ori [1], the ℓ-modes of these fields, denoted $\Phi^{(ret/S)}_{\ell}(x) = \sum_{m=-\ell}^{\ell} \Phi^{(ret/S)}_{\ell m}(r,t) Y_{\ell m}(\theta,\phi)$, are finite at the $x \to z$ limit and the singular nature of the self-fields only shows up when one sums over ℓ. Thus, if one obtains the ℓ modes of both retarded and singular fields, the regularization can be performed at the level of the ℓ-modes. More specifically, the regular self-field is calculated in the mode sum regularization scheme as

$$\Phi^R(z) = \sum_{\ell=0}^{\infty} \left[\Phi^{ret}_{\ell}(z) - \Phi^S_{\ell}(z)\right]. \tag{2.43}$$

Or, alternatively, one can perform the regularization at the level of the modes of the self-force,

$$F^R_{\alpha} = \sum_{\ell=0}^{\infty} \left(F_{\alpha,\ell} - F^S_{\alpha,\ell}\right), \tag{2.44}$$

where $F_{\alpha,\ell}$ and $F^S_{\alpha,\ell}$ are the ℓ-modes of the F_α and $F^S_{\alpha,\ell}$, respectively. It is important to note that all of these modes can be directly evaluated at position of the particle, but the subtraction must be performed before the ℓ-sum to achieve its convergence.

Since the singular self-force (and self-field) can be completely determined from the "local picture" calculations outlined in Section 2.3 (supplied with expansions for the biscalars $U(x,x')$ and $V(x,x')$), computation of its ℓ modes can be achieved by considering multipole expansions of the results obtained in that manner. The ℓ-modes of the singular part of the self-force in Schwarzschild were shown [1; 25; 26] to have the following general structure:

$$F^S_{\alpha,\ell} = \pm A_\alpha L + B_\alpha + D_{\alpha,\ell}. \tag{2.45}$$

Where $L \equiv \ell + 1/2$ and the quantities A_α, B_α and $D_{\alpha,\ell}$ are called the regularization parameters. The parameters A_α and B_α depend only on the particle's trajectory and the $\pm$ carried by the A_α term comes from a discontinuity when evaluating the field gradient at the position of the particle [25]. Besides depending on the trajectory, $D_{\alpha,\ell}$ is at most $O(\ell^{-2})$ and satisfies $\sum_{\ell=0}^{\infty} D_{\alpha,\ell} = 0$. Therefore, it does not contribute to F^S_α. However, the inclusion of higher order contributions coming from $D_{\alpha,\ell}$ was shown to speed the numerical convergence of the ℓ sum dramatically [21]. The general form of equation (2.45) also holds for the EM and gravitational self-force [26], though the parameters may depend on the spin of the field in question.

2.4.2 Decoupled Field Equations

The task that remains unaddressed is the determination of the ℓ-modes of the retarded field. To obtain these, one can substitute (2.42) into the Klein-Gordon equation and obtain a partial differential equation for the modes $\Phi^{ret}_{\ell m}(r,t)$. Alternatively, a further decomposition into Fourier harmonics,

$$\Phi^{ret}(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} e^{-i\omega t} \Phi^{ret}_{\ell m\omega}(r) Y_{\ell m}(\theta,\phi), \tag{2.46}$$

can be taken in order to obtain an ordinary differential equation for the $\Phi^{ret}_{\ell m\omega}(r)$ modes. A general formalism for studying scalar, vector and tensor perturbations in Kerr spacetime exists in the form of the Teukolsky formalism [27]. Here, it is worth briefly presenting this formalism, as methods developed in the context of general perturbations in Kerr will be used to obtain the $\Phi^{ret}_{\ell m\omega}(r)$ modes.

In its most general form, the Teukolsky equation [27] is a master equation for Newman-Penrose [28] scalars in Kerr spacetime. The specialization of this equation to Schwarzschild spacetime reads

$$\frac{r^4}{\Delta}\frac{\partial^2 \Psi_s}{\partial t^2} - \left[\frac{1}{\sin^2\theta}\right]\frac{\partial^2\Psi_s}{\partial\varphi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Psi_s}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi_s}{\partial\theta}\right) + \\ - 2is\frac{\cos\theta}{\sin^2\theta}\frac{\partial\Psi_s}{\partial\varphi} - 2s\left[\frac{r^2 M}{\Delta} - r\right]\frac{\partial\Psi_s}{\partial t} + \left[s^2\cot^2\theta - s\right]\Psi_s = 4\pi r^2 T_s, \tag{2.47}$$

where $\Delta \equiv r^2 f(r)$. The quantities Ψ_s and T_s are the Teukolsky master variable and a master source term, which have different meaning depending on the value of the spin-weight parameter s. The Klein-Gordon equation (2.1) in Schwarzschild is recovered from the $s = 0$ case of this master equation. For $s = \pm 1$ and $s = \pm 2$, one obtains the equations for NP scalars associated with independent components of the electromagnetic and Weyl tensors, respectively. The exact expressions for Ψ_s and T_s arising from the NP formulation of Maxwell's ($s = \pm 1$) and linearized gravity equations ($s = \pm 2$) are found in Teukolsky's orginal work [27]. The Schwarzschild specialized Teukolsky master equation (2.47) is a decoupled second-order partial differential equation that admits a separation of variables of the form

$$\Psi_s = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \; {}_sR_{\ell m\omega}(r) \; {}_sY_{\ell m}(\theta, \phi), \tag{2.48}$$

$$4\pi r^2 T_s = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \; {}_sT_{\ell m\omega}(r) \; {}_sY_{\ell m}(\theta, \phi), \tag{2.49}$$

where ${}_sY_{\ell m}(\theta, \phi)$ are the spin-weighted spherical harmonics and the radial functions ${}_sR_{\ell m\omega}(r)$ satisfy the Teukolsky radial equation in Schwarzschild spacetime:

$$\Delta^{-s} \frac{d}{dr}\left(\Delta^{s+1} \frac{dR_{\ell m\omega}}{dr}\right) + \left(\frac{r^2\omega^2}{f(r)} - \frac{2is\omega(r - 3M)}{f(r)} - \ell(\ell+1)\right) \; {}_sR_{\ell m\omega}(r) = \; {}_sT_{\ell m\omega}(r). \tag{2.50}$$

The modes $\Phi^{(ret)}_{\ell m\omega}(r)$ of the retarded scalar field in equation (2.46) are then given by solutions of the equation above with $s = 0$, where the right-hand side is given by ${}_0T_{\ell m\omega} = -\delta(r - r')/r^2$. It is worth mentioning that the in the axially symmetric Kerr spacetime, a generalization of the spherical harmonic basis of angular functions is required.

To obtain solutions to the Teukolsky radial equation (2.50) the Green's function method shall be adopted. The causal field generated by point source must have the properties of being purely ingoing at the event horizon and purely outgoing at infinity. The study of the asymptotic behaviour of the radial equation (2.50), carried by Teukolsky [27], reveals the existence of homogeneous solutions with the following properties:

$${}_sR^{in}_{\ell m\omega}(r) \sim \; {}_sB^{trans}_{\ell m\omega} \Delta^{-s} e^{-i\omega r_\star}, \qquad r \to 2M; \tag{2.51}$$

$${}_sR^{up}_{\ell m\omega}(r) \sim \; {}_sC^{trans}_{\ell m\omega} r^{-2s-1} e^{i\omega r_\star}, \qquad r \to \infty. \tag{2.52}$$

Where $B^{trans}_{l\omega s}$ and $C^{trans}_{l\omega s}$ are transmission coefficients (which are constant in r). The coordinate

$$r_* \equiv r + r_s \log\left(\frac{r - r_s}{r_s}\right), \tag{2.53}$$

where $r_s = 2M$ is the Schwarzschild radius, is called a *tortoise coordinate* and has the effect of placing the horizon at $r_* = -\infty$. When combined with the Fourier mode $e^{-i\omega t}$, ${}_sR^{in}_{\ell m\omega}$ has the "ingoing at the horizon" property and shall be referred to as the *in* solution. Similarly, ${}_sR^{up}_{\ell m\omega}$ has the property of being purely outgoing at infinity and shall be called the *up* solution. Then, provided that one can obtain these homogeneous solutions, a Green's function to the Teukolsky radial equation (2.50) with the desired causal properties can be constructed from them. In Chapter 4, a method for obtaining these homogeneous solutions in series of special functions will be presented. For the scalar case, a Green's function for the radial equation that is compatible with the causal behaviour described is written as

$$g_{\ell m\omega}(r, r') = \frac{-1}{W^{in/up}_{\ell m\omega}}\left({}_0R^{in}_{\ell m\omega}(r)\, {}_0R^{up}_{\ell m\omega}(r')\Theta(r' - r) + {}_0R^{up}_{\ell m\omega}(r)\, {}_0R^{in}_{\ell m\omega}(r')\Theta(r - r')\right),$$

$$W^{in/up}_{\ell m\omega} \equiv \Delta \mathcal{W}({}_0R^{in}_{\ell m\omega}, {}_0R^{up}_{\ell m\omega}), \tag{2.54}$$

where $\mathcal{W}(R^{in}_{\ell m\omega}, R^{up}_{\ell m\omega})$ is the Wronskian of the two solutions,

$$\mathcal{W}({}_0R^{in}_{\ell m\omega}, {}_0R^{up}_{\ell m\omega}) \equiv {}_0R^{in}_{\ell m\omega}\frac{\mathrm{d}}{\mathrm{d}r}\left({}_0R^{up}_{\ell m\omega}\right) - {}_0R^{up}_{\ell m\omega}\frac{\mathrm{d}}{\mathrm{d}r}\left({}_0R^{in}_{\ell m\omega}\right), \tag{2.55}$$

and $W^{in/up}_{\ell m\omega}$ is constant in r.

In Schwarzschild spacetime, homogeneous solutions to the Teukolsky radial equation are related to solutions of the Regge-Wheeler (RW) equation [29],

$$\partial^2_{r_*}\, {}_sX_{\ell m\omega} + \left[\omega^2 - V_{\ell\omega s}(r)\right] {}_sX_{\ell m\omega} = 0, \tag{2.56}$$

$$V_{\ell s}(r) \equiv f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M(1 - s^2)}{r^3}\right), \tag{2.57}$$

by the Chandrasekhar transformation [30]. The main advantage of working with the Regge-Wheeler equation (2.56) is the fact that solutions to this equation with the desired ingoing and outgoing properties exist in the form of regular waves:

$${}_sX^{in}_{\ell m\omega}(r_*) \sim {}_sB^{trans}_{\ell m\omega} e^{-i\omega r^\star}, \qquad r_* \to -\infty; \tag{2.58}$$

$${}_sX^{up}_{\ell m\omega}(r_*) \sim {}_sC^{trans}_{\ell m\omega} e^{i\omega r^\star}, \qquad r_* \to \infty. \tag{2.59}$$

For the specific case of the scalar field, the Chandrasekhar transformation reduces to a simple division by r,

$$ {}_0R^h_{\ell m\omega}(r) = \frac{{}_0X^h_{\ell m\omega}(r)}{r}. \tag{2.60} $$

Here h labels a generic homogeneous solution of the respective equation. From now on, the label referring to $s = 0$ shall be dropped. Thus, the radial Green's function modes in equation (2.54) can be easily expressed in terms of solutions to the homogeneous Regge-Wheeler equation. The retarded Green's function to the Klein-Gordon equation in Schwarzschild spacetime is written explicitly as

$$ G^{ret}(x,x') = 2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} g_{\ell m\omega}(r,r')Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi'), \tag{2.61} $$

$$ = \frac{L}{\pi}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} g_{\ell m\omega}(r,r')P_\ell(\cos\gamma), \tag{2.62} $$

where the spherical harmonic addition theorem has been used and γ is defined by

$$ \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\big(\phi-\phi'\big), \tag{2.63} $$

which reduces to $\gamma = \phi - \phi'$ for $\theta = \theta' = \pi/2$. The radial Green's function modes $g_{\ell m\omega}(r,r')$ are to be obtained by solving either one of the radial equations and the retarded field is obtained integrating this expression over the worldline as in equation (2.4).

Chapter 3

WKB SOLUTIONS TO THE REGGE-WHEELER EQUATION

In this chapter, approximate solutions to the homogeneous Regge-Wheeler equation for large-ℓ are obtained with the use of the Wentzel–Kramers–Brillouin (WKB) method. This method consists in a powerful tool for obtaining approximate solutions to linear differential equations whose exact solutions exhibit rapid oscillations or rapid exponential growth or decay in a certain region. A detailed self-contained discussion of WKB and other related methods can be found in the book by C. Bender & S. Orszag [31]. Throughout the first section of this Chapter, their construction of the WKB solutions to a generic auxiliary equation is followed. When applying this method to the construction of asymptotic solutions to the Regge-Wheeler equation for large ℓ, I acknowledge the work notes shared by N. Zilberman [32] regarding the obtention of the leading-ℓ expressions for homogeneous solution satisfying the ingoing at the horizon boundary condition.

3.1 THE GENERAL WKB SOLUTIONS

Instead of directly tackling the Regge-Wheeler equation (2.56), WKB solutions to the following auxiliary equation shall be obtained:

$$\lambda^2 \partial_{r_*}^2 X = UX. \tag{3.1}$$

Here λ is a formal small parameter and X and U are functions of r_*. The WKB solutions to this equation are obtained by considering the following ansatz:

$$X \sim \exp\left\{\sum_{n=0}^{\infty} \delta^{n-1} S_n(r_*)\right\} \qquad \text{with } \delta \to 0, \tag{3.2}$$

where δ is a formal expansion parameter and $S_n(r_*)$ are functions to be determined by perturbativelly solving the differential equation. The substitution of this ansatz into equation (3.1) yields a differential equation for the functions $S_n(r_*)$:

$$\lambda^2 \left(\sum_{n=0}^{\infty} \delta^{n-1} S_n''(r_*) + \left(\sum_{n=0}^{\infty} \delta^{n-1} S_n'(r_*)\right)^2\right) = U. \tag{3.3}$$

Here $'$ stands for the derivative with respect to r_*. This equation shall be solved in a perturbative fashion for small λ and δ.

The dominant term on the left-hand side of equation (3.3) is a term proportional to λ^2/δ^2. The fact that the right-hand side is $O(1)$ implies that λ/δ must be $O(1)$. Then, without loss of generality, δ can be set to equal λ. The equation arising from the $O(\lambda^0)$ terms in (3.3) is

$$S_0'^2(r_*) = U(r_*), \tag{3.4}$$

and its integration yields

$$S_0(r_*) = \pm \int^{r_*} dr_*' \sqrt{U(r_*')} + C_0, \tag{3.5}$$

where U has been written explicitly as a function of r_* and C_0 is a constant of integration. The next-order terms in equation (3.3) are $O(\lambda)$ and collecting them yields a differential equation for the function $S_1(r_*)$:

$$S_0'' + 2S_1'S_0' = 0. \tag{3.6}$$

To solve this equation one needs to substitute the derivatives of $S_0(r_*)$, which are obtained by differentiating equation (3.5). The solution for the function $S_1(r_*)$ is found to be

$$S_1(r_*) = -\frac{1}{4}\log U(r_*) + C_1, \tag{3.7}$$

where C_1 is again a constant of integration. Similarly, the $O(\lambda^n)$ terms in equation (3.3) yield a differential equation for the function $S_n(r_*)$ that depend on the derivatives of functions up to $S_{n-1}(r_*)$. For a generic $n \geq 2$, this equation reads:

$$2S_0'(r_*)S_1'(r_*) + S_{n-1}''(r_*) + \sum_{j=1}^{n-1} S_j'(r_*)S_{n-j}'(r_*) = 0 \qquad (n \geq 2). \tag{3.8}$$

Expressions for the first few higher-order terms, taken from [31], are given:

$$S_2(r_*) = \pm \int^{r_*} \left[\frac{U''}{8U^{3/2}} - \frac{5(U')^2}{32U^{5/2}} \right] dr_*', \tag{3.9}$$

$$S_3(r_*) = -\frac{U''}{16U^2} + \frac{5U'^2}{64U^3}, \tag{3.10}$$

$$S_4(r_*) = \pm \int^{r_*} \left[\frac{d^4U/dr_*'^4}{32U^{5/2}} - \frac{7U'U'''}{32U^{7/2}} - \frac{19(U'')^2}{128U^{7/2}} + \frac{221U''(U')^2}{256U^{9/2}} - \frac{1,105(U')^4}{2,048U^{11/2}} \right] dr_*', \tag{3.11}$$

$$S_5(r_*) = -\frac{d^4U/dr_*'^4}{64U^3} + \frac{7U'U'''}{64U^4} + \frac{5(U'')^2}{64U^4} - \frac{113(U')^2U''}{256U^5} + \frac{565(U')^4}{2,048U^6}. \tag{3.12}$$

Here the arguments of $U(r_*')$ and its derivatives have been omitted for cleaner notation. One can check from (3.8) that all $S_n(r_*)$ functions of even n inherit the $\pm$ sign from $S_0(r_*)$. For the convergence of the WKB series, these functions must satisfy

$$|S_n(r_*)| \gg \lambda |S_{n+1}(r_*)|. \tag{3.13}$$

At this point, the connection between WKB solutions to the auxiliary equation (3.1) and WKB solutions to the homogeneous Regge-Wheeler equation (2.56) shall be made. First, the generic $U(r_*)$ is replaced by its counterpart in the Regge-Wheeler equation,

$$U(r_*) \to U_{\ell\omega s}(r) \equiv -\omega^2 + V_{\ell s}(r), \tag{3.14}$$

where r is taken to depend implicitly on r_*. Next, λ is set to 1 and the conditions for the convergence of WKB series are imposed on the functions $S_n(r_*)$ themselves. Namely, it is required that

$$\left| \frac{S_{n+1}(r)}{S_n(r)} \right| \ll 1. \tag{3.15}$$

If the above condition holds, two independent WKB solutions to the homogeneous Regge-Wheeler equation can be written as

$$ {}_sX_{\ell\omega}^{\mathrm{WKB},+}(r) \equiv \exp\left\{\sum_{n=0}^{\infty} S_n^{\ell\omega s}(r)\right\}, \tag{3.16} $$

$$ {}_sX_{\ell\omega}^{\mathrm{WKB},-}(r) \equiv \exp\left\{\sum_{n=0}^{\infty}(-1)^{n+1} S_n^{\ell\omega s}(r)\right\}, \tag{3.17} $$

where $S_n^{\ell\omega s}(r)$ is defined by replacing $U(r_*)$ by $U_{\ell\omega s}(r)$ in the expression for the function $S_n(r_*)$ with the choice of the positive sign in every $\pm$ sign. General WKB solutions the homogeneous Regge-Wheeler equation are then expressed as

$$ {}_sX_{\ell\omega}^{\mathrm{WKB}} \equiv C_{\ell\omega s}^{+}\ {}_sX_{\ell\omega}^{\mathrm{WKB},+}(r) + C_{\ell\omega s}^{-}\ {}_sX_{\ell\omega}^{\mathrm{WKB},-}(r), \tag{3.18} $$

where $C_{\ell\omega s}^{(\pm)}$ are constants.

The substitution of the expression for $U_{\ell\omega s}(r)$ into equations (3.5), (3.7), (3.9)-(3.12) reveals the following asymptotic behaviour of the $S_n^{\ell\omega s}(r)$ functions as $\ell \to \infty$:

$$ S_0^{\ell\omega s}(r) = O(\sqrt{\ell}), \qquad S_1^{\ell\omega s}(r) = O(\log \ell), \qquad (\text{if } \lim_{\ell\to\infty} \omega^2/V_{\ell s}(r) < 1) \tag{3.19} $$

$$ S_2^{\ell\omega s}(r) = O(\ell^{-1}), \qquad S_3^{\ell\omega s}(r) = O(\ell^{-2}), \qquad (\text{if } \lim_{\ell\to\infty} \omega^2/V_{\ell s}(r) < 1) \tag{3.20} $$

$$ S_4^{\ell\omega s}(r) = O(\ell^{-3}), \qquad S_5^{\ell\omega s}(r) = O(\ell^{-4}). \qquad (\text{if } \lim_{\ell\to\infty} \omega^2/V_{\ell s}(r) < 1) \tag{3.21} $$

Thus, as long as the condition $\lim_{\ell\to\infty} \omega^2/V_{\ell s}(r) < 1$ is satisfied, the WKB series resembles a generalized asymptotic series as $\ell \to \infty$ that is uniform in ω and r. In this case, $U_{\ell\omega s}(r) \to +\infty$ as $\ell \to \infty$, so that $S_0^{\ell\omega s}(r)$ is, subject to a choice of C_0, real-valued as $\ell \to \infty$ and the independent WKB solutions (3.16) and (3.17) are characterized by exponential growth and decay. Near $r = r_s$ and $r = \infty$, $V_{\ell s}(r) \to 0$ and the behaviour described in the above equations does not hold. This implies that *in* and *up* boundary conditions cannot be directly enforced on WKB solutions obtained in the region of $U_{\ell\omega s}(r) > 0$. Instead, *in* and *up* WKB solutions in this region will be constructed by matching the general WKB solutions (3.18) to other asymptotic solutions valid at the horizon and at infinity.

3.2 LEADING-ORDER APPROXIMATION AND ENFORCEMENT OF BOUNDARY CONDITIONS

The leading-order WKB approximation to a general homogeneous solution to the Regge-Wheeler equation is obtained by considering terms up to $O(S_1(n))$ in the exponents of the independent solutions,

$$ {}_sX^{\text{WKB},+}_{\ell\omega}(r) \sim {}_sX^{\text{WKB},+}_{\ell\omega,(0)}(r) \equiv U_{\ell\omega s}(r)^{-1/4} e^{S_0^{\ell\omega s}(r)}, \qquad (\text{for } S_2^{\ell\omega s}(r) \to 0) \tag{3.22} $$

$$ {}_sX^{\text{WKB},-}_{\ell\omega}(r) \sim {}_sX^{\text{WKB},-}_{\ell\omega,(0)}(r) \equiv U_{\ell\omega s}(r)^{-1/4} e^{-S_0^{\ell\omega s}(r)}. \qquad (\text{for } S_2^{\ell\omega s}(r) \to 0) \tag{3.23} $$

where $S_0^{\ell\omega s}(r)$, is explicitly given by

$$ S_0^{\ell\omega s}(r) \equiv \int^{r_*} dr'_* \sqrt{U_{\ell\omega s}(r(r'_*))} = \int^{r} dr' \frac{\sqrt{U_{\ell\omega s}(r')}}{f(r')}. \tag{3.24} $$

This and other integrals for the functions $S_n^{\ell\omega s}(r)$ of even n show in equations (3.9) and (3.11) have representations in terms of combinations of Elliptic functions with complicated arguments (see Appendix A), which shall not be directly used in the upcoming analytical calculations. For $\lim_{\ell\to\infty} \omega^2/V_{\ell s}(r) < 1$, the first neglected term, $S_2^{\ell\omega s}(r)$, is $O(\ell^{-1})$, so that the leading-order WKB approximation captures[1] the leading-ℓ behaviour as $\ell \to \infty$. In this section, the leading-order WKB solutions (3.16) and (3.17) will be matched to asymptotic solutions near the horizon and for large r satisfying *in* and *up* boundary conditions, respectively. As shall be seen, performing the matching of just the leading-order solutions gives enough information for the construction of asymptotic expansions for the *in* an *up* solutions beyond leading order except for an overall normalization constant, as in both cases the contributions from one of the two independent solutions are shown to vanish exponentially as $\ell \to \infty$.

3.2.1 THE WKB *in* SOLUTION

Now, asymptotic solutions to the Regge-Wheeler in the near-horizon limit shall be obtained. To do so, the potential $V_{\ell s}(r)$ is expanded for small displacement away from $r = r_s$. This is achieved by defining $\delta r \equiv r - r_s$ and expanding $V_{\ell s}(r)$ for small $\delta r/r_s$,

$$ V_{\ell s}(r) = \underbrace{\left(\frac{\ell(\ell+1)}{r_s^3} + \frac{(1-s^2)}{r_s^3} \right)}_{v_{\ell s} \equiv} \delta r + O\big(\delta r^2/r_s^2\big). \tag{3.25} $$

[1] Here, the use of the word "captures" is intentional, as the leading-order WKB approximation also contains terms of sub-leading order with respect to the large-ℓ asymptotics.

Approximate solutions to the Regge-Wheeler equation accurate up to $O(\delta r/r_s)$ are sought. To obtain these, the displacement δr is re-expressed in terms of r_* by expanding the definition of the tortoise coordinate (2.53) for $\delta r/r_s \ll 1$,

$$r_* = r_s + 2M \log\left(\frac{\delta r}{r_s}\right) + O(\delta r/r_s) \quad \Longrightarrow \quad \delta r \sim r_s e^{k(r_*-r_s)} \quad \text{as} \quad \delta r/r_s \to 0, \tag{3.26}$$

where $k \equiv 1/r_s$. Thus, considering terms up to linear order in the expansion, the Regge-Wheeler equation reads:

$$\partial^2_{r_*}\ {}_sX_{\ell\omega} + \left(\omega^2 - v_{\ell s} r_s e^{k(r_*-r_s)}\right)\ {}_sX_{\ell\omega} = O(\delta r^2). \tag{3.27}$$

An asymptotic general solution to this equation in the limit $\delta r/r_s \to 0$ can be written as a combination of modified Bessel functions [33, Chapter 10]:

$${}_sX_{\ell\omega} \sim C^{H,+}_{\ell\omega s} I_{+\alpha}\left(2r_s\sqrt{v_{ls}e^{k(r_*-r_s)}}\right) + C^{H,-}_{\ell\omega s} I_{-\alpha}\left(2r_s\sqrt{v_{ls}e^{k(r_*-r_s)}}\right), \ (\delta r \ll r_s) \tag{3.28}$$

where $\alpha \equiv 2i\omega r_s$, $I_\alpha(x)$ is the modified Bessel function of the first kind and $C^{H,\pm}_{\ell\omega s}$ are constants.

To obtain a purely ingoing solution, one needs to examine the asymptotic behaviour of the modified Bessel functions as their arguments approach zero ($r_* \to -\infty$). Using the relation [33, (10.30.1)]

$$I_\alpha(z) \sim 2^{-\alpha}\frac{z^\alpha}{\Gamma(1+\alpha)}, \qquad (\text{for } z \ll 1) \tag{3.29}$$

one finds that, as $r_* \to -\infty$, the two independent solutions that make up equation (3.28) behave like *ingoing* and *outgoing* waves,

$$I_{\pm 2i\omega r_s}\left(2r_s\sqrt{v_{ls}r_s e^{k(r_*-r_s)}}\right) \sim \frac{r_s^{\pm 3i\omega r_s} v_{\ell s}^{\pm i\omega r_s}}{\Gamma(1 \pm 2i\omega r_s)} e^{\pm i\omega(r_*-r_s)} \qquad \text{as } r_* \to -\infty. \tag{3.30}$$

Therefore, a normalized purely ingoing asymptotic solution is obtained by taking $C^{H,+}_{\ell\omega s} = 0$ and

$$C^{H,-}_{\ell\omega s} = \frac{\Gamma(1-2i\omega r_s)e^{-i\omega r_s}}{r_s^{-3i\omega r_s} v_{\ell s}^{-i\omega r_s}}. \tag{3.31}$$

Near the horizon a normalized *in* solution to the Regge-Wheeler equation is approximated by

$${}_sX^{in}_{\ell\omega} \sim\ {}_sX^{H,in}_{\ell\omega} \equiv C^{H,-}_{\ell\omega s} I_{-\alpha}(2r_s\sqrt{v_{\ell s}r_s e^{k(r_*-r_s)}}). \qquad (\text{for } \delta r/r_s \ll 1) \tag{3.32}$$

Next, an expansion for the asymptotic *in* solution in the region of validity of the WKB solutions is sought — that is, ${}_sX^{H,in}_{\ell\omega}$ shall be expanded for large positive $U_{\ell\omega s}(r)$. This can be achieved by re-expressing r_* in terms of δr and taking the leading asymptotics of the modified Bessel function [33, (10.40.5)] in equation (3.32) as $v_{\ell s} r_s^2 \delta r \to \infty$ (i.e. for large $r_s^2 V_{\ell s}(r)$),

$$ {}_sX^{H,in}_{\ell\omega} \sim \frac{e^{2r_s\sqrt{v_{\ell s}\delta r}}}{(4\pi r_s\sqrt{v_{\ell s}\delta r})^{1/2}} + ie^{-2\pi\omega r_s}\frac{e^{-2r_s\sqrt{v_{\ell s}\delta r}}}{(4\pi r_s\sqrt{v_{\ell s}\delta r})^{1/2}}, \quad \left(\text{for } v_{\ell s} r_s^2 \delta r \to \infty\right). \tag{3.33} $$

To obtain an asymptotic expansion for the leading-order WKB solutions near the horizon, the functions $S_0^{\ell\omega s}(r)$ and $S_1^{\ell\omega s}(r)$ are expanded for small $\delta r/r_s$ with $v_{\ell s} r_s^2 \delta r \to \infty$. One obtains:

$$ S_0^{\ell\omega s}(r) \sim \int^{\delta r} d\delta r' r_s \frac{\sqrt{v_{\ell s}\delta r'}}{\delta r'} = 2r_s\sqrt{v_{\ell s}\delta r}, \qquad (\delta r/r_s \to 0,\ v_{\ell s} r_s^2 \delta r \to \infty) \tag{3.34} $$

$$ S_1^{\ell\omega s}(r) \sim \log\left(v_{\ell s}\delta r\right)^{-1/4} \qquad (\delta r/r_s \to 0,\ v_{\ell s} r_s^2 \delta r \to \infty). \tag{3.35} $$

Thus, a leading-order WKB solution that is matched to a purely ingoing asymptotic near the horizon is given by:

$$ {}_sX^{\text{WKB},in}_{\ell\omega,(0)}(r) \equiv \frac{1}{\sqrt{4\pi r_s}}\Big({}_sX^{\text{WKB},+}_{\ell\omega,(0)}(r) + ie^{-2\pi\omega r_s}\ {}_sX^{\text{WKB},-}_{\ell\omega,(0)}(r)\Big). \tag{3.36} $$

Since the term containing ${}_sX^{\text{WKB},-}_{\ell\omega,(0)}$ decays exponentially with $S_0^{\ell\omega s}(r) = O(\ell)$, it gives negligible contribution to ${}_sX^{\text{WKB},in}_{\ell\omega}(r)$ away from the horizon. Thus, one can write

$$ {}_sX^{\text{WKB},in}_{\ell\omega,(0)}(r) \sim \frac{1}{\sqrt{4\pi r_s}}\ {}_sX^{\text{WKB},+}_{\ell\omega,(0)}(r). \qquad (\text{for } \ V_{\ell s}(r)/\omega^2 \gg 1) \tag{3.37} $$

Note that ${}_sX^{\text{WKB},in}_{\ell\omega}(r)$ receives negligible contributions from ${}_sX^{\text{WKB},-}_{\ell\omega}(r)$ still holds if one consider higher-order terms in the WKB series. As the leading term in ${}_sX^{\text{WKB},-}_{\ell\omega}(r)$ falls faster than any higher-order contribution from the other independent solution as $\ell \to \infty$ with $\lim_{\ell\to\infty}\omega^2/V_{\ell s}(r) < 1$. Then, an expansion for the unnormalized *in* can be written as

$$ {}_sX^{in}_{\ell\omega}(r) \sim\ {}_sX^{\text{WKB},in}_{\ell\omega}(r) \equiv \exp\left\{\sum_{n=0}^{\infty} S_n^{\ell\omega s}(r)\right\} \qquad (\ell \to \infty,\ \text{for } \lim_{\ell\to\infty}\omega^2/V_{\ell s}(r) < 1). \tag{3.38} $$

The accuracy of this expression compared to results obtained with the *Black Hole Perturbation Toolkit* [6] implementation of the MST method [2] (which will be discussed in the following chapter) for *Wolfram's Mathematica* is depicted in Figure 3.1.

3.2.2 THE WKB *up* SOLUTION

An analogous procedure is followed to obtain an *up* solution. A leading asymptotic expansion for the potential as $r_s/r \to 0$ is considered,

$$V_{\ell s}(r) \sim V_\ell^{\text{far}}(r) \equiv \frac{\ell(\ell+1)}{r_*^2} \qquad (r_s/r \to 0). \tag{3.39}$$

Asymptotic solutions of the Regge-Wheeler equation as $r/r_s \to \infty$ are obtained by solving

$$\partial_{r_*}^2 \; {}_sX_{\ell\omega}^{\text{far}} + \left(\omega^2 - V_\ell^{\text{far}}(r)\right) \; {}_sX_{\ell\omega}^{\text{far}} = 0. \tag{3.40}$$

for $X_{\ell\omega}^{far}$, so that solutions to the Regge-Wheeler equation are given by ${}_sX_{\ell\omega} \sim X_{\ell\omega}^{far}$ as $r/r_s \to \infty$. Solutions to this equation are also given in terms of Bessel functions. The general solution to (3.40) is obtained to be:

$$X_{\ell\omega}^{\text{far}} = \sqrt{r_*}(c_J J_L(\omega r_*) + c_Y Y_L(\omega r_*)), \tag{3.41}$$

where $J_L(x)$ and $Y_L(x)$ with $L = \ell + 1/2$ are Bessel functions of the first and second kind, respectively, and c_J and c_Y are constants. The particular combination of these functions that behaves like an outgoing wave at infinity is called the Hankel function of the first kind, and is given by:

$$H_L(\omega r_*) = J_L(\omega r_*) + iY_L(\omega r_*). \tag{3.42}$$

This function possesses the following asymptotic behaviour for large argument [33, (10.17.5)]:

$$H_L(\omega r_*) \sim -ie^{-i\ell\pi/2}\sqrt{\frac{2}{\pi\omega r_*}}e^{i\omega r_*} \qquad (\omega r_* \to \pm\infty). \tag{3.43}$$

Thus, an asymptotic solution to the Regge-Wheeler equation that behaves as a normalized outgoing wave at infinity is written as:

$${}_sX_{\ell\omega}^{up} \sim X_{\ell\omega}^{\text{far},up}(r_*) \equiv ie^{i\ell\pi/2}\sqrt{\frac{\pi\omega r_*}{2}}H_L(\omega r_*) \qquad (r_s/r \to 0). \tag{3.44}$$

To perform the matching with the WKB leading-order expressions, this solution shall be expanded for $r_s^2 V_{\ell s}(r) \to \infty$. Looking at the potential in equation (3.39), one can see that this corresponds to expanding for $L \gg \omega r_*$. Here the asymptotic expansions for large order of $J_L(\omega r_*)$ and $Y_L(\omega r_*)$ [33, (10.19)] are used,

$$\sqrt{r_*}J_L(\omega r_*) \sim \frac{\sqrt{r_*}}{\sqrt{2\pi L}}\left(\frac{e\omega r_*}{2L}\right)^L \qquad (L\to\infty), \tag{3.45}$$

$$\sqrt{r_*}Y_L(\omega r_*) \sim -\sqrt{\frac{2r_*}{\pi L}}\left(\frac{e\omega r_*}{2L}\right)^{-L} \qquad (L\to\infty). \tag{3.46}$$

Lastly, the leading asymptotic expressions for the functions $S_0^{\ell\omega s}(r)$ and $S_1^{\ell\omega s}(r)$ for $r_s/r \to 0$ and $L\to\infty$ (with $Lr_s/r\to\infty$) are obtained as

$$\exp\left\{S_0^{\ell\omega s}(r)\right\} \sim \exp\left\{\int^r dr'\sqrt{\frac{\ell(\ell+1)}{r_*^2}}\right\} \sim r_*^L \qquad \left(\frac{r_s}{r}\to 0, \frac{Lr_s}{r}\to\infty\right). \tag{3.47}$$

$$S_1^{\ell\omega s}(r) \sim \log\sqrt{\frac{r_*}{L}} \qquad \left(\frac{r_s}{r}\to 0, \frac{Lr_s}{r}\to\infty\right). \tag{3.48}$$

One can see that ${}_sX^{\mathrm{WKB},+}_{\ell\omega,(0)}$ and ${}_sX^{\mathrm{WKB},-}_{\ell\omega,(0)}$ are matched with $\sqrt{r_*}J_L(\omega r_*)$ and $\sqrt{r_*}Y_L(\omega r_*)$, respectively. From equation (3.45), one can see that ${}_sX^{\mathrm{WKB},+}_{\ell\omega,(0)}$ is multiplied by L^{-L}. Thus, once again, in the limit $\ell\to\infty$ (with $\lim_{\ell\to\infty}\omega^2/V_{\ell s}(r)<1$) one of the independent WKB solutions gives negligible contribution to the solution sought. An unnormalized *up* solution can then be written as:

$${}_sX^{up}_{\ell\omega}(r) \sim {}_sX^{\mathrm{WKB},up}_{\ell\omega}(r) \equiv \exp\left\{\sum_{n=0}^{\infty}(-1)^{n+1}S_n^{\ell\omega s}(r)\right\} \quad \left(\ell\to\infty,\ \text{for}\ \lim_{\ell\to\infty}\omega^2/V_{\ell s}(r)<1\right). \tag{3.49}$$

In Figure 3.1, a comparison of this expression to numerical results obtained using the *Black Hole Perturbation Toolkit* [6] is seen.

3.3 THE GREEN'S FUNCTION MODES FROM THE WKB SOLUTIONS

Lastly, an asymptotic expansion for the radial Green's function modes in equation (2.54) for $\ell\to\infty$ that is uniformly valid in ω and r as long as $\lim_{\ell\to\infty}\omega^2/V_{\ell s}(r)<1$ is written. To do so, the solutions to the homogeneous radial scalar field equation are recovered (2.60) from the WKB Regge-Wheeler solutions,

$$R^{in,\mathrm{WKB}}_{\ell\omega} \equiv \frac{1}{r}\exp\left\{\sum_{n=0}^{\infty}S_n^{\ell\omega}(r)\right\}, \tag{3.50}$$

$$R^{up,\mathrm{WKB}}_{\ell\omega} \equiv \frac{1}{r}\exp\left\{\sum_{n=0}^{\infty}(-1)^{n+1}S_n^{\ell\omega}(r)\right\}, \tag{3.51}$$

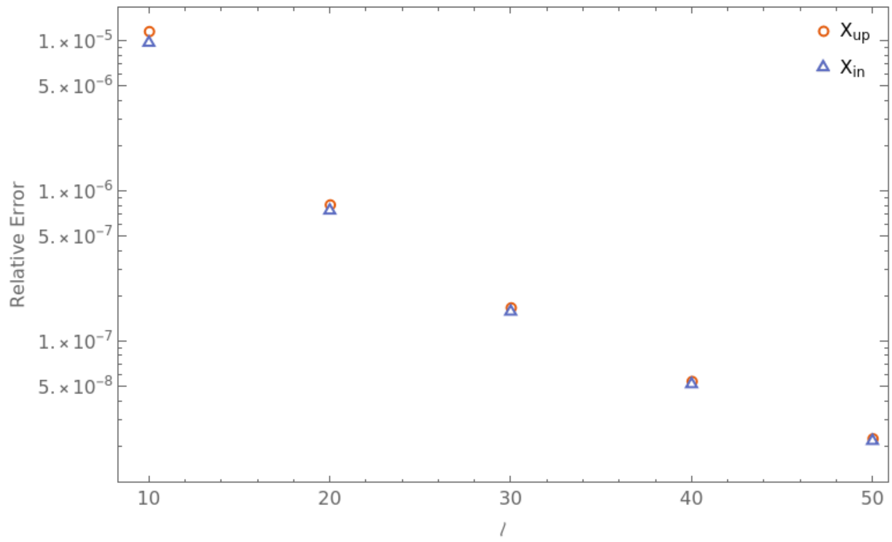

FIGURE 3.1: Relative error of the WKB large-ℓ solutions in equations (3.38) and (3.49) truncated at $n = 3$ compared to solutions obtained from the *Black Hole Perturbation Toolkit* [6] for $s = 0$, $r = 6M$ and $\omega M = \ell/20$.

where $S_n^{\ell\omega}(r) \equiv S_n^{\ell\omega 0}(r)$. The Wronskian of these solutions, calculated as in equation (2.55), is

$$\mathcal{W}(R_{\ell\omega}^{in,\text{WKB}}, R_{\ell\omega}^{up,\text{WKB}}) = -\frac{2}{r^2}\left(\sum_{n=0}^{\infty} S_{2n}^{\ell\omega}{}'(r)\right) \times \exp\left(2\sum_{n=0}^{\infty} S_{2n+1}^{\ell\omega}(r)\right). \tag{3.52}$$

Then, a generalized asymptotic expansion as $\ell \to \infty$ of the radial Green's function modes in equation (2.54) is

$$g_{\ell m\omega}(r,r') \sim g_{\ell m\omega}^{\text{WKB}}(r,r') \equiv \frac{-1}{rr'\mathcal{W}^{\text{WKB}}}\left(\exp\left\{\sum_{n=0}^{\infty}\left(S_n^{\ell\omega}(r) + (-1)^{n+1}S_n^{\ell\omega}(r')\right)\right\}\Theta(r'-r)+ \right.$$
$$\left. + \exp\left\{\sum_{n=0}^{\infty}\left(S_n^{\ell\omega}(r') + (-1)^{n+1}S_n^{\ell\omega}(r)\right)\right\}\Theta(r-r')\right), \tag{3.53}$$

for $\lim_{\ell\to\infty}\omega^2/V_\ell(r) < 1$, where $V_\ell(r) \equiv V_{\ell 0}(r)$ $(s = 0)$ and $W^{\text{WKB}} \equiv \Delta\mathcal{W}(R_{\ell\omega}^{in,\text{WKB}}, R_{\ell\omega}^{up,\text{WKB}})$. It is worth highlighting that the definition of these Green's function modes in dependent of the overall normalization constants of the homogeneous solutions, since any overall constant appearing in the numerator is canceled out by the same constant in the Wronskian. The obtained homogeneous solutions show great agreement with the MST solutions for large ℓ while being computationally cheap to evaluate (especially if compared to full numerical solutions, which are particularly expensive to compute for large ℓ).

Chapter 4

MST SOLUTIONS TO THE TEUKOLSKY EQUATION

The most well established method for obtaining analytical solutions to the homogeneous Teukolsky radial equation in Kerr spacetime is the one developed by Mano, Suzuki and Takasugi (MST) [2; 34]. Their method builds on a previous study of solutions to spheroidal equations as series of different special functions carried out by Leaver [35]. In his work, Leaver obtains a solution to this type of equation in the form of a series of Coulomb wave functions that is valid everywhere except at the horizon. Another solution to Teukolsky's equation valid everywhere except at $r \to \infty$ in the form of a series of hypergeometric functions was obtained by Mano, Suzuki and Takasugi [2]. In the region $r_s < r < \infty$, the matching of the two solution is possible. Later, the same three autors applied an analogous method to obtain series solutions to the homogeneous Regge-Wheeler equation [36].

In this chapter, the MST method for obtaining solutions the Schwarzschild-specialized Teukolsky radial equation (2.50) is reviewed based on the original work of Mano, Suzuki and Takasugi (MST) [2; 34] and on the later review by Sasaki & Tagoshi [37]

4.1 SOLUTION IN SERIES OF HYPERGEOMETRIC FUNCTIONS

A second-order differential equation with at most three regular singular points can be transformed into a hypergeometric equation,

$$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0, \tag{4.1}$$

where a, b and c are some real or complex parameters. One can check that the Teukolsky radial equation possesses an irregular singular point at infinity. Therefore, its solutions are not the hypergeometric functions that solve equation (4.1). However, one may expect the further one is from the irregular singularity (i.e. the smaller the r), the more the exact solutions to the Teukolsky radial equation should resemble solutions to the hypergeometric equation (given the appropriate transformations). This indicates that a solution to Teukolsky's equation valid everywhere except at $r \to \infty$ may be constructed as series of hypergeometric functions. The fact that such a solution is valid at the horizon, implies that it can be constructed to satisfy the *ingoing* boundary condition in equation (2.51).

To pursue a series solution to the homogeneous Teukolsky radial equation satisfying the *in* boundary condition, the following transformation is done to Teukolsky's radial function:

$$ {}_sR_{\ell m\omega}(x) = (-x)^{-s-i\epsilon} e^{i\epsilon x} P_{in}(x). \tag{4.2}$$

Where $x \equiv -\omega(r - r_s)/\epsilon$ and $\epsilon \equiv 2M\omega$ are dimensionless quantities. Subject to this transformation, the Teukolsky Radial Equation (2.50) takes the form:

$$\begin{aligned} x(1-x)P_{in}'' + [1 - s - 2i\epsilon - 2(1-i\epsilon)x]P_{in}' + [i\epsilon(1-i\epsilon) + \ell(\ell+1)]P_{in} = \\ = 2i\epsilon\left[-x(1-x)P_{in}' + (1-s)xP_{in}'\right] + \left[\epsilon^2 - i\epsilon\kappa(1-2s)\right]P_{in}. \end{aligned} \tag{4.3}$$

One can see that the left-hand side of this equation matches exactly the form of the generic hypergeometric equation (4.1) and that the extra terms on the right-hand side are explicitly of $O(\epsilon)$. Hence, a solution to this equation can be proposed as a series of hypergeometric functions with ϵ as some sort of expansion parameter. When proposing such a solution, an important aspect of MST and Leaver's derivations is the inclusion of a free parameter ν, called the *Renormalized Angular Momentum.* As shall be made clear later, the introduction of this parameter will be crucial for guaranteeing the convergence of the series. By adding $[\nu(\nu+1) - \ell(\ell+1)]P_{in}$ to both sides of the equation (4.3), one arrives at:

$$\begin{aligned} x(1-x)P_{in}'' + [1 - s - 2i\epsilon - 2(1-i\epsilon)x]P_{in}' + [i\epsilon(1-i\epsilon) + \nu(\nu+1)]P_{in} = \\ = 2i\epsilon\left[-x(1-x)P_{in}' + (1-s)xP_{in}\right] + \\ + \left[\epsilon^2 - i\epsilon\kappa(1-2s) + \nu(\nu+1) - \ell(\ell+1)\right]P_{in}. \end{aligned} \tag{4.4}$$

It shall be required that $\nu \to \ell$ in the limit $\epsilon \to 0$, so that one recovers the original differential equation at zero-th order. Also, one can check that this equation is unchanged by the transformation $\nu \to -\nu - 1,\ \ell \to -\ell - 1$. Considering the form this equation, the following ansatz is proposed for the series solution:

$$P_{in}(x) = \sum_{n=-\infty}^{\infty} c_n^\nu p_{n+\nu}(x) \tag{4.5}$$

$$p_{n+\nu}(x) = {}_2F_1(n+\nu+1-i\epsilon, -n-\nu-i\epsilon, 1-s-i\epsilon; 1-s-2i\epsilon; x). \tag{4.6}$$

With the use of recurrence relations for the hypergeometric function and its derivatives [33, Section 15.5],

$$xp_{n+\nu} = -\frac{(n+\nu+1-s-i\epsilon)(n+\nu+1-i\epsilon)}{2(n+\nu+1)(2n+2\nu+1)} p_{n+\nu+1} + \tag{4.7}$$

$$+\frac{1}{2}\left[1+\frac{i\epsilon(s+i\epsilon)}{(n+\nu)(n+\nu+1)}\right] p_{n+\nu} - \frac{(n+\nu+s+i\epsilon)(n+\nu+i\epsilon)}{2(n+\nu)(2n+2\nu+1)} p_{n+\nu-1}, \tag{4.8}$$

$$x(1-x)p'_{n+\nu} = \frac{1}{2}(s+i\epsilon)\left[1+\frac{i\epsilon(1-i\epsilon)}{(n+\nu)(n+\nu+1)}\right] p_{n+\nu} + \tag{4.9}$$

$$\frac{(n+\nu+i\epsilon)(n+\nu+1-i\epsilon)}{(2n+2\nu+1)}\left[\frac{(n+\nu+1-s-i\epsilon)}{2(n+\nu+1)} p_{n+\nu+1} - \frac{(n+\nu+s+i\epsilon)}{2(n+\nu)} p_{n+\nu-1}\right],$$

the differential equation (4.4) is turned into a three-term recurrence relation for the series coefficients c_n^ν:

$$\alpha_n^\nu c_{n+1}^\nu + \beta_n^\nu c_n^\nu + \gamma_n^\nu c_{n-1}^\nu = 0, \tag{4.10}$$

$$\alpha_n^\nu = \frac{i\epsilon(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)(n+\nu+1+i\epsilon)}{(n+\nu+1)(2n+2\nu+3)}; \tag{4.11}$$

$$\beta_n^\nu = -\ell(\ell+1) + (n+\nu)(n+\nu+1) + 2\epsilon^2 + \frac{\epsilon^2\left(s^2+\epsilon^2\right)}{(n+\nu)(n+\nu+1)}; \tag{4.12}$$

$$\gamma_n^\nu = -\frac{i\epsilon(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)(n+\nu-i\epsilon)}{(n+\nu)(2n+2\nu-1)}. \tag{4.13}$$

Like second order differential equations, three-term recurrence relations admit two independent solutions in the form of sequences $\{c_n^{(1)}\}$ and $\{c_n^{(2)}\}$ [38]. A sequence $\{c_n^{(1)}\}$ is called *minimal as* $n \to \infty$ if it satisfies $\lim_{n\to\infty} c_n^{(1)}/c_n^{(2)} = 0$. Conversely, a sequence $\{c_n^{(1)}\}$ is called *minimal as* $n \to -\infty$ if it satisfies $\lim_{n\to-\infty} c_n^{(1)}/c_n^{(2)} = 0$. In both of these cases, the sequence $c_n^{(2)}$ is called a dominant sequence. The ratio between successive terms belonging to a sequence solution can be expressed recursively in the form of continued fractions:

$$R_n^\nu \equiv \frac{c_n^\nu}{c_{n-1}^\nu} = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}^\nu}; \tag{4.14}$$

$$L_n^\nu \equiv \frac{c_n^\nu}{c_{n+1}^\nu} = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}^\nu}. \tag{4.15}$$

R_n^ν and L_n^ν are often called the *right-mover* and *left-mover*. The behaviour of solutions of the three-term recurrence relation for large $|n|$ is determined by the asymptotic behaviour of the ratios β_n^ν/α_n^ν and $\gamma_n^\nu/\alpha_n^\nu$ as $n \to \pm\infty$ [38]. By calculating these and evoking the theorems from the asymptotic theory of difference equations (namely, theorems 2.1 through 2.3 in [38]), one can prove the existence of two independent solutions to the three-term recurrence relation with the following asymptotic behaviour as $n \to \infty$:

$$\lim_{n\to\infty} \frac{a_n^{\nu(min,+)}}{a_{n-1}^{\nu(min,+)}} = \frac{i\epsilon}{2n}, \qquad \lim_{n\to\infty} \frac{a_n^{\nu(dom,+)}}{a_{n-1}^{\nu(dom,+)}} = \frac{2in}{\epsilon}. \tag{4.16}$$

The sequence $\{a_n^{\nu(min,+)}\}$ corresponds to a minimal solution as $n \to \infty$. Similarly, the asymptotic behaviour of the recurrence relation as $n \to -\infty$ implies the existence of two independent solutions satisfying:

$$\lim_{n\to-\infty} \frac{a_n^{\nu(min,-)}}{a_{n+1}^{\nu(min,-)}} = -\frac{i\epsilon}{2n}, \qquad \lim_{n\to-\infty} \frac{a_n^{\nu(dom,-)}}{a_{n+1}^{\nu(dom,-)}} = -\frac{2in}{\epsilon}. \tag{4.17}$$

The solution $\{a_n^{\nu(min,-)}\}$ corresponds to a minimal solution as $n \to -\infty$. These minimal solutions are unique except for a overall normalization factor. Thus, one can use this freedom to set $\{a_0^{\nu(min,\pm)}\} = 1$.

It is at this point that the previously introduced parameter ν becomes important. In general, the minimal solutions $\{a_n^{\nu(min,+)}\}$ and $\{a_n^{\nu(min,-)}\}$ do not coincide. As shall be discussed soon, it is mandatory that coefficients of the series of hypergeometric functions are minimal for both $n \to -\infty$ and $n \to \infty$ in order for the series to be convergent in both directions. Selecting either one of the $\{a_n^{\nu(min,\pm)}\}$ sequences at this stage would then lead to a diverging series as $n \to \mp\infty$. To fix this problem, the parameter ν shall be chosen to make the two minimal sequences match. That is, ν is to be chosen by requiring that the coefficients calculated from the *left* and *right* mover continued fractions agree for any given n. Mathematically, this corresponds to solving

$$R_{n+1}^\nu L_n^\nu = 1, \tag{4.18}$$

for ν for any n (usually one solves it for $n = 0$). Solving this transcendental equation requires either the use of numerical methods or of some sort of approximation, such as post-Newtonian theory. From now on, it shall be assumed that the coefficients chosen are those of the minimal solution with, which will be simply denoted by $\{a_n^\nu\} = \{a_n^{\nu(min,\pm)}\}$ with the choice of normalization $a_0^\nu = 1$. The series coefficients are given by products of the continued fractions R_n^ν and L_n^ν, i.e. for $n > 2$,

$$a_n^\nu = R_n^\nu R_{n-1}^\nu \ldots R_1^\nu \tag{4.19}$$

$$a_{-n}^\nu = L_{-n}^\nu L_{-n+1}^\nu \ldots L_{-1}^\nu. \tag{4.20}$$

The asymptotic behaviour of the hypergeometric functions $p_{n+\nu}$ for large $|n|$ can be extracted from the large-parameter asymptotic expansions given in [33, (15.12.5)]. The ratio between two consecutive $p_{n+\nu}$ in the directions $n \to \pm\infty$ for large $|n|$ is determined to be:

$$\lim_{n\to\infty} \frac{p_{n+\nu}}{p_{n+\nu-1}} = \lim_{n\to-\infty} \frac{p_{n+\nu}}{p_{n+\nu+1}} = 1 - 2x + \sqrt{(1-2x)^2 - 1}. \tag{4.21}$$

Therefore, the adoption of the coefficients from the minimal solution guarantees that the ratio between consecutive (in the directions $n \to \pm\infty$) decays with n^{-1} as $|n| \to \infty$.

Moreover, one can check that the solution constructed indeed satisfies the *ingoing* condition at the horizon by taking the $x \to 0$ limit:

$${}_sR_{\ell m\omega}(x) \sim (-x)^{-s-i\epsilon} e^{i\epsilon x} \sum_{n=-\infty}^{\infty} a_n^\nu. \tag{4.22}$$

For now on this solution shall be denoted R_{in}^ν. For the purpose of the later matching with the Coulomb type solutions, it will be useful to re-express this solution as combination of two independent series solutions to the Teukolsky equation.

The Teukolsky radial equation (2.50) is invariant under the transformation $\ell \to -\ell - 1$. By noting that $\alpha_n^\nu = \gamma_{-n}^{-\nu-1}$ and $\alpha_{-n}^{-\nu-1} = \gamma_n^\nu$, one can see that the coefficients of the series satisfy $a_n^\nu = a_{-n}^{-\nu-1}$. Therefore, the R_{in}^ν (and $p_{n+\nu}$) solution constructed preserves the symmetry $\nu \to -\nu - 1$ ($\ell \to -\ell - 1$) satisfied by equation (4.4). By evoking the following identity [33, (15.8.3)],

$$\frac{\sin(\pi(b-a))}{\pi} {}_2F_1(a,b;c;x) = \frac{(1-x)^{-a}}{\Gamma(b)\Gamma(c-a)} {}_2F_1\left(a, c-b; a-b+1; \frac{1}{1-x}\right) + \tag{4.23}$$

$$- \frac{(-x)^{-b}}{\Gamma(a)\Gamma(c-b)} {}_2F_1\left(b, c-a; b-a+1; \frac{1}{1-x}\right), \tag{4.24}$$

one can make explicit the symmetry in $\nu \to -\nu - 1$ by expressing R^{ν}_{in} as combination of two independent solutions:

$$R^{\nu}_{in} = R^{\nu}_{0} + R^{-\nu-1}_{0}; \tag{4.25}$$

$$R^{\nu}_{0} = e^{-i\hat{z}+\epsilon}\left(-1+\frac{\hat{z}}{\epsilon}\right)^{-s-i\epsilon}\left(\frac{\hat{z}}{\epsilon}\right)^{\nu+i\epsilon}\sum_{n=-\infty}^{\infty} a^{\nu}_{n}\frac{\Gamma(1-s-2i\epsilon)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\epsilon)\Gamma(n+\nu+1-s-i\epsilon)} \times \left(\frac{\hat{z}}{\epsilon}\right)^{n} {}_2F_1(-n-\nu-i\epsilon, -n-\nu+s+i\epsilon; -2n-2\nu; \epsilon/\hat{z}). \tag{4.26}$$

Where $\hat{z} \equiv \omega r$ so that $1 - x = \hat{z}/\epsilon$. The other independent solution is obtained by substituting $\nu \to -\nu - 1$ in the equation above. These solutions are, of course, valid everywhere except at $r \to \infty$.

4.2 SOLUTIONS IN SERIES OF COULOMB FUNCTIONS

In order to construct a solution possessing the *outgoing at infinity* property (2.52), a solution valid at $r = \infty$ is sought. Such a solution was first obtained by Leaver as a series of Coulomb wave functions. A generic Coulomb equation has the form:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\rho^2} + \left(1 - \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right) w = 0. \tag{4.27}$$

where η is a complex parameter. In the same spirit as the previous section, a transformation that rearranges the Teukolsky radial equation (2.50) highlighting a Coulomb equation on the left-hand side is performed. This transformation can be taken to be:

$${}_sR_{\ell m\omega} = \hat{z}^{-1-s}\left(1 - \frac{\epsilon}{\hat{z}}\right)^{-s-i\epsilon} f_c(\hat{z}). \tag{4.28}$$

Subject to this transformation, the Teukolsky radial equation becomes:

$$\hat{z}^2 f''_c + \left[\hat{z}^2 + 2(\epsilon+is)\hat{z} - \ell(\ell+1)\right] f_c = \epsilon\hat{z}(f''_c + f_c) - \epsilon\kappa(1-s-2i\epsilon) f'_c + + \frac{\epsilon(1-s-i\epsilon)(1-i\epsilon)}{\hat{z}} f_c - [2\epsilon^2 - \epsilon(\epsilon+is) f_c]. \tag{4.29}$$

From now on, a procedure very similar to the one in the previous section is followed. One can see that the left-side of the above equation has the same form of the coulomb equation (4.27) except for a global $\hat{z}^2$ factor and that right-hand side is $O(\epsilon)$. A solution to this equation as a series of Coulomb wave functions shall be proposed. Once again, the convergence of such a series solution will require the introduction of the renormalized angular momentum. By adding the term $[\nu(\nu+1)-\ell(\ell+1)]f_c$ to both sides of equation (4.29), one obtains

$$\hat{z}^2 f_c'' + \left[\hat{z}^2 + 2(\epsilon+is)\hat{z} - \nu(\nu+1)\right] f_c = \epsilon\hat{z}(f_c''+f_c) - \epsilon\kappa(1-s-2i\epsilon)f_c' + \\ + \frac{\epsilon(1-s-i\epsilon)(1-i\epsilon)}{\hat{z}} f_c + [\ell(\ell+1)-\nu(\nu+1)+-2\epsilon^2-\epsilon(\epsilon+is)]f_c. \tag{4.30}$$

The following ansatz is taken for the series solution:

$$f_c(\hat{z}) = \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu+1+i\eta)_n}{(\nu+1-i\eta)_n} b_n^\nu F_{n+\nu}(-is-\epsilon,\hat{z}), \tag{4.31}$$

where the Coulomb functions $F_{n+\nu}(-is-\epsilon,\hat{z})$ can be expressed in terms of the confluent hypergeometric function ${}_1F_1(a,b;\hat{z})$,

$$F_{n+\nu}(\eta,\hat{z}) = e^{-i\hat{z}} 2^{n+\nu} \hat{z}^{n+\nu+1} \frac{\Gamma(n+\nu+1-i\eta)}{\Gamma(2n+2\nu+2)} \, {}_1F_1(n+\nu+1-i\eta, 2n+2\nu+2; 2i\hat{z}), \tag{4.32}$$

with $\eta = -is-\epsilon$. The notation $(x)_n \equiv \Gamma(x+n)/\Gamma(x)$ stands for the Pochhammer symbol. The reason for the inclusion of the Pochhammer symbols in the ansatz (4.31), is that, by taking this particular form of ansatz, the coefficients b_n^ν are shown to satisfy the same recurrence relation as the coefficients a_n^ν. That is, through the use of the recurrence relations for the Coulomb wave functions [33, (33.17)],

$$\frac{1}{\hat{z}} F_{n+\nu} = \frac{(n+\nu+1+s-i\epsilon)}{(n+\nu+1)(2n+2\nu+1)} F_{n+\nu+1} + \frac{is+\epsilon}{(n+\nu)(n+\nu+1)} F_{n+\nu} \\ + \frac{(n+\nu-s+i\epsilon)}{(n+\nu)(2n+2\nu+1)} F_{n+\nu-1}, \tag{4.33}$$

$$F_{n+\nu}' = -\frac{(n+\nu)(n+\nu+1+s-i\epsilon)}{(n+\nu+1)(2n+2\nu+1)} F_{n+\nu+1} + \frac{is+\epsilon}{(n+\nu)(n+\nu+1)} F_{n+\nu} \\ + \frac{(n+\nu+1)(n+\nu-s+i\epsilon)}{(n+\nu)(2n+2\nu+1)} F_{n+\nu-1}, \tag{4.34}$$

one obtains the three-term recurrence relation for the coefficients b_n^ν to be exactly the same one as in equation (4.10). This fact is crucial for the later matching of the two solutions.

A solution to the Teukolsky radial equation valid everywhere except at $r = r_s$ can then be written as:

$$R_C^\nu = (2\hat{z})^\nu e^{-i\hat{z}} \hat{z}^{-s} \Big(1 - \frac{\epsilon}{\hat{z}}\Big)^{-s-i\epsilon} \sum_{n=-\infty}^{\infty} \Bigg[a_n^\nu \frac{(\nu+1+i\eta)_n}{(\nu+1-i\eta)_n} \frac{\Gamma(n+\nu+1-i\eta)}{\Gamma(2n+2\nu+2)} \times \\ \times (-2i\hat{z})^n \; {}_1F_1(n+\nu+1-i\eta, 2n+2\nu+2; 2i\hat{z}) \Bigg]. \tag{4.35}$$

It is still necessary to check, whether the choice of minimal coefficients is sufficient to guarantee the convergence of the series. The large $|n|$ behaviour of the Coulomb functions can be investigated through the use of [33, (33.5.IV)], subsequent Coulomb functions are found to satisfy:

$$\lim_{n\to\infty} \frac{F_{n+\nu}}{F_{n+\nu-1}} = \lim_{n\to-\infty} \frac{F_{n+\nu}}{F_{n+\nu+1}} = \frac{2}{\hat{z}} \tag{4.36}$$

Thus, the choice of minimal coefficients guarantees the convergence of the series everywhere except at $\hat{z} = 0$ $(r = r_s)$.

Next, the *outgoing* boundary conditions at infinity shall be enforced on the series solution obtained. To do so, it is useful to re-express the confluent hypergeometric function ${}_1F_1(a, b, \hat{z})$ as a combination of irregular confluent hypergeometric functions $U(a, b, \hat{z})$ by use of the following relation [33, (13.2.41)]:

$$\frac{1}{\Gamma(b)} \; {}_1F_1(a, b, 2i\hat{z}) = \frac{\mathrm{e}^{-a\pi \mathrm{i}}}{\Gamma(b-a)} U(a, b, \hat{z}) + \frac{\mathrm{e}^{(b-a)\pi \mathrm{i}}}{\Gamma(a)} \mathrm{e}^{2i\hat{z}} U\left(b-a, b, -\hat{z}\right) \tag{4.37}$$

The solution R_C^ν can then be expressed as:

$$R_C^\nu = R_{C,in}^\nu + R_{C,out}^\nu; \tag{4.38}$$

$$R_{C,in}^\nu = e^{-i\hat{z}} 2^\nu e^{i\pi(\nu+1-s+i\epsilon)} \frac{\Gamma(\nu+1+s+i\epsilon)}{\Gamma(\nu+1-s-i\epsilon)} \hat{z}^{\nu+i\epsilon} (\hat{z}-\epsilon)^{-s-i\epsilon} \\ \times \sum_{n=-\infty}^{\infty} i^n a_n^\nu (2\hat{z})^n U(n+\nu+1-s+i\epsilon, 2n+2\nu+2; 2i\hat{z}); \tag{4.39}$$

$$R_{C,out}^\nu = 2^\nu e^{-\pi\epsilon} e^{-i\pi(\nu+1+s)} e^{i\hat{z}} \hat{z}^{\nu+i\epsilon} (\hat{z}-\epsilon)^{-s-i\epsilon} \\ \times \sum_{n=-\infty}^{\infty} i^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} f_n^\nu (2\hat{z})^n U(n+\nu+1+s-i\epsilon, 2n+2\nu+2; -2i\hat{z}). \tag{4.40}$$

The asymptotic behaviour of these functions at infinity is obtained from the large-argument asymptotics of $U(a, b, \pm x)$ [33, (13.7.3)]

$$U(a, b, \pm x) \sim x^{-a} \qquad (x \to \infty). \tag{4.41}$$

Therefore the function $R^{\nu}_{C,out}$ is found to be purely outgoing at infinity,

$$R^{\nu}_{C,out} \sim C_{out} \hat{z}^{-1-2s} e^{i\hat{z}} \qquad (\hat{z} \to \infty), \tag{4.42}$$

where C_{out} is independent of $\hat{z}$. An expression for this function in terms of independent solutions to the Teukolsky equation is sought. Another independent homogeneous solution to the radial equation can be found by replacing $\nu \to -\nu - 1$ onto the equation for R^{ν}_{c} (4.35). By also re-expressing $R^{-\nu-1}_{c}$ with the use of equation (4.37), one writes:

$$R^{-\nu-1}_{C} = R^{-\nu-1}_{C,in} + R^{-\nu-1}_{C,out}, \tag{4.43}$$

where the functions $R^{-\nu-1}_{C,in}$ and $R^{-\nu-1}_{C,up}$ are obtained by replacing $\nu \to -\nu-1$ in equations (4.39) and (4.40). With the use of Kummer's identity [33, (13.7.40)],

$$U(a, b, \hat{z}) = \hat{z}^{1-b} U(a - b + 1, 2 - b, \hat{z}) \tag{4.44}$$

One can express $R^{-\nu-1}_{C,in/up}$ in terms of $R^{\nu}_{C,in/up}$,

$$R^{-\nu-1}_{Cin} = - ie^{-i\pi\nu} \frac{\sin\pi(\nu - s + i\epsilon)}{\sin\pi(\nu + s - i\epsilon)} R^{\nu}_{Cin}, \tag{4.45}$$

$$R^{-\nu-1}_{Cout} = ie^{i\pi\nu} R^{\nu}_{Cout}, \tag{4.46}$$

which allows for the writing of an expression for $R^{\nu}_{up} = R^{\nu}_{Cout}$ as a combination of the two independent solutions,

$$R^{up}_{C} = \gamma_\nu R^{\nu}_{C} + \delta_\nu R^{-\nu-1}_{C}; \tag{4.47}$$

$$\gamma_\nu = \left(e^{2i\pi\nu} + \frac{\sin\pi(\nu - s + i\epsilon)}{\sin\pi(\nu + s - i\epsilon)}\right)^{-1} \frac{\sin\pi(\nu - s + i\epsilon)}{\sin\pi(\nu + s - i\epsilon)}; \tag{4.48}$$

$$\delta_\nu = -\left(e^{2i\pi\nu} + \frac{\sin\pi(\nu - s + i\epsilon)}{\sin\pi(\nu + s - i\epsilon)}\right)^{-1} ie^{i\pi\nu}. \tag{4.49}$$

4.3 MATCHING OF THE TWO SOLUTIONS

Finally, the two types of solutions shall be matched in their common domain of validity, $r_s < r < \infty$. To do so, the functions ${}_2F_1(a, b; c; \hat{z})$ and ${}_1F_1(a, b; \hat{z})$ shall be expressed in their series representation and the matching shall be achieved by demanding that each power of $\hat{z}$ carries the same coefficient. Starting with R_0^ν, the $\hat{z}$ dependent terms inside the sum in equation (4.26) are collected and expressed as:

$$\left(\frac{\hat{z}}{\epsilon}\right)^n {}_2F_1(a,b;c;\epsilon/\hat{z}) = \left(\frac{\hat{z}}{\epsilon}\right)^n \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{(c)_j j!}\hat{z}^{n-j} = \sum_{j=0}^{\infty} f_j(n)\hat{z}^{n-j}. \tag{4.50}$$

Where the parameters for the hypergeometric functions in (4.26) were abbreviated as a, b and c and the coefficients $f_j(n)$ implicitly defined in the last equality depend on n through a, b and c. Explicitly, $f_j(n)$ is given by:

$$f_j(n) = \frac{(-n-\nu-i\epsilon)_j(-n-\nu+s+i\epsilon)_j}{j!\epsilon^{n-j}(-2n-2\nu)_j}. \tag{4.51}$$

Subject to these definitions, R_0^ν is re-expressed as

$$R_0^\nu = e^{-i\hat{z}+\epsilon}\left(-1+\frac{\hat{z}}{\epsilon}\right)^{\nu-s}\left(\frac{\hat{z}}{\epsilon}\right)^{\nu+i\epsilon}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty} a_n^\nu \frac{\Gamma(1-s-2i\epsilon)\Gamma(2n+2\nu+1)f_j(n)}{\Gamma(n+\nu+1-i\epsilon)\Gamma(n+\nu+1-s-i\epsilon)}\hat{z}^{n-j}, \tag{4.52}$$

or, by collecting all the $\hat{z}$-independent terms,

$$R_0^\nu = e^{-i\hat{z}+\epsilon}\left(-1+\frac{\hat{z}}{\epsilon}\right)^{-s-i\epsilon}\left(\frac{\hat{z}}{\epsilon}\right)^{\nu+i\epsilon}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{\infty} C_{n,j}^0\hat{z}^{n-j}, \tag{4.53}$$

$$C_{n,j}^0 \equiv a_n^\nu \frac{\Gamma(1-s-2i\epsilon)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\epsilon)\Gamma(n+\nu+1-s-i\epsilon)}f_j(n). \tag{4.54}$$

To explicitly write this expression as a sum over powers of $\hat{z}$, one can define a new index $k = n - j$ and consistently re-write the sums,

$$R_0^\nu = e^{-i\hat{z}+\epsilon}\left(-1+\frac{\hat{z}}{\epsilon}\right)^{-s-i\epsilon}\left(\frac{\hat{z}}{\epsilon}\right)^{\nu+i\epsilon}\sum_{k=-\infty}^{\infty}\sum_{n=k}^{\infty} C_{n,n-k}^0\hat{z}^k. \tag{4.55}$$

From the expression above, one can find that the pre-factor to each power $\hat{z}^k$ in the sum is given by:

$$A_{0,k}^\nu = e^{-i\hat{z}+\epsilon}\left(-1+\frac{\hat{z}}{\epsilon}\right)^{-s-i\epsilon}\left(\frac{\hat{z}}{\epsilon}\right)^{\nu+i\epsilon}\sum_{n=k}^{\infty} C_{n,n-k}^0. \tag{4.56}$$

Next, a similar procedure is followed to obtain the coefficients of each power of $\hat{z}$ in the sum present in the solution R_C^ν. The $\hat{z}$ dependent terms inside the sum are collected and the confluent hypergeometric function is written in its representation:

$$\hat{z}^n \ {}_1F_1(\mathrm{a}, \mathrm{b}; \hat{z}) = \sum_{j=0}^{\infty} \frac{(\mathrm{a})_j}{(\mathrm{b})_j j!} \hat{z}^{n+j} = \sum_{j=0}^{\infty} g_j(n) \hat{z}^{n+j}. \tag{4.57}$$

The explicit expression for the coefficients $g_j(n)$ is obtained by substituting a and b for the actual parameters of the confluent hypergeometric functions in equation (4.35):

$$g_j(n) = \frac{(n+\nu+1+s-i\epsilon)_j}{(2n+2\nu+2)_j j!}. \tag{4.58}$$

Collecting all $\hat{z}$-independent terms inside the sum in a single coefficient, R_C^ν is expressed as:

$$R_C^\nu = (2\hat{z})^\nu e^{-i\hat{z}} \hat{z}^{-s} \left(1 - \frac{\epsilon}{\hat{z}}\right)^{-s-i\epsilon} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} C_{n,j}^C \hat{z}^{n+j}, \tag{4.59}$$

$$C_{n,j}^C = (-2i)^n a_n^\nu \frac{\Gamma(\nu+1-i\eta)}{\Gamma(\nu+1+i\eta)} \frac{\Gamma(n+\nu+1+i\eta)}{\Gamma(2n+2\nu+2)} g_j(n). \tag{4.60}$$

Again, the indices are redefined to obtain a explicit sum in powers of $\hat{z}$:

$$R_C^\nu = (2\hat{z})^\nu e^{-i\hat{z}} \hat{z}^{-s} \left(1 - \frac{\epsilon}{\hat{z}}\right)^{-s-i\epsilon} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{k} C_{n,k-n}^C \hat{z}^k. \tag{4.61}$$

Thus, the pre-factor to k-th power of $\hat{z}$ in the sum is:

$$A_{C,k}^\nu = (2\hat{z})^\nu e^{-i\hat{z}} \hat{z}^{i\epsilon} \epsilon^{-s-i\epsilon} \left(-1 + \frac{\hat{z}}{\epsilon}\right)^{-s-i\epsilon} \sum_{n=-\infty}^{k} C_{n,k-n}^C. \tag{4.62}$$

Using the obtained results, the ratio between $A_{0,k}^\nu$ and $A_{C,k}^\nu$ for any given k is calculated to be:

$$K_\nu^k \equiv \frac{A_{0,k}^\nu}{A_{C,k}^\nu} = 2^{-\nu} \epsilon^{s-\nu} e^{\epsilon} \frac{\left(\sum_{n=k}^{\infty} C_{n,n-k}^0\right)}{\left(\sum_{n=-\infty}^{k} C_{n,k-n}^C\right)}. \tag{4.63}$$

Though it is not shown here, the ratio K_ν^k is independent of k [37]. Therefore, one can set $K_\nu \equiv K_\nu^k$ for any (integer) k and the matching of the solutions is expressed as:

$$R_0^\nu = K_\nu R_C^\nu. \tag{4.64}$$

In the upcoming chapters, the Coulomb type of solution will be adopted for the calculation of the self-force. Thus, it will be useful to express the R_{in}^{ν} solution given in equation (4.25) in terms of R_C^{ν} and $R_C^{-\nu-1}$,

$$R_C^{in} = K_\nu R_C^{\nu} + K_{-\nu-1} R_C^{-\nu-1}. \tag{4.65}$$

Chapter 5

REGULARIZATION PARAMETERS FOR A CIRCULAR GEODESIC

In this chapter, the WKB expressions constructed in Chapter 3 are used obtain the A_α and B_α regularization parameters for the self-force for scalar charge in a circular geodesic in Schwarzschild spacetime. After finishing this thesis, another derivation [39] of these regularization parameters for circular geodesics in Schwarzschild using WKB solutions to the Regge-Wheeler equation was noticed in the literature by the author. The main difference of the work done by Bini, D., Damour, T. (2015) [39] to the calculations performed in this chapter, is the fact that they take the specialization to a circular geodesic from the start and derive WKB expression for the ℓm modes of the field, while in this work expressions for the Green's function $\ell m\omega$ modes are written and the integration over the source is carried out at the end.

To derive the expressions for the necessary regularization parameters, large-ℓ asymptotic expressions for the self-force modes are obtained by following the multi-scale analysis in [1; 25] to evaluate integrals over small proper-time intervals and for large-ℓ.

5.1 SPECIALIZATION TO CONSTANT ORBITAL RADIUS

An equatorial circular geodesic at radius r_0 in Schwarzschild spacetime is described by the following relations in standard Schwarzschild coordinates:

$$z(\tau) = \Big\{z^t(\tau), z^r(\tau), z^\theta(\tau), z^\phi(\tau)\Big\} = \Big\{u^t\tau, r_0, \pi/2, u^\phi\tau\Big\}, \tag{5.1}$$

where τ is the proper-time parameter and the four-velocity along the geodesic is $u^\alpha = \{u^t, 0, 0, u^\phi\}$ with

$$u^t = \sqrt{\frac{r_0}{r_0 - 3M}}, \qquad u^\phi = \frac{1}{r_0}\sqrt{\frac{M}{r_0 - 3M}}. \tag{5.2}$$

Here the choice of $z^\theta = \pi/2$ is made without loss of generality due to the spherical symmetry of Schwarzschild spacetime. The ℓ-modes of the Green's function are defined from equation (2.61) as

$$G_\ell^{ret}(x, x') \equiv \frac{L}{\pi} P_\ell(\cos\gamma) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} g_{\ell m\omega}(r, r'), \tag{5.3}$$

where $G^{ret}(x, x') = \sum_{\ell=0}^{\infty} G_\ell^{ret}(x, x')$. Similarly, the ℓ-modes of the full self-force are defined from equation (2.8) as

$$F_{\alpha,\ell} \equiv q^2(\delta_\alpha^\beta + u_\alpha u^\beta) \lim_{x\to z(0)} \nabla_\beta \int_{-\infty}^{0_+} d\tau G_\ell^{ret}(x, z(\tau)), \tag{5.4}$$

where the evaluation point of the self-force is chosen to be $x = z(\tau = 0)$ and $F_\alpha = \sum_{\ell=0}^{\infty} F_{\alpha,\ell}$. Motivated by the separation done in Hikida et al [3], the radial Green's function modes are separated into symmetric and anti-symmetric contibutions,

$$g_{\ell m\omega}(r, r') = g_{\ell m\omega}^{(+)}(r, r') + \operatorname{sign}(r - r') g_{\ell m\omega}^{(-)}(r, r'), \tag{5.5}$$

$$g_{\ell m\omega}^{(\pm)}(r, r') \equiv \frac{-1}{2W^{in/up}} \big(R_{\ell\omega}^{in}(r) R_{\ell\omega}^{up}(r') \pm R_{\ell\omega}^{in}(r') R_{\ell\omega}^{up}(r)\big), \tag{5.6}$$

and the symmetric and anti-symmetric ℓ-modes of the Green's function are similarly defined as

$$G_\ell^{(+)}(x, x') \equiv L P_\ell(\cos\gamma) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} g_{\ell m\omega}^{(+)}(r, r'), \tag{5.7}$$

$$G_\ell^{(-)}(x, x') \equiv L P_\ell(\cos\gamma) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \operatorname{sign}(r - r') g_{\ell m\omega}^{(-)}(r, r'). \tag{5.8}$$

Lastly, the contribution of the $(+)$ and $(-)$ Green's function modes to the self-force ℓ-modes are defined as:

$$F^{(-)}_{\alpha,\ell} \equiv q^2(\delta^\beta_\alpha + u_\alpha u^\beta) \lim_{x\to z(0)} \nabla_\beta \int_{-\infty}^{0_+} d\tau G^{(-)}_\ell(x, z(\tau)), \tag{5.9}$$

$$F^{(+)}_{\alpha,\ell} \equiv q^2(\delta^\beta_\alpha + u_\alpha u^\beta) \lim_{x\to z(0)} \nabla_\beta \int_{-\infty}^{0_+} d\tau G^{(+)}_\ell(x, z(\tau)), \tag{5.10}$$

The A_α and B_α regularization parameters for the circular geodesic shall be shown to be obtainable from the leading-ℓ of $F^{(-)}_{\alpha,\ell}$ and $F^{(+)}_{\alpha,\ell}$, respectively. When evaluating expressions regarding the $(-)$ modes, the convention of taking the limit $r \to r'$ "from above" $(r - r' = 0_+)$ is adopted. Choosing the opposite convention would change the resulting expressions for $F^{(-)}_{\alpha,\ell}$ by an overall minus sign coming from $\mathrm{sign}(r - r')$.

For a orbit of constant radius $z^r(\tau) = r_0$, one is allowed to bring the $x^r \to z^r$ part of the limit inside the integral. Then, by explicitly expressing $G^{(\pm)}_\ell(x, x')$ in terms of its radial modes (5.6), the components of the self-force ℓ-modes in equation (5.9) can be written as

$$F^{(\pm)}_{r,\ell} \equiv q^2 \lim_{x\to z(0)} \int_{-\infty}^{0_+} d\tau \frac{L}{\pi} P_\ell(\cos\gamma) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-z^t)} \partial_r g^{(\pm)}_{\ell m\omega}(r, r_0)\bigg|_{r=r_0}, \tag{5.11}$$

$$F^{(\pm)}_{\alpha,\ell} \equiv q^2(\delta^\beta_\alpha + u_\alpha u^\beta) \lim_{x\to z(0)} \nabla_\beta \int_{-\infty}^{0_+} d\tau \frac{L}{\pi} P_\ell(\cos\gamma) \times \times \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-z^t)} g^{(\pm)}_{\ell m\omega}(r_0, r_0) \qquad (\alpha \neq r). \tag{5.12}$$

where the τ dependent quantities are $z^t = z^t(\tau)$ and $\gamma = \phi - z^\phi(\tau)$. For the purpose of obtaining the regularization parameters for the circular geodesic, large-ℓ expansions of the above expressions are sought.

Expressions for the anti-symmetric radial Green's function modes and its derivative at $r = r_0$ are easily written as

$$g^{(-)}_{\ell m\omega}(r_0, r_0) = 0 \qquad \text{and} \qquad \partial_r g^{(-)}_{\ell m\omega}(r, r_0)\bigg|_{r=r_0} = \frac{1}{2r_0^2 f(r_0)}. \tag{5.13}$$

For large ℓ, the $g^{(+)}_{\ell m\omega}(r, r')$ modes and its derivative can be written in terms of WKB solutions obtained in equations (3.50) and (3.51),

$$g^{(+)}_{\ell m\omega}(r_0, r_0) \sim g^{\mathrm{WKB},(+)}_{\ell m\omega}(r_0, r_0) \equiv -\frac{1}{2r_0^2 f(r_0)\left(\sum_{n=0}^\infty S^{\ell\omega}_{2n}{}'(r_0)\right)}, \tag{5.14}$$

$$\partial_r g^{(+)}_{\ell m\omega}(r, r_0)|_{r=r_0} \sim \partial_r g^{\mathrm{WKB},(+)}_{\ell m\omega}(r, r_0)|_{r=r_0} = -\frac{1}{2r_0^3 f(r_0)}\left(\frac{-1 + \sum_{n=0}^\infty r_0 S^{\ell\omega}_{2n+1}{}'(r_0)}{\sum_{n=0}^\infty S^{\ell\omega}_{2n}{}'(r_0)}\right). \tag{5.15}$$

The above asymptotic relations hold for $\ell \to \infty$ with $\lim_{\ell\to\infty} \omega^2/V_\ell(r) < 1$. Due to the evaluation at $r = r_0$, the above quantities end up depending only on derivatives of the functions $S_n^{\ell\omega}(r)$, which simplifies a lot the calculations as one does not need to work with the integrals coming from the even n functions (i.e. equations (3.9) and (3.11)).

Next, terms that are of the same order with respect to the WKB expansion are collected by expanding (5.14) and (5.15) for $S_n^{\ell\omega} \gg S_{n+1}^{\ell\omega}$,

$$g_{\ell m\omega}^{\mathrm{WKB},(+)}(r_0, r_0)\Big|_{r=r_0} = \frac{1}{2r_0^2 f(r_0)} \sum_{k=0}^{\infty} \mathcal{S}_k(r_0), \tag{5.16}$$

$$\partial_r g_{\ell m\omega}^{\mathrm{WKB},(+)}(r_0, r_0)\Big|_{r=r_0} = \frac{1}{2r_0^3 f(r_0)} \sum_{k=0}^{\infty} \mathcal{T}_k^{\ell\omega}(r_0). \tag{5.17}$$

where the functions $\mathcal{S}_k(r_0)$ and $\mathcal{T}_k^{\ell\omega}(r_0)$ are of order $O(\ell^{-2k-1})$. The first two terms in each expansion are given here:

$$\mathcal{S}_0^{\ell\omega}(r_0) = \frac{1}{S_0^{\ell\omega\prime}(r_0)} = \frac{f(r_0)}{\sqrt{U_{\ell\omega s}(r_0)}}, \tag{5.18}$$

$$\mathcal{S}_1^{\ell\omega}(r_0) = -\frac{S_2^{\ell\omega\prime}(r_0)}{S_0^{\ell\omega\prime}(r_0)^2} = \frac{f\left(f'V_\ell' + fV_\ell''\right)}{8U_{\ell\omega}(r)^{5/2}} - \frac{5f^2V_\ell'^2}{32U_{\ell\omega}^{7/2}}, \tag{5.19}$$

$$\mathcal{T}_0^{\ell\omega}(r_0) = \frac{-1 + r_0 S_1^{\ell\omega\prime}(r_0)}{S_0^{\ell\omega\prime}(r_0)} = -f(r_0)\left(\frac{1}{U_{\ell\omega}^{1/2}} + \frac{1}{4}\frac{r_0 V_\ell'}{U_{\ell\omega}^{3/2}}\right), \tag{5.20}$$

$$\begin{aligned}\mathcal{T}_1^{\ell\omega}(r_0) &= \frac{r_0 S_3^{\ell\omega\prime}(r_0)}{S_0^{\ell\omega\prime}(r_0)} + \frac{(1 - r_0 S_1^{\ell\omega\prime}(r_0))S_2^{\ell\omega\prime}(r_0)}{S_0^{\ell\omega\prime}(r_0)^2} = \\ &= -\frac{35 f(r_0)^3 V_\ell'^3}{128 U_{\ell\omega}^{9/2}} - \frac{5f^2V_\ell'\left(f\left(V_\ell' - 2r_0V_\ell''\right) - 2r_0 f'V_\ell'\right)}{32U_{\ell\omega}^{7/2}} \\ &- \frac{f\left(r_0 f'^2 V_\ell' + f\left(r_0 f'' V_\ell' + f'\left(3r_0 V_\ell'' - 2V_\ell'\right)\right) + f^2\left(r_0 V_\ell''' - 2V_\ell''\right)\right)}{16U_{\ell\omega}^{5/2}}.\end{aligned} \tag{5.21}$$

Here the argument r_0 has been omitted from various functions for cleaner notation. All derivatives are taken with respect to r evaluated at $r = r_0$. The notation $U_{\ell\omega}(r)$ refers to the specialization of $U_{\ell\omega s}(r)$ to $s = 0$.

5.1.1 Fourier Transforms

Now, the radial modes at $r = r_0$, which corresponds to the specialization needed for the calculation regarding a circular geodesic, shall be transformed into the time-domain. Since, $\partial_r g^{(-)}_{\ell m\omega}(r_0, r_0)$ is ω-independent, it trivially transforms to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-z^t)}\partial_r g^{(-)}_{\ell m\omega}(r_0, r_0) = \frac{(t-z^t)}{2r_0^2 f(r_0)}. \tag{5.22}$$

To perform the Fourier transform of $g^{(+)}_{\ell m\omega}(r_0, r_0)$ and its derivative, one must be aware of the fact that the WKB solutions are not uniform in ω, which implies that a full representation of the ℓ-modes of the Green's function in the time-domain cannot be obtained from those expressions. Instead, the more humble task of obtaining the contribution to the Green's function ℓ-modes coming from the $\omega^2/V_\ell(r_0) < 1$ region of the spectrum is accepted. The quantities $\mathcal{S}_0^{\ell\omega}(r_0)$ and $\mathcal{T}_0^{\ell\omega}(r_0)$ are transformed to the time-domain by re-expressing Fourier integrals as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} f(\omega) \to \frac{1}{2\pi}\int_{-\omega_{cut}}^{\omega_{cut}} d\omega e^{-i\omega t} f(\omega), \tag{5.23}$$

$$\omega_{cut} \equiv \sqrt{V_\ell(r_0)}. \tag{5.24}$$

The contributions coming from this restricted region of the frequency spectrum will be proven to be enough (discussion at the end of this chapter) for the obtention of regularization parameters for the circular geodesic.

From equations (5.18)–(5.21), one can see that each term in $g^{(+)}_{\ell m\omega}(r_0, r_0)$ and $\partial_r g^{(+)}_{\ell m\omega}(r_0, r_0)$ depends on ω through some negative half-integer powers of $U_{\ell\omega}$. To perform the ω integration of such terms, one can write

$$\int_{-\omega_{cut}}^{\omega_{cut}} \frac{d\omega}{2\pi} e^{-i\omega(t-z^t)}\left(V_\ell(r) - \omega^2\right)^{-p/2} = V_\ell(r)^{(1-p)/2}\int_{-1}^{1}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{t}}\left(1-\bar{\omega}^2\right)^{-p/2}, \tag{5.25}$$

where $\bar{\omega} \equiv \omega/\omega_{cut}$, $\bar{t} \equiv \omega_{cut}(t - z^t)$ and p is some integer. The integral on the right-hand side of the above equation is a Hankel integral [40],

$$\int_{-1}^{1} d\bar{\omega} e^{-i\bar{\omega}\bar{t}}\left(1-\bar{\omega}^2\right)^{-p/2} = 2^{(1-p)/2}\sqrt{\pi}\Gamma(1-p/2)x^{(p-1)/2}J_{(1-p)/2}(\bar{t}). \tag{5.26}$$

Combining the two equations above, a general formula for integrating $U_{\ell\omega}^{-p/2}$ over frequency is obtained:

$$\int_{-\omega_{cut}}^{\omega_{cut}} \frac{d\omega}{2\pi} e^{-i\omega(t-z^t)} U_{\ell\omega}^{-p/2} = V_\ell^{(1-p)/2} 2^{-(p+1)/2} \pi^{-1/2} \Gamma(1-p/2) \bar{t}^{(p-1)/2} J_{(1-p)/2}(\bar{t}). \tag{5.27}$$

The expressions for the integrals of the leading-terms $\mathcal{S}_0^\ell$ and $\mathcal{T}_0^\ell$ are written here:

$$\mathcal{S}_0^\ell(r_0, t, z^t) \equiv \int_{-\omega_{cut}}^{\omega_{cut}} \frac{d\omega}{2\pi} e^{-i\omega(t-z^t)} \mathcal{S}_0^{\ell\omega}(r_0) = \frac{J_0(\bar{t})}{2}, \tag{5.28}$$

$$\mathcal{T}_0^\ell(r_0, t, z^t) \equiv \int_{-\omega_{cut}}^{\omega_{cut}} \frac{d\omega}{2\pi} e^{-i\omega(t-z^t)} \mathcal{T}_0^{\ell\omega}(r_0) = -\frac{f(r_0)}{2} \Big(J_0(\bar{t}) + \frac{r_0 V_\ell'}{4V_\ell} \bar{t} J_1(\bar{t}) \Big). \tag{5.29}$$

5.2 LARGE-ℓ EXPANSION AND INTEGRATION OVER THE TRAJECTORY

In possession of the large-ℓ expansions of the radial modes of the Green's function at $r = r_0$ in the time-domain, one can tackle the evaluation of the $F_{\alpha,\ell}^{(+)}$ and $F_{\alpha,\ell}^{(-)}$ contributions to the full self-force modes for a circular geodesic. Equations (5.13) and (5.22), imply that the only non-vanishing contribution coming from the $(-)$ part of the modes is

$$F_{r,\ell}^{(-)} = q^2 \lim_{\substack{x\to z(0)\\ \varepsilon\to 0_+}} \int_{-\infty}^{\varepsilon} d\tau L P_\ell(\cos\Big(\phi - z^\phi(\tau)\Big) \frac{\delta(t - z^t(\tau))}{r_0^2 f(r_0)} = \frac{q^2 L}{u^t r_0^2 f(r_0)}. \tag{5.30}$$

This term is divergent when summed over ℓ and is identified with $A_\alpha L$ ($\alpha = r$) in equation (2.45) (as shall be seen, terms coming from the symmetric part are atmost $O(L^0)$),

$$A_r \equiv \lim_{L\to\infty} \frac{F_{r,\ell}^{(-)}}{L} = \frac{q^2}{u^t r_0^2 f(r_0)} \qquad \text{for constant } z^r = r_0. \tag{5.31}$$

Now the assumption that the large-ℓ of $F_{\alpha,\ell}^{(+)}$ comes solely from the integration of the radial Green's function modes over the restricted frequency region (5.23) is made. This assumption is justified at the end of the this chapter for the case of a circular geodesic. In terms of (5.28) and (5.29), one can write

$$F^{(+)}_{r,\ell} \sim \frac{q^2 L}{r_0^3 f(r_0)} \lim_{\substack{x\to z(0)\\ \varepsilon\to 0_+}} \int_{-\infty}^{\varepsilon} d\tau P_\ell(\cos\gamma)\mathcal{T}^\ell_0(r_0,t,z^t), \tag{5.32}$$

$$F^{(+)}_{\alpha,\ell} \sim \frac{q^2 L}{r_0^2 f(r_0)}(\delta^\beta_\alpha + u_\alpha u^\beta) \lim_{\substack{x\to z(0)\\ \varepsilon\to 0_+}} \nabla_\beta \int_{-\infty}^{\varepsilon} d\tau P_\ell(\cos\gamma)\mathcal{S}^\ell_0(r_0,t,z^t), \tag{5.33}$$

for $\ell\to\infty$ with $\lim_{\ell\to\infty} V_\ell/\omega^2 < 1$, perform the τ integrals and take the leading ℓ at the end to get rid of any sub-leading terms. Instead, the evaluation of the above integrals is avoided by adopting the multi-scale expansion approach introduced in [1; 25]. Recalling the discussion in Chapter 2, the singular (as $x\to z$) contributions to Φ^{ret} and Φ^S are those coming form the "direct" part of the field (as opposed to the "tail" part), which at the $x\to z$ limit are sourced on the particle's position itself (i.e. Φ^S has no tail). Thus, the large-ℓ behaviour of $F^{(+)}_{r,\ell}$ must be obtainable by instead carrying out the τ integral over a small interval that includes the particle's current position. With this in mind, the following quantities are defined:

$$F^{(+)}_{r,\ell\varepsilon} \equiv \frac{Lq^2}{r_0^3 f(r_0)} \lim_{x\to z(0)} \int_{-\varepsilon}^{\varepsilon} d\tau P_\ell(\cos\gamma)\mathcal{T}^\ell_0(r_0,t,z^t), \tag{5.34}$$

$$F^{(+)}_{\alpha,\ell\varepsilon} \equiv \frac{Lq^2}{r_0^2 f(r_0)}(\delta^\beta_\alpha + u_\alpha u^\beta) \lim_{x\to z(0)} \nabla_\beta \int_{-\varepsilon}^{\varepsilon} d\tau P_\ell(\cos\gamma)\sum_{k=0}^{\infty}\mathcal{S}^\ell_0(r_0,t,z^t). \tag{5.35}$$

for a small positive ε. Inside the domain of integration, τ is small and one can expand quantities in powers of τ. However, since the ultimate goal is to obtain the leading-ℓ of the above expressions, one must be cautious about not losing information about the large-ℓ the self-force modes.

The definitions in [1; 25] are adopted for the construction of expansions for small τ and large ℓ. First, a new variable is defined as

$$\Lambda \equiv -L\tau. \tag{5.36}$$

Quantities are then expanded for large L and small τ while keeping Λ fixed. For large-ℓ, the Legendre function can be expanded as [41]

$$P_\ell(\cos\gamma) = \sqrt{\frac{\gamma}{\sin\gamma}}\left[J_0(\gamma L) + \frac{1}{8}\left(\cot\gamma - \frac{1}{\gamma}\right)\frac{J_1(\gamma L)}{L}\right] + O(L^{-2}). \tag{5.37}$$

By taking $\gamma = \phi - z^\phi(\tau) = -u^t\tau$, the above equation is further approximated by writing $\tau = -\Lambda/L$ and expanding for large L,

$$P_\ell(\cos(\phi - z^\phi(\tau)) = J_0(\Lambda u^\phi) + \frac{1}{24}\frac{2(\Lambda u^\phi)^2 J_0(\Lambda u^\phi) - \Lambda u^\phi J_1(\Lambda u^\phi)}{L^2} + O\left(L^{-3}\right). \tag{5.38}$$

The radial quantities, which depend on ℓ through $V_\ell(r)$ (and its derivatives) and on τ and ℓ through $\bar{t}$, are expanded by re-expressing this potential in terms of L,

$$V_\ell(r_0) = f(r)\frac{L^2}{r^2} - f(r)\frac{(r-8M)}{4r^2} \tag{5.39}$$

and by writing $(t - z^t) = -u^t\tau = \Lambda/L$. The functions $\mathcal{S}_0^\ell(r_0, t, z^t)$ and $\mathcal{T}_0^\ell(r_0, t, z^t)$ are expanded as

$$2\mathcal{S}_0^\ell(r_0, t, z^t) = J_0\left(\Lambda\tilde{u}^t\right) + \frac{(r_0 - 8M)}{8L^2 r_0}\tilde{u}^t\Lambda J_1\left(\tilde{u}^t\Lambda\right) + O\big(L^{-4}\big), \tag{5.40}$$

$$2\mathcal{T}_0^\ell(r_0, t, z^t) = -f(r)J_0\left(\Lambda\tilde{u}^t\right) - \frac{1}{4}\tilde{u}^t\left(rf' - 2f\right)\Lambda J_1\left(\tilde{u}^t\Lambda\right) + O\big(L^{-2}\big), \tag{5.41}$$

where $\tilde{u}^t \equiv u^t\sqrt{f(r_0)}/r_0$. Important information can be obtained by noticing that the expansions constructed for $\mathcal{S}_0^\ell(r_0, t, z^t)$, $\mathcal{T}_0^\ell(r_0, t, z^t)$ and $P_\ell(\cos\gamma)$ contain only terms that are even in Λ. By applying either ∇_t or ∇_ϕ to the integrand in equation (5.35), it becomes odd and vanishes when integrated over the symmetric interval. Thus, only $F^{(+)}_{r,\ell\epsilon}$ is non-vanishing. By changing the integration variable to Λ, one can write

$$\begin{aligned} F^{(+)}_{r,\ell\epsilon} \sim -\frac{q^2}{2r_0^3 f(r_0)}\Bigg[&\int_{-L\varepsilon}^{L\varepsilon} d\Lambda J_0(\Lambda u^\phi)\big(f(r)J_0\left(\Lambda\tilde{u}^t\right)\big) + \qquad (L\to\infty) \\ &+ \frac{1}{4}\tilde{u}^t\left(rf' - 2f\right)\int_{-L\varepsilon}^{L\varepsilon}\Lambda J_0(\Lambda u^\phi)J_1\left(\tilde{u}^t\Lambda\right)\Bigg]. \end{aligned} \tag{5.42}$$

The above expression depends on L only through the integration limits. Then, its leading-ℓ is obtained by taking the limit $L \to \infty$ with ε fixed (i.e. by replacing the integration limits with $\pm\infty$). Using the results in [33, (10..22.5),(10.43.26)] for the integrals involving products of Bessel functions, one can write

$$\int_{-\infty}^{\infty} d\Lambda\ J_0(\bar{u}^t\Lambda)J_0(u^\phi\Lambda) = \frac{2}{\bar{u}^t}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(u^\phi)^2}{(\bar{u}^t)^2}\right), \tag{5.43}$$

$$\int_{-\infty}^{\infty} d\Lambda\ \Lambda\ J_0(\bar{u}^t\Lambda)J_1(u^\phi\Lambda) = \frac{2}{\bar{u}^{t2}}\ {}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 1; \frac{(u^\phi)^2}{(\bar{u}^t)^2}\right). \tag{5.44}$$

Then, by collecting the pre-factors in equation (5.42), the parameter B_r for the circular geodesic is obtained from the $L \to \infty$ limit of the right-hand side of equation, as it yields an $O(L^0)$ contribution for the self-force modes,

$$\begin{aligned} B_r &\equiv \lim_{L\to\infty} F^{(+)}_{r,\ell\epsilon} \\ &= -\frac{q^2}{u^t\sqrt{f(r_0)}r_0^2}\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(r_0\Omega)^2}{f(r_0)}\right) - \frac{1}{2(u^t)^2 f(r_0)}\ {}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 1; \frac{(r_0\Omega)^2}{f(r_0)}\right)\right], \end{aligned} \tag{5.45}$$

where $\Omega \equiv u^\phi/u^t = \sqrt{M/r_0^3}$ is the orbital frequency of the particle. This result agrees with the results found in the literature [1; 42]. The remaining components of B_α vanish, since only $F^{(+)}_{r,\ell\epsilon}$ is non-vanishing.

Now, the reason why the restricted frequency integral gives correct results for the B_r regularization parameter is discussed. By integrating $G^{ret}(x, z(\tau))$ as in equation (2.61) over τ before performing the frequency integral, one is faced with the following integral:

$$\int_{-\infty}^{\infty} d\tau e^{i\omega z^t(\tau)} Y^*_{\ell m}\left(\frac{\pi}{2}, z^\phi(\tau)\right) = \frac{1}{u^t}\int_{-\infty}^{\infty} dt' e^{i(\omega - m\Omega)t'} Y^*_{\ell m}(\pi/2, 0), \tag{5.46}$$

$$= \frac{2\pi Y^*_{\ell m}(\pi/2, 0)}{u^t}\delta(\omega - m\Omega). \tag{5.47}$$

The last equality implies that the ℓ-th mode of the field receives contributions from frequencies satisfying $\omega = m\Omega \leq \ell\Omega$, which are verified to satisfy

$$\lim_{\ell\to\infty} \frac{\omega^2}{V_\ell(r_0)} \leq \frac{(r_0\Omega)^2}{f(r_0)} < 1 \quad \text{for} \quad r_0 > 3M, \tag{5.48}$$

so that all the relevant frequencies for stable circular geodesics lie inside the region where the large-ℓ asymptotics constructed from the WKB solutions are uniformly valid in ω. The derivation shown in this chapter also holds for circular orbits that are not geodesics, as long as the above condition is satisfied. Moreover, it is a fact that the regularization parameters obtained in this chapter through analysis of the retarded Green's function must match ones that would be obtained from an analysis of the Detweiler-Whiting Singular Green's function (2.12), as the self-fields associated with these two Green's functions must possess the same singular structure.

Chapter 6

SEMI-ANALYTICAL SCALAR SELF-FORCE REGULARIZATION IN SCHWARZSCHILD SPACETIME

A challenging aspect regarding the application of the mode-sum regularization method to the calculation of the self-force is the fact that, while the regularization itself is carried out at the level of the ℓ-modes of the self-force (i.e. in the time domain), the field modes that need to be regularized are usually obtained by solving decoupled field equations in the frequency domain. In particular, the radial Green's function modes (2.54) in the frequency domain can be written in terms of the MST *in* and *up* homogeneous solutions given in equations (4.47) and (4.65) (with $s = 0$). Due to the complicated ω dependence of these solutions, transformation of these Green's function modes into the time domain is not possible in analytical closed-form. Alternatively, one can numerically evaluate the ℓ-modes of the full self-force — i.e. the self-force calculated from the retarded field, prior to regularization (2.8) — and try to perform the regularization by numerically subtracting from them the regularization parameters. In principle, the subtraction of only the A_α and B_α parameters, which for the case of a circular geodesic are given by (5.31) and (5.45), is enough to obtain a convergent ℓ-sum for the self-force. However, in practice, this ℓ-sum exhibits very slow convergence if one does not include higher-order regularization parameters [21]. To overcome this problem without including these higher-order terms, the analytical post-Newtonian (pN) regularization method introduced by Hikida et al [3] will be applied to evaluate the contribution to the self-force coming from the self-force modes of *large enough* ℓ. As shall be made clear, for a calculation accurate to N pN orders, all self-force ℓ-modes satisfying $\ell \geq \mathrm{N}+1$ are described by a single general pN expression, which can be summed analytically from $\ell = N+1$ to $\ell = \infty$.

In the following sections, the method introduced by Hikida et al [3] is reviewed and their results regarding the application of this method to the calculation of the scalar self-force for a particle in a circular geodesic in Schwarzschild [43] are reproduced. The calculation done in this work differs slightly from the one in Hikida et al. [43] by the fact that the self-force modes that are not covered by the generic pN expression (i.e. the modes with $\ell < N+1$) are evaluated numerically, while in their work, individual pN expansions are written for each one of these modes.

6.1 FRAMEWORK FOR ANALYTICAL REGULARIZATION

Throughout this chapter, the MST solutions in series of Coulomb functions satisfying the *in* (4.65) and *up* (4.47) boundary conditions are adopted for the construction of the radial modes of the retarded Green's function (2.54). The radial modes in equation (2.54) are re-expressed by explicitly writing ${}_0R^{in}_{\ell m\omega}(r) = R^{in}_C(r)$ and ${}_0R^{up}_{\ell m\omega}(r) = R^{up}_C(r)$,

$$g_{\ell m\omega}(r,r') = \frac{-1}{W^{in/up}_{\ell m\omega}}\big(R^{in}_C(r)R^{up}_C(r')\Theta(r'-r) + R^{up}_C(r)\ R^{in}_C(r')\Theta(r-r')\big), \tag{6.1}$$

where

$$R^{in}_C(r) = K_\nu R^\nu_C(r) + K_{-\nu-1}R^{-\nu-1}_C(r), \tag{6.2}$$

$$R^{up}_C(r) = \gamma_\nu R^\nu_C(r) + \delta_\nu R^{-\nu-1}_C(r). \tag{6.3}$$

Even though the same notation is maintained, all of the MST quantities discussed in this chapter are to be considered specializations for $s = 0$ of the ones constructed in Chapter 4. By substituting the above equations into (6.1), one can express the radial Green's function modes in terms of the two independent Coulomb series solutions R^ν_C and $R^{-\nu-1}_C(r)$. When doing so, due to the fact that R^{in}_C and R^{up}_C are linear combinations of the same two basis functions, many of the terms that arise end up not carrying Theta functions (i.e. the identity $\Theta(x)+\Theta(-x) = 1$ appears). Then, by collecting these terms, the following separation of the Green's function modes is introduced [3]:

$$g_{\ell m\omega}(r,r') = g^{\tilde{R}}_{\ell m\omega}(r,r') + g^{\tilde{S}}_{\ell m\omega}(r,r'), \tag{6.4}$$

$$g^{\tilde{R}}_{\ell m\omega}(r,r') \equiv \frac{-1}{(K_\nu \delta_\nu - K_{-\nu-1}\gamma_\nu)W^{\nu/-\nu-1}} \Big\{ \delta_\nu K_{-\nu-1} R_C^{-\nu-1}(r) R_C^{-\nu-1}(r') + \tag{6.5}$$

$$+ \gamma_\nu K_\nu R_C^\nu(r) R_C^\nu(r') + \gamma_\nu K_{-\nu-1}\big(R_C^{-\nu-1}(r) R_C^\nu(r') + R_C^{-\nu-1}(r') R_C^\nu(r)\big) \Big\},$$

$$g^{\tilde{S}}_{\ell m\omega}(r,r') \equiv \frac{-1}{W^{\nu/-\nu-1}} \big(R_C^\nu(r) R_C^{-\nu-1}(r')\Theta(r'-r) + R_C^{-\nu-1}(r) R_C^\nu(r')\Theta(r-r') \big), \tag{6.6}$$

where

$$W^{\nu/-\nu-1} \equiv \Delta \mathcal{W}(R_C^\nu, R_C^{-\nu-1}), \tag{6.7}$$

and $\mathcal{W}(R_C^\nu, R_C^{-\nu-1})$ is the Wronskian of the two independent solutions calculated as in equation (2.55). One can see that $g^{\tilde{R}}_{\ell m\omega}(r,r')$ is a linear combination of homogeneous solutions to the radial equation and, therefore, is itself a homogeneous solution. Thus, contributions from this part of the Green's function modes to the self-force must be regular.

In the context of the separation of the radial Green's function modes shown in equations (6.4)–(6.6), the full self-force is expressed as

$$F_\alpha = F_\alpha^{\tilde{R}} + F_\alpha^{\tilde{S}}, \tag{6.8}$$

where $F_\alpha^{\tilde{R}}$ and $F_\alpha^{\tilde{S}}$ are the contributions to the self-force calculated from $g^{\tilde{R}}_{\ell m\omega}(r,r')$ and $g^{\tilde{S}}_{\ell m\omega}(r,r')$, respectively. These are defined by replacing $g_{\ell m\omega}(r,r')$ for either $g^{\tilde{R}}_{\ell m\omega}(r,r')$ or $g^{\tilde{S}}_{\ell m\omega}(r,r')$ in equation (2.61) and calculating the respective self-force contributions according to equation (2.7). Next, this separation is compared to the Detweiler-Whiting decomposition presented in Chapter 2, in which the full self-force is separated as in equation (2.40),

$$F_\alpha = F_\alpha^R + F_\alpha^S. \tag{6.9}$$

Since F_α is independent of the particular scheme adopted, the physical regularized self-force, F_α^R, can be expressed in terms of the remaining quantities by equating (6.8) and (6.9),

$$F_\alpha^R = F_\alpha^{\tilde{R}} + F_\alpha^{\tilde{S}} - F_\alpha^S. \tag{6.10}$$

The fact that both F_α^R and $F_\alpha^{\tilde{R}}$ are regular implies that the singular contributions coming from F_α^S must be completely canceled out by those coming from $F_\alpha^{\tilde{S}}$. That is, $F_\alpha^{\tilde{S}-S} \equiv F_\alpha^{\tilde{S}} - F_\alpha^S$ must be regular. This fact constrains the ℓ-modes of $F_\alpha^{\tilde{S}}$ to have the following structure:

$$F_{\alpha,\ell}^{\tilde{S}} = \pm A_\alpha L + B_\alpha + \tilde{D}_{\alpha,\ell}, \tag{6.11}$$

where A_α and B_α are exactly the same regularization parameters as those coming from $F_\alpha^{\tilde{S}}$ in equation (2.45). The quantity $\tilde{D}_{\alpha,\ell}$ is at most $O(\ell^{-2})$ and is not constrained to be equal to the parameter $D_{\alpha,\ell}$ in equation (2.45), since the ℓ-sums of both of these quantities converge. In fact, since $\sum_{\ell=0}^{\infty} D_{\alpha,\ell} = 0$, the $\tilde{D}_{\alpha,\ell}$ terms give the only non-vanishing contributions to $F_\alpha^{\tilde{S}-S}$,

$$F_\alpha^{\tilde{S}-S} \equiv \sum_{\ell=0}^{\infty} \left(F_{\alpha,\ell}^{\tilde{S}} - F_{\alpha,\ell}^S \right) = \sum_{\ell=0}^{\infty} \tilde{D}_{\alpha,\ell}. \tag{6.12}$$

Similar to the separation done in the previous chapter, the modes of $g_{\ell\omega}^{\tilde{S}}(r,r')$ will be split into symmetric and anti-symmetric parts,

$$g_{\ell\omega}^{\tilde{S}}(r,r') = \text{sign}(r-r') g_{\ell\omega}^{\tilde{S}(-)}(r,r') + g_{\ell\omega}^{\tilde{S}(+)}(r,r'), \tag{6.13}$$

$$g_{\ell\omega}^{\tilde{S}(\pm)}(r,r') \equiv \frac{-1}{2W_{\ell\omega}^{\nu/-\nu-1}} \left(R_C^{\nu}(r) R_C^{-\nu-1}(r') \pm R_C^{\nu}(r') R_C^{-\nu-1}(r) \right). \tag{6.14}$$

As proven in [3, Appendix B], the contribution to self-force modes coming from the $g_{\ell\omega}^{\tilde{S}(-)}(r,r')$ part of the radial Green's function modes yield exactly the A_α regularization parameter for any orbit. This allows for $F_\alpha^{\tilde{S}-S}$ to also be expressed as

$$F_\alpha^{\tilde{S}-S} = \sum_{\ell=0}^{\infty} F_{\alpha,\ell}^{\tilde{S}(+)} - B_\alpha, \tag{6.15}$$

where $F_{\alpha,\ell}^{\tilde{S}(+)}$ is the self-force contribution calculated from the $g_{\ell\omega}^{\tilde{S}(+)}(r,r')$ modes, which is led by B_α for large ℓ. Thus, the obtention of only the $F_{\alpha,\ell}^{\tilde{S}(+)}$ modes together with B_α is enough for the calculation of the $F_\alpha^{\tilde{S}-S}$ contribution to F_α^R.

6.2 POST-NEWTONIAN EXPANSIONS

With the goal of rendering possible the analytical integration of the Green's function modes over frequency, post-Newtonian expansions for these quantities shall be constructed. A post-Newtonian expansion is said to be of order N (and may also referred to as an N PN expansion) if contains terms up $O(v^{2\mathrm{N}})$, where v is the velocity of the particle. In the slow motion approximation, one takes $\hat{z} = \omega r \sim \Omega r = O(v)$ (for a circular geodesic, the second relation becomes an equality). This construction also implies that $\epsilon/\hat{z} = 2M/r = O(v^2)$, which corresponds to a *weak-field* expansion.

At this point, it is convenient to perform a change in the normalization of the homogeneous solutions so that, except for the overall factor of $\hat{z}^\nu$ in equation (4.35), the leading order in the pN expansion of the solutions with the new normalization is unit. Solutions with the new normalization are defined as:

$$\phi_C^\nu \equiv \frac{\Gamma(2\nu+2)}{\Gamma(\nu+1+i\epsilon)} R_C^\nu, \tag{6.16}$$

and the new normalization for the other independent solution is found by replacing $\nu \to -\nu-1$ in the above equation. The Green's functions modes in equations (6.5)–(6.6) can easily be re-expressed in terms of the solutions with this new normalization. Next, the following quantities are defined:

$$\Phi^\nu \equiv \sum_{n=-\infty}^{\infty} a_n^\nu \Phi_n^\nu, \qquad\qquad \phi_C^\nu = (2\hat{z})^\nu \Phi^\nu. \tag{6.17}$$

$$\Phi_n^\nu \equiv e^{-i\hat{z}}\left(1-\frac{\epsilon}{\hat{z}}\right)^{-i\epsilon}\left[\frac{(\nu+1-i\epsilon)_n}{(2\nu+2)_{2n}}(-2i\hat{z})^n \; {}_1F_1(n+\nu+1+i\epsilon, 2n+2\nu+2; 2i\hat{z})\right]. \tag{6.18}$$

The analogous quantities referring to the other independent solution are obtained by replacing $\nu \to -\nu-1$ in the above expressions.

6.2.1 Properties of the three-term recurrence relation

To start tackling the construction of pN expansions for the independent MST solutions and ultimately for the Green's function modes, the three-term recurrence relation (4.10) that determines the coefficients a_n^ν and the renormalized angular momentum ν shall be studied. Following the analysis in [2; 37], expansions for a_n^ν and ν in powers of ϵ can be obtained by perturbatively solving the recurrence relation about $\epsilon = 0$ subject to the requirement that $\nu \to \ell$ for $\epsilon \to 0$. An important aspect of this analysis is that generic series expansions for the recurrence relation coefficients (4.11)–(4.13) are not uniform in n and ℓ, meaning that a general formula for the expansions of a_n^ν and ν valid for every n and ℓ cannot be written. The behaviour of these coefficients shall now be studied for fixed integer $\ell > 0$ (as shall be seen, $\ell = 0$ needs to be handled separately).

The following statements can be made about the coefficients:

$$\alpha_n^\nu = i\epsilon\frac{(n+\ell+1)^2}{2\ell+2n+3} + O(\epsilon^2) = O(\epsilon) \qquad \text{for } n \neq -\ell-1, \tag{6.19}$$

$$\beta_n^\nu = n(n+2\ell+1) + O(\epsilon) = O(1) \qquad \text{for } n \neq -2\ell-1,\ n\neq 0, \tag{6.20}$$

$$\gamma_n^\nu = i\epsilon\frac{(\ell+n)^2}{2\ell+2n-1} + O(\epsilon^2) = O(\epsilon). \qquad \text{for } n \neq -\ell. \tag{6.21}$$

Whenever the coefficients are given by the non-special cases in the above equations, it will be said that they are *regular behaved.* For $n > 0$, no special value arises and the continued fraction equation (4.14) schematically acquires the form:

$$R_n^\nu = \frac{O(\epsilon)}{O(1) + O(\epsilon)\frac{O(\epsilon)}{O(1)+\ldots}} = O(\epsilon) \qquad \text{for } n > 0, \tag{6.22}$$

which implies that every next coefficient of the series in the direction of positive n acquires a factor of ϵ,

$$a_n^\nu = O(\epsilon^n) \qquad \text{for } n \geq 0. \tag{6.23}$$

In this case, the coefficients a_n^ν and the continued fraction R_n^ν will also be said to be *regular behaved.*

Before tackling the study of the coefficients for $n < 0$, a constrain on ν can be obtained by demanding that equation (4.18) is satisfied for $n = 0$,

$$R_1^\nu L_0^\nu = 1, \tag{6.24}$$

Since $R_1^\nu = O(\epsilon)$, one must have

$$L_0^\nu = -\frac{\alpha_0^\nu}{\beta_0^\nu + \gamma_0^\nu L_{-1}^\nu} = O(\epsilon^{-1}) \tag{6.25}$$

for this equality to hold. Furthermore, the fact that $\alpha_0^\nu = O(\epsilon)$ (6.19), implies that the denominator of this continued fraction equation must be $O(\epsilon^2)$. Accounting for the expressions for these coefficients, this can only happen in two scenarios: either $\beta_0^\nu = O(\epsilon)$ and $L_{-1}^\nu = O(1)$ but the leading order terms in the denominator cancel out (for any ℓ) or β_0^ν and L_{-1}^ν are at most $O(\epsilon^2)$ and $O(\epsilon)$, respectively, and no cancellation is required. The latter case will be taken as assumption and later checked for consistency. The fact that $\beta_0^\nu = O(\epsilon^2)$ implies that $-\ell(\ell+1)+\nu(\nu+1)$ in equation (4.12) is $O(\epsilon^2)$ or, in other words,

$$\nu = \ell + O(\epsilon^2). \tag{6.26}$$

Using this relation, one can determine the leading order in ϵ of the coefficients and of L_n^ν for the special values of n in (6.21). The behaviour of L_n^ν for any given $n < 0$ and for $\ell \neq 0$ is found to be [2]:

$$L_n^\nu = \begin{cases} O(\epsilon^2) & \text{for } n = -\ell - 1, \\ O(1/\epsilon) & \text{for } n = -2\ell - 1, \\ O(\epsilon) & \text{for all other } n < 0. \end{cases} \tag{6.27}$$

One can check that the assumption $L_{-1}^\nu = O(\epsilon)$ is consistent with these expressions. Then, each next coefficient in the negative n direction acquires a factor of ϵ until it reaches an *irregular behaved* $L_{-\ell-1}^\nu$,

$$a_n^\nu = O(\epsilon^{-n}) \qquad\qquad \text{for } -\ell - 1 < n < 0. \tag{6.28}$$

As in the positive case, these are said to have *regular behaviour*. An important fact is that one can calculate coefficients up to $a_{-\ell}^\nu = O(\epsilon^\ell)$ before encountering coefficients with *irregular behaviour*. Then, if one is to write an expression that needs expansions of the coefficients up to $O(\epsilon^\ell)$, no knowledge of the *irregular behaved* coefficients is needed, as they are of too high order. When building the pN expansions for the independent MST solutions, this fact will allow for the writing of a general pN formula for Φ^ν valid for *large enough* ℓ compared to the desired pN order, meaning ℓ values for which the contributions associated with *irregular behaved* coefficients appear beyond the given pN order.

6.2.2 Expansions for *large enough* ℓ

Now, the construction of generic pN expansions for $g_{\ell\omega}^{\tilde{S}}(r, r')$ and $g_{\ell\omega}^{\tilde{R}}(r, r')$ is discussed. To do so, it will be useful to re-write equation (6.18) for Φ_n^ν by expressing the confluent hypergeometric function in its series representation (4.57),

$$\Phi_n^\nu = e^{-i\hat{z}}\left(1 - \frac{\epsilon}{\hat{z}}\right)^{-i\epsilon}\left[(-2i\hat{z})^n \frac{(\nu+1-i\epsilon)_n}{(2\nu+2)_{2n}} \sum_{j=0}^{\infty} \frac{(n+\nu+1+i\epsilon)_j}{(2n+2\nu+2)_j j!}(2i\hat{z})^j\right]. \tag{6.29}$$

One can note that the leading term in the pN expansion of this quantity is proportional to $\hat{z}^n = O(v^n)$. For $n \geq 0$, equation (6.23) states that $a_n^\nu = O(\epsilon^n) = O(v^{3n})$, which implies $a_n^\nu \Phi_n^\nu = O(v^{4n})$. Thus, to obtain an expansion for Φ^ν containing terms up to $O(v^{2\mathrm{N}})$ the sum over positive n can be halted at $n = \mathrm{N}/2$. For negative n, the last *regular behaved* coefficient is $a_{-\ell}^\nu$. In the interval $-\ell \leq n < 0$, one has $a_{-n}^\nu = O(\epsilon^n) = O(v^{3n})$, which implies $a_{-n}^\nu \Phi_{-n}^\nu = O(v^{2n})$. Therefore, an expansion for Φ^ν obtained only with knowledge of the *regular behaved* coefficients is accurate up to the ℓ-th pN order. For $\Phi^{-\nu-1}$, one order of accuracy is lost due to factors of $1/\epsilon$ arising from expanding the Pochhammer symbols about negative integer values. In this case, the expansion containing only *regular behaved* coefficients is accurate to $(\ell-1)$-th pN order. Thus, for a calculation accurate to the N-th pN order, all modes satisfying $\ell \geq N+1$ are considered *large enough* so that pN expansions of both independent solutions contain only contributions from *regular behaved* coefficients.

To obtain expansions for the renormalized angular momentum, one can express the three-term recurrence relation (4.10) for $n=0$ as a continued fraction equation,

$$\alpha_0^\nu R_1^\nu + \beta_0^\nu + \gamma_0^\nu L_{-1}^\nu = 0, \tag{6.30}$$

expand it in powers of ϵ and solve it for the expansion coefficients of ν. General expansions can be obtained by assuming *regular behaviour* up to a given negative n. For example, by assuming *regular behaviour* up to $n=-2$, a generic expansion for ν accurate to $O(\epsilon^4)=O(v^{12})$ is obtained:

$$\nu = \ell + \sum_{k=1}^{2} \nu_{2k}\epsilon^{2k}, \tag{6.31}$$

$$\nu_2 = -\frac{15\ell^2+15\ell-11}{2(2\ell-1)(2\ell+1)(2\ell+3)}, \tag{6.32}$$

$$\nu_4 = -\frac{1}{8\ell(\ell+1)(2\ell-3)(2\ell-1)^3(2\ell+1)^3(2\ell+3)^3(2\ell+5)}\Big(18480\ell^{10}+92400\ell^9+9800\ell^8+ \\ -235200\ell^7-382305\ell^6+64365\ell^5+278260\ell^4-9955\ell^3-73892\ell^2+8733\ell+3240\Big),$$

These expressions of course do not hold for $\ell=0$ and $\ell=1$, since either L_{-1}^ν or L_{-2}^ν are not *regular behaved* in these cases. The inclusion of higher-order terms in this expansion is relatively easy, but for each next-order correction added, the general formula becomes invalid for the next ℓ-mode (e.g. to obtain a general expression for the $O(\epsilon^6)$ correction, one would have to assume that L_{-3}^ν is *regular behaved*, thus rendering the general form of this correction invalid for $\ell \leq 2$).

In possession of an expansion for ν, expansions for the series coefficients a_n^ν are obtained by expanding equations (4.19) and (4.20) in powers of ϵ. To obtain generic pN expansions for Φ^ν valid for $N \leq \ell$, one can set $\hat{z} \to v\hat{z}$ and $\epsilon \to v^3\epsilon$ and expand Φ^ν in powers of v, truncating at order $O(v^{2\mathrm{N}})$. Here, the generic formula for the expansion of

Φ^ν including terms up to third pN order (3PN) is explicitly written:

$$\Phi^\nu = 1 - \left[\frac{\hat{z}^2}{2(2\ell+3)} + \frac{\ell\epsilon}{2\hat{z}}\right] + \left[\frac{\hat{z}^4}{8(2\ell+3)(2\ell+5)} + \frac{\left(\ell^2-5\ell-10\right)\hat{z}\epsilon}{4(\ell+1)(2\ell+3)} + \right.$$
$$\left. + \frac{\ell(\ell-1)^2\epsilon^2}{4(2\ell-1)\hat{z}^2}\right] + \left[\frac{\left(4\ell^2+46\ell+135\right)\hat{z}^6}{720(2\ell+3)(2\ell+5)} - \frac{\left(3\ell^3-27\ell^2-142\ell-136\right)\hat{z}^3\epsilon}{48(\ell+1)(\ell+2)(2\ell+3)(2\ell+5)} + \right.$$
$$\left. - \frac{\left(\ell^3-18\ell^2+17\ell-4\right)\epsilon^2}{8(2\ell-1)^2} - \frac{\ell(\ell-2)^2(\ell-1)\epsilon^3}{24(2\ell-1)\hat{z}^3}\right] + O(v^8) \tag{6.33}$$

Terms of higher order are given in Appendix B. This particular expression is valid for $\ell \geq 3$. An expansion for $\Phi^{-\nu-1}$ valid for $\ell \geq 4$ is easily obtained by replacing $\ell \to -\ell-1$ in the above expression.

Next, the construction of pN expansions for $g^{\tilde{S}}_{\ell m\omega}(r,r')$ and $g^{\tilde{R}}_{\ell m\omega}(r,r')$ from the expansions of Φ^ν and $\Phi^{-\nu-1}$ is discussed. From the fact that $\Phi^\nu = O(1)$ and $\Phi^{-\nu-1} = O(1)$, one has $\phi^\nu_C = O(v^\ell)$ and $\phi^{-\nu-1}_C = O(v^{-\ell-1})$. When recovering the original normalization of the homogeneous solutions through the inversion of equation (6.16), the solution $R^{-\nu-1}_C$ acquires a factor of ϵ. Thus, $R^\nu_C = O(v^\ell)$ and $R^{-\nu-1}_C = O(v^{-\ell+2})$. The leading contribution to the Wronskian is found to be:

$$\frac{\mathcal{W}(R^\nu_C, R^{-\nu-1}_C)}{\omega} = -\frac{\Gamma(\nu+1+i\epsilon)}{\Gamma(2\nu+2)}\frac{\Gamma(-\nu+i\epsilon)}{\Gamma(-2\nu)}\left[\frac{(2\ell+1)}{\hat{z}^2} + O(\epsilon^2)\right] = O(v) \tag{6.34}$$

Then, the denominator in the equation for $g^{\tilde{S}}_{\ell\omega}(r,r')$ (6.6) satisfies $\omega W^{\nu/-\nu-1} = O(\epsilon)$, which implies:

$$g^{\tilde{S}}_{\ell m\omega}(r,r') = O(1). \tag{6.35}$$

This fact implies that all ℓ-modes of $g^{\tilde{S}}_{\ell m\omega}(r,r')$ up to $\ell = \infty$ are of the same order in the pN expansion, meaning that the ℓ-sum cannot be truncated at some finite ℓ. This shall not be a problem since expansions for all modes satisfying $\ell \geq \mathrm{N}+1$ can be retrieved from the generic pN formulas for the homogeneous solutions as the one shown in equation (6.33).

To understand how many ℓ-modes of $g^{\tilde{R}}_{\ell m\omega}(r,r')$ need to be computed, one needs to obtain expansions for the remaining quantities in equation (6.5). By replacing ν with $\nu = \ell + O(\epsilon^2)$ in equations (4.48)–(4.49) and (4.63) and expanding for small ϵ, the following relations are obtained:

$$\gamma_\nu = O(1); \qquad \delta_\nu = O(1); \qquad K_\nu = O(v^{-3\ell-6}); \qquad K_{-\nu-1} = O(v^{3\ell}). \tag{6.36}$$

With the use of these relations, the leading term in the pN expansion of $g^{\tilde{R}}_{\ell m\omega}(r,r')$ is found to be

$$g^{\tilde{R}}_{\ell m\omega}(r,r') \sim -\frac{\gamma_\nu R^\nu_C(r) R^\nu_C(r')}{\delta_\nu W^{\nu/-\nu-1}} = O(v^{2(\ell-1)}). \tag{6.37}$$

Thus, one needs to obtain $\mathrm{N}+1$ ℓ-modes of $g^{\tilde{R}}_{\ell\omega}(r,r')$ for a calculation accurate to N pN orders.

At this point, it is worth highlighting a remarkable property of $g^{\tilde{S}}_{\ell m\omega}(r,r')$. As seen in equation (6.33), apart from the factors of $\hat{z}^\nu$ and $\hat{z}^{-\nu-1}$, the pN expansions for the homogeneous Coulomb series solutions depend on the Fourier-frequency only through integer (and also even) powers of ω. While the overall factors of $\hat{z}^\nu$ and $\hat{z}^{-\nu-1}$ contain terms like $\log(\hat{z})$ in their pN expansions, no $\log(\omega)$ term appears in the pN expansion of $g^{\tilde{S}}_{\ell m\omega}(r,r')$, since these modes only depend on the homogeneous solutions through combinations of the form $R^\nu_C(r)R^{-\nu-1}_C(r')$, meaning that the factors of ω^ν and $\omega^{-\nu}$ cancel out (i.e. $\hat{z}^\nu \hat{z}'^{-\nu-1} = \omega^{-1} r^\nu r'^{-\nu-1}$). This implies that pN expansions of $g^{\tilde{S}}_{\ell m\omega}(r,r')$ are easily transformed analytically to the time domain. The other part of the radial Green's function modes, $g^{\tilde{R}}_{\ell\omega}(r,r')$, do not share this property, since it contains other combinations of $R^\nu_C(r)$ and $R^{-\nu-1}_C(r')$. However, since $g^{\tilde{R}}_{\ell\omega}(r,r')$ is associated with regular contributions to the self-force, there is no issue in performing a full numerical evaluation of the ℓ-modes of $F^{\tilde{R}}_\alpha$, as the ℓ-sum of these should converge exponentially.

6.3 THE SELF-FORCE FOR A PARTICLE IN A CIRCULAR GEODESIC

In addition to a general formula for the pN expansion of $\Phi^{(\nu/-\nu-1)}$ for *large enough* ℓ, a full analytical evaluation of $F^{\tilde{S}-S}_\alpha$ accurate to N pN orders requires the obtention of other N+1 expansions for the modes that are not covered by the generic formula. Instead of obtaining such expansions, a semi-analytical calculation scheme shall be adopted. This approach differs from Hikida et al (2005) [43] by the fact that they obtain individual pN expansions for each one of the $\ell < \mathrm{N}+1$ modes, while these are evaluated and regularized numerically in this work. In principle, one can start the analytical evaluation with the use of the generic pN formulas at $\ell = N+1$. However, by instead starting at any $\ell \geq \mathrm{N}+2$, the computation of pN expansions for $g^{\tilde{R}}_{\ell m\omega}(r,r')$ is bypassed, since equation (6.37) implies that the leading term in the pN expansion of these modes will be at most $O(v^{2\mathrm{N}+2})$. With this in mind, the following regularization procedure is proposed:

$$F^R_\alpha = F^{R,\mathrm{num}}_\alpha + F^{R,\mathrm{pN}}_\alpha, \tag{6.38}$$

$$F_\alpha^{R,\mathrm{num}} \equiv \sum_{\ell=0}^{\ell_0-1}\bigg(F_{\alpha,\ell} - A_\alpha L - B_\alpha\bigg), \tag{6.39}$$

$$F_\alpha^{R,\mathrm{pN}} \equiv \sum_{\ell=\ell_0}^{\infty} F_{\alpha,\ell}^{\tilde{S}-S} = \sum_{\ell=\ell_0}^{\infty}\bigg(F_{\alpha,\ell}^{\tilde{S}(+)} - B_\alpha\bigg), \tag{6.40}$$

where ℓ_0 is some integer satisfying $\ell_0 \geq \mathrm{N}+2$. The evaluation of $F_\alpha^{R,\mathrm{pN}}$ shall be carried out analytically with the use of the general pN formulas, while the contributions from the low-ℓ modes, $F_\alpha^{R,\mathrm{num}}$, are to be evaluated numerically from equation (5.4) with the Green's function radial modes given in terms of the MST series solutions, which are truncated after some desired precision is reached. For the circular geodesic calculation, the regularization of the low ℓ-modes of the r-component of the self-force will be performed by numerically subtracting the A_r and B_r parameters given in equations (5.31) and (5.45). The remaining components require no regularization, meaning that they can be trivially calculated numerically and shall not be discussed in this work.

To obtain a pN expansion for $F_{\alpha,\ell}^{\tilde{S}(+)}$, $g_{\ell m\omega}^{\tilde{S}(+)}(r,r')$ is expressed as a sum over powers of ω by collecting terms from all pN orders,

$$g_{\ell m\omega}^{\tilde{S}(+)}(r,r') = \sum_{k=0}^{\infty}\omega^{2k}\mathrm{g}_{\ell mk}^{\tilde{S}(+)}(r,r'), \tag{6.41}$$

where $\mathrm{g}_{\ell mk}^{\tilde{S}(+)}(r,r')$ has been implicitly defined in the above equation as the coefficient to the $2k$-th power of ω in the pN expansion of $g_{\ell m\omega}^{\tilde{S}(+)}(r,r')$. These modes are transformed into the time-domain by using the formula:

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\omega^{2k}\mathrm{g}_{\ell mk}^{\tilde{S}(+)}(r,r') = (-1)^k\delta^{(k)}(t-t')\mathrm{g}_{\ell mk}^{\tilde{S}(+)}(r,r'), \tag{6.42}$$

where $\delta^{(n)}(x)$ stands for the n-th derivative of the Delta function, $\delta^{(n)}(x) \equiv \partial_x^n\delta(x)$. For a generic geodesic orbit, the $F_{\alpha,\ell}^{\tilde{S}(+)}$ contribution to the self-force modes at $x = z(\tau = 0)$ is given by

$$F_{\alpha,\ell}^{\tilde{S}(+)} = 4\pi q^2(\delta_\alpha^\beta + u_\alpha u^\beta)\lim_{x\to z(0)}\nabla_\beta\sum_{m=-\ell}^{\ell}\sum_{k=0}^{\infty}\int_{-\infty}^{0_+}d\tau\Bigg((-1)^k\delta^{(k)}(t-z^t(\tau)) \times \mathrm{g}_{\ell mk}^{\tilde{S}(+)}(r,z^r(\tau))Y_{\ell m}(\theta,\varphi)Y_{\ell m}^*(z^\theta(\tau),z^\varphi(\tau))\Bigg), \tag{6.43}$$

where the worldline integration is easily performed by changing variables to z^t,

$$F^{\tilde{S}(+)}_{\alpha,\ell} = \lim_{x\to z(0)} 4\pi q^2(\delta^\beta_\alpha + u_\alpha u^\beta)\nabla_\beta \sum_{m=-\ell}^{\ell}\sum_{k=0}^{\infty}\left\{(-1)^k\partial_t^{2k}\left[\frac{d\tau(t)}{dt}\times \right.\right. \tag{6.44}$$
$$\left.\left. \times\, \mathrm{g}^{\tilde{S}(+)}_{\ell mk}(r, z^r(\tau))Y_{\ell m}(\theta,\varphi)Y^*_{\ell m}(z^\theta(t), z^\varphi(t))\right]\right\}.$$

Now, equation (6.44) is specialized to a circular equatorial geodesic described by the relations (5.1),

$$F^{\tilde{S}(+)}_{\alpha,\ell} = \lim_{x\to z(0)} \frac{4\pi q^2}{u^t}(\delta^\beta_\alpha + u_\alpha u^\beta)\nabla_\beta \sum_{m=-\ell}^{\ell}\sum_{k=0}^{\infty}\left\{(-1)^k \mathrm{g}^{\tilde{S}(+)}_{\ell k}(r, r_0)\right. \tag{6.45}$$
$$\left. \times\, \partial_t^{2k}Y_{\ell m}(\pi/2,\varphi)Y^*_{\ell m}(\pi/2,\Omega t)\right\}.$$

To perform the summation over m, the formula given in [3] (with the correction found in [44]) is used:

$$\sum_{m=-\ell}^{\ell} m^{2n}\left|Y_{\ell m}\left(\frac{\pi}{2},\varphi\right)\right|^2 = \lambda_n(\ell), \qquad \text{for integer } n, \tag{6.46}$$
$$\sum_{m=-\ell}^{\ell} m^{2n+1}\left|Y_{\ell m}\left(\frac{\pi}{2},\varphi\right)\right|^2 = 0, \qquad \text{for integer } n, \tag{6.47}$$

where $\lambda_n(\ell)$ is implicitly defined by Taylor expanding

$$\sum_{n=0}^{\infty}\frac{\lambda_n(\ell)x^{2n}}{(2n)!} = \frac{2\ell+1}{4\pi}e^{\ell x}{}_2F_1\left(1/2,-\ell;1;1-e^{-2x}\right) \tag{6.48}$$

about $x = 0$ and equating the coefficients to each power of x. One can see that the t and φ components of the self-force modes in equation (6.45) vanish, since one would be taking an odd amount of derivatives of the spherical harmonic functions. Thus, the only non-vanishing component of $F^{\tilde{S}(+)}_{\alpha,\ell}$ is

$$F^{\tilde{S}(+)}_{r,\ell} = \frac{4\pi q^2}{u^t}\sum_{m=-\ell}^{\ell}\sum_{k=0}^{\infty}\Omega^{2k}\left\{\partial_r \mathrm{g}^{\tilde{S}(+)}_{\ell k}(r,r_0)|_{r=r_0} m^{2k}|Y_{\ell m}(\pi/2,\Omega t)|^2\right\}.$$
$$= \frac{4\pi q^2}{u^t}\sum_{k=0}^{N}\Omega^{2k}\lambda_k(\ell)\partial_r \mathrm{g}^{\tilde{S}(+)}_{\ell k}(r,r_0)|_{r=r_0} \tag{6.49}$$

Here, the first few orders of the pN expansion of $F_{r,\ell}^{\tilde{S}(+)}$, calculated from the general pN formula (6.33), are written

$$F_{r,\ell}^{\tilde{S}(+)} = -\frac{q^2}{2r_0^2u^t}\left[1-\left(\frac{2M}{r_0}-\frac{\ell(\ell+1)}{(2\ell-1)(2\ell+3)}\right)(r_0\Omega)^2+ \left(\frac{6\left(3\ell^2+3\ell-2\right)}{(4\ell^2+4\ell-3)}\left(\frac{M}{r_0}\right)^2-\frac{9\ell(\ell+1)\left(3\ell^2+3\ell-2\right)}{4(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}\right)(r_0\Omega)^4\right]+O(v^6). \tag{6.50}$$

Since $F_{r,\ell}^{\tilde{S}(+)}$ is led by B_r for large ℓ. A pN expansion for B_r can be obtained by calculating $\lim_{\ell\to\infty}F_{\alpha,\ell}^{\tilde{S}(+)}$,

$$B_r = -\frac{q^2}{2r_0^2u^t}\left[1-2\left(\frac{M}{r_0}-\frac{(r_0\Omega)^2}{4}\right)+\left(\frac{9M^2}{2r_0^2}-\frac{27}{64}(r_0\Omega)^4\right)\right]+O(v^6). \tag{6.51}$$

Then, by subtracting the above terms from $F_{\alpha,\ell}^{\tilde{S}(+)}$, a pN expansion for $F_{\alpha,\ell}^{\tilde{S}-S}$ is obtained,

$$F_{r,\ell}^{\tilde{S}-S} = \frac{q^2}{r_0^2u^t}\left[\frac{3(r_0\Omega)^2}{8(2\ell-1)(2\ell+3)}-\frac{3}{4\left(4\ell^2+4\ell-3\right)}\left(\frac{M}{r_0}\right)^2+ +\left(\frac{9\left(184\ell^2+184\ell-135\right)}{128(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}\right)r_0^4\Omega^4\right]+O(v^6). \tag{6.52}$$

In Appendix B, pN expansions for the above quantities accurate up to five pN orders are given. The 8PN expansions that are used for the upcoming calculations shall be made available in a *Mathematica* notebook by the date of this thesis defense.

To estimate the error of a N-th pN order expansion of $F_{r,\ell}^{\tilde{S}-S}$, one can calculate the (N + 1)-th order correction to that expression. In Figure, 6.1 a plot of the estimated error of a 7PN expansion of $F_{r,\ell}^{\tilde{S}-S}$ as a function of ℓ and $q=\sqrt{4\pi}$ and $r_0=6M$ is shown. One can see that significant accuracy in the final result may be gained by starting the analytical evaluation at a slightly higher ℓ_0 value than $\ell_0 = \mathrm{N}+2$.

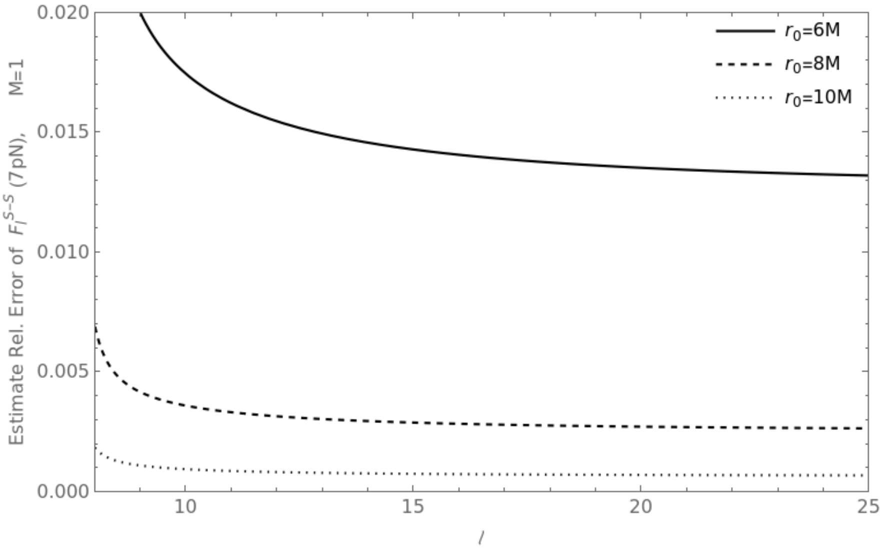

FIGURE 6.1: Estimated relative error of a 7PN expansion of $F^{\tilde{S}-S}_{\alpha,\ell}$, obtained by calculating the $8PN$ correction to that expression, as a function of ℓ for $M = 1$, $q = \sqrt{4\pi}$ at various values of r_0.

To evaluate $F_r^{R,\text{pN}}$ in equation (6.38), an 8PN general expansion for $F^{\tilde{S}-S}_{r,\ell}$ is summed analytically to infinity starting from different values of $\ell_0 \geq \text{N}+2$, while the remaining modes in $F_r^{R,\text{num}}$ are evaluated numerically with the use of the *Black Hole Perturbation Toolkit* [6] implementation of the MST method. The results obtained for the self-force F_α^R for various values of ℓ_0 and r_0 are shown in Table 6.1. These are compared to the results in N. Warburton & L. Barack (2010) [45], which were obtained by performing a numerical evaluation of the self-force modes up to $\ell = 50$ and applying a fitting technique to evaluate the contributions from the $\ell > 50$ modes.

r_0	ℓ_0	$F_r^{R,\text{RpN}}$	F_r^R	Ref. value [45]	| Rel. error |
	10	1.212213×10^{-4}	1.654451×10^{-4}		0.01
$6M$	15	7.947239×10^{-5}	1.664123×10^{-4}	1.677283×10^{-4}	0.008
	20	5.927987×10^{-5}	1.667781×10^{-4}		0.006
	10	1.659632×10^{-5}	1.377700×10^{-5}		0.0005
$10M$	15	1.097126×10^{-6}	1.378068×10^{-5}	1.378448×10^{-5}	0.0003
	20	8.205166×10^{-6}	1.378172×10^{-5}		0.0002
	10	5.078874×10^{-6}	2.719935×10^{-6}		5×10^{-5}
$14M$	15	3.366070×10^{-6}	2.719996×10^{-6}	2.720083×10^{-6}	3×10^{-5}
	20	2.519512×10^{-6}	2.720020×10^{-6}		2×10^{-5}

TABLE 6.1: Results for the self-force for particle in a circular geodesic at different r_0 radius and for $q^2 = 4\pi$ and $M = 1$, calculated according to (6.38) with the use of 8PN expansions for $F_r^{R,\text{RpN}}$ and for different ℓ_0 values. In the last columm, one can see the relative error of the expressions obtained compared to the results in [45] rounded to the first significant figure.

As expected, the results obtained for F_r^R in Table 6.1 show great agreement with the results taken from the literature the larger the orbital radius is. However, even for the innermost stable circular orbit (ISCO) at $r_0 = 6M$, where the pN series is expected to converge more slowly, one can still get a relatively accurate result using the 8PN expansions by calculating more ℓ-modes numerically.

Chapter 7

CONCLUSION

In this thesis a rather self-contained calculation of the scalar self-force for a circular geodesic in Schwarzschild spacetime has been performed through the review and application of various techniques found in the literature. Two different methods for obtaining homogeneous solutions to the field equations were discussed, in the form of the approximate large-ℓ WKB solutions derived in Chapter 3 and of the MST [2] solutions reviewed in Chapter 4, and used in conjunction to calculate the self-force for circular geodesic using an adaptation of the regularization method taken from Hikida et al (2004) [3]. The obtained results showed good agreement with the reference values, even though as few as 20 ℓ-modes have been explicitly computed numerically (compared to the 50 ℓ-modes computed in the results taken from the literature) and the chosen pN order for evaluation of the remaining modes was relatively low (8PN).

An immediate generalization to the work done in this thesis would be to attempt to obtain higher-order regularization parameters for the circular geodesic in Schwarzschild spacetime with the use of the WKB solutions obtained. A generalization of such results to other geodesic orbits would also be of great value.

Appendix A

EXPRESSION FOR THE $S_0^{\ell\omega 0}(r)$ TERM OF THE WKB SERIES

For the purpose of illustrating the complexity of the even n functions in the WKB series solution to the Regge-Wheeler equation, the full expression for $S_0^{\ell\omega 0}(r)$ is given:

$$
\begin{aligned}
S_0^{\ell\omega 0}(r) = {}& \frac{(r-x_2)^2(x_1-x_4)\sqrt{\frac{(r-x_3)(x_1-x_2)}{(r-x_2)(x_1-x_3)}}\sqrt{\frac{(r-x_1)(r-x_4)(x_1-x_2)(x_2-x_4)}{(r-x_2)^2(x_1-x_4)^2}}}{r^2\sqrt{U_{\ell\omega 0}(r)}x_2(x_2-2M)(x_2-x_1)(x_2-x_4)} \times \\
& \times \Bigg(2(2M-x_2)(l(l+1)x_2+2M) + x_2\omega^2(x_4(x_2-2M)(x_2-x_1)+ \\
& + x_2(2M(x_2-x_1)+x_2(x_1+x_2)))\Bigg) F\left(\sin^{-1}\left(\sqrt{p(r)}\right)\Big| q\right) + \\
& \frac{\omega^2(r-x_2)(r-x_3)(x_1-x_4)\sqrt{\frac{(r-x_1)(r-x_4)(x_1-x_2)(x_2-x_4)}{(r-x_2)^2(x_1-x_4)^2}}}{r^2\sqrt{U_{\ell\omega 0}(r)}\sqrt{\frac{(r-x_3)(x_1-x_2)}{(r-x_2)(x_1-x_3)}}} E\left(\sin^{-1}\left(\sqrt{p(r)}\right)\Big| q\right) + \\
& \frac{1}{r^2\sqrt{U_{\ell\omega 0}(r)}}\Bigg\{\frac{(r-x_2)^2(x_1-x_4)\sqrt{\frac{(r-x_3)(x_1-x_2)}{(r-x_2)(x_1-x_3)}}\sqrt{\frac{(r-x_1)(r-x_4)(x_1-x_2)(x_2-x_4)}{(r-x_2)^2(x_1-x_4)^2}}}{x_1x_2(x_1-2M)(2M-x_2)(x_2-x_4)} + \\
& -\omega^2(r-x_1)(r-x_3)(r-x_4) \\
& \Bigg[4M(2M-x_1)(2M-x_2)\Pi\left(\frac{x_2(x_1-x_4)}{x_1(x_2-x_4)}; \sin^{-1}\left(\sqrt{p(r)}\right)\Big| q\right) \\
& x_1x_2\omega^2(2M-x_1)(2M-x_2)(4M+x_1+x_2+x_3+x_4)\times \\
& \qquad \times \Pi\left(\frac{x_1-x_4}{x_2-x_4}; \sin^{-1}\left(\sqrt{p(r)}\right)\Big| q\right) + \\
& -16M^3x_1x_2\omega^2\Pi\left(\frac{(x_2-2M)(x_4-x_1)}{(x_1-2M)(x_4-x_2)}; \sin^{-1}\left(\sqrt{p(r)}\right)\Big| q\right)\Bigg]\Bigg\}. + C_0
\end{aligned}
$$

Here,

$$q \equiv \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_3)(x_2 - x_4)}, \tag{A.1}$$

$$p(r) \equiv \frac{(r - x_1)(x_2 - x_4)}{((r - x_2)(x_1 - x_4)}, \tag{A.2}$$

and $(x|y)$ and $\Pi(x; y|z)$ are complete elliptic integrals of the second and third kind, respectively, and $F(x|y)$ denotes the incomplete elliptic integral of the first kind. The quantities x_i are (complex) solutions to the quartic equation:

$$U_{\ell\omega 0}(x) = 0. \tag{A.3}$$

Appendix B

PN EXPANSIONS

In the equations below 5PN expansions for various quantities defined in Chapter 6 are written.

$$\Phi^{\nu} = \sum_{k=0}^{10} p_k \tag{B.1}$$

$$p_0 = 1 \tag{B.2}$$

$$p_2 = -\frac{z^2}{4\ell+6} - \frac{\ell\epsilon}{2z} \tag{B.3}$$

$$p_4 = \frac{z^4}{32\ell^2+128\ell+120} + \frac{\left(\ell^2-5\ell-10\right)z\epsilon}{4(\ell+1)(2\ell+3)} + \frac{(\ell-1)^2\ell\epsilon^2}{(8\ell-4)z^2} \tag{B.4}$$

$$p_6 = -\frac{z^6}{48(2\ell+3)(2\ell+5)(2\ell+7)} + \frac{\left(-3\ell^3+27\ell^2+142\ell+136\right)z^3\epsilon}{48(\ell+1)(\ell+2)(2\ell+3)(2\ell+5)} + \\ -\frac{\left(\ell^3-18\ell^2+17\ell-4\right)\epsilon^2}{8(1-2\ell)^2} - \frac{(\ell-2)^2(\ell-1)\ell\epsilon^3}{(48\ell-24)z^3} \tag{B.5}$$

$$p_8 = \frac{(\ell-3)^2(\ell-2)^2(\ell-1)\ell\epsilon^4}{96\left(4\ell^2-8\ell+3\right)z^4} + \frac{\left(5\ell^4-60\ell^3-625\ell^2-1548\ell-1108\right)z^5\epsilon}{480(\ell+1)(\ell+2)(\ell+3)(2\ell+3)(2\ell+5)(2\ell+7)} + \\ \frac{\left(2\ell^6-61\ell^5+53\ell^4+386\ell^3-286\ell^2-4\ell+24\right)\epsilon^3}{48(1-2\ell)^2\ell(2\ell+1)z} + \frac{z^8}{384(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)} + \\ + \frac{z^2\epsilon^2}{96(1-2\ell)^2(\ell+1)(\ell+2)(2\ell+1)(2\ell+3)^3(2\ell+5)}(48\ell^9-1152\ell^8-7040\ell^7+ \\ -8212\ell^6+10953\ell^5+15745\ell^4-10867\ell^3-7749\ell^2+6930\ell-768) \tag{B.6}$$

$$p_9 = \frac{3i\left(3\ell^2+3\ell-2\right)\epsilon^3}{2\ell(\ell+1)(2\ell-1)(2\ell+3)} \tag{B.7}$$

$$
\begin{aligned}
p_{10} = &-\frac{(\ell-4)^2(\ell-3)^2(\ell-2)(\ell-1)\ell\epsilon^5}{960\,(4\ell^2-8\ell+3)\,z^5} - \frac{z^{10}}{3840(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)(2\ell+11)}+ \\
&- \frac{(\ell(7\ell(5\ell((\ell-14)\ell-253)-5822)-73032)-43968)z^7\epsilon}{26880(\ell+1)(\ell+2)(\ell+3)(\ell+4)(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)}+ \\
&- \frac{z^4\epsilon^2}{960(1-2\ell)^2(\ell+1)(\ell+2)(\ell+3)(2\ell+1)(2\ell+3)^3(2\ell+5)^2(2\ell+7)}(160\ell^{11}-5200\ell^{10}+ \\
&- 53840\ell^9 - 74872\ell^8 + 715258\ell^7 + 3065539\ell^6 + 4173300\ell^5 + 569492\ell^4 - 2743668\ell^3 + \\
&- 883399\ell^2 + 690870\ell + 37080)+ \\
&- \frac{z\epsilon^3}{192(1-2\ell)^2\ell(\ell+1)^2(2\ell+1)(2\ell+3)^3(2\ell+5)}(6480+59568\ell+-115776\ell^2+ \\
&- 278316\ell^3 + 67017\ell^4 + 453521\ell^5 + 405867\ell^6 + 169443\ell^7 + 31236\ell^8 - 672\ell^9 + \\
&- 768\ell^{10} + 16\ell^{11}) - \frac{\epsilon^4}{192(1-2\ell)^2\ell(2\ell-3)(2\ell+1)(2\ell+3)z^2}(4\ell^9 - 188\ell^8 + 483\ell^7 + \\
&+ 3127\ell^6 - 6795\ell^5 - 4211\ell^4 + 13208\ell^3 - 4404\ell^2 - 936\ell + 432)
\end{aligned}
\tag{B.8}
$$

All of the other p_k with $k \leq 10$ that are not shown in the above expressions vanish.

The $F_\ell^{\tilde{S}(+)}$ modes obtained from this expansion reads:

$$
F_\ell^{\tilde{S}(+)} = -\frac{q^2}{2u^t r_0^2}\sum_{k=0}^{5} f_{2k}^{\tilde{S}(+)} \tag{B.9}
$$

$$
f_0^{\tilde{S}(+)} = 1 \tag{B.10}
$$

$$
f_2^{\tilde{S}(+)} = \frac{2M}{r} - \frac{\ell(\ell+1)r^2\Omega^2}{(2\ell-1)(2\ell+3)}, \tag{B.11}
$$

$$
f_4^{\tilde{S}(+)} = \frac{6\left(3\ell^2+3\ell-2\right)M^2}{(4\ell^2+4\ell-3)\,r^2} - \frac{9\ell(\ell+1)\left(3\ell^2+3\ell-2\right)r^4\Omega^4}{4(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}, \tag{B.12}
$$

$$
\begin{aligned}
f_6^{\tilde{S}(+)} = &-\frac{8\left(-5\ell^2-5\ell+3\right)M^3}{(4\ell^2+4\ell-3)\,r^3} - \frac{25\ell(\ell+1)\left(5\ell^4+10\ell^3-5\ell^2-10\ell+8\right)r^6\Omega^6}{4(2\ell-5)(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)(2\ell+7)}+ \\
&+ \frac{\ell\left(120\ell^5+360\ell^4+186\ell^3-228\ell^2-107\ell+67\right)M\Omega^2}{(1-2\ell)^2(2\ell+1)^2(2\ell+3)^2}+ \\
&- \frac{5\left(9\ell^6+27\ell^5-27\ell^4-99\ell^3-4\ell^2+50\ell-12\right)Mr^3\Omega^4}{2(\ell-1)(\ell+2)(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
f_8^{\tilde{S}(+)} = & \frac{10\left(35\ell^4+70\ell^3-115\ell^2-150\ell+72\right)M^4}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)r^4}+ \\
& -\frac{245\ell(\ell+1)\left(35\ell^6+105\ell^5-35\ell^4-245\ell^3+168\ell^2+308\ell-272\right)r^8\Omega^8}{64(2\ell-7)(2\ell-5)(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)}+ \\
& +\frac{2\left(280\ell^6+840\ell^5+490\ell^4-420\ell^3-265\ell^2+85\ell+18\right)M^3\Omega^2}{(1-2\ell)^2(2\ell+1)^2(2\ell+3)^2r}+ \\
& -\frac{7Mr^5\Omega^6}{(\ell-2)(\ell-1)(\ell+2)(\ell+3)(2\ell-5)(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)(2\ell+7)}\times \\
& \times(25\ell^{10}+125\ell^9-175\ell^8-1450\ell^7-565\ell^6+3905\ell^5+2115\ell^4-4220\ell^3+ \\
& -728\ell^2+2312\ell-480)-\frac{M^2r^2\Omega^4}{2(\ell-1)(\ell+2)(2\ell-3)(2\ell+1)^2(2\ell+5)\left(4\ell^2+4\ell-3\right)^3}\times \\
& \times(5040\ell^{12}+30240\ell^{11}+33960\ell^{10}-107400\ell^9-222265\ell^8+87980\ell^7+ \\
& +380541\ell^6+49973\ell^5-214188\ell^4-27901\ell^3+49762\ell^2-42\ell-2700)
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
f_{10}^{\tilde{S}(+)} = & \frac{12\left(63\ell^4+126\ell^3-203\ell^2-266\ell+120\right)M^5}{\left(16\ell^4+32\ell^3-56\ell^2-72\ell+45\right)r^5}+ \\
& -\frac{567\ell(\ell+1)r^{10}\Omega^{10}}{64(2\ell-9)(2\ell-7)(2\ell-5)(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)(2\ell+11)}\times \\
& \times\left(63\ell^8+252\ell^7-42\ell^6-1008\ell^5+567\ell^4+3108\ell^3-2508\ell^2-4272\ell+3968\right)+ \\
& +\frac{6M^4\Omega^2}{(2\ell-3)(2\ell+1)^2(2\ell+5)\left(4\ell^2+4\ell-3\right)^3r^2}(5040\ell^{10}+25200\ell^9+22680\ell^8-60480\ell^7+ \\
& -103305\ell^6+7605\ell^5+67846\ell^4+2057\ell^3-16733\ell^2+1146\ell+810)+ \\
& -\frac{27Mr^7\Omega^8}{32(\ell-3)(\ell-2)(\ell-1)(\ell+2)(\ell+3)(\ell+4)(2\ell-7)(2\ell-5)(2\ell-3)(2\ell-1)(2\ell+3)}\times \\
& \times\frac{1}{(2\ell+5)(2\ell+7)(2\ell+9)}[1225\ell^{14}+8575\ell^{13}-18375\ell^{12}-221725\ell^{11}-121765\ell^{10}+ \\
& +1628025\ell^9+1905715\ell^8-4934615\ell^7-5687416\ell^6+9853452\ell^5+6824328\ell^4+ \\
& -13192144\ell^3-2471712\ell^2+7290432\ell-1370880]+ \\
& -\frac{3M^2r^4\Omega^6}{4(3-2\ell)^2(\ell-2)(\ell-1)(\ell+2)(\ell+3)(2\ell-5)(2\ell+1)^2(2\ell+5)^2(2\ell+7)\left(4\ell^2+4\ell-3\right)^3}\times \\
& \times[201600\ell^{18}+1814400\ell^{17}+2156000\ell^{16}-23878400\ell^{15}-68525800\ell^{14}+82133800\ell^{13}+ \\
& +467565250\ell^{12}+116541700\ell^{11}-1207160341\ell^{10}-830800255\ell^9+1516390102\ell^8+ \\
& +1181435438\ell^7-1211990527\ell^6-74(5040\ell^{10}+25200\ell^9+22680\ell^8-60480\ell^7+ \\
& -103305\ell^6+7605\ell^5+67846\ell^4+2057\ell^3-16733\ell^2+1146\ell+810)4605125\ell^5+ \\
& +620195828\ell^4+193336002\ell^3-152171712\ell^2-9317160\ell+952560]
\end{aligned}
\tag{B.15}
$$

(B.16)

From the $\ell \to \infty$ limit of the above result, one finds an expansion for B_r:

$$B_r = \frac{q\ 2}{2u^t r^2} \sum_{k=0}^{5} b_{2k} \tag{B.17}$$

$$b_0 = 1 \tag{B.18}$$

$$b_2 = \frac{2M}{r} - \frac{r^2\Omega^2}{4} \tag{B.19}$$

$$b_4 = \frac{9M^2}{2r^2} - \frac{27r^4\Omega^4}{64} \tag{B.20}$$

$$b_6 = -\frac{125}{256} r^6\Omega^6 - \frac{45r^3\Omega^4}{32} + \frac{10M^3}{r^3} + \frac{15M^2\Omega^2}{8} \tag{B.21}$$

$$b_8 = -\frac{8575r^8\Omega^8}{16384} - \frac{175Mr^5\Omega^6}{64} + \frac{175}{8r^4} - \frac{315M^2r^2\Omega^4}{128} + \frac{35M^3\Omega^2}{4r} \tag{B.22}$$

$$b_{10} = -\frac{35721r^{10}\Omega^{10}}{65536} - \frac{33075Mr^7\Omega^8}{8192} + \frac{189M^5}{4r^5} - \frac{4725Mr^4\Omega^6}{512} + \frac{945M^6\Omega^2}{32r^2} \tag{B.23}$$

The $F_r^{R,\mathrm{pN}}$ contribution to the regularized self-force up to 5PN and for a given ℓ_0 is obtained by subtracting the expansion for B_r from the one for $F_\ell^{\tilde{S}(+)}$ and summing from a given ℓ_0 to infinity.

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