Master

Dissertation

The vacuum energy with non-ideal boundary conditions via approximate functional equation

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"THE VACUUM ENERGY WITH NON-IDEAL BOUNDARY CONDITIONS VIA APPROXIMATE FUNCTIONAL EQUATION"

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“Before I came here I was confused about this subject. Having listened to your lecture I am still confused. But on a higher level.”

- Enrico Fermi

Resumo

Apresentamos em linhas gerais o esquema de quantização canônica para o campo eletromagnético e a descrição de sua energia de ponto-zero. Esta energia é divergente e métodos matemáticos de regularização e renormalização são necessários para torná-la finita. Devido à semelhança com o caso eletromagnético, discutiremos um desses métodos aplicados a energia do vácuo de um campo escalar quantizado na presença de superfícies onde o campo satisfaz condições de contorno ideais. Definimos domínios limitados $\Omega \subset \mathbb{R}^d$, nos quais o campo pode satisfazer condições de contorno ideais ou não ideais. Chamamos isso de condições de contorno *de Dirichlet ideais para altas frequências*. Utilizando um procedimento de regularização analítica, obtemos a energia do vácuo para um campo escalar sem massa a temperatura zero na presença de duas supercífices planas definindo o domínio $\Omega = \mathbb{R}^{d-1} \times [0, L]$ com condições de contorno de Dirichlet. Para abordar o caso de condições de contorno não ideais, utilizamos uma expansão assintótica baseada em uma equação funcional aproximada para a função zeta de Riemann, onde são definidas somas finitas fora do domínio original de convergência. No contexto eletromagnético, mostramos que esta situação descreve a correção de condutividade finita para a energia de ponto-zero. Finalmente, para obter a energia de Casimir para um campo escalar sem massa na presença de uma caixa retangular, com comprimentos L_1 e L_2, ou seja, $\Omega = [0, L_1] \times [0, L_2]$ com condições de contorno não ideais, usamos uma equação funcional aproximada da função zeta de Epstein.

Palavras-chave: Energia de Casimir, condições de contorno não-ideais, expansão assintótica.

Abstract

We present in broad strokes the canonical quantization scheme for the electromagnetic field and the description of its zero-point energy. This energy gives rise to divergences, and mathematical methods of regularization and renormalization are necessary to render them finite. In one such method, due to its similarity to the electromagnetic case, we discuss the application of this method to the vacuum energy of a quantized scalar field in the presence of classical surfaces. We define bounded domains $\Omega \subset \mathbb{R}^d$, where the field can satisfy either ideal or non-ideal boundary conditions. We call it *ideal high-pass Dirichlet* boundary conditions. Employing an analytical regularization procedure, we obtain the vacuum energy for a massless scalar field at zero temperature in the presence of a slab geometry $\Omega = \mathbb{R}^{d-1} \times [0, L]$ with Dirichlet boundary conditions. To address the case of non-ideal boundary conditions, we use an asymptotic expansion based on an approximate functional equation for the Riemann zeta function, where finite sums are defined outside the original domain of convergence. In the electromagnetic context, this situation describes the finite conductivity correction to the zero-point energy. Finally, to derive the Casimir energy for a massless scalar field in the presence of a rectangular box, with lengths L_1 and L_2, *i.e.*, $\Omega = [0, L_1] \times [0, L_2]$ with non-ideal boundary conditions, we employ an approximate functional equation for the Epstein zeta function.

Keywords: Casimir Energy, Non-Ideal Boundary Conditions, Asymptotic Expansion.

Contents

Chapter 1

Introduction

Quantum fields are fundamental mathematical objects in the description of natural phenomena. With the construction of a Fock space, we may interpret excitation from a vacuum state in an asymptotic limit as point-like particles. In this context, in Minkowski spacetime, fields can be represented using annihilation and creation operators and particles manifest after an operation within a vacuum state. The vacuum expectation value of a free quantum scalar field, for example, is zero in the absence of spontaneous symmetry break (SSB). However, the expectation value of the square of a scalar field in a vacuum state diverges. Since the energy density for a massive scalar field has this quadratic term contributing, this quantity also diverges. As measurable quantities are always finite, some procedures must be implemented to deal with divergent contributions.

To address this issue, assuming that the field satisfies classical boundary conditions, we introduce boundaries, consequently modifying the zero-point energy. The Casimir effect is a measurable macroscopic manifestation of this result [1–8] and it has been measured in different geometric configurations [9–12]. The physical origin of the effect lies in the change in the vacuum modes associated with the quantized electromagnetic field by the presence of macroscopic surfaces. Moreover, the Casimir effect is not exclusive to electromagnetic fields; it can also manifest with other constrained fields, such as massless fermionic fields, due to the interaction of quantum field vacuum modes with idealized surfaces where the field satisfies classical boundary conditions [13]. Additionally, in the quasi-particle landau scenario, the literature has been discussing the phononic Casimir effect. In this case, the speed of light is replaced by the speed of sound in the medium [14].

As we discussed, in the canonical formalism for quantum fields, the vacuum energies are divergent. To render these quantities physically meaningful, different approaches have been developed. These approaches can be categorized into local [15–21] and global methods. In this context, our focus relies on the global approach, which investigates the total energy of the quantized field with idealized boundary conditions [22, 23]. This approach uses two natural methods to regularize and renormalize the divergent vacuum energy. The first one is the cut-off method, where an ultraviolet regulator function is introduced in the divergent sum of the eigenfrequencies. On general grounds, the regularized vacuum energy exhibits Weyl's terms with a geometric origin, cut-off independent contributions, and terms that vanish as the cut-off is removed. With these geometric terms in hand, we can implement a renormalization procedure by introducing auxiliary boundaries and subtracting the regularized energies of different configurations. The second set is analytic regularization procedures, with a noteworthy method being zeta function regularization. We can use its special values to interpret divergent series or products without the necessity of their removal. Although the cutoff method with the auxiliary configurations and the analytic regularization discussed above are quite different in their grounds, it is possible to compare them and show to be analytically equivalent in some specific situations [24–27].

On the other hand, on physical grounds, the preceding discussion considered ideal boundary conditions. For the electromagnetic field modes, this means perfect conductivity and it is an idealization. However, metallic plates often behave as dielectric for high-frequency modes, and as conductors for infrared modes. Following the original formulation, the issue of determining the conductivity correction to the electromagnetic Casimir force arises. To derive this correction, Lifshitz proposed a model treating the electromagnetic field as a classical field, where attractive or repulsive forces arise from the fluctuating charges and currents of the boundaries [28]. In this dissertation, we obtain the correction to the Casimir force for the case of non-ideal boundary conditions using a different method. Instead of discussing the nonlinear problem of the microscopic modeling of finite conductivity *i.e.*, nonideal boundary conditions, we confine ourselves to make use of the spectral theory of elliptic differential operators and the correction to the Casimir energy can be discussed using analytic regularization procedure and approximate functional equations.

In our methodology, we use the fact that the total renormalized energy of scalar fields in the presence of bounded domains can be derived using an analytic regularization procedure, where the Dirichlet and Neumann Laplacian are useful technical devices. It's known that the vacuum energy in the slab geometry $\mathbb{R}^{d-1} \times [0, L]$ with Dirichlet boundary conditions can be written in terms of the Riemann zeta function. To calculate its correction due to nonideal boundary conditions, we represent the energy density using an asymptotic expansion derived by Hardy and Littlewood [29]. They obtained an approximate functional equation for the Riemann zeta function written as finite sums beyond their original domain of convergence. Next, we generalize the previous result in the case of a field in the presence of a rectangular box with lengths L_1 and L_2 with non-ideal boundary conditions. This work is based on the paper [30]

This dissertation is organized as follows. In Chap. 2, we present the canonical quantization scheme for the electromagnetic field and its zero-point energy. In the next chapter, Chap. 3, we discuss how to obtain the renormalized vacuum energy for a massless scalar field at zero temperature in the presence of perfect mirrors, using the cutoff method and the analytic extension method. Then, in Chap. 4, we present the correction to the Casimir energy in the presence of dielectrics. Following that, we use an approximate functional equation to obtain the same correction to the renormalized vacuum energy due to nonideal boundary conditions for a slab geometry $\mathbb{R}^{d-1} \times [0, L]$. This method is then applied to obtain it in a rectangular box, with lengths L_1 and L_2, considering nonideal boundary conditions.

Conclusions are provided in the last chapter, Chap.5. Here, we use $\hbar = c = k_B = 1$.

Chapter 2

Quantum Field Theory

2.1 Canonical Quantization and Fock space

The formulation of a quantum field theory introduced new aspects that are lacking in a classical field theory. The canonical field quantization scheme establishes a correspondence between classical quantities and quantum operators, making it easier to work with. However, as we shall see later, this formulation encounters a divergence in the zero-point energies. Thus, the quantum field theory based on the operator concept, in principle, requires additional prescriptions to deal with these infinities and become a well-defined theory, such as Wick's normal ordering, which is an approach to maintaining the vacuum expectation value equal to zero in an infinite volume. Despite the divergence, after some manipulations, this abstract mathematical construction leads us to many experimental manifestations. Our first goal is to comprehend this quantization scheme and provide a brief overview of the associated calculations, with a particular emphasis on the electromagnetic field and defining the vacuum state. In order to achieve this, we first must recall some basic facts about the theory, see Refs [31, 32].

2.1.1 Field quantization and the vacuum energy

To begin, in order to describe an infinite number of degrees of freedom, our classical dynamical variables are now considered as continuous fields in space and time, denoted by $\phi(x,t)$, with a corresponding Lagrangian only depending on the field variable and their first derivatives, namely,

$$L(t) = \int d^3x \mathcal{L}[\phi(x,t), \partial_\mu \phi(x,t)]. \tag{2.1}$$

where, $\mathcal{L}$ is the lagrangian density. Analogous to the particle mechanic, we define the momentum canonically conjugate to the field variables as

$$\Pi(x,t) = \frac{\partial L(t)}{\delta(\partial_t \phi(x,t))} \tag{2.2}$$

Once we have defined the classical dynamical variables of a physical system, the quantization proceeds by replacing the classical fields with Hermitian operators. These operators must satisfy canonical equal-time commutation relations given by

$$[\hat{\phi}(x,t), \hat{\Pi}(x',t)] = i\delta(x-x'); \quad [\hat{\phi}(x,t), \hat{\phi}(x',t)] = [\hat{\Pi}(x,t), \hat{\Pi}(x',t)] = 0 \tag{2.3}$$

Since we are working from now exclusively with operators, we can drop the "hat" notation. Thus, $\hat{\phi}(x,t) = \phi(x,t)$ and $\hat{\Pi}(x,t) = \Pi(x,t)$.

If we want to construct an explicit representation of these operators, we can use a complete set of "classical" solutions. Since our focus is to discuss the vacuum energy, it is more convenient to adopt the momentum picture. Then, we expand our field operators on such a basis in terms of a generalized Fourier decomposition in mode functions

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \phi_k(t), \tag{2.4}$$

$$\Pi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \Pi_k(t), \tag{2.5}$$

where $k \cdot x = k^0 t - \mathbf{k} \cdot \mathbf{x}$. The mode functions are also promoted to operators and inherit their commutation relations. By performing Fourier transforms in x and x', after some algebra, we find

$$[\phi_k(t), \Pi_{k'}(t)] = i\delta(k+k'), \tag{2.6}$$

where the plus sign shows that the variable which is conjugate to ϕ_k is $\Pi_{-k} = \Pi_k^\dagger$. An important fact that has to be considered in the mathematical description is that a physical field has to satisfy certain boundary conditions. Therefore, it is useful to consider the field not in an infinite space, but inside a finite cubic box of volume V. At any given time, the operators $\phi(x,t)$ and $\Pi(x,t)$ can be expanded in terms of the Fourier series:

$$\phi(x,t) = \sum_k \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_k(t), \tag{2.7}$$

$$\Pi(x,t) = \sum_k \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}} \Pi_{-k}(t). \tag{2.8}$$

The following step is to develop a Hamiltonian description. Once we have the classical Hamiltonian description of a physical system, the quantization of such a system is quite straightforward. In terms of the operators ϕ and Π the field Hamiltonian H, formally identical with a classical Hamilton function, is obtained as

$$H = \int d^3x \left(\Pi \partial_t \phi - \mathcal{L}(\phi, \partial_\mu \phi)\right). \tag{2.9}$$

At this point, only the fundamental postulate for the quantization of fields has been employed. Unfortunately, this postulate alone is not sufficient to determine a measurable energy in a satisfactory way. As we shall observe, singularities arise leading to divergent vacuum expectation values. Then, to eliminate this result, the physical field theory requires renormalizabilty as a fundamental property. In order to make this more transparent and to see how zero-point energies appear in the theory, let us now consider the electromagnetic field, and then define its physical vacuum energy.

2.1.1.1 The zero-point energy of the electromagnetic field

We turn now to the discussion of the zero-point energy of the electromagnetic field. To start, let us describe the theory. The Maxwell's equation in vacuum are

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0; \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0; \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{\partial t}$$

where **E** and **B** are the electric and the magnetic fields respectively. To the quantization scheme, it is more convenient to rewrite Maxwell's equations in a Lorentz covariant form. This is achieved by the fact that the zero divergence of **B** and Faraday's Law allows the introduction of the four-dimensional vector potential $A^\mu = (A^0, \mathbf{A})$ which characterizes the fields by

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A};$$
$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} A_0$$

We now define the four-dimensional curl of this potential by an antisymmetric tensor of rank two called the field strength tensor, written by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.10}$$

We note that

$$F_{i0} = \partial_0 A_i - \partial_i A_0 = E_i, \tag{2.11}$$

where **E** denotes the electric field. Similarly,

$$F_{ij} = \partial_i A_j - \partial_j A_i = -e_{ijk} B_k, \tag{2.12}$$

where e_{ijk} denotes the three dimensional Levi-Civita tensor and **B** denotes the magnetic field vector. In a matrix representation, we have

$$F\mu\nu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{2.13}$$

Although this formulation is useful for further calculations, the presented potential is not a directly observable quantity or uniquely determined. The introduction of a local gauge transformation, expressed as

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x), \tag{2.14}$$

ensures that the values of the field strength tensor remain unaltered. This gauge invariance in the fields leads to peculiarities in the Maxwell field theory, and we need to assume a gauge fixing. One particular choice of gauge condition that can be used is the Coulomb gauge, also known as the transverse gauge. This choice requires the three-dimensional divergence to vanish

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0, \tag{2.15}$$

i.e., the vector potential $\boldsymbol{A}(x,t)$ is a spatially transverse field, with polarization vectors orthogonal to the direction of propagation. Initially, the theory has presented a vector field with four degrees of freedom. However, as known from electrodynamics, the photon has only two polarization states. Consequently, after the gauge fixing, the two "unphysical" degrees of freedom of the electromagnetic field are eliminated, leaving only transverse photons.

Now that we have outlined the theory, we can study the dynamics and then proceed to the quantization scheme. Let us consider the free field Lagrangian as

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2.16}$$

with A_μ as the independent dynamical variable. We can define the conjugate momentum fields as

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\mu)} = -F^{0\mu}, \tag{2.17}$$

and the first peculiarity of the Maxwell field theory appears. The time-like component Π^0 vanishes

$$\Pi^0(x) = -F^{00} = 0, \tag{2.18}$$

and as a result, the $A_0(x)$ component of the field would commute with all the field variables in the theory. Therefore, $A_0(x)$ would act as a classical function even if we consider it as an operator. Without loss of generality, we can set it equal to zero

$$A_0(x) = 0, \tag{2.19}$$

and once again, we are choosing a particular gauge for the theory, known as the temporal (axial) gauge. Furthermore, in this gauge, the spatial components of the momentum coincide with the electric field

$$\Pi^i = -F^{0i} = -(\partial^0 A^i - \partial^i A^0) = E_i, \tag{2.20}$$

The next step in the quantization scheme is to propose the following commutation rules

$$\begin{aligned} [A_i(x,t), A_j(x',t)] &= [\Pi_i(x,t), \Pi_j(x',t)] = 0, \\ [A_i(x,t), \Pi_j(x',t)] &= i\delta^{(tr)}_{ij}(x - x'), \end{aligned} \tag{2.21}$$

where, in order to maintain the transversality, in the last relation, the usual δ-function is replaced by the divergence-free transverse δ-function, which is a non-local operator given by

$$i\delta^{(tr)}_{ij}(x - x') = (\delta_{ij} - \Delta^{-1}\partial_i\partial_j)\delta(x - x'). \tag{2.22}$$

As we can see, this both gauge choices,

$$A_0(x) = 0; \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0, \tag{2.23}$$

lead us to the Maxwell's equations reduced to

$$\Box \boldsymbol{A} = 0, \tag{2.24}$$

and, therefore to represent **A** and **Π** in terms of Fourier series, it is convenient to introduce a particular four-dimensional vector basis $\{\epsilon_k^{(\lambda)}(\lambda = 1, 2); \mathbf{k}/|\mathbf{k}|\}$ that represents the polarization. Since the vector potential is transverse, this basis must satisfy the conditions

$$\epsilon_k^{(\lambda)} \cdot \mathbf{k} = 0, \quad \epsilon_k^{(\lambda)} \cdot \epsilon_k^{(\lambda')} = \delta_{\lambda\lambda'}. \tag{2.25}$$

Defining the polarization vectors with respect to the direction of the momentum vector simplifies things. Therefore, the vector nature of the field variable is now characterized by the polarization vectors. Additionally, for a completeness relation for the basis, it requires that

$$\sum_{\lambda} \epsilon_{k,i}^{(\lambda)} \epsilon_{k,j}^{(\lambda)} + \frac{k_i k_j}{|\mathbf{k}|^2} = \delta_{ij}. \tag{2.26}$$

Once again, our focus is to discuss the vacuum energy. For this reason, the Fourier representation of the fields $A(x)$ and $\Pi(x)$ is written in a finite box as

$$\mathbf{A}(x,t) = \sum_{k,\lambda} \frac{1}{\sqrt{V}} \exp(i\mathbf{k}\cdot\mathbf{x}) \epsilon_k^{(\lambda)} A_k^{(\lambda)}(t), \tag{2.27}$$

and

$$\boldsymbol{\Pi}(x,t) = \sum_{k,\lambda} \frac{1}{\sqrt{V}} \exp(i\mathbf{k}\cdot\mathbf{x}) \epsilon_k^{(\lambda)} \Pi_{-k}^{(\lambda)}(t), \tag{2.28}$$

and, due to the hermiticity of A and Π, we get the relations

$$\epsilon_{-k}^{(\lambda)} A_{-k}^{(\lambda)}(t) = \epsilon_k^{(\lambda)} A_k^{(\lambda)+}(t); \quad \epsilon_{-k}^{(\lambda)} \Pi_k^{(\lambda)+}(t) = \epsilon_k^{(\lambda)} \Pi_{-k}^{(\lambda)}(t), \tag{2.29}$$

which allow us to derive the commutation rules, in the momentum space, for the operators $A_k^{(\lambda)}(t)$ and $\Pi_k^{(\lambda)}(t)$ as

$$[A_k^{(\lambda)}(t), A_{k'}^{(\lambda')}(t)] = [\Pi_k^{(\lambda)}(t), \Pi_{k'}^{(\lambda')}(t)] = 0; \tag{2.30}$$

$$[A_k^{(\lambda)}(t), \Pi_{k'}^{(\lambda')}(t)] = i\delta_{\lambda\lambda'}\delta(k - k'). \tag{2.31}$$

Now, the Hamiltonian for electromagnetic field theory, in its canonical quantization form, yields

$$H = \frac{1}{2}\int_V d^3x \{\Pi^2 + \mathbf{A}\cdot(-\boldsymbol{\nabla}^2\boldsymbol{A})\}, \tag{2.32}$$

where we used the identity, in the Coulomb gauge, $\int_V d^3x (\boldsymbol{\nabla} \times \boldsymbol{A})^2 = \int_V d^3x \boldsymbol{A} \cdot (-\boldsymbol{\nabla}^2 \boldsymbol{A})$. In terms of the Fourier components, the Hamiltonian becomes an infinite sum of uncoupled harmonic oscillators

$$H = \frac{1}{2}\sum_{\mathbf{k}\lambda} \{\Pi_k^{(\lambda)+}\Pi_k^{(\lambda)} + \omega_k^2 A_k^{(\lambda)+} A_k^{(\lambda)}\}. \tag{2.33}$$

Then, we define the creation and annihilation operators $a_k^{(\lambda)+}$ and $a_k^{(\lambda)}$ in terms of $A^{(\lambda)}$ and $\Pi^{(\lambda)}$

$$a_k^{(\lambda)+} = \sqrt{\frac{\omega_k}{2}} \left(A_k^{(\lambda)} - \frac{i}{\omega} \Pi_k^{(\lambda)+} \right), \tag{2.34}$$

and

$$a_k^{(\lambda)} = \sqrt{\frac{\omega_k}{2}} \left(A_k^{(\lambda)} + \frac{i}{\omega} \Pi_k^{(\lambda)+} \right). \tag{2.35}$$

We expect these operators to satisfy the commutation relations

$$[a_k^{(\lambda)}, a_{k'}^{(\lambda')+}] = \delta_{\lambda\lambda'} \delta(k - k'); \qquad [a_k^{(\lambda)+}, a_{k'}^{(\lambda')+}] = [a_k^{(\lambda)}, a_{k'}^{(\lambda')}] = 0 \tag{2.36}$$

This allows us to write the Hamiltonian in the form

$$H = \sum_{k,\lambda} \omega_k \left(n_k^{(\lambda)} + \frac{1}{2} \right), \tag{2.37}$$

where the energy of the confined field has a pure point spectrum and levels are determined by the photon number operators $n_k^{(\lambda)} = a_k^{(\lambda)+} a_k^{(\lambda)}$ and the Fock representation, which characterizes the states concerning the ground state, the vacuum state $|0\rangle$. We can now define the vacuum of the quantized electromagnetic field by $a_k^{(\lambda)} |0\rangle = 0$. Consequently, the quantized free electromagnetic field also carries an infinite zero-point energy,

$$E_0 = \langle 0| \mathcal{H} |0\rangle = \frac{1}{2} \sum_{k,\lambda} \omega_k. \tag{2.38}$$

In this context, it is noteworthy that we can arrive at the equivalent expression for the vacuum energy through a more straightforward approach by assuming a massless scalar field with Dirichlet boundary conditions. Consequently, in our calculations for the Casimir energy in a quantum scenario, we will consider this particular case.

Chapter 3

The Quantum Vacuum

3.1 The divergent zero-point energy

A free quantized bosonic or fermionic field is mathematically equivalent to an infinite set of uncoupled one-dimensional harmonic oscillators, each characterized by the frequency ω_k. In the vacuum state, the summation over increasingly higher frequencies, each possessing a zero-point energy of $\frac{1}{2}\omega_k$, results in a divergent quantity. However, infinities are not physically meaningful; therefore, there exist several mathematical methods of regularization and renormalization that render them finite. In this chapter, we present two examples of such approaches.

3.1.1 Casimir's original approach - the cut-off method

First, we follow the Casimir's original paper [1]. Let us define a bosonic field confined in a finite cubic cavity of volume L^3 *i.e.* bounded by perfectly conducting walls. To calculate the vacuum energy, we put a perfect conducting square plate with side L at an adjustable distance parallel to the xy plane. The physical vacuum energy of quantized fields can be understood as the difference between two configurations. To calculate in this case, the plate is put first at a small distance a from the xy plane and then at a considerable distance, let us say, for example, $L/2$, as in Fig. 1

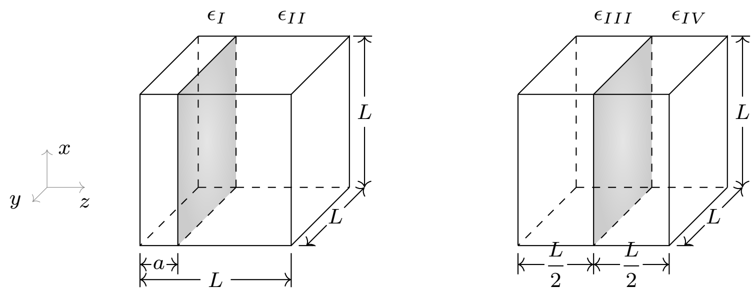


Figure 1 – Different configuration of a box with a conducting parallel plate

We note that the summation extended over all frequencies, in both cases, is a divergent quantity, but after some manipulations, the difference will be shown to have a well-defined meaning to be interpreted as the interaction between the plate and the xy face.Inside this box, the field's eigenmodes are determined by requiring the following boundary conditions for the electric and magnetic field, $\mathbf{n} \cdot \mathbf{B} = \mathbf{0}$ and $\mathbf{n} \times \mathbf{E} = \mathbf{0}$, on the walls. The field's possible vibrations are defined by

$$0 \leq x \leq L; \qquad 0 \leq y \leq L; \qquad 0 \leq z \leq a; \tag{3.1}$$

with the wave numbers given by

$$k_x = \frac{\pi}{L} n_x; \qquad k_y = \frac{\pi}{L} n_y; \qquad k_z = \frac{\pi}{a} n_z \tag{3.2}$$

where $n_i = 0, 1, 2...$, with $i = x, y, z$. The frequencies are given from elementary theory as

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{\varkappa^2 + k_z^2}. \tag{3.3}$$

There are two normal modes for each k_i from the two independent polarizations, for this reason, we should multiply by a factor of 2. If a single k_i is zero, we have only one independent polarization. Since we assume that L is very large, k_x and k_y can be treated as a continuous variable. Thus, we find

$$\begin{aligned} \epsilon_k &= \sum_{k_x,k_y,k_z} \frac{1}{2} k = \sum_{k_x,k_y,k_z} \frac{1}{2} \sqrt{k_x^2 + k_y^2 + k_z^2} \\ &= \left(\frac{L}{\pi}\right)^2 \int_{k_x=0}^{\infty} dk_x \int_{k_y=0}^{\infty} dk_y \left(\frac{1}{2} \sqrt{k_x^2 + k_y^2} + \sum_{n_z=1}^{\infty} \sqrt{k_x^2 + k_y^2 + \left(\frac{\pi}{a} n_z\right)^2} \right). \end{aligned} \tag{3.4}$$

Introducing polar coordinates in the k_x,k_y plane,

$$\int dk_x dk_y = \int_0^{\infty} d\varkappa \, \frac{\pi}{2} \varkappa, \tag{3.5}$$

where the angular part ranges from 0 to $\frac{\pi}{2}$ for $k_x, k_y > 0$. We have

$$\epsilon_k = \frac{L^2}{2\pi} \int_0^{\infty} d\varkappa \, \varkappa \left(\frac{\varkappa}{2} + \sum_{n_z=1}^{\infty} \sqrt{\varkappa^2 + \left(\frac{\pi}{a} n_z\right)^2} \right). \tag{3.6}$$

In the situation in which a is large, the summation in n_z may be treated as an integral. Then, with these two different configurations in hand, we can see the Casimir energy as

$$\epsilon = \frac{L^2}{2\pi} \left[\int_0^{\infty} d\varkappa \, \varkappa \left(\frac{\varkappa}{2} + \sum_{n_z=1}^{\infty} \sqrt{\varkappa^2 + \left(\frac{\pi}{a} n_z\right)^2} \right) - \int_0^{\infty} \left(\frac{a}{\pi} dk_z\right) \int_0^{\infty} d\varkappa \, \varkappa \left(\sqrt{\varkappa^2 + k_z^2} \right) \right], \tag{3.7}$$

In this expression, the first term on the right-hand side represents the energy between the plates satisfying the discussed boundary conditions. We then subtract from his the energy in the same region but without imposing the boundary conditions. This allows k_z to be treated as a continuous variable.

As discussed earlier, these summations are divergent quantities. To obtain a finite result, we need to introduce a cutoff function $f(k/k_c)$ which satisfies $f(k/k_c) \approx 1$ for $k \leq k_c$ and tends to zero sufficiently rapidly for $k \gg k_c$ and follows the condition $f(1) = \frac{1}{2}$. For this regularization to be legitimized on physical grounds we can interpret that the high-frequency mode does not interact with the plates and therefore the zero point energy for these frequencies will not be influenced by the position of the plate.

We define a new variable $u = a^2\varkappa^2/\pi^2$ and introduce the $f(k/k_c)$, whith $k = \sqrt{\varkappa^2 + k_z^2}$.

$$\epsilon = \frac{\pi^2 L^2}{4a^3}\left[\int_0^\infty du \left(f\left(\frac{\pi\sqrt{u+n_z^2}}{ak_c}\right)\frac{\sqrt{u}}{2} + \sum_{n_z=1}^{\infty}\sqrt{u+n_z^2}f\left(\frac{\pi\sqrt{u+n_z^2}}{ak_c}\right)\right) + \right.$$
$$\left. - \int_0^\infty dn_z \int_0^\infty du \left(\sqrt{u+n_z^2}\right) f\left(\frac{\pi\sqrt{u+n_z^2}}{ak_c}\right)\right]. \tag{3.8}$$

Identifying $F(n)$ as

$$F(n) = \int_0^\infty du \left(\sqrt{u+n_z^2}\right) f\left(\frac{\pi\sqrt{u+n_z^2}}{ak_c}\right), \tag{3.9}$$

we can apply the Euler-Maclaurin formula, for $p > 0 \in \mathbb{Z}$ and a function $F(x)$ is p times continuously differentiable on the interval [m,n], known as

$$S - I = \sum_m^n F(x) - \int_m^n F(x)dx = \sum_{k=1}^p \frac{B_k}{k!}\left(F^{(k-1)}(n) - F^{(k-1)}(m)\right), \tag{3.10}$$

where the B_k is the *kth* Bernouli number. Therefore, in our case, we have

$$\sum_0^\infty F(n) - \int_0^\infty F(n)dn = \sum_{k=1}^p \frac{B_k}{k!}\left(-F^{(k-1)}(0)\right)$$
$$= -\frac{1}{2}F^{(0)}(0) - \frac{1}{12}F^{(1)}(0) + \frac{1}{30\cdot 24}F^{(3)}(0) + ... \tag{3.11}$$

Introducing the variable $w = u + n_z^2$, we write

$$F(n) = \int_{n^2}^\infty dw \underbrace{\left(\sqrt{w}\right) f\left(\frac{\pi\sqrt{w}}{ak_c}\right)}_{F(w)}, \tag{3.12}$$

Now, we can use the Leibniz rule to know the value of the derivatives. We have

$$F^{(1)}(n) = \frac{d}{dn}\left(\int_{n^2}^\infty F(w)dw\right) = -F(n^2)\frac{d}{dn}n^2 + \int_{n^2}^\infty \cancelto{0}{\frac{\partial}{\partial n}F(w)}dw \tag{3.13}$$

whence,

$$F^{(1)}(n) = -2n^2 f\left(\frac{\pi n}{ak_c}\right)$$
$$F^{(1)}(0) = 0$$
$$F^{(3)}(0) = -4. \tag{3.14}$$

Thus, we find the energy per unit area to be

$$\epsilon_d(a) = \frac{\epsilon}{L^2} = -\frac{\pi^2}{720}\cdot\frac{1}{a^3} \tag{3.15}$$

as long as $ak_c \gg 1$. We can see that there exists a negative energy between two perfectly conducting metal plates for low frequencies independent of the material, regardless of the material, even in a vacuum situation.

3.1.2 Analytic extension method

Another method used in order to obtain a finite result for the vacuum energy is the analytic regularization procedure. A direct advantage of this method is that it allows shorter calculations, as we shall see. We aim to obtain the Casimir energy for a massless scalar field at zero temperature in the presence of a slab geometry $\Omega = \mathbb{R}^{d-1} \times [0, L]$ with Dirichlet boundary conditions. This particular scenario can be seen as a general description and the results can be extended to the electromagnetic field in a three-dimensional manifold. First, let us assume a free neutral scalar field defined in a $(d+1)$-dimensional flat space-time. Its field equation, the Klein-Gordon equation, reads

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m_0^2\right)\varphi(t, \mathbf{x}) = 0. \tag{3.16}$$

To proceed, we restrict the field to a d-dimensional box with lengths $(L_1 \times L_2 \times ... \times L_{d-1} \times L_d)$. Assuming Dirichlet boundary conditions, the total energy of the quantized field in the vacuum state inside the box is $\langle 0|\mathcal{H}|0\rangle = U_d(L_1, ..., L_{d-1}, L_d)$ *i.e.* the vacuum energy. Using the condition $L_d \ll L_i$ for $(i = 1, 2, ..., d-1)$, and defining $L_d = L$, the unrenormalized vacuum energy can be written as

$$U_d(L_1, ..., L_{d-1}, L) = \frac{1}{(2\pi)^{d-1}}\left(\prod_{i=1}^{d-1} L_i\right)\int\prod_{i=1}^{d-1}\mathrm{d}q_i\sum_{n=1}^{\infty}\left(q_1^2 + ... + q_{d-1}^2 + \left(\frac{n\pi}{L}\right)^2 + m_0^2\right)^{\frac{1}{2}}. \tag{3.17}$$

To discuss the case similar to the electromagnetic field let us assume $m_0^2 = 0$ and the unrenormalized vacuum energy per unit area is defined as

$$\epsilon_d(L) = \frac{U_d(L_1, ..., L_{d-1}, L)}{\left(\prod_{i=1}^{d-1} L_i\right)}, \tag{3.18}$$

which is a divergent expression. Once we have the angular part of the integral over $(d-1)$-dimensional k space, the product can be written as

$$\epsilon_d(L) = \frac{(4\pi)^{\frac{1-d}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\sum_{n=1}^{\infty}\int_0^{\infty}\mathrm{d}r\, r^{d-2}\left[r^2 + \left(\frac{n\pi}{L}\right)^2\right]^{\frac{1}{2}}. \tag{3.19}$$

introducing the variable change $x = L^2 r^2/\pi^2 n^2$ and after a straightforward calculation, we get

$$\epsilon_d(L) = \frac{(4\pi)^{\frac{1-d}{2}}}{2\Gamma\left(\frac{d-1}{2}\right)}\left(\frac{\pi}{L}\right)^d\int_0^{\infty}\mathrm{d}x\, x^{\frac{d-3}{2}}(1+x)^{\frac{1}{2}}\sum_{n=1}^{\infty} n^d. \tag{3.20}$$

In the limit $L \to \infty$ we should obtain the fundamental result that the vacuum is a Lorentz invariant state of zero energy. Using the definition of the Beta function

$$\mathcal{B}(m, n) = \int_0^{\infty}\frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.21}$$

and an analytic continuation principle, the vacuum energy per unit area is given by

$$\epsilon_d(L) = -\frac{\pi^{\frac{d}{2}}\Gamma\left(-\frac{d}{2}\right)}{2(2L)^d}\zeta(-d), \tag{3.22}$$

where $\zeta(s)$ the analytic extension of the Riemann zeta function which is a function of the complex variable $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$. To give some sense in this divergent summation, as we indicated above, the major interest in the $\zeta(s)$ as a function of a complex variable is to continue the definition beyond the domain of convergence of the series. It is originally defined in the half-plane Re$(s) > 1$ through the Euler's product

$$\zeta(s) = \prod_{p\in\mathbb{P}} (1 - p^{-s})^{-1}, \tag{3.23}$$

where $\mathbb{P}$ denotes the set of all primes. Using the canonical decomposition of natural numbers, the above expression, in the region where it converges, can be written as a Dirichlet series [33].

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3.24}$$

The series is defined by summing over the set of natural numbers $n \in \mathbb{N}$ and can be extended to the complex plane as a meromorphic function using the Poisson summation formula with a simple pole at $s = 1$. Additionally, it is possible to show that Riemann zeta-function $\zeta(s)$ satisfies a functional equation valid for $s \in \mathbb{C} \setminus \{0, 1\}$, as we shall see. To start, in order to be as clear as possible, we rely on the Poisson Summation formula, defining the Fourier transform of an integral function f on $\mathbb{R}$ by

$$\tilde{f}(x) = \int_{-\infty}^{\infty} dy\ f(y) e^{-2\pi ixy}. \tag{3.25}$$

Then

$$\sum_{-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \tilde{f}(n), \tag{3.26}$$

which both sides converge absolutely. Applying the transformation on $f(x) = \exp\{-x^2\pi\nu\}$, we have

$$\begin{aligned} \tilde{f}(n) &= \int_{-\infty}^{\infty} dx\ e^{-x^2\pi\nu} e^{-2\pi inx} \\ &= \frac{1}{\sqrt{\nu}} e^{-\frac{\pi n^2}{\nu}}, \end{aligned} \tag{3.27}$$

on completing the square and substituting $y = x + in/\nu$. This provides that ν is real and positive. Then we can see from the Poisson Summation Formula, the relation given by

$$\theta(\nu) = \frac{1}{\sqrt{\nu}} \theta\left(\frac{1}{\nu}\right), \tag{3.28}$$

where the theta function is

$$\theta(\nu) = \sum_{n=-\infty}^{\infty} \exp\{-n^2\pi\nu\}. \tag{3.29}$$

Before considering the zeta function, we introduce

$$\psi(\nu) = \frac{\theta(\nu) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2\nu}; \qquad 2\psi(\nu) + 1 = \frac{1}{\sqrt{\nu}}\{2\psi\left(\frac{1}{\nu}\right) + 1\}, \tag{3.30}$$

and its Mellin transform is for $\frac{s}{2}$

$$\psi^*\left(\frac{s}{2}\right) = \left(\sum_{n=1}^{\infty}(\pi n^2)^{-\frac{s}{2}}\right)\mathcal{M}\left[e^{-t};\frac{s}{2}\right] = \pi^{-\frac{s}{2}}\left(\sum_{n=1}^{\infty} n^{-s}\right)\int_0^{\infty} t^{\frac{s}{2}-1}e^{-t}dt = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \quad (3.31)$$

We can identify that, by splitting the range of integration in the transformation integral at the point $\nu = 1$ and applying the relation 3.28 for $\psi(\nu)$, we get

$$\begin{aligned}\psi^*\left(\frac{s}{2}\right) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^1 \psi(\nu)\nu^{\frac{s}{2}-1}d\nu + \int_1^{\infty}\psi(\nu)\nu^{\frac{s}{2}-1}d\nu \\ &= \int_0^1 \nu^{\frac{s}{2}-1}\left[\frac{1}{2\sqrt{\nu}}\{2\psi\left(\frac{1}{\nu}\right)+1\} - \frac{1}{2}\right]d\nu + \int_1^{\infty}\psi(\nu)\nu^{\frac{s}{2}-1}d\nu \\ &= \int_1^{\infty} y^{\frac{(1-s)}{2}-1}\psi(y)dy - \frac{1}{s(1-s)} + \int_1^{\infty}\psi(\nu)\nu^{\frac{s}{2}-1}d\nu \end{aligned} \quad (3.32)$$

where in the last step we have changed $y = 1/\nu$. We therefore conclude that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^{\infty}(\nu^{\frac{s}{2}-1} + \nu^{\frac{(1-s)}{2}-1})\psi(\nu)d\nu - \frac{1}{s(1-s)}. \quad (3.33)$$

The analytic extension of the zeta function has a simple pole in $s = 1$. The integral contribution converges due to the exponential decay of $\psi(\nu)$. This expression allows us to define $\zeta(s)$ in whole complex plane. Notably, the right-hand side remains invariant under the interchange of s with $1 - s$ in the critical strip and exhibits a point of symmetry at $s = 1/2$, thus

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \quad (3.34)$$

using some properties of the Gamma function and defining $\vartheta(s)$ as

$$\vartheta(s) = \frac{(2\pi)^s\Gamma(1-s)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)}, \quad (3.35)$$

where $\vartheta(s)$ has meaning for every complex value of s not equal to a positive odd integer. We get a reflection formula for the Riemann zeta-function

$$\zeta(s) = \vartheta(s)\zeta(1-s). \quad (3.36)$$

This functional equation allows us to extend the $\zeta(s)$ to the half-plane $\sigma \leq \frac{1}{2}$. Now, we have the zeta function defined for the whole complex plane. The above calculations are an intermediate step crucial to discussing the modifications in the renormalized vacuum energy of a scalar field in the presence of surfaces where the scalar field satisfies non-ideal boundary conditions. As we can see, for the three-dimensional manifold, *i.e.*, for $d = 3$, we have $\Gamma\left(\frac{-3}{2}\right) = \frac{4\sqrt{\pi}}{3}$ and $\zeta(-3) = \frac{1}{120}$ we obtain the same result as that found with the cutoff method.

Chapter 4

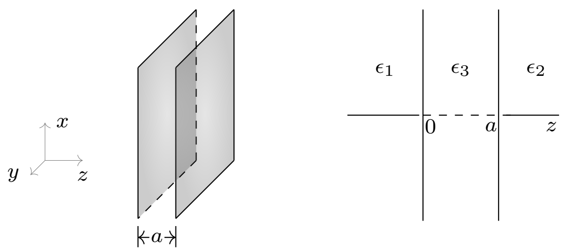
Casimir Effect with non-ideal boundary conditions

4.1 Corrections to the vacuum energy

In experiments on the vacuum energy between two plates, the simplifying assumption of perfect conductivity at all field frequencies is unrealistic, and to get a physical result of the effect the Casimir expression must contain the dielectric properties of the media. Lifshitz [28] developed a macroscopic theory to describe these properties; however, his theory is fairly complicated, and occasional doubts have been raised regarding its validity. His physical basis, in effect, assumes the interaction of the objects as regarded as occurring through the medium of the fluctuating electromagnetic field which is present both the interior and extends beyond its boundaries. Rather than presenting the details of Lifshitz's derivation, for simplicity, we follow a somewhat different approach that ultimately leads to the same expression. This idea was first used by van Kampen, Nijboer and Schram [34] and in detail in Milonni's book [35].

4.1.1 Milloni's Approach

We study the interaction between the plates occurring by a classical electromagnetic field that extends beyond its boundaries, and the media is filled with dielectric constants, which have different materials in each region. The interior of those plates has a dielectric constant $\epsilon_3(\omega)$, the half-space $z < 0$ has a $\epsilon_1(\omega)$ and the region $z > a$ has $\epsilon_2(\omega)$. This approach can be seen as a generalization of the Casimir calculation, although the source of the effect is now the current and the polarization fluctuations in the plates. Analogous to Casimir's approach, we picture the interacting bodies bounded by conducting walls separating three media with conducting plates at a distance a.



We aim to find the natural frequencies ω that satisfy the Maxwell equations with the appropriate boundary conditions. By the translation symmetry along the xy plane, we expect that the electric and magnetic fields as free waves in this plane and only depend on coordinates z. Let us assume solutions in the form

$$\mathbf{E}_0(\mathbf{r}) = [e_x(z)\hat{x} + e_y(z)\hat{y} + e_z(z)\hat{z}]e^{i(k_x x + k_y y)}, \tag{4.1}$$

$$\mathbf{B}_0(\mathbf{r}) = [b_x(z)\hat{x} + b_y(z)\hat{y} + b_z(z)\hat{z}]e^{i(k_x x + k_y y)}, \tag{4.2}$$

and for this plane symmetry, given a mode of the field, even though generic, we can make $k_y = 0$ by a particular choice of coordinates without loss of generality. The Gauss law in the electric displacement field implies

$$ike_x + \frac{de_z}{dz} = 0, \qquad k \equiv k_x. \tag{4.3}$$

Using our solution, we have that the components of the electric field satisfy

$$\frac{d^2 e_i}{dz^2} - K^2 e_i = 0, \tag{4.4}$$

where the index $i = \{x, y, z\}$ and we have defined

$$K^2 = k^2 - \epsilon(\omega)\omega^2, \qquad K^2 > 0. \tag{4.5}$$

Thus, all the boundary conditions are satisfied if (1) ϵe_z and de_z/dz are continuous and if (2) e_y and de_y/dz are continuous. From equation 4.4 for the z component we have, avoiding unphysical exponentially growing solutions,

$$e_z(z) = \begin{cases} Ae^{K_1 z}, & z < 0 \\ Be^{K_3 z} + Ce^{-K_3 z}, & 0 \le z \le a \\ De^{-K_2 z}, & z > a \end{cases} \tag{4.6}$$

where $K_j \equiv \sqrt{k^2 - \epsilon_j(\omega)\omega^2}$. The boundary condition (1) at $z = 0$ and $z = a$ lead us to four linear algebraic equations for A, B, C, D. In a matrix representation, we have

$$\begin{pmatrix} \epsilon_1 & -\epsilon_3 & -\epsilon_3 & 0 \\ K_1 & -K_3 & K_3 & 0 \\ 0 & \epsilon_3 e^{K_3 a} & \epsilon_3 e^{-K_3 a} & -\epsilon_2 e^{-K_2 a} \\ 0 & K_3 e^{K_3 a} & -K_3 e^{-K_3 a} & K_2 e^{-K_2 a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.7}$$

The nontrivial solution to these equations is achieved when the determinant of the matrix of coefficients vanishes. Bearing this in mind, a straightforward calculation leads us to the expression

$$\frac{(\epsilon_3 K_1 + \epsilon_1 K_3)(\epsilon_3 K_2 + \epsilon_2 K_3)}{(\epsilon_3 K_1 - \epsilon_1 K_3)(\epsilon_3 K_2 - \epsilon_2 K_3)} e^{2K_3 a} - 1 = 0. \tag{4.8}$$

We can do an analogous approach for the boundary condition (2), and we get

$$\frac{(K_1 + K_3)(K_2 + K_3)}{(K_1 - K_3)(K_2 - K_3)} e^{2K_3 a} - 1 = 0. \tag{4.9}$$

These equations provide us with all the allowed frequencies ω that need to be considered in the vacuum energy summation. Since these modes depend on the continuum variable k, associated with the plane waves in the xy-plane and on the discrete index n, to calculate the total zero-point energy

$$\epsilon(a) = \frac{E(a)}{L^2} = \frac{1}{4\pi} \int_0^\infty dk\, k \left[\sum_n \omega_n^{(1)}(k) + \sum_n \omega_n^{(2)}(k) \right], \tag{4.10}$$

where L is the length of the unbounded and non-compact dimensions and the $\omega_n^{(1)}$ and $\omega_n^{(2)}$ are the frequencies associated with the boundary conditions (1) and (2), respectively. Although our primary focus is on the Casimir energy, it is more straightforward to calculate the force and subsequently perform integration to obtain the energy. For this purpose, examining the left-hand side of equations 4.8 and 4.9 and after a cumbersome calculation we get the force per unit area as

$$\begin{aligned} f(a) = -\frac{\partial}{\partial a}\epsilon(a) = &-\frac{1}{2\pi^2} \int_0^\infty dk\, k \int_0^\infty d\xi K_3 \times \\ &\times \left(\left[\frac{(\epsilon_3 K_1 + \epsilon_1 K_3)(\epsilon_3 K_2 + \epsilon_2 K_3)}{(\epsilon_3 K_1 - \epsilon_1 K_3)(\epsilon_3 K_2 - \epsilon_2 K_3)} e^{2K_3 a} - 1 \right]^{-1} + \right. \\ &\left. + \left[\frac{(K_1 + K_3)(K_2 + K_3)}{(K_1 - K_3)(K_2 - K_3)} e^{2K_3 a} - 1 \right]^{-1} \right). \end{aligned} \tag{4.11}$$

Despite this not being precisely the Lifshitz result, a slight manipulation can bring us to it. By introducing a variable change to define $k^2 = \epsilon_3 \xi^2 (p^2 - 1)$, $K_3 = \sqrt{\epsilon_3} \xi p$, $K_{1,2}^2 = \epsilon_3 \xi^2 s_{1,2}^2$, and $\epsilon_3 = 1$, since we have a vacuum between the plates, we agree exactly with Lifshitz's result in a conceptually simpler way based on the zero-point energy.

$$\begin{aligned} f(a) = &-\frac{1}{2\pi^2} \int_1^\infty dp\, p^2 \int_0^\infty d\xi\, \xi^3 \times \\ &\times \left(\left[\frac{(s_1 + \epsilon_1 p)(s_2 + \epsilon_2 p)}{(s_1 - \epsilon_1 p)(s_2 - \epsilon_2 p)} e^{2\xi p a} - 1 \right]^{-1} + \left[\frac{(s_1 + p)(s_2 + p)}{(s_1 - p)(s_2 - p)} e^{2\xi p a} - 1 \right]^{-1} \right). \end{aligned} \tag{4.12}$$

That's not enough to obtain the correction for the Casimir effect. It's important to stress that this effect arises from frequencies ξ in the range of approximately $\xi \approx 1/a$. Consequently, the predominant frequencies fall within the infrared and visible regions of the electromagnetic spectrum. For these frequencies, the dielectric constant in first and second order is

$$\epsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}, \tag{4.13}$$

where ω_p is the plasma frequency of the material. We want to study identical imperfect conductors separated by vacuum, *i.e.* $s_{1,2} = s$. Then, our expression for the force becomes

$$f(a) = -\frac{1}{2\pi^2}\int_1^\infty dp\, p^2 \int_0^\infty d\xi\, \xi^3 \left(\left[\frac{(s+\epsilon p)^2}{(s-\epsilon p)^2} e^{2\xi p a} - 1 \right]^{-1} + \left[\frac{(s+p)^2}{(s-p)^2} e^{2\xi p a} - 1 \right]^{-1} \right). \tag{4.14}$$

For perfect conductors, we have ${}^{(s+\epsilon p)^2}/_{(s-\epsilon p)^2} = {}^{(s+p)^2}/_{(s-p)^2} = 1$. However, in an approximation scenario for imperfect conductivity, we get a slight deviation from the perfect case. We get, after some manipulations, therefore

$$f(a) = f_c(a) \left[1 - \frac{16}{3\omega_p a} \right], \tag{4.15}$$

where we can identify the Casimir force for perfect boundary conditions $f_c(a)$. After calculating the Casimir force, we can now integrate to obtain the energy

$$\epsilon(a) = \epsilon_c(a) \left[1 - \frac{4}{\omega_p a} \right]. \tag{4.16}$$

The Casimir energy $E_c(a)$ has a factor a^{-3}, then the effect of imperfect conductivity is therefore to diminish the Casimir energy by a factor a^{-4}.

4.1.2 Analytic extension approach for non-ideal boundary conditions

Following the structure outlined in the previous chapter, we now employ the analytic extension method to derive the same correction to the Casimir energy as obtained by Lifshitz's approach.

4.1.2.1 Driving in a guide wave

In our case, we are discussing the vacuum energy of a quantized scalar field in the presence of boundaries, where the field satisfies non-ideal boundary conditions. Those can be understood as finite conductivity conditions. We can call it *ideal high-pass Dirichlet boundary condition*. To clarify, our boundary condition is over the frequencies, we can think of them as the following: for frequencies smaller than some ω_{k_c} we do have the usual Dirichlet boundary conditions, otherwise, the plates are transparent for the field. However, the crucial point behind the need for this approach, is that it is not convenient to simply calculate the correction to the renormalized vacuum energy separating the effects of the low-energy vacuum modes from the high-energy modes using a sharp cut-off, once that is a sum of positive terms and it always yields to a positive energy density, *i.e.*,

$$\epsilon_d^{\text{f.c.}}(L) = \sum_{k=1}^{k_c} \omega_k > 0, \tag{4.17}$$

where ω_{k_c+1} is plasma frequence of the material.

We start using an analytic regularization procedure and the fact that for Dirichlet boundary conditions the eigenvalues vary continuously under a smooth deformation of the domain (spectral stability of elliptic operator under domain deformation) and the minimax principle says that the eigenvalues monotonously decrease when the domain is enlarger,

$$\sigma_m(\Omega_1) \geq \sigma_m(\Omega_2), \;\; \Omega_1 \subset \Omega_2. \tag{4.18}$$

By the above arguments, we can use approximate functional equation that expresses the Riemann zeta function as finite sums, outside their original domain of convergence. We have presented so far the main results of the zeta function and its properties. Despite that, the need for our work claims for more advanced topics. We shall be concerned now with approximate functional equations. This result relies on the description of the zeta function as a certain finite sum at the points s and $1-s$.

There are several methods to achieve the approximate functional equation, but we follow the derivation discussed in more detail in Ref. [36]. Initially, we used a classical result by Hardy and Littlewood. Let us write the Riemann zeta function as

$$\begin{aligned}
\zeta(s) &= \sum_{n\leq n_c} n^{-s} + \sum_{n>n_c} n^{-s} \\
&= \sum_{n\leq n_c} n^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}x\, x^{s-1}\left(\sum_{n>n_c} e^{-nx}\right) \\
&= \sum_{n\leq n_c} n^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}x\, \frac{x^{s-1}\, e^{-n_c x}}{e^x-1},
\end{aligned} \tag{4.19}$$

where the absolute convergence justifies the inversion of the order of summation and integration. To proceed, we analyze the following integral $I(s)$. We have

$$I(s) = \int_C \mathrm{d}z\, \frac{z^{s-1}\, e^{-n_c z}}{e^z - 1}, \tag{4.20}$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction excluding the points $\pm 2\pi i, \pm 4\pi i, ...$ and returns to infinity. We obtain

$$I(s) = \left(e^{2\pi\, is} - 1\right)\int_0^\infty \mathrm{d}x\, \frac{x^{s-1}\, e^{-n_c x}}{e^x - 1}. \tag{4.21}$$

Using the analytic continuation principle, if s is not a positive integer, we can write

$$\zeta(s) = \sum_{n\leq n_c} n^{-s} + \frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}\int_C \mathrm{d}z\, \frac{z^{s-1}\, e^{-n_c z}}{e^z-1}. \tag{4.22}$$

From the above equation, after a specific choice of variable, we can replace the contour C with straight lines C_1,C_2, C_3, C_4, and use the residue theorem. Therefore we have

$$\zeta(s) = \sum_{n\leq x}\frac{1}{n^s} + \vartheta(s)\sum_{n\leq y}\frac{1}{n^{1-s}} + \frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right), \tag{4.23}$$

Following integration and some approximations, we obtain an approximate representation of the zeta function in terms of finite sums

$$\zeta(s) = \sum_{n\leq x}\frac{1}{n^s} + \vartheta(s)\sum_{n\leq y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(t^{\frac{1}{2}-\sigma}\, y^{\sigma-1}), \tag{4.24}$$

for $0 \leq \sigma < 1$ holds for given $x, y, t > C > 0$ satisfying $2\pi xy = t$ where $t \gg 1$. This is known as an approximate functional equation and it is crucial for our main results.

Before proceeding, we need to be clear about the choice for the plasma frequency. For simplicity, using the approximate functional equation, we discuss the case of a slab geometry $\mathbb{R}^{d-1} \times [0, L]$. Making a parallel with the electromagnetic case, in the scalar field scenario, we define the plasma frequency ω_p and the plasma wavelength $\lambda_p = 2\pi/w_p$. Next, we define a "critical" mode index n_c, which will be related to the plasma wavelength. In order to find an adequate maximum number of states n_c for a single compactified direction, we need to introduce first the notion of the density of states $\rho(k)$ in the phase space and the number of states $dN = \rho(k)d^dk$ that lies between k and $k + dk$. In our d-dimensional space, where all the directions are finite and have lengths $L_1, L_2, ..., L_{d-1}, L$, then the density of states is simply

$$\rho(k) = \left(\frac{L}{\pi^d}\right) \prod_{i=1}^{d-1} L_i \, , \tag{4.25}$$

we can find the number of states inside a volume that possess the maximum value of moment k_{max} as

$$N(k_{max}) = \int_{|k|<k_{max}} \mathrm{d}^d k \rho(k) = \rho \, \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} k_{max}^d \, , \tag{4.26}$$

where we have used the definitions of the volume of a sphere in d-dimensions. On the other side, we are interested in obtaining the maximum number of states in a single compactified direction n_c. We have that

$$N(k_{max}) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} n_c^d. \tag{4.27}$$

Therefore we identified $n_c^d = \rho k_{max}^d$. Now, we relate the maximum wave number with the plasma frequency of the material in such a manner that $k_{max} = 2\pi/\lambda_p$. With all this, after some algebra, we conclude

$$n_c = 2\left(\frac{L^{1/d}}{\lambda_p}\right) \prod_{i=1}^{d-1} L_i^{1/d}. \tag{4.28}$$

Since all the directions L_i from $i = \{1, 2, ..., d-1\}$ are much larger than L, the only dependence of the maximum number of states is of the form

$$n_c(L) \equiv \left(\frac{L}{\lambda_p}\right)^{1/d} \, . \tag{4.29}$$

In the Hardy and Littlewood approximate functional equation, Eq. 4.24, we choose

$$x = y = \left(\frac{L}{\lambda_p}\right)^{1/d} = n_c \quad \Rightarrow \quad t = 2\pi \left(\frac{L}{\lambda_p}\right)^{2/d} = 2\pi n_c^2 \, . \tag{4.30}$$

Using the asymptotic expansion, we get the Casimir energy as

$$\epsilon_d(L) = -\frac{\pi^{\frac{d}{2}}\Gamma\left(-\frac{d}{2}\right)}{2(2L)^d} \left[H_{n_c}(-d) + \vartheta(-d)H_{n_c}(d+1)\right]. \tag{4.31}$$

The quantities $H_n(s)$ are the generalized harmonic numbers. Once the Eq. (4.31) only makes sense as an analytic continuation, those finite sums must be understood as such. Moreover, we stress the fact that the equality holds by analytic continuation outside the strip $0 < \sigma < 1$. This can be shown using an analytic continuation of the asymptotic expansion.

Each generalized harmonic number has an expression for its domain of interest in the complex plane. Lets us start from the second term in the sum, $H_{n_c}(d+1)$. Formally, this quantity is given by

$$H_{n_c}(d+1) \equiv \sum_{n=1}^{n_c} \frac{1}{n^{d+1}}. \tag{4.32}$$

However, since we start from Eq. (3.22), which is an analytic continuation, the finite sum should be taken in the range of interest. In such a situation, we can use a known expression [37]

$$H_{n_c}(d+1) = \zeta(d+1) + \frac{(-1)^d}{d!}\psi_d(n_c+1), \tag{4.33}$$

which holds for $n_c \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}$ and $d \in \mathbb{N}$, and $\psi_m(x)$ is the polygamma function. Using a recurrence relation and an expression for large arguments, we can write the polygamma function as

$$\psi_d(n_c+1) = \frac{(-1)^d d!}{n_c^{d+1}} + (-1)^{d+1} \sum_{k=0}^{\infty} \frac{(k+d-1)!}{k!} \frac{B_k}{n_c^{d+k}}\,, \tag{4.34}$$

where B_k are the Bernoulli numbers and we take the convention $B_1 = 1/2$. Using the definition of n_c and in the limit of $L/\lambda_p \gg 1$ we can write

$$\psi_d(n_c+1) \approx (-1)^{d+1} \left(\frac{\lambda_p}{L}\right) \left[(d-1)! - \frac{1}{2} d! \left(\frac{\lambda_p}{L}\right)^{\frac{1}{d}}\right], \tag{4.35}$$

which allows us to write the $H_{n_c}(d+1)$ in powers of λ_p/L. For the first term of Eq. 4.31, we formally have

$$H_{n_c}(-d) \equiv \sum_{n=1}^{n_c} \frac{1}{n^{-d}}, \tag{4.36}$$

and an analytic continuation can be obtained using some elementary operations and the uniqueness of the analytic continuation, is straightforward to see that

$$H_{n_c}(-d) = \zeta(-d) - \zeta_H(-d; n_c+1), \tag{4.37}$$

where $\zeta_H(-d; n_c+1)$ is the Hurwitz zeta-function, defined by

$$\zeta_H(s; a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}. \tag{4.38}$$

Let us define the Casimir energy per unit area with non-ideal boundary conditions, *i.e.*, finite conductivity $\epsilon_d^{\text{f.c.}}$ by

$$\epsilon_d^{\text{f.c.}}(L) \equiv -\frac{1}{L^d} \frac{\pi^{d/2}}{2^{d+1}} \Gamma\left(-\frac{d}{2}\right) \zeta_H(-d; n_c+1). \tag{4.39}$$

Once this is performed, we can identify the contribution from the ideal boundary conditions, and the remaining part can be regarded as a correction term. We get

$$\epsilon_d^{\text{f.c.}}(L) = \epsilon_d(L) + \frac{\Gamma(1+d)\lambda_p}{2\,\Gamma\left(1+\frac{d}{2}\right)} \left(\frac{1}{4\sqrt{\pi}}\right)^d \left[\frac{1}{L^{d+1} d} - \frac{\lambda_p^{\frac{1}{d}}}{2L^{d+1+\frac{1}{d}}}\right]. \tag{4.40}$$

As we have observed, in the slab geometry, the Casimir energy is a negative quantity ($\epsilon_d(L) < 0$), while the second contribution in the above equation is positive diminishing the Casimir energy. Note that our first finite conductivity correction to the electromagnetic Casimir energy in a three-dimensional manifold is the same as the correction obtained using the Lifshitz calculations. In contrast, the second correction is smaller, with the Lifshitz formula giving a second correction as L^{-5}, whereas ours gives $L^{-\frac{13}{3}}$. Therefore, we have succeeded in deriving the Casimir energy per unit area with non-ideal boundary conditions.

The basic assumption that needs to be carefully investigated is the discussion of vacuum energy in a bounded domain. In order to get a more complete result, in the next section, we generalize the above result to the $d = 2$ dimensional case for a finite volume box.

4.1.2.2 Living in a box

Let us discuss now the eigenvalues of a second-order elliptic self-adjoint partial differential operator on scalar functions on a bounded domain. We consider the eigenvalues of $-\Delta$ on a connected open set Ω in Euclidean space $\mathbb{R}^2$. We assume that the massless scalar field is confined in a rectangular box, with lengths L_1 and L_2 obeying Dirichlet boundary conditions. The eigenfrequencies that we use to expand the field operator are given by

$$\omega_{n_1 n_2} = \left[\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2\right]^{\frac{1}{2}}; \quad n_1, n_2 = 1, 2, ... \tag{4.41}$$

The unrenormalized vacuum energy in this case is

$$U(L_1, L_2) = \frac{1}{2}\sum_{n_1,n_2=1}^{\infty} \omega_{n_1 n_2}. \tag{4.42}$$

Making use of an analytic regularization procedure, the divergent expression can be written as

$$E(L_1, L_2, s) = \frac{1}{2}\sum_{n_1,n_2=1}^{\infty} \omega_{n_1 n_2}^{-2s}, \tag{4.43}$$

for $s \in \mathbb{C}$. Observe that, the vacuum energy is obtained when $s = -\frac{1}{2}$. The above double series converges absolutely and uniformly for $Re(s) > 1$. An analytic function, which plays an important role in algebraic number theory is the Epstein zeta-functions associated with quadratic forms [38] , and its approximate equation can be derived as follows. Let's start by assuming that

$$\phi(a, b, c; x, y) = ax^2 + cxy + by^2, \tag{4.44}$$

where a, b and $c \in \mathbb{R}$ and $a > 0$ and $\eta = 4ab - c^2 > 0$. Lets us define the function $\mathcal{A}(s)$ by the series

$$\mathcal{A}(a, b, c; s) = \sum_{n_1,n_2=-\infty}^{\infty}{}' \phi^{-s}(a, b, c; n_1, n_2), \tag{4.45}$$

The above series defines an analytic function for $s = \sigma + it$, ($\sigma \in \mathbb{R}$ and $t \in \mathbb{R}$) and $\sigma > 1$, where we adopt the notation that the prime sign in the summation means that the contribution $n_1 = n_2 = 0$ (the origin of the mode space) must be excluded. This particular case of the Epstein zeta-function can be continued analytically to the whole complex plane, except for a simple pole at $s = 1$ [39]. This double series exhibits a functional equation that can be obtained using properties of the theta function or the Poisson summation formula. Potter [40] showed that $\mathcal{A}(s)$, like, $\zeta(s)$, can be continued analytically throughout the s-plane and that it satisfies the functional equation

$$\mathcal{A}(a, b, c; s) = X(s)\mathcal{A}\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; 1 - s\right), \tag{4.46}$$

where we define

$$X(s) = \left(\frac{2\pi}{\sqrt{\eta}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}. \tag{4.47}$$

If A is a positive constant, and

$$-\frac{1}{8} \leq \sigma \leq 1\frac{1}{8}, \quad x > A, \quad y > A, \quad 4\pi^2 xy = \Delta t^2, \tag{4.48}$$

then

$$\mathcal{A}(s) = \sum_{\phi \leq x}{}' \phi^{-s} + X(s) \sum_{\phi \leq y}{}' \phi^{s-1} + O\left\{x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{|t|}\right)^{\frac{1}{2}} \log|t|\right\}. \tag{4.49}$$

We are interested in the case where $c = 0$, henceforth we take $\mathcal{A}(a, b, 0; s) \equiv \mathcal{A}(a, b; s)$ and similar for ϕ. Let us define the function $Z\left(\frac{1}{L_1}, \frac{1}{L_2}; s\right)$ by

$$Z\left(\frac{1}{L_1}, \frac{1}{L_2}; s\right) = \sum_{n_1, n_2 = -\infty}^{\infty}{}' \left(\frac{n_1^2}{L_1} + \frac{n_2^2}{L_2}\right)^{-s}. \tag{4.50}$$

We can find the vacuum energy written as

$$E(L_1, L_2; s) = \frac{1}{8} Z\left(\frac{\pi^2}{L_1^2}, \frac{\pi^2}{L_2^2}; s\right) - \frac{1}{4}\left[\left(\frac{\pi}{L_1}\right)^{-2s} + \left(\frac{\pi}{L_2}\right)^{-2s}\right]\zeta(2s). \tag{4.51}$$

As it was discussed, $E(L_1, L_2, s)$ is analytic in $s \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$. Through the analytic continuation of the Epstein and the Riemann zeta function, the vacuum energy $U(L_1, L_2) = E(L_1, L_2; s = -1/2)$ for the system with Dirichlet boundary conditions is written as

$$U(L_1, L_2) = \frac{\pi}{48}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) - \frac{L_1 L_2}{32\pi} \sum_{n_1, n_2 = -\infty}^{\infty}{}' \left(n_1^2 L_1^2 + n_2^2 L_2^2\right)^{-\frac{3}{2}}. \tag{4.52}$$

The next step involves discussing the scalar case similar to the electromagnetic of imperfect conductors, characterized by a plasma frequency ω_p. To obtain the correction to the Casimir energy via asymptotic series, using the same approach discussed in the previous section, we will need to use the Hatree-Littlewood approximate functional equation for the Riemann zeta function and also the Potter approximate functional equation for the Epstein zeta function.

Let's start analyzing the Epstein zeta function. It's convenient to introduce a λ_p term in our expression in order to only have adimensional quantities and establish a parallel with the Casimir energy in a finite conductivity scenario. In this case, we have

$$\mathcal{A}\left(\frac{\pi^2\lambda_p^2}{L_1^2},\frac{\pi^2\lambda_p^2}{L_2^2};s\right)=\sum_{\Phi\leq x}{}'\,\Phi_{12}^{-s}+X(s)\sum_{\Phi\leq y}{}'\,\Phi_{12}^{s-1}\,, \tag{4.53}$$

in order to the notation be lightened, we defined

$$\begin{aligned}\Phi_{12} &\equiv \phi\left(\frac{\pi^2\lambda_p^2}{L_1^2},\frac{\pi^2\lambda_p^2}{L_2^2};n_1,n_2\right)\\ &= \frac{\pi^2\lambda_p^2}{L_1^2}n_1^2+\frac{\pi^2\lambda_p^2}{L_2^2}n_2^2\,,\end{aligned} \tag{4.54}$$

once $4\pi^2 xy=\eta\, t^2$ and

$$\eta=4\left(\frac{\pi^2\lambda_p^2}{L_1L_2}\right)^2\Rightarrow xy=\left(\frac{\pi\lambda_p^2}{L_1L_2}\right)^2 t^2. \tag{4.55}$$

Since

$$X(s)=\left(\frac{L_1L_2}{\pi\lambda_p^2}\right)^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}, \tag{4.56}$$

using a similar argument that we used before, but now all dimensions remain compact, we can define the quantities

$$\begin{aligned}n_c^{(1)} &\equiv \left(\frac{L_1}{\lambda_p}\right)^{1/2}\ \text{and}\ n_c^{(2)}\equiv\left(\frac{L_2}{\lambda_p}\right)^{1/2}\\ \Rightarrow\ xy &= \left[\frac{\pi}{\left(n_c^{(1)}n_c^{(2)}\right)^2}\right]^2 t^2,\end{aligned} \tag{4.57}$$

which, considering the fact that we do not have a preferred direction, indicate to us that the natural choice for t should be

$$t=\frac{1}{\pi}\left(n_c^{(1)}n_c^{(2)}\right)^2\Rightarrow x=y=n_c^{(1)}n_c^{(2)}. \tag{4.58}$$

Looking back to the Eq. 4.53, we see that the sums are over all modes inside the ellipse defined by

$$\frac{n_1^2}{L_1n_c^{(1)}n_c^{(2)}}+\frac{n_2^2}{L_2n_c^{(1)}n_c^{(2)}}=\left(\frac{1}{\pi\lambda_p}\right)^2=\ \text{constant}, \tag{4.59}$$

in the (n_1,n_2)-plane with the origin removed.

For the Riemann zeta-function contributions in Eq. 4.51, we have

$$\zeta(2s)=\sum_{n\leq u}\frac{1}{n^{2s}}+\vartheta(2s)\sum_{n\leq v}\frac{1}{n^{1-2s}}, \tag{4.60}$$

for $\alpha\gg 1$ where $2\pi uv=\alpha$. Proceeding exactly as in the slab bag geometry case, we find that

$$u=v\equiv n_c^{(i)}=\left(\frac{L_i}{\lambda_p}\right)^{1/2}\Rightarrow\alpha=2\pi\frac{L_i}{\lambda_p};\quad i=1,\ 2. \tag{4.61}$$

Continuing from the previous section, we employ an analogous method using the same harmonic number definitions, once the range in the complex plane will be the same. Considering the case where $s = -1/2$ and manipulating the equations, is possible to find that

$$E(L_1, L_2; s) = \frac{\lambda_p^{2s}}{8} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}}' \Phi_{12}^{-s} + \left(\frac{L_1 L_2}{\pi \lambda_p^2}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{\lambda_p^{2s}}{8} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}}' \Phi_{12}^{s-1} + \\ -\frac{\lambda_p^{2s}}{4} \sum_{i=1}^{2} \left\{ \left(\frac{\lambda_p}{L_i}\right)^{-2s} \left[2\zeta(2s) - \zeta_H(2s; n_c^{(i)}+1) \right] + (-1)^{-4s+1} \vartheta(2s) \left[\frac{1}{2s} \left(\frac{\lambda_p}{L_i}\right)^{-3s} - \frac{1}{2} \left(\frac{\lambda_p}{L_i}\right)^{\frac{-6s+1}{2}} \right] \right\}. \tag{4.62}$$

We define the vacuum energy for finite conductivity $E^{\text{f.c.}}$ as

$$E^{\text{f.c.}}\left(L_1, L_2, s = -\frac{1}{2}\right) = U^{\text{f.c.}}(L_1, L_2) \equiv \frac{1}{8\lambda_p} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}} \Phi_{12}^{\frac{1}{2}} - \frac{1}{4} \sum_{i=1}^{2} \frac{1}{L_i} \left[\zeta_H(-1; n_c^{(i)} + 1) - \frac{1}{6} \right]. \tag{4.63}$$

Therefore

$$U^{\text{f.c.}}(L_1, L_2) = U(L_1, L_2) - \frac{\pi^2 \lambda_p^3}{32(L_1 L_2)^2} \sum_{\Phi \leq n_c^{(1)} n_c^{(2)}} \Phi_{12}^{-\frac{3}{2}} + \frac{1}{2\lambda_p (2\pi)^2} \sum_{i=1}^{2} \left[\left(\frac{\lambda_p}{L_i}\right)^{3/2} - \frac{1}{2} \left(\frac{\lambda_p}{L_i}\right)^{2} \right] \tag{4.64}$$

is the Casimir energy for a rectangular box with non-ideal boundary conditions.

Chapter 5

Conclusions

This work aims to show that the vacuum energy of quantum fields satisfying nonideal boundary conditions can be calculated using spectral theory. First, we investigated the total energy of a quantized massless scalar field satisfying ideal boundary conditions using an analytic regularization procedure. Next, we extend the above result to the case of "imperfect conductor" boundary conditions, which we denote as ideal high-pass Dirichlet boundary conditions. In this scenario, the crucial point is that it is not convenient to simply calculate the correction to the renormalized vacuum energy by separating the effects of the low-energy from the high-energy vacuum modes using a sharp cut-off. This procedure effectively discards high-energy frequencies giving wrong results. Furthermore, we discussed the Casimir energy associated with a massless scalar field assuming a slab geometry $\mathbb{R}^{d-1} \times [0, L]$. To obtain the correction to Casimir force in the case of ideal high-pass Dirichlet boundary conditions, we use an approximate functional equation. This approach considers all the infinite modes through an analytic continuation. To accomplish this, we represent the energy density using finite sums outside the original domain of convergence of the Dirichlet series. Finally, we demonstrate how it is possible to obtain the correction to the Casimir force generated by a massless scalar field in three-dimensional spacetime in the presence of a rectangular box, with lengths L_1 and L_2.

We have discussed how zero-point fluctuations manifest as divergent vacuum energy. The Casimir effect has been observed in laboratory settings, yet certain challenges persist, notably the cosmological constant problem [41]. By introducing a cosmological constant as a free parameter in Einstein's equation and evaluating the stress-energy tensor at the vacuum state, we obtain the energy density of the vacuum, which receives contributions from the zero-point fluctuations of all quantum fields present in the universe. This energy density yields a divergent quantity, and as discussed, regularization of these infinities becomes a crucial concern. Initially, following the approach proposed in the finite conductivity case, we can introduce a sharp cut-off at $k = M$, where the physical interpretation of M represents the scale at which the effective theory utilized earlier encounters limitations [42]. However, similar to the plasma frequency case, this approach yields erroneous outcomes. Imposing a cut-off solely on spatial momentum disrupts Lorentz invariance. Thus, to address this issue, the regularization scheme must preserve it [43]. A possible choice is the analytic regularization with a noteworthy method being zeta function regularization. However, in this scenario, the challenges extend beyond the choice of a regularization scheme. Even when all conditions are adhered to, the theoretical expectation value for the cosmological constant often significantly deviates from experimental observations. It is noteworthy that various observational findings, not only in cosmology indicate disparities between theoretical predictions and empirical measurements [44, 45]. We aim to use our method of using asymptotic expansion and approximate functional equations to shed light on this problem as in the case of the conductivity correction of the Casimir effect.

The primary insight of this work is that, in the case of imperfect boundary conditions, we have to consider the ultraviolet modes by employing analytic continuation. Subsequently, an approximate functional equation can be utilized to determine the correction to the Casimir energy arising from imperfect boundary conditions. The analogy with the cosmological constant problem is evident. An alternative approach to addressing the cosmological constant problem involves considering the cosmological de Sitter model, characterized by a static cosmological model featuring a non-zero cosmological constant. Within this framework, an Euclidean section in the de Sitter spacetime can be identified and can assume a massive scalar field. To determine the vacuum energy via an analytic continuation procedure, it is necessary to derive a suitable form of zeta function. Similar to the previous scenario, if an approximate functional equation can be obtained, one can then explore the correction of the vacuum energy responsible for generating the cosmological constant within the de Sitter spacetime. This question deserves further investigations. Another possibility is to compare the result obtained using a approximate functional equation with the Milloni's method for the case of a electromagnetic field in the interior of a rectangular waveguide.

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