CBPF – Centro Brasileiro de Pesquisas Físicas

Coordenação de Cosmologia, Astrofísica e Interações Fundamentais

A novel method for renormalization in quantum field theory in curved space-time

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Supervisor: Marc Casals

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Abstract

In quantum field theory in curved space-time, two important physical quantities are the expectation value of the stress-energy tensor $\langle \hat{T}_{\mu\nu} \rangle$ and of the square of the field operator $\langle \hat{\varphi}^2 \rangle$. These expectation values must be renormalized, which is usually performed via the so-called point-splitting method. There is in the literature a practical implementation of this method for static, spherically-symmetric space-times. However, it does not readily generalize to other types of space-time. In this thesis, we present a novel implementation of the point-splitting procedure, which may be used in the future for more general space-times. As an example, we apply our method to the renormalization of $\langle \hat{\varphi}^2 \rangle$ for a massless scalar field in flat space-time and in Bertotti-Robinson space-time. We also briefly review the theory of quantum fields in curved space-time and the standard approaches to renormalization.

Resumo

Na teoria quântica de campos em espaço-tempo curvo, duas importantes quantidades físicas são o valor esperado do tensor de momento-energia $\langle \hat{T}_{\mu\nu} \rangle$ e do quadrado do operador de campo $\langle \hat{\varphi}^2 \rangle$. Esses valores esperados devem ser renormalizados, o que normalmente é realizado segundo o método de *point-splitting*. Há na literatura uma implementação prática desse método para espaços-tempos estáticos e esfericamente simétricos. Contudo, ela não se generaliza para outros tipos de espaço-tempo. Nessa dissertação, apresentamos uma nova implementação do procedimento de *point-splitting*, que pode ser utilizada no futuro para espaços-tempos mais gerais. Como um exemplo, aplicamos nosso método para a renormalização de $\langle \hat{\varphi}^2 \rangle$ para um campo escalar não massivo no espaço-tempo plano e no espaço-tempo de Bertotti-Robinson. Também revisamos de maneira breve a teoria de campos quânticos em espaços-tempos curvos e as abordagens tradicionais para a renormalização.

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Introduction

Even though the literature on quantum fields in curved space-time goes back to earlier times, the discovery by Hawking [1] that black holes emit thermal radiation of quantum origin was a major drive to that field of study. Since then, the interest in the quantum properties of black holes has grown steadily.

The motivation for such interest is the fact that black holes are physical systems in which both quantum and gravitational effects are important. Therefore, the theory of quantum fields in black hole space-times may hint at properties of a full theory of quantum gravity. In particular, the information paradox [2] is still an open question that has gathered much attention in the last decades. Furthermore, the detection of gravitational waves by LIGO [3, 4, 5] (which match the theoretical waveform for the coalescence of a pair of black holes) gives strong support for the existence of black holes in the Universe.

In the study of quantum field theory in curved space-time, a quantity of great importance is the expectation value of the stress-energy tensor operator. Indeed, it appears on the right-hand side of the semiclassical Einstein's equation, being a source for the backreacted metric. Moreover, the stress-energy tensor may be useful by itself, since it may contain important information about the system. For instance, in the case of black hole space-times, it carries information about the flux of particles at the horizon and at infinity. Another physical quantity in the theory is the expectation value of the square of the field operator.

However, these quantities are ill-defined without recourse to a procedure of renormalization. That is a consequence of their being quadratic on the field, while the field is,

in fact, an operator- valued distribution. In the last decades, methods for the renormalization of the stress-energy tensor and the square of the field in curved space-time have been studied. A practical implementation for such renormalization has been achieved in static, spherically-symmetric space-times with recourse to Euclideanization and the WKB approximation for the mode solutions [7]–[12]. Unfortunately, that approach is not straightforwardly generalizable to other space-times, such as the astrophysically important example of rotating black holes.

The main objective of the present work is to describe a novel method of renormalization, which may be used in more general space-times. We start by reviewing the theory of quantum fields in curved space-time in Chapter 1. Specifically, we review canonical quantization in flat space-time, emphasizing the properties which will be useful in the generalization to curved space-time. Then, we review the canonical quantization procedure in curved space-time, and the two-point functions which are going to be used throughout the text.

Chapter 2 deals with the renormalization of the expectation value of the stress-energy tensor and the square of the field. We describe the point-splitting method, and the standard implementation of this prescription in static, spherically-symmetric space-times. Finally, we discuss recent developments on the renormalization in other space-times such as Kerr.

In Chapter 3, we describe in detail the proposed novel method. Then, it is applied in Chapter 4 to the renormalization of the square of the field operator in flat space-time and in Bertotti-Robinson space-time.

Throughout the text, we use units $\hbar = 1$, $c = 1$, $8\pi G = 1$ and $\varepsilon_0 = 1$. Our signature convention for the space-time metric is $(-+++)$.

Chapter 1

Quantum field theory in curved space-time

Since the development of quantum mechanics in the first decades of the 20th century, much effort has been put into applying the principles of quantum physics to the description of the fundamental interactions and matter. In such approach, both the fundamental interactions and the matter particles are described by fields, which are quantized in a fixed, flat space-time. This quantum field theory achieved great success in the description of the elementary particles, giving rise to the well-known Standard Model of Particle Physics.

Still, the Standard Model does not include the oldest known fundamental interaction, gravity. The most successful description of the gravitational interaction, to this date, is Einstein's General Relativity. Yet, General Relativity is a classical theory, and there seems to be fundamental difficulties in its quantization[1].

Nevertheless, there are physical settings in which both gravitational and quantum effects are important. In scenarios in which both the quantum effects and gravity are sufficiently strong, General Relativity is expected to break down. Examples include the physics near space-time singularities, such as inside black holes and the primordial, hot and dense Universe.

[1]For a detailed review of the difficulties and proposals to the problem of the quantization of gravity, see, for instance, Ref. [13].

The existence of such important physical phenomena in which both gravitational and quantum effects are relevant, together with the difficulty in quantizing gravity directly, encourage the development of a semiclassical approach to the problem. One may quantize non-gravitational fields in a curved, yet fixed, background. This space-time is not quantized, but is a solution to Einstein's field equations of General Relavity.

In this chapter, we shall review the quantum field theory in curved space-time, and highlight some of the main differences with the quantization in Minkowski space-time. We also review the process of renormalization according to the so-called point-splitting method. The review given here is brief. For a more thorough review, the reader is referred to standard textbooks, such as Wald [14].

1.1 Canonical quantization in flat space-time

The process of canonical quantization in curved space-times is based on the same idea as in flat space-time. As we shall see, the main differences arise because the traditional process of quantization makes extensive use of the Poincaré symmetry in flat space-time. The presence of such a symmetry allows one to define a preferred vacuum state. On the other hand, in curved space-time the metric has no pre-determined symmetry. Let us start by pointing out where the Poincaré symmetry is invoked in the process of canonical quantization of a scalar field in flat space-time.

Let $\varphi(x)$ be a classical real scalar field in flat space-time $\mathcal{M}$, with mass m. The metric is the Minkowski metric $\eta_{\mu\nu}$. Then, the action is given by

$$\mathcal{S}[\varphi] = \int_{\mathcal{M}} \mathrm{d}^4x \mathcal{L}(\varphi(x), \partial_\mu\varphi(x)), \qquad \mathcal{L} = -\frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2\right), \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density (the negative sign comes from the choice of signature for the metric).

The extremizing of this action gives rise to the Klein-Gordon field equation:

$$\left(\Box - m^2\right)\varphi = 0, \qquad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{1.2}$$

In order to quantize this system via canonical quantization, we must first formulate its Hamiltonian structure. That requires a foliation of the space-time along a time coordinate, which allows one to define a time evolution. That is quite natural in the case of flat space-time, since we have global inertial coordinates $(t, \vec{x})$. Thus, we can write the action (1.1) as[2]

$$\mathcal{S}[\varphi] = \int_{\mathbb{R}} \mathrm{d}t\, L(\varphi(t,\vec{x}), \dot{\varphi}(t,\vec{x}), \nabla\varphi(t,\vec{x})), \tag{1.3}$$

by defining the Lagrangian L from the Lagrangian density $\mathcal{L}$ as

$$L(\varphi(t,\vec{x}), \dot{\varphi}(t,\vec{x}), \nabla\varphi(t,\vec{x})) = \int_{\Sigma_t} \mathrm{d}^3\vec{x} \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 \right), \tag{1.4}$$

where the dot represents time derivative, ∇ is the gradient in Cartesian coordinates and Σ_t is a hypersurface of constant t.

Then, the conjugate momentum to the configuration variable $\varphi(x)$ is

$$\pi(x) = \frac{\delta L}{\delta\dot{\varphi}(x)} = \dot{\varphi}(x). \tag{1.5}$$

Now, the field equation (1.2) has a unique solution if we specify $\varphi(x)$ and $\pi(x)$ at some time t. Hence, the phase space of this system is given by the pair $(\varphi(x), \pi(x))$ at some time t. These functions are assumed to be smooth, real and with compact support.

We are now in a position to define the Poisson brackets in this system. Given two functionals F and G of the phase space variables $\varphi(x)$ and $\pi(x)$, their Poisson bracket at time t is given by

$$\{F, G\} = \int_{\Sigma_t} \mathrm{d}^3\vec{x} \left(\frac{\delta F}{\delta\varphi(t,\vec{x})} \frac{\delta G}{\delta\pi(t,\vec{x})} - \frac{\delta G}{\delta\varphi(t,\vec{x})} \frac{\delta F}{\delta\pi(t,\vec{x})} \right). \tag{1.6}$$

In particular, one can take the Poisson brackets of the canonical variables themselves, at time t. These are the *fundamental Poisson brackets*, which can be directly calculated

[2] In this work, we take x to represent a point in 4-dimensional space-time, and $\vec{x}$ a point in 3-dimensional space.

from Eq. (1.6):

$$\begin{gathered}\{\varphi(t,\vec{x}),\varphi(t,\vec{x'})\}=0, \qquad \{\pi(t,\vec{x}),\pi(t,\vec{x'})\}=0,\\ \{\varphi(t,\vec{x}),\pi(t,\vec{x'})\}=\delta^{(3)}(\vec{x}-\vec{x'}).\end{gathered} \tag{1.7}$$

The process of canonical quantization consists of promoting the observables (which are functions, or rather functionals, in the phase space), including φ and π, to Hermitian operators on a Hilbert space $\mathcal{H}$. Similarly, the Poisson bracket is promoted to a commutator, according to the rule $\{\cdot,\cdot\}\mapsto -i[\cdot,\cdot]$. Then, the fundamental Poisson brackets become the *equal-time canonical commutation relations*:

$$\begin{gathered}[\hat{\varphi}(t,\vec{x}),\hat{\varphi}(t,\vec{x'})]=0, \qquad [\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{x'})]=0,\\ [\hat{\varphi}(t,\vec{x}),\hat{\pi}(t,\vec{x'})]=i\delta^{(3)}(\vec{x}-\vec{x'}).\end{gathered} \tag{1.8}$$

Note that $\hat{\varphi}(x)$ and $\hat{\pi}(x)$ are, in fact, operator-valued distributions. Distributions, or generalized functions, only make sense as functionals on the space of test functions (i.e., smooth functions with compact support). Accordingly, we define the *smeared field* as the field operator acting on a test function f:

$$\hat{\varphi}(f)=\int_{\mathcal{M}} \mathrm{d}^4x\,\hat{\varphi}(x)f(x). \tag{1.9}$$

Using this definition, a more rigorous version of Eq. (1.8) shall be given in section 1.3 (Eq. (1.33)).

We are still required to build the Hilbert space $\mathcal{H}$. In order to gain some insight, let us turn back to the Klein-Gordon equation (1.2). Once again taking advantage of Poincaré symmetry, we separate the solution in Cartesian coordinates[3] $(t,\,\vec{x})$. That gives us plane-wave mode solutions, $\exp\left\{i\left(\vec{k}\cdot\vec{x}\pm\omega_{\vec{k}}t\right)\right\}$, with $\omega_{\vec{k}}=\sqrt{|\vec{k}|^2+m^2}$, labeled by the vector $\vec{k}\in\mathbb{R}^3$.

The (complexified) space of solutions to the Klein-Gordon equation accepts a scalar

[3]Note that in this section we assume Cartesian coordinates, but the same process described in here could be carried out using arbitrary spatial coordinates. Indeed, in section 4.1 we quantize a scalar field in flat space-time using spherical coordinates.

product, given by

$$(\varphi_1, \varphi_2) = i \int_{\Sigma_t} \mathrm{d}^3\vec{x} \left(\varphi_1^*(x) \partial_t \varphi_2(x) - \varphi_2(x) \partial_t \varphi_1^*(x) \right), \tag{1.10}$$

where φ_1 and φ_2 are (complex) solutions to the Klein-Gordon equation and, once again, Σ_t is a hypersurface of constant t. This scalar product is conserved by time evolution, i.e., it is independent of the value of the time t. That is a consequence of the field equation (together with Gauss' theorem), as can be seen directly by taking the time derivative of Eq. (1.10).

Now, we normalize the mode solutions under the scalar product defined above as

$$u_{\vec{k}}(x) = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \exp\left\{ i \left(\vec{k} \cdot \vec{x} - \omega_{\vec{k}} t \right) \right\}. \tag{1.11}$$

These modes form a complete set of (complex) solutions to the Klein-Gordon equation when taken together with their complex conjugates, $u_{\vec{k}}^*(x)$. In other words, $\{u_{\vec{k}}(x), u_{\vec{k}}^*(x) \mid \vec{k} \in \mathbb{R}^3\}$ spans the space of complex solutions. These modes are orthonormal in the following sense:

$$\begin{gathered} (u_{\vec{k}}, u_{\vec{k}'}) = \delta^{(3)}(\vec{k} - \vec{k}'), \qquad (u_{\vec{k}}^*, u_{\vec{k}'}^*) = -\delta^{(3)}(\vec{k} - \vec{k}'), \\ (u_{\vec{k}}, u_{\vec{k}'}^*) = 0. \end{gathered} \tag{1.12}$$

It should not come as a surprise that these orthonormality relations are related to the canonical commutation relations, after quantization.

Under the requirement that the field $\varphi(x)$ be real, we finally find a general solution to Eq. (1.2):

$$\varphi(x) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k} \Big(u_{\vec{k}}(x) a_{\vec{k}} + u_{\vec{k}}^*(x) a_{\vec{k}}^* \Big). \tag{1.13}$$

Now the quantization is straightforward. One recognizes that the system may be interpreted as a continuous collection of harmonic oscillators labeled by $\vec{k}$, each one with frequency $\omega_{\vec{k}}$. Then, the coefficients $a_{\vec{k}}$ and $a_{\vec{k}}^*$ are promoted to, respectively, annihilation

and creation operators, $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$. The field operator becomes

$$\hat{\varphi}(x) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\Big(u_{\vec{k}}(x)\hat{a}_{\vec{k}} + u^*_{\vec{k}}(x)\hat{a}^\dagger_{\vec{k}}\Big). \tag{1.14}$$

This equation should be taken with caution, since the integral does not converge in general. Again, this is a consequence of the fact that $\hat{\varphi}(x)$ is an operator-valued distribution.

That finally allows us to construct the Hilbert space for this quantized system. We *define* the Minkowski vacuum $|0\rangle$ to be the state that is annihilated by every $\hat{a}_{\vec{k}}$, that is, $\hat{a}_{\vec{k}}|0\rangle = 0 \ \ \forall\vec{k}$. From the vacuum we build the one-particle states $|\vec{k}\rangle \equiv \hat{a}^\dagger_{\vec{k}}|0\rangle$. These form the *one-particle Hilbert space*, $\mathcal{H}_1$.

Now, the Hilbert space of the theory can be defined as the Fock space built from $\mathcal{H}_1$. Define the n-particle Hilbert $\mathcal{H}_n$ space as $\mathcal{H}_n \equiv \bigotimes_{j=1}^{n}{}_S\mathcal{H}_1$, i.e., the symmetric nth-tensor power of $\mathcal{H}_1$. Then, the Fock space is the (infinite, but countable) direct sum $\mathcal{H} = \bigoplus_{j=0}^{\infty}\mathcal{H}_j$. $\mathcal{H}_0$ is defined as the complex one-dimensional space $\mathbb{C}$, which is the space spanned by the vacuum $|0\rangle$.

That is sufficient for canonical quantization in flat space-time. However, as we have noted before, the existence of Poincaré symmetry was implicitly assumed thoughout this construction. Thus, before we move to curved space-time, it is wise to reformulate the process of quantization in a way that can be generalized to a setting without Poincaré symmetry.

1.1.1 Positive- and negative-frequency modes

Besides the foliation along the time coordinate (which, as we shall see, generalizes to globally hyperbolic space-times), the other crucial point in which we used the existence of Poincaré symmetry was in the expansion of the solution of the Klein-Gordon equation in plane-wave modes. That would not be possible in a general curved space-time. Let us see how to overcome this.

First of all, notice that we were careful not to use the term "inner product" for the map defined by Eq. (1.10). That might seem unrelated to the problem of quantization, but is

actually *essential* to it. The issue is that such a scalar product is *not* positive-definite in the whole complexified space of solutions (which we call $\mathcal{S}_{\mathbb{C}}$), as Eq. (1.12) shows in the case of $u^*_{\vec{k}}(x)$. On the other hand, when restricted to the space spanned by the positive-frequency solutions $u_{\vec{k}}(x)$ (henceforth called $\mathcal{P}$), then Eq. (1.10) *is* positive-definite, and we may call it the Klein-Gordon inner product.

Note that we say that a solution is *positive-frequency (negative-frequency)* with respect to the time coordinate t when its Fourier transform is zero for negative (positive) frequencies. It is clear from the form of $u_{\vec{k}}(x)$, given by Eq. (1.11), that solutions spanned by $u_{\vec{k}}(x)$ are positive-frequency with respect to t and solutions spanned by $u^*_{\vec{k}}(x)$ are negative-frequency with respect to t.

Given any real solution to the Klein-Gordon equation, we may extract its positive-frequency part by expanding it as in Eq. (1.13), and selecting the part spanned by $u_{\vec{k}}(x)$. Similarly, given a positive-frequency solution $\varphi_+(x)$, it can be written as a linear combination of $u_{\vec{k}}(x)$, as in

$$\varphi_+(x) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\; u_{\vec{k}}(x) a_{\vec{k}}. \tag{1.15}$$

Then, the coefficients $a_{\vec{k}}$ obtained in Eq. (1.15) may be substituted into Eq. (1.13) in order to find a real solution of the field equation. Therefore, we find a one-to-one map between the space of real solutions, $\mathcal{S}$, and the space of positive-frequency solutions, $\mathcal{P}$. In other words, these spaces may be identified.

Note that while $\mathcal{S}$ is the space of real solutions, the solutions in $\mathcal{P}$ may be complex. If the complexification of $\mathcal{S}$ were a finite-dimensional space, we might describe "extracting the positive-frequency part" of a solution as an operation of projection defined in $\mathcal{S}_{\mathbb{C}}$, into its subspace $\mathcal{P}$. Then $\mathcal{S}_{\mathbb{C}}$ could be written as a direct sum of orthogonal complements $\mathcal{P} \oplus \widetilde{\mathcal{P}}$, where $\widetilde{\mathcal{P}}$ is the space spanned by the negative-frequency solutions $u^*_{\vec{k}}(x)$. In infinite-dimensional spaces, however, the scenario is more intricate.[4]

Now, since the Klein-Gordon inner product is positive-definite and well-defined in $\mathcal{P}$,

[4] An important caveat is that in the infinite-dimensional case, $\mathcal{P}$ should not be seen as a subspace of $\mathcal{S}_{\mathbb{C}}$. Rather, one could see its completion under the Klein-Gordon inner product (which, as we show, is $\mathcal{H}_1$), as a subspace of the completion of $\mathcal{S}_{\mathbb{C}}$. On the other hand, the completion of $\mathcal{S}_{\mathbb{C}}$ depends on the inner product defined in $\mathcal{H}_1$, which gives a sort of circular argument. For more details, see Wald [14], section 3.2.

we may complete the space and obtain a Hilbert space. That is the one-particle Hilbert space $\mathcal{H}_1$ defined above. Then, there is an identification between positive-frequency solutions of the field equation (or real solutions, since we have seen that $\mathcal{S} \sim \mathcal{P}$) and the one-particle Hilbert space $\mathcal{H}_1$.[5]

Indeed, such a map is given by $\langle 0|\hat{\varphi}(x)$. Via direct application of Eq. (1.14), it is not hard to see that, given a general $|\psi\rangle \in \mathcal{H}_1$, we have that $\psi_+(x) \equiv \langle 0|\hat{\varphi}(x)|\psi\rangle$ is a general combination of positive-frequency solutions to the Klein-Gordon equation. Reciprocally, given a real solution $\psi(x)$ to the Klein-Gordon equation, there is only one $|\psi\rangle \in \mathcal{H}_1$ such that $\psi_+(x) = \langle 0|\hat{\varphi}(x)|\psi\rangle$, where $\psi_+(x)$ is the positive-frequency part of $\psi(x)$. That gives us an alternative road to constructing the one-particle Hilbert space $\mathcal{H}_1$. Then, the Hilbert space of the theory is the Fock space built from $\mathcal{H}_1$, as described above.

The advantage of this formulation of the canonical quantization over the one described before is that it can be generalized to curved space-times. Indeed, we do not employ the Poincaré symmetry in this construction. The only requirement is to be able to separate between positive- and negative-frequency solutions.

That leads to the question of whether a different choice of separation between positive- and negative-frequency modes would have given rise to different Hilbert spaces. *Different*, in this case, means *unitarily inequivalent*, since unitarily equivalent Hilbert spaces, for all physical purposes, describe the same phenomena.

The answer is yes – and may be seen even in flat space-time. A famous example is the Rindler vacuum. A Rindler observer is an observer following a path of constant acceleration in flat space-time. If we define modes which are positive frequency with respect to the proper time of such an observer, instead of the inertial time t, then the Hilbert space built as above is inherently different from the space built from the Minkowksi vacuum. Indeed, a particle detector following the worldline of a Rindler observer detects a thermal bath of particles when the field is in the Minkowski vacuum (see, e.g., Ref. [15], section 3.3).

On the other hand, in flat space-time there is an obvious choice of "preferred" vacuum:

[5]The mapping from $\mathcal{H}_1$ to the completion of $\mathcal{P}$, however, is not surjective in general, since an inner product space is, in general, a dense subspace of its completion.

the one which respects the symmetries of the space-time, i.e., the one which is Poincaré invariant. That uniquely determines the Minkowski vacuum. The existence of this preferred choice means that many of the difficulties of the quantization in curved space-times may be avoided in flat space-time.

The ambiguity in the choice of vacuum (equivalently, the choice of Hilbert space, or the separation between positive- and negative-frequency modes) becomes even more apparent in curved space-time. In general, there is no symmetry to distinguish a preferred vacuum. In fact, one may argue that this is the main difference between quantization in flat and curved space-times. Some authors compare that to the inexistence of a preferred coordinate system in General Relativity.

In particular, the notion of a particle becomes blurred in curved space-times. If one cannot distinguish a preferred vacuum, then one cannot distinguish a preferred type of particles. The same configuration of a field might be seen as containing different particle contents depending on the vacuum to which one compares. That is not an inconsistency – rather, it means that the concept of "particle" is not fundamental in curved space-times. There are some scenarios, however, in which a notion of particles may still be useful (e.g., past and future null infinities in an asymptotically-flat space-time).

It is pertinent to mention here that there is a mathematically rigorous approach to quantum field theory, known as algebraic quantum field theory, in which one finds a common setting for describing these various unitarily inequivalent Hilbert spaces arising in the theory. We shall not pursue such an approach in this work, though. The interested reader is again referred to textbooks (e.g., [16]).

1.2 Canonical quantization in curved space-time

Given all these remarks, we return to the formulation of the construction described above in the case of curved space-times.

Remember that we assume a curved space-time with a general metric $g_{\mu\nu}$, but this metric is assumed to be *given and fixed*. This means that, for now, we are not considering the dynamics of the metric itself, as given by Einstein's equation. This approach is

analogous to the quantization of a point electrical charge under an external, fixed electromagnetic field – as a first approach, we do not consider the effect of the charge upon the electromagnetic field itself.

Of course, after quantization one may obtain the (renormalized) expectation value of the stress-energy tensor of the quantized field. That can be substituted into the right-hand side of Einstein's equation in order to find the effect of *backreaction* on the metric. We shall discuss more about this in Chapter 2.

First of all, the action given by Eq. (1.1) is generalized according to the minimal coupling prescription – the Minkowski metric is replaced by a general metric $g_{\mu\nu}$ and the partial derivatives by covariant derivatives compatible with the metric:

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + m^2\varphi^2\right). \tag{1.16}$$

Thus, the field equation retains the same Klein-Gordon form, $(\Box - m^2)\varphi = 0$, but with the D'Alembertian operator now given by $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Now, the well-posedness of the initial value problem for the field equation requires some restriction to the space-time. What we need is the existence of a Cauchy surface, defined as an achronal set whose domain of dependence is the whole space-time (i.e., every inextendible causal curve through any point in the space-time must cross such a surface).

In space-times which admit a Cauchy surface (called *globally hyperbolic*), the Klein-Gordon equation has a unique solution defined on the whole space-time, given initial data on a Cauchy surface – see [17], section 7.4 (specifically, proposition 7.4.5), and [14] (theorem 4.1.2). By "initial data" we mean the value of the field and its derivative normal to the surface. For an explicit expression of the solution to this initial value problem, see Eq. (1.32) below.

Moreover, globally hyperbolic space-times may be foliated by Cauchy surfaces in much a similar way to the one described for flat space-time. The coordinate t, however, represents a generalized time coordinate, and not the global, inertial Minkowski time.

Together with the time coordinate t, we may define spatial coordinates x^i (the Roman spatial indices i, j run from 1 to 3) such that the vectors ∂_i are tangent to the

Cauchy surface Σ_t. Then, we perform a standard $(3+1)$-decomposition of the space-time, decomposing the time vector ∂_t at the Cauchy surface Σ_t as

$$(\partial_t)^\mu = N n^\mu + N^\mu, \tag{1.17}$$

where n^μ is the unit normal to Σ_t. N is called the *lapse function* and N^μ the *shift vector*, which is tangent to Σ_t. Given such a decomposition, the metric is written as

$$\begin{gathered} g_{00} = -N^2 + N^i N_i, \qquad g_{0i} = N_i \\ g^{00} = -1/N^2, \qquad g^{0i} = N^i/N^2, \qquad g^{ij} = \gamma^{ij} - N^i N^j/N^2, \end{gathered} \tag{1.18}$$

where we defined $\gamma_{ij} \equiv g_{ij}$ (the induced metric on Σ_t). Therefore, the Lagrangian, which in flat space-time was given by Eq. (1.4), becomes

$$L(\varphi, \dot{\varphi}, \nabla_i\varphi) = \int_{\Sigma_t} \mathrm{d}^3x \sqrt{\gamma} N \left(\frac{1}{2N^2} \left(\dot{\varphi} - N^i \nabla_i \varphi\right)^2 - \frac{1}{2}\gamma^{ij}\nabla_i\varphi\nabla_j\varphi - \frac{1}{2}m^2\varphi^2 \right), \tag{1.19}$$

where $\gamma = \det \gamma_{ij}$ (note that $\sqrt{-g} = N\sqrt{\gamma}$). Having obtained the Lagrangian, it is straightforward to find the conjugate momentum as in Eq. (1.5), and define the Poisson brackets as in Eq. (1.6). The relations (1.7) are also still true.

That gives the Hamiltonian structure of the theory. As usual, the canonical quantization is given by promoting observables to Hermitian operators on a Hilbert space, with commutation relations given by the map $\{\cdot,\cdot\} \mapsto -i[\cdot,\cdot]$. Consequently, the equal-time canonical commutation relations are still given by Eq. (1.8).

When one is able to perform separation by variables in the field equation, the main difference in the quantization procedure comes from the definition of the mode solutions. As discussed before, in a general curved space-time there is no natural choice of separation between positive- and negative-frequency modes.

Note that the Klein-Gordon scalar product (Eq. 1.10) may be generalized to curved space-times by writing it in a covariant form,

$$(\varphi_1, \varphi_2) = i \int_{\Sigma_t} \mathrm{d}^3x \, \sqrt{\gamma} \, n^\mu \Big(\varphi_1^*(x) \partial_\mu \varphi_2(x) - \varphi_2(x) \partial_\mu \varphi_1^*(x) \Big), \tag{1.20}$$

As before, this scalar product is independent of the time, i.e., of the choice of Cauchy surface Σ_t. Then, the modes are required to be orthonormal in the sense of Eq. (1.12).

Given a complete, orthonormal set of mode solutions, the quantization procedure is carried out analogously to the case of flat space-time. One expands the field as in Eq. (1.13), and quantizes by promoting the coefficients to annihilation and creation operators as in Eq. (1.14). The vacuum state $|0\rangle$ arising from this construction is defined via $\hat{a}_\Lambda|0\rangle = 0,\ \forall\Lambda$ (Λ is a generic index for labeling the modes; e.g., in flat space-time we had $\vec{k}$). Finally, the Hilbert space is built via the Fock construction described above.

Note that for each choice of complete set of mode solutions, we arrive at different vacuum states and, consequently, different Hilbert spaces. These are potentially unitarily inequivalent, as explained before.

This process may be easier to understand by giving a concrete example. Take the space-time to be the exterior of a Schwarzschild black hole with mass M. We define different mode solutions by imposing boundary conditions on the characteristic initial-data surface $\mathscr{I}^- \cup \mathscr{H}^-$, where $\mathscr{I}^-$ is past null infinity and $\mathscr{H}^-$ is the past horizon. Define the "time" coordinates $u = -e^{\frac{r-t}{4M}}\sqrt{\frac{r}{2M}-1}$ and $v = e^{\frac{r+t}{4M}}\sqrt{\frac{r}{2M}-1}$, where t and r are the standard Schwarzschild time and radial coordinates.

Then, *up*-modes $u^{\rm up}_{\ell m\omega}$ are defined by imposing that $u^{\rm up}_{\ell m\omega}(x) \sim 0$ at $\mathscr{I}^-$ and $u^{\rm up}_{\ell m\omega}(x) \sim \frac{1}{r}\mathcal{N}^{\rm up}_\omega Y^m_\ell(\theta,\phi)e^{-i\omega u}$ at $\mathscr{H}^-$, where $\mathcal{N}^{\rm up}_\omega$ is a normalization constant, $Y^m_\ell(\theta,\phi)$ are the spherical harmonics, ω is a non-negative real, ℓ is a non-negative integer, and m is an integer between $-\ell$ and ℓ. These modes are purely outgoing waves and positive-frequency with respect to t at $\mathscr{H}^-$.

Next, define *in*-modes $u^{\rm in}_{\ell m\omega}$ by imposing $u^{\rm in}_{\ell m\omega}(x) \sim 0$ at $\mathscr{H}^-$ and $u^{\rm in}_{\ell m\omega}(x) \sim \frac{1}{r}\mathcal{N}^{\rm in}_\omega \times Y^m_\ell(\theta,\phi)e^{-i\omega v}$ at $\mathscr{I}^-$. These modes are purely incoming waves and positive-frequency with respect to t at $\mathscr{I}^-$.

Finally, $\{u^{\rm up}_{\ell m\omega}, u^{\rm in}_{\ell m\omega}, u^{\rm up\,*}_{\ell m\omega}, u^{\rm in\,*}_{\ell m\omega}\}$ form a complete set of solutions. We expand the field in this basis and quantize as described above:

$$\hat{\varphi}(x) = \int_0^\infty {\rm d}\omega \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(u^{\rm up}_{\ell m\omega}(x)\hat{a}^{\rm up}_{\ell m\omega} + u^{\rm in}_{\ell m\omega}(x)\hat{a}^{\rm in}_{\ell m\omega} + {\rm h.c.}\Big). \tag{1.21}$$

The vacuum state defined by $\hat{a}^{\text{in/up}}_{\ell m\omega}|B\rangle = 0$ ($\forall\ \ell, m, \omega$) is called the *Boulware vacuum*. It is usually interpreted as the vacuum state perceived by a static observer (i.e., observers following integral curves of the time vector ∂_t).

Note that this example shows a case in which we have to use more than one type of mode solution (in this case, *in*- and *up*-modes) in order to obtain a complete set.

Although we are not going to detail further, it is important to say that different choice of modes would give rise to the different vacua which are usually found in the literature on Schwarzschild space-time, namely the Unruh vacuum and Hartle-Hawking vacuum.

1.3 Two-point functions

In quantum field theory, one frequently encounters expectation values of the product of field operators. For instance, in the path integral formulation of quantum field theory, the so-called Feynman propagator is crucial to the calculation of internal lines of Feynman diagrams and, consequently, scattering amplitudes.

Quantum field theory in curved space-time is no different in this regard: functions of the type $\langle 0|\hat{\varphi}(x)\hat{\varphi}(x')|0\rangle$ (where $|0\rangle$ is a vacuum state) are crucial to the theory, such as in the response function of particle detectors and, within the point-splitting method, in the calculation of the expectation value of the stress-energy tensor (more on this later).

Let us investigate these two-point functions. Given a quantum scalar field $\hat{\varphi}(x)$, the most simple of these is the *Wightman function*, defined by

$$G_+(x, x') = \langle\psi|\hat{\varphi}(x)\hat{\varphi}(x')|\psi\rangle, \tag{1.22}$$

for some state $|\psi\rangle$. Clearly, this function is a solution of the homogeneous field equation, i.e., $(\Box - m^2)G_+(x, x') = 0$, both for x and x'. Its complex conjugate is denoted by $G_-(x, x') \equiv (G_+(x, x'))^* = \langle\psi|\hat{\varphi}(x')\hat{\varphi}(x)|\psi\rangle$. From this, a number of other two-point functions may be defined. We give here the definition of some of them:

- The *Pauli-Jordan function* $G(x, x')$, defined as the commutator of the field operator

at different points:

$$iG(x,x') = \langle\psi|[\hat{\varphi}(x),\hat{\varphi}(x')]|\psi\rangle = G_+(x,x') - G_-(x,x'). \tag{1.23}$$

- The *retarded and advanced Green functions*, respectively $G_{\text{ret}}(x,x')$ and $G_{\text{adv}}(x,x')$, given by

$$\begin{aligned} G_{\text{ret}}(x,x') &= -G(x,x')\theta(t-t') \\ G_{\text{adv}}(x,x') &= G(x,x')\theta(t'-t), \end{aligned} \tag{1.24}$$

  where $\theta(\cdot)$ is the Heaviside step function. Clearly, Eq. (1.24) gives the following relationship:

$$G(x,x') = G_{\text{adv}}(x,x') - G_{\text{ret}}(x,x'). \tag{1.25}$$

- The *Feynman Green function* $G_F(x,x')$, given by

$$\begin{aligned} -iG_F(x,x') &= \langle\psi|T\{\hat{\varphi}(x)\hat{\varphi}(x')\}|\psi\rangle \\ &= G_+(x,x')\theta(t-t') + G_-(x,x')\theta(t'-t), \end{aligned} \tag{1.26}$$

  where $T\{\hat{\varphi}(x)\hat{\varphi}(x')\}$, the *time-ordered product*, is defined as

$$T\{\hat{\varphi}(x)\hat{\varphi}(x')\} \equiv \begin{cases} \hat{\varphi}(x)\hat{\varphi}(x'), & \text{if } t > t' \\ \hat{\varphi}(x')\hat{\varphi}(x), & \text{if } t' > t \end{cases}. \tag{1.27}$$

Note that Eqs. (1.24) and (1.26) require a previous definition of a time coordinate. As explained before, in globally hyperbolic space-times we can always foliate the space-time via Cauchy surfaces and define a time coordinate accordingly. Throughout this work, we will be concerned mostly about the Feynman Green function and the Wightman function.

We have that G_+, G_- and G are solutions of the homogeneous field equation. However, as the name suggests, G_{ret}, G_{adv} and G_F are Green functions of the field equation. Indeed, by direct application of the Klein-Gordon operator to Eqs. (1.24) and (1.26), one finds

that

$$(\Box - m^2)G_A(x,x') = \frac{\delta^{(4)}(x-x')}{\sqrt{-g}}, \tag{1.28}$$

where G_A stands for $G_{\rm ret}$, $G_{\rm adv}$ or $-G_F$ (the negative sign in G_F is a standard convention). Eq. (1.28) is the equation for a Green function.

Since all of these are Green functions, one might wonder how to make a distinction between them. In order to understand this, it is instructive to return to the case of flat space-time. Let us investigate Eq. (1.28) in this case. The D'Alembertian is simply $\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2$. By taking the Fourier transform of both sides, we arrive at

$$(\omega^2 - k^2 - m^2)\widetilde{G}_A(\omega,\vec{k};x') = e^{i\omega t'}e^{-i\vec{k}\cdot\vec{x'}}, \tag{1.29}$$

where $\widetilde{G}_A$ is the Fourier transform of G_A with respect to x. Thus, we find an integral representation for G_A, namely

$$G_A(x,x') = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\; e^{i\vec{k}\cdot(\vec{x}-\vec{x'})}\int_{\mathbb{R}} \mathrm{d}\omega\; \frac{e^{-i\omega(t-t')}}{\omega^2-\omega_{\vec{k}}^2}, \tag{1.30}$$

where $\omega_{\vec{k}}^2 \equiv k^2 + m^2$.

Now, the ω-integral may be calculated with the help of a contour on the complex ω plane. However, notice that there are two poles on the real line: $\pm\omega_{\vec{k}}$. The key point is the choice of contour around these poles. Each Green function corresponds to a different contour – actually, even the homogeneous two-point functions may be given in that form (but with a closed contour around these poles, instead of a contour over the real line).

Indeed, a path that runs above the poles in the complex plane gives the retarded Green function. A path that runs below the poles gives the advanced Green function. A path that runs below the negative pole ($-\omega_{\vec{k}}$) and above the positive pole ($+\omega_{\vec{k}}$) gives the Feynman Green function.

For the homogeneous two-point functions: a positively-oriented closed contour around the positive pole gives $-iG_+(x,x')$. A positively-oriented closed contour around the negative pole gives $iG_-(x,x')$. Finally, a positively-oriented closed contour around both poles gives the Pauli-Jordan function, $G(x,x')$.

It is important to mention that an equivalent way of characterizing the Feynman Green function is by adding a small imaginary part $i\varepsilon$ to the denominator of Eq. (1.30), and taking the limit $\varepsilon \to 0^+$ after calculating the integral. This is called the *Feynman prescription*, and is clearly equivalent to taking the contour described above. Explicitly, Eq. (1.30) becomes, for the Feynman Green function:

$$G_F(x,x') = \frac{1}{(2\pi)^4} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\; e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} \int_{\mathbb{R}} \mathrm{d}\omega\; \frac{e^{-i\omega(t-t')}}{\omega^2 - \omega_{\vec{k}}^2 + i\varepsilon}, \tag{1.31}$$

That gives us a characterization of these various two-point functions. Note that Eqs. (1.30) and (1.31) are valid only in flat space-time.

As can be seen immediately from the Hadamard form, in general $G_+(x,x')$ and $G_F(x,x')$ are non-zero even when the points x and x' are space-like separated. Note that it can be considered as a correlation between points which are not causally connected. However, there is no violation of causality. Indeed, it is not hard to show, from Green's identities, that a homogeneous solution of the Klein-Gordon equation may be written from the initial value defined on a Cauchy surface Σ_t as

$$\varphi(x) = \int_{\Sigma_t} \mathrm{d}^3y\, \sqrt{\gamma}\, n^\mu \Big(\varphi(t,y^i)\partial_\mu G(x;t,y^i) - G(x;t,y^i)\partial_\mu\varphi(t,y^i)\Big), \tag{1.32}$$

where γ_{ij} is the induced metric on Σ_t and n^μ is the oriented unit normal to Σ_t. Note that x is a space-time point, and y^i is a spatial point. Eq. (1.32), of course, is related to our previous discussion about the well-posedness of the initial value problem in globally hyperbolic space-times.

The Pauli-Jordan function $G(x,x')$ does not depend on the quantum state (that can be seen from Eq. (1.33) below). Moreover, it has support only when x and x' are causally connected (i.e., time-like or null separated). It follows that the retarded Green function has support only on the region where x' lies in the causal future of x (i.e., it is inside or on the future light cone of x), and the advanced Green function has support where x' lies in the causal past of x (inside or on the past light cone of x).

On the other hand, the Wightman function and the Feynman Green function at a

given point generally have support on the whole space-time, and depend on the quantum state in which they are calculated – hence, they also depend on the separation between positive- and negative-frequency modes.

Before we end this section, we should mention that, since the commutator of the field operators is proportional to the identity operator, then Eq. (1.23) can be given in a stronger form,

$$[\hat{\varphi}(x), \hat{\varphi}(x')] = iG(x, x')\,\hat{\mathbf{1}}, \tag{1.33}$$

where $\hat{\mathbf{1}}$ is the identity operator. Eq.(1.33) is the generalization of the equal-time canonical commutation relations that we promised before. It is sometimes called the *covariant canonical commutation relation*. Its advantage is the fact that it does not rely on the definition of a time coordinate.

Again, we remind the reader that Eqs. (1.32) and (1.33) should be understood in terms of distributions. A mathematically rigourous version of these equations should be given in terms of the smeared field.

We end here our brief discussion about two-point functions. For more details and proofs, the reader is again referred to standard textbooks (e.g., [18], chapter 4; or [15], section 2.7).

Chapter 2

Renormalization of the stress-energy tensor

Now that we have described the formulation of the quantum theory, we are in a position to calculate expectation values of operators. A particularly important operator is the stress-energy tensor, since this quantity, at least classically, appears on the right hand side of Einstein's equation.

Let us formulate this explicitly. Einstein's equation is derived from the Einstein-Hilbert action coupled to a matter action:

$$\mathcal{S} = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}R + \int_{\mathcal{M}} \mathrm{d}^4x\,\mathcal{L}_M, \tag{2.1}$$

where $\mathcal{M}$ is the space-time manifold, R is the Ricci scalar, g is the determinant of the metric $g_{\mu\nu}$, and $\mathcal{L}_M$ is the matter Lagrangian density. The variation of this action yields the field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{2.2}$$

where the *stress-energy tensor* $T_{\mu\nu}$ is given by[1]

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{2.3}$$

It is symmetric in its indices and is covariantly conserved, i.e.,

$$\nabla_\mu T^\mu{}_\nu = 0. \tag{2.4}$$

From Eq. (2.3), the stress-energy tensor for a minimally-coupled real scalar field in curved space-time, described by the Lagrangian (1.16), is found to be

$$T_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\left(\nabla_\alpha\varphi\nabla^\alpha\varphi + m^2\varphi^2\right). \tag{2.5}$$

Now, as explained before, the idea of quantum field theory in curved space-time is to quantize the field (in our case, $\varphi(x)$) on a fixed background metric. The background metric is a solution to Einstein's equation (2.2) *without* the field that is being quantized (of course, there may be other background fields on the right side of Einstein's equation – e.g., a classical electromagnetic field).

However, after quantization, a *stress-energy tensor operator* $\hat{T}_{\mu\nu}(x)$ may be formally built from the field operator $\hat{\varphi}(x)$ according to Eq. (2.5). Then, in a semiclassical approach, one may consider the effect of the *backreaction* of the quantum field on the original background metric, giving a modified metric $\bar{g}_{\mu\nu}$. Such backreaction would be given by the expectation value of the stress-energy tensor operator in a state $|\psi\rangle$, according to the so-called *semiclassical Einstein's equation*,

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = \langle\psi|\hat{T}_{\mu\nu}|\psi\rangle, \tag{2.6}$$

where $\bar{R}_{\mu\nu}$ and $\bar{R}$ are respectively the Ricci tensor and scalar curvature calculated in the backreacted metric $\bar{g}_{\mu\nu}$.

[1]Note that the stress-energy tensor defined by Eq. (2.3) may be different from the canonical stress-energy tensor derived from $\mathcal{L}_M$ via Noether's theorem, $\theta_{\mu\nu}$. In particular, $\theta_{\mu\nu}$ may not be symmetric. However, one can always obtain $T_{\mu\nu}$ from $\theta_{\mu\nu}$ via Belinfante's construction [19].

This kind of approach may be considered as a first approximation towards a true theory of quantum gravity. Of course, one expects such an approach to have a limited range of validity. In particular, when the quantum nature of the gravitational field itself becomes important, the semiclassical theory is expected to break down. Still, it seems to be valid for some relevant physical applications. As mentioned before, the most important examples are the physics of black holes and primordial cosmology, away from Planck scales.

2.1 Point-splitting method

Nevertheless, as is usual in quantum field theory, one stumbles upon infinities appearing in the theory. The expectation value on the right side of Eq. (2.6) is, in general, formally divergent. That should not come as a surprise – we can see from Eq. (2.5) that $\hat{T}_{\mu\nu}(x)$ is quadratic in the field operator $\hat{\varphi}(x)$. On the other hand, $\hat{\varphi}(x)$ is an operator-valued distribution and, as is well-known, the product of a distribution with itself *at the same point* x is mathematically ill-defined. Thus, some process of renormalization is necessary in order to make the stress-energy tensor operator well-defined. Similarly, the two-point functions defined above diverge when $x' \to x$. That happens for the same reason as the divergence in $\hat{T}_{\mu\nu}(x)$: in the coincidence limit they become quadratic in $\hat{\varphi}(x)$, while $\hat{\varphi}(x)$ is an operator-valued distribution.

It should be noted that even in flat space-time one needs to impose some sort of regularization procedure because of this reason. In that case, one finds the Hamiltonian operator:

$$\begin{aligned} \hat{H} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\, \frac{\omega_{\vec{k}}}{2}\Big(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\Big) \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\, \omega_{\vec{k}}\Big(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2}\delta^{(3)}(0)\Big), \end{aligned} \tag{2.7}$$

where in the second line we used the commutation relation for the annihilation and creator operators, $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k}')$. Now, this Hamiltonian is clearly divergent, because of

the $\delta^{(3)}(0)$ term. Even its vacuum expectation value is divergent:

$$\langle 0|\hat{H}|0\rangle = \frac{\delta^{(3)}(0)}{2}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\;\omega_{\vec{k}}. \tag{2.8}$$

However, since the divergence comes from the second term in Eq. (2.7), we have that, even if the expectation values are divergent, the *difference* between expectation values in different states is well-defined.

In fact, one can *define* the regularized expectation value of $\hat{H}$ in a state $|\psi\rangle$ to be the difference between that expectation value and the vacuum expectation value:

$$\langle\psi|:\hat{H}:|\psi\rangle \equiv \langle\psi|\hat{H}|\psi\rangle - \langle 0|\hat{H}|0\rangle. \tag{2.9}$$

Consequently, the renormalized expectation value in the vacuum itself is zero.

This prescription is called *normal ordering*, since it is equivalent to writing all the annihilation operators to the right of creation operators in the first line of Eq. (2.7). Indeed, if one defines the normal-ordered Hamiltonian by

$$:\hat{H}: = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\;\omega_{\vec{k}}\,\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}, \tag{2.10}$$

then the expectation value $\langle\psi|:\hat{H}:|\psi\rangle$ is equivalent to the one defined by Eq. (2.9).[2]

We would like to have a similar prescription to renormalize $\hat{T}_{\mu\nu}(x)$ in curved space-time. Again, the problem is that we do not have a preferred vacuum state, as is the case in flat space-time. Hence there is no obvious choice of state $|0\rangle$ in order to define the renormalized expectation value as in Eq. (2.9).

Still, differences between expectation values in different states are well-defined, for a large class of states (we will soon define what we mean by "large class of states"). E.g., for two such states $|\psi_1\rangle$ and $|\psi_2\rangle$ the difference

$$\langle\hat{T}_{\mu\nu}\rangle_{\psi_1-\psi_2} \equiv \langle\psi_1|\hat{T}_{\mu\nu}|\psi_1\rangle - \langle\psi_2|\hat{T}_{\mu\nu}|\psi_2\rangle \tag{2.11}$$

[2]Note, however, that there is still an infinity coming from the volume of the space. That can be solved by considering the field in a box of volume V, and afterwards setting $V \to \infty$.

is finite. That is a consequence of the fact that the expectation value of the stress-energy tensor has the same diverging behaviour, independently of the state. Thus, one might consider a regularization procedure in order to make explicit such behaviour.

It has already been noted that the reason for this divergence is that $\hat{T}_{\mu\nu}$ is quadratic in the field operator, while products of distributions at the same point are ill-behaved. On the other hand, the product of distributions at different points is well-defined.

Therefore, the standard regularization approach is to define an expression for $\langle\psi|\hat{T}_{\mu\nu}(x,x')|\psi\rangle$, at points separated. In the literature, this regularization scheme is called *point-splitting*. In general, the "point-split" stress-energy tensor can be written as a differential operator both in x and x' acting on the Feynman Green function. As an example: from Eq. (2.5) we can see that in the case of a minimally-coupled real scalar field one may define

$$\begin{aligned}\langle\psi|\hat{T}_{\mu\nu}(x,x')|\psi\rangle &= -i\left(g^{\nu'}_{\nu}(x,x')\nabla_{\mu}\nabla_{\nu'} - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha}_{\alpha'}(x,x')\nabla_{\alpha}\nabla^{\alpha'} + m^2\right)\right)G_F(x,x') \\ &\equiv -i\hat{D}_{\mu\nu}G_F(x,x'),\end{aligned} \tag{2.12}$$

where $G_F(x,x') = i\langle\psi|T\{\hat{\varphi}(x)\hat{\varphi}(x')\}|\psi\rangle$ is the Feynman Green function in the state $|\psi\rangle$, defined in section 1.3. The covariant derivatives with primed indices act on the point x'. Also note that the bivector of parallel displacement $g^{\mu}_{\nu'}$ is defined as a bivector which, when acting on a vector at x', gives the parallel-transported vector at x.

Another quantity which is quadratic in the field, and hence needs to be renormalized, is the expectation value of the square of the field operator, $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$. Since $G_F(x,x') = i\langle\psi|T\{\hat{\varphi}(x)\hat{\varphi}(x')\}|\psi\rangle$, we see that $G_F(x,x')$ is a "point-split" analogue of $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$. Following the spirit of normal ordering in flat space-time, the idea in order to renormalize $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$ is to subtract from $G_F(x,x')$ a biscalar $G_{\text{sing}}(x,x')$ which contains all of its singularity structure, so that in the coincidence limit we have:

$$\langle\psi|\hat{\varphi}^2(x)|\psi\rangle_{\text{ren}} = -i\lim_{x'\to x}\Big(G_F(x,x') - G_{\text{sing}}(x,x')\Big). \tag{2.13}$$

Similarly, $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle$ is renormalized according to

$$\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle_{\rm ren} = -i\lim_{x'\to x}\hat{D}_{\mu\nu}\Big(G_F(x,x') - G_{\rm sing}(x,x')\Big), \tag{2.14}$$

where $\hat{D}_{\mu\nu}$ is the operator defined in Eq. (2.12). Clearly, a procedure to renormalize $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$ is a prototype to the renormalization of $\langle\psi|\hat{T}_{\mu\nu}(x,x')|\psi\rangle$. Therefore, in this work we will be mainly concerned with the renormalization of $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$.

The problem now is to find a way to obtain $G_{\rm sing}(x,x')$. It may be given by the Hadamard form of the Feynman Green function: the Green function equation (1.28) admits local solutions of the form [20]

$$G_F(x,x') = \lim_{\varepsilon\to 0^+}\frac{i}{8\pi^2}\left\{\frac{\Delta^{\frac{1}{2}}(x,x')}{\sigma(x,x')+i\varepsilon} + V(x,x')\ln(\sigma(x,x')+i\varepsilon) + W(x,x')\right\}, \tag{2.15}$$

where $\sigma(x,x')$, called *Synge's world function*, is defined as one-half of the square of the geodesic distance between x and x', $\Delta(x,x') \equiv -\det(-\nabla_\mu\nabla_{\nu'}\sigma(x,x'))$ is the *van Vleck-Morette determinant*, $V(x,x')$ and $W(x,x')$ are regular and symmetric biscalars, and '$i\varepsilon$' represents the Feynman prescription, described in section 1.3.

We see that the singularity behaviour of $G_F(x,x')$ as $x'\to x$ (in which case $\sigma(x,x')\to 0$) is entirely contained in the first two terms of Eq. (2.15), which we call the *singular part* [3] of the Hadamard form.

Clearly, a solution of the form (2.15) can only be given if $\sigma(x,x')$ is well-defined, i.e., if there exists a unique geodesic connecting x and x'. That is not always the case. Obvious examples include: (i) antipodal points on a two-sphere, which are connected by an infinite number of geodesics of equal length; (ii) consider an otherwise flat space-time, but with the origin removed; there is no geodesic connecting points which are directly opposite along a line that passes through the origin, since that point is removed.

We say that a region N_x is a *normal neighborhood* of x if every point in it can be joined to x along a unique geodesic which lies entirely in N_x. In such a neighborhood, $\sigma(x,x')$

[3] A note must be given about this naming convention. Despite we calling the first two terms in Eq. (3.8) *singular*, they contain parts which are not divergent as $\sigma\to 0$. Indeed, $\lim_{\epsilon\to 0^+}\ln(\sigma+i\varepsilon) = \ln|\sigma| + i\pi\theta(\sigma)$. The last term is not divergent as $\sigma\to 0$, but it is still non-smooth.

is well-defined. As a consequence of this definition, we see that Eq. (2.15) only makes sense when x' is in a normal neighborhood of x. However, we are only concerned with the behaviour of $G_F(x, x')$ as x' approaches x. In such a case, we may simply assume that x' is in a normal neighborhood of x.

Another caveat is that not always can $G_F(x, x')$ be given in a form as Eq. (2.15), even as $x' \to x$. Only for some states will this singularity structure be valid. These are known as *Hadamard states*.

It has been proved[4] that, in globally hyperbolic space-times, Hadamard states exist and form a dense subset of the Hilbert space of the theory. This is why we said earlier that Eq. (2.11) is finite for a "large class of states". Usually, one assumes that all physically meaningful states in the theory are Hadamard states.

Moreover, it can be shown that the first two terms in Eq. (2.15) are purely geometrical (i.e., depend only on the space-time), while $W(x, x')$ depends on the quantum state. Indeed, one may expand $V(x, x')$ and $W(x, x')$ in terms of σ, as in

$$\begin{aligned} V(x, x') &= \sum_{n=0}^{\infty} V_n(x, x')\sigma^n(x, x') \\ W(x, x') &= \sum_{n=0}^{\infty} W_n(x, x')\sigma^n(x, x'), \end{aligned} \tag{2.16}$$

where $V_n(x, x')$ and $W_n(x, x')$ are regular biscalars. Note that, despite appearances, Eq. (2.16) is not a Taylor series[5].

By substituting Eq. (2.16) into the Green function equation, one finds recurrence relations for the V_n and W_n. The V_n are completely determined by the recurrence relations in terms of geometrical quantities (i.e., the metric, the world function, etc.). On the other hand, while every W_n can be written in terms of W_0, we have that W_0 itself is, in principle, arbitrary. This arbitrariness amounts to the choice of quantum state in which the Green function is calculated. This is why we say that $W(x, x')$ depends on the quantum state.

[4] See Wald [14], p. 95.

[5] It is shown in Ref. [21] (see Theorem 6.2.1 therein) that there always exists a subregion of the normal neighborhood of x such that, for x' in that subregion, the series in Eq. (2.16) converge uniformly. Then again, the convergence of these series are not strictly relevant to our case, since we are only interested in the singular behaviour of $G_F(x, x')$.

Now we are in a position to renormalize $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$. We may construct the counter-term $G_{\rm sing}(x,x')$ which appears in Eq. (2.13) using the Hadamard form, with an appropriate choice for $W(x,x')$. Since the singularity structure of $G_{\rm sing}$ and G_F must be the same (because it only depends on the geometry), then the limit in Eq. (2.13) will be well-defined.

Indeed, Wald (1977) [22] proposes to use $G_{\rm sing}$ as the function built from Eq. (2.15) with $W_0 = 0$. This is proposed in analogy to the case of flat space-time. The only caveat, pointed out in Wald (1978) [23] is that the function $W(x,x')$ built from $W_0 = 0$ may not be symmetric. As a consequence, $G_{\rm sing}(x,x')$ would not satisfy the equation for x', and because of this the renormalized stress-energy tensor would not not be conserved. Hence, one needs to add a term "by hand" in order to recover conservation of the stress-energy tensor. The addition of this term is consistent with the method; the reason shall be clear at the end of this section.

We should also mention that the inclusion of this term gives rise to the trace anomaly. Classically, in a conformally-invariant theory the trace of the stress-energy tensor must be null. However, one finds that, after quantization, the trace of the renormalized expectation value of the stress-energy tensor is not null anymore. That shows that the conformal symmetry is broken in the process of quantization. This is called the trace, or conformal, anomaly. The same is found to happen in flat space-time, and is related to the appearance of a mass scale in the process of renormalization.

A related proposal is suggested by Brown and Ottewill (1986) [24]. Based on the fact that differences between expectation values of the stress-energy tensor are well defined independently of renormalization, they note that

$$\begin{aligned}\langle\psi_1|\hat{T}_{\mu\nu}|\psi_1\rangle - \langle\psi_2|\hat{T}_{\mu\nu}|\psi_2\rangle &= -i\lim_{x'\to x}\hat{D}_{\mu\nu}\Big(G_F^{(1)}(x,x') - G_F^{(2)}(x,x')\Big) \\ &= -i\lim_{x'\to x}\hat{D}_{\mu\nu}\Big(W^{(1)}(x,x') - W^{(2)}(x,x')\Big),\end{aligned} \tag{2.17}$$

where $G_F^{(1)}$ is the Green function calculated in the state $|\psi_1\rangle$, $W^{(1)}$ is defined as in Eq. (2.15), and similarly for $G_F^{(2)}$ and $W^{(2)}$. Thus, one may simply define the renor-

malized value of $\langle\psi|\hat{\varphi}(x)|\psi\rangle$ to be

$$\langle\psi|\hat{\varphi}^2(x)|\psi\rangle_{\rm ren} = -i \lim_{x'\to x} W(x,x'). \tag{2.18}$$

Clearly, this is the same as Eq. (2.13) with $G_{\rm sing}$ being the singular part of the Hadamard form (i.e., the first two terms in Eq. (2.15)). Since this term is purely geometrical, it may be calculated only once for each space-time.

Note that, again, in this prescription it may be necessary to add a term to the expectation value of the stress-energy tensor in order to recover the conservation of $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle_{\rm ren}$, and that introduces the trace anomaly.

A different prescription is to obtain an expansion for the Feynman Green function from the Schwinger-DeWitt expansion of the heat kernel. That is a local expansion bearing the short distance behaviour of the Green function and, as such, does not capture global information such as the quantum state. From that, one may extract the terms which do not vanish as $\sigma \to 0$, and then derive the terms which are to be subtracted from $\langle\psi|\hat{\varphi}(x)\hat{\varphi}(x')|\psi\rangle$ and $\langle\psi|\hat{T}_{\mu\nu}(x,x')|\psi\rangle$ in order to renormalize $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$ and $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle$, respectively. Such an approach was carried out by Christensen (1976) [25]. In the literature, this is the most commonly followed procedure.

In that approach, one finds an expression for the Feynman Green function in a form similar to Eq. (2.15). Therefore, it may be understood as a method for building the Hadamard form with a particular choice of W_0.[6] That automatically renders $W(x,x')$ symmetric. Because of this, the renormalized expectation value of the stress-energy tensor derived from this method is already conserved and has the right trace anomaly, with no need to add another term by hand. A disadvantage of this method is that the expansion is found in terms of inverse powers of the mass of the field. Accordingly, one finds difficulties in applying this method for massless fields.

With so many different prescriptions, one may ask whether all of these constructions are equivalent. The answer to that may be found in Wald's axioms, also proposed in

[6]However, note that the DeWitt-Schwinger expansion gives a representation of $W(x,x')$ as a series which, in fact, does not converge. This is not a problem, since we are only interested in the first terms of this expansion, which are to be used as counter-terms for the renormalization.

Ref. [22]. Wald proposes four[7] axioms in order to define the renormalized stress-energy tensor (the first axiom is presented in here as given by Fulling [18], p. 187):

1. given normalized states $|\psi_1\rangle$ and $|\psi_2\rangle$, $\langle\psi_1|\hat{T}_{\mu\nu}|\psi_1\rangle_{\text{ren}} - \langle\psi_2|\hat{T}_{\mu\nu}|\psi_2\rangle_{\text{ren}}$ agrees with the formal expression given by $\langle\psi_1|\hat{T}_{\mu\nu}|\psi_1\rangle - \langle\psi_2|\hat{T}_{\mu\nu}|\psi_2\rangle$ (since, in that case, the formal expression yields a finite value).

2. In flat space-time with the standard $\mathbb{R}^4$ topology, the renormalization prescription is equivalent to normal ordering.

3. $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ is conserved, i.e., $\nabla^\mu\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = 0$

4. Causality: consider a class of asymptotically static space-times with the same inital spatial geometry, so that the initial states of the field may be identified. For a fixed "in" state, $\langle in|\hat{T}_{\mu\nu}|in\rangle_{\text{ren}}$ at point x depends only on the space-time geometry to the causal past of x. Similarly, consider a class of space-times with the same final geometry, so that the final states of the field may be identified. For a fixed "out" state, $\langle out|\hat{T}_{\mu\nu}|out\rangle_{\text{ren}}$ at point x depends only on the space-time geometry to the causal future of x.

In that paper, it is shown that a renormalized expectation value of the stress-energy tensor given by a renormalization prescription that satisfies these four axioms is unique in the folllowing sense: given $\langle\psi|\hat{T}^{(1)}_{\mu\nu}|\psi\rangle_{\text{ren}}$ and $\langle\psi|\hat{T}^{(2)}_{\mu\nu}|\psi\rangle_{\text{ren}}$ obtained from different prescriptions, but which satisfy the four axioms, then their difference $t_{\mu\nu}$ must be a "local curvature term" which can be absorbed in the gravitational Lagrangian. By that, we mean that $t_{\mu\nu}$ is a tensor that can be derived in the form of Eq. (2.3) from a "Lagrangian" composed of terms proportional to a constant, to the scalar curvature R, or higher order terms in the curvature such as R^2 or $R_{\mu\nu}R^{\mu\nu}$. The constant and scalar curvature terms may be absorbed in the Einstein-Hilbert Lagrangian as a redefinition of, respectively, Newton's constant G and the cosmological constant[8] Λ.

[7] Actually, five axioms are proposed, but in Ref. [23] it is shown that the fifth axiom cannot be satisfied. That is related to appearance of the trace anomaly and the introduction of a length scale in the process of renormalization.

[8] We did not include the cosmological constant term in the Einstein-Hilbert Lagrangian (2.1), but

The terms proportional to R^2 and $R_{\mu\nu}R^{\mu\nu}$ are not straightforwardly absorbed in the Einstein-Hilbert Lagrangian, but they could be interpreted as being present in an effective semiclassical Lagrangian arising from a full theory of quantum gravity. Indeed, these terms are the same as the counter-terms found in the one-loop renormalization of gravity as given by 't Hooft and Veltman (1974) (see Eq. 5.24 in Ref. [26]).

The prescriptions described above satisfy Wald's axioms. Consequently, they are equivalent up to local curvature terms. The equivalence of the "axiomatic renormalization scheme" of Brown and Ottewill [24] and the "traditional scheme" using the DeWitt-Schwinger expansion is explicitly proven in Ref. [27]. It is shown that both prescriptions give exactly the same stress-energy tensor in the electromagnetic case. In the scalar case, they differ by terms proportional to $G_{\mu\nu}$ and $g_{\mu\nu}$, and thus may be absorbed by, respectively, constant and R terms in the gravitational Lagrangian (and this difference vanishes in the massless case). Given this equivalence, we shall follow the renormalization prescription of Brown and Ottewill throughout this text.

In that sense, the point-splitting procedure described in this section makes contact with the common process of renormalization in quantum field theory, via counter-terms added to the "bare" Lagrangian.

2.2 Implementation of the renormalization procedure

We have seen how one is able to renormalize $\langle\hat{\varphi}^2(x)\rangle$ and $\langle\hat{T}_{\mu\nu}(x)\rangle$ via point-splitting. However, the practical implementation of Eqs. (2.13) and (2.14) is not straightforward. Typically, $G_F(x,x')$ is obtained from Eq. (1.26) as a mode sum via expanding $\hat{\varphi}(x)$ as in Eq. (1.21). However, the mode solutions are usually known only numerically, since the mode equation may not have a closed solution in terms of known functions. On the other hand, as we have seen, the counter-term to be subtracted from $G_F(x,x')$ is usually written in terms of geometric quantities such as the world function $\sigma(x,x')$ and the bivector of parallel displacement $g^{\mu}_{\nu'}$, and not as a mode sum.

one could easily have included it as $\mathcal{L}_{\mathrm{E-H}} = \frac{1}{16\pi G}\sqrt{-g}(R-2\Lambda)$ (where we write Newton's constant G explicitly).

Because of these difficulties, one needs a method to carry out in practice the subtraction in Eqs. (2.13) and (2.14), so that the mode sum converges when the coincidence limit is taken. In this section, we briefly review the standard method used to first calculate $\langle\hat{\varphi}^2(x)\rangle_{\rm ren}$ and $\langle\hat{T}_{\mu\nu}(x)\rangle_{\rm ren}$ in static, spherically-symmetric space-times such as Schwarzschild. Then, we cite recent results regarding the calculation of these quantities in more general space-times, such as Kerr.

2.2.1 Static, spherically-symmetric space-times

With the idea of renormalization via point-splitting established, the next natural step is to calculate the renormalized values of $\langle\hat{\varphi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$ in physically interesting space-times, such as Schwarzschild.

In 1980, Candelas [6] studied the asymptotic behaviour of $\langle\hat{\varphi}^2(x)\rangle_{\rm ren}$ and $\langle\hat{T}_{\mu\nu}(x)\rangle_{\rm ren}$ in various states for a massless, conformally coupled scalar field $\hat{\varphi}(x)$ in Schwarzschild as $r \to 2M$ and $r \to \infty$, and obtained their value on the event horizon. In 1984, Candelas and Howard [7] numerically obtained $\langle\hat{\varphi}^2(x)\rangle_{\rm ren}$ for a massless, conformally coupled scalar field in Schwarzschild in the Hartle-Hawking vacuum, in the region exterior to the event horizon (i.e., $r > 2M$).

Their method consisted in using a WKB approximation for the mode solutions in order to separate the expression for $\langle\hat{\varphi}^2(x)\rangle_{\rm ren}$ in two parts: one that could be calculated analytically, and one mode sum that could be performed numerically. In this approach, one uses a Wick rotation in time (i.e., $t \mapsto -i\tau$), Euclideanizing the space-time.

Subsequently, Howard and Candelas [8, 9] used the same approach in order to calculate $\langle\hat{T}_{\mu\nu}(x)\rangle_{\rm ren}$, again for a massless, conformally coupled scalar field in Schwarzschild in the Hartle-Hawking vacuum, in the region exterior to the event horizon. The method was generalized by Anderson [10] and Anderson, Hiscock and Samuel [11, 12], respectively for $\langle\hat{\varphi}^2(x)\rangle_{\rm ren}$ and $\langle\hat{T}_{\mu\nu}(x)\rangle_{\rm ren}$. This generalization allows one to perform the renormalization of the stress-energy tensor in a general static, spherically-symmetric space-time, for a general scalar field (i.e., any mass m and any coupling constant ξ to the scalar curvature). The field may be in a zero-temperature vacuum state, or in a thermal state with

temperature T.

Let us investigate this method more closely. In the rest of this subsection, we closely follow Ref. [12]. They start with a general static, spherically-symmetric space-time. In the Euclidean space approach, one sets $\tau \equiv it$. That means the space is now Riemannian (i.e., positive signature both in time and space). Using spherical coordinates on the spatial sector, the metric is

$$\mathrm{d}s^2 = f(r)\mathrm{d}\tau^2 + h(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \tag{2.19}$$

where $f(r)$ and $h(r)$ are positive-definite functions of r (if the space is asymptotically flat, they must be constant for $r \to \infty$). Note the absence of a negative sign in the time direction. The reason behind such Euclideanization is discussed at the end of this subsection.

In this space-time, consider a scalar field $\varphi(x)$ with mass m and coupling ξ to the scalar curvature. It may be quantized as described in section 1.2 (the generalization to arbitrary coupling ξ is straightforward). Let us assume that the field is in the Euclidean vacuum[9]. That is the state that, in the Lorentzian sector, is defined in terms of positive-frequency modes with respect to the time-like coordinate t. From this expansion in modes, one finds an expression for the Euclidean Green function G_E, which is the correspondent of the Feynman Green function in the Euclidean sector:

$$G_E(x,x') = \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\omega\cos(\omega(\tau-\tau'))\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)\mathcal{N}_{\ell\omega}p_{\ell\omega}(r_<)q_{\ell\omega}(r_>), \tag{2.20}$$

where $r_<$ and $r_>$ are respectively the lesser and the greater of r and r'. As before, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$, and $\mathcal{N}_{\ell\omega}$ is a normalization constant. The functions $p_{\ell\omega}(r)$ and $q_{\ell\omega}(r)$ are solutions to the homogeneous radial mode equation in this

[9]Ref. [12] considers both the zero-temperature Euclidean vacuum and a thermal state with temperature T (e.g., the Hartle-Hawking state for $T = \kappa/2\pi$, where κ is the surface gravity of the black hole). However, for simplicity, we review here only the zero-temperature case. Computationally, the main difference is that in the nonzero temperature case the ω-integrals become infinite sums over a discrete variable n.

space-time, namely

$$\left\{\frac{1}{h}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\left(\frac{2}{rh}+\frac{f'}{2fh}-\frac{h'}{2h^2}\right)\frac{\mathrm{d}}{\mathrm{d}r}-\left(\frac{\omega^2}{f}+\frac{\ell(\ell+1)}{r^2}+m^2+\xi R\right)\right\}y=0, \tag{2.21}$$

where the prime means derivative with respect to r. $p_{\ell\omega}(r)$ is defined as the solution which is well behaved at the event horizon (or $r=0$ if there is no event horizon), and it is divergent at infinity. $q_{\ell\omega}(r)$ is defined the way around.

Then, an unrenormalized expression for the expectation value of the stress-energy tensor, $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{unren}}$, may be obtained from a generalization of Eq. (2.12) to arbitrary ξ. In this case, G_E should be used instead of G_F.

In order to renormalize the expectation value of the stress-energy tensor, the counter-terms given by Christensen [25] are used. As described in section 2.1, these are obtained from a DeWitt-Schwinger expansion of the Feynman Green function. Hence, the renormalized expression for $\langle\hat{T}_{\mu\nu}\rangle$ via point-splitting is given by

$$\langle\hat{T}_{\mu\nu}(x)\rangle_{\mathrm{ren}}=\lim_{x'\to x}\Big(\langle\hat{T}_{\mu\nu}(x,x')\rangle_{\mathrm{unren}}-\langle\hat{T}_{\mu\nu}(x,x')\rangle_{\mathrm{DS}}\Big), \tag{2.22}$$

where $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{DS}}$ are Christensen's counter-terms. The limit $x'\to x$ is taken along the time direction, i.e., we set $r'=r$, $\theta'=\theta$, $\phi'=\phi$ and $\tau'=\tau-\epsilon$, with $\epsilon>0$. With this point-splitting, the geometrical terms $\sigma(x,x')$ and $g^{\mu}_{\nu'}(x,x')$ in $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{DS}}$ may be expanded in terms of ϵ.

However, one still needs to handle the first term in Eq. (2.22) in order to carry out the subtraction properly. This is where the WKB approximation is used. The trick, developed in the papers cited in the beginning of this subsection, is to add and subtract a term $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{WKBdiv}}$ obtained from the WKB expansion of the mode solutions, separating $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{ren}}$ in two parts:

$$\begin{aligned}\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{ren}}&=\lim_{x'\to x}\Big((\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{unren}}-\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{WKBdiv}})+(\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{WKBdiv}}-\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{DS}})\Big)\\&=\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{numeric}}+\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{analytic}}.\end{aligned} \tag{2.23}$$

Both parts, $\langle\hat{T}_{\mu\nu}\rangle_{\rm numeric}$ and $\langle\hat{T}_{\mu\nu}\rangle_{\rm analytic}$, are finite. As the name suggests, $\langle\hat{T}_{\mu\nu}\rangle_{\rm analytic}$ can be calculated analytically, by carrying out the ℓ-sums and ω-integrals in $\langle\hat{T}_{\mu\nu}\rangle_{\rm WKBdiv}$. Also, $\langle\hat{T}_{\mu\nu}\rangle_{\rm numeric}$ is well-defined and can be calculated numerically. Moreover, $\langle\hat{T}_{\mu\nu}\rangle_{\rm analytic}$ dominates over $\langle\hat{T}_{\mu\nu}\rangle_{\rm numeric}$, at least in the region where the numerical calculation was performed, which is the exterior to the event horizon. Therefore, $\langle\hat{T}_{\mu\nu}\rangle_{\rm analytic}$ it may be used as an analytical approximation to $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$.

Let us describe briefly how $\langle\hat{T}_{\mu\nu}\rangle_{\rm WKBdiv}$ is obtained. First, we review how to apply a standard WKB approach to the mode equation (2.21). The functions $p_{\ell\omega}$ and $q_{\ell\omega}$ are re-expressed in terms of a function $\mathcal{W}$ as

$$\begin{aligned} p_{\ell\omega}(r) &= \frac{1}{(2r^2\mathcal{W})^{1/2}}\exp\left\{\int \mathrm{d}r\,\mathcal{W}\,\frac{h^{1/2}}{f^{1/2}}\right\} \\ q_{\ell\omega}(r) &= \frac{1}{(2r^2\mathcal{W})^{1/2}}\exp\left\{-\int \mathrm{d}r\,\mathcal{W}\,\frac{h^{1/2}}{f^{1/2}}\right\}, \end{aligned} \tag{2.24}$$

and substituting into Eq. (2.21) we find an equation for $\mathcal{W}$:

$$\mathcal{W}^2 = \Omega^2(r) + V_1(r) + V_2(r) + \frac{1}{2}\left\{\frac{f}{h}\frac{\mathcal{W}''}{\mathcal{W}} + \left(\frac{f'}{h} - \frac{fh'}{h^2}\right)\frac{\mathcal{W}'}{2\mathcal{W}} - \frac{3}{2}\frac{f}{h}\left(\frac{\mathcal{W}'}{\mathcal{W}}\right)^2\right\}, \tag{2.25}$$

where

$$\begin{aligned} \Omega^2(r) &\equiv \omega^2 + m^2 f + (2\ell+1)^2\frac{f}{4r^2} \\ V_1(r) &\equiv \frac{f'}{2rh} - \frac{fh'}{2rh^2} - \frac{f}{4r^2} \\ V_2(r) &\equiv \xi R f, \end{aligned} \tag{2.26}$$

and R is the scalar curvature in this space-time.

The idea of the WKB approximation is to solve Eq. (2.25) iteratively. Essentially, one takes one order higher for each radial derivative, so that Ω^2 is order zero, V_1 is order one, V_2 is order two, and the term under curly brackets in Eq. (2.25) is order two. Thus, for example, the zeroth-order solution is $\mathcal{W} = \Omega$, and the first-order solution is $\mathcal{W} = \Omega + \frac{V_1}{2\Omega}$.

Then, the nth-order solution of $\mathcal{W}$ is substituted into Eq. (2.24) in order to find an approximation for the modes. These are substituted into Eq. (2.20), which gives a WKB

approximation to G_E. Finally, from this one finds the nth-order WKB approximation to $\langle\hat{T}_{\mu\nu}\rangle$. Note that it is written as a mode sum over ℓ and integral over ω.

Now, one is interested in capturing the ultraviolet divergences of the mode sum, so that the divergences in $\langle\hat{T}_{\mu\nu}\rangle_{\rm unren}$ are canceled by $\langle\hat{T}_{\mu\nu}\rangle_{\rm WKBdiv}$. Hence, $\langle\hat{T}_{\mu\nu}\rangle_{\rm WKBdiv}$ is obtained by explicitly carrying out the ℓ-sums in the WKB representation of $\langle\hat{T}_{\mu\nu}\rangle$. The result is expanded in terms of inverse powers of ω, and terms up to order ω^{-1} are retained. The details of these calculations are given in Ref. [12] (see the Appendices in there). If the WKB approximation is carried out to fourth order[10], then $\langle\hat{T}_{\mu\nu}\rangle_{\rm WKBdiv}$ constructed as above contains all the divergences in $\langle\hat{T}_{\mu\nu}\rangle_{\rm unren}$, rendering $\langle\hat{T}_{\mu\nu}\rangle_{\rm numeric}$ finite. Therefore, the coincidence limit $\epsilon \to 0$ can be taken before the mode sums are computed. Now, the rest of the calculations to obtain $\langle\hat{T}_{\mu\nu}\rangle_{\rm numeric}$ may be performed numerically.

As for $\langle\hat{T}_{\mu\nu}\rangle_{\rm analytic}$, the ω-integrals in $\langle\hat{T}_{\mu\nu}\rangle_{\rm WKBdiv}$ are explicily calculated, giving an expansion in powers of ϵ. Since $\langle\hat{T}_{\mu\nu}\rangle_{\rm DS}$ is already expressed as an expansion in powers of ϵ, then their difference can be readily calculated. The divergences are canceled out analytically. At last, the limit $\epsilon \to 0$ may be taken, giving the result for $\langle\hat{T}_{\mu\nu}\rangle_{\rm analytic}$.

Note that the possibility of using a WKB approximation is crucial to the method described above. However, in the Lorentzian sector, Ω would be given by $\Omega^2 = -\omega^2 + m^2 f + (2\ell+1)^2 \frac{f}{4r^2}$. Thus, given ω and ℓ, there may exist an r_0 such that $\Omega(r_0) = 0$. This is a turning point for Eq. (2.25). As is well known[11], the WKB approximation breaks down near a turning point, and the solution behaves differently for $r > r_0$ and for $r < r_0$. Near r_0, one needs to match the two solutions, which makes the WKB approximation hard to implement even for low-order calculations. That means that the renormalization method described above is impractical in the Lorentzian case. This is the main reason for Euclideanizing the space-time, since in that case there are no turning points.

In static, spherically-symmetric space-times, Euclideanization can always be performed. In other words, one can always perform the Wick rotation $\tau = it$, carry out the calculations, and then set back $t = -i\tau$. In more general space-times, however, that may not be

[10]For the renormalization of $\langle\hat{\varphi}^2\rangle$, only a second-order WKB approximation is necessary.

[11]For an example of the handling of turning points in the semiclassical WKB approximation in basic quantum mechanics, see Landau and Lifshitz [28], § 47.

case. Consequently, the method described above may not generalize straightforwardly to other types of space-time.

2.2.2 More general space-times

In the previous subsection, we reviewed a method to implement the renormalization of $\langle\hat{\varphi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$ in static, spherically-symmetric space-times. However, many interesting physical settings do not belong to this category. Indeed, in order to accurately describe a black hole evaporating via Hawking radiation, one would need a dynamical (i.e., non-static) metric. Having a renormalization procedure in this case might be useful, for instance, to investigate open questions such as the black hole information paradox. Another example are astrophysical black holes. One expects that the black holes in the Universe possess non-zero angular momentum. Therefore, a metric such as Kerr, which is stationary but not spherically symmetric, should be used.

As explained above, the method described in the previous subsection may be unsuitable to other types of space-time. For this reason, it would be useful to have a renormalization procedure which does not make use of Euclideanization or the WKB approximation. The objetive of the present work is to describe an example of such a method (Chapter 3). We then apply it to the simple cases of a scalar field in Minkowski and Bertotti-Robinson space-times.

In the course of this project, other works in a similar direction have been published. In particular, Levi and Ori [29, 30, 31, 32] developed a new method, which has since been employed to the renormalization of the stress-energy tensor in Kerr space-time for a massless, minimally-coupled scalar field in the Unruh vacuum [33]. Their method, which they call *pragmatic mode-sum regularization*, requires that the background admits one symmetry (i.e., one Killing vector field). It has been presented in three variants, according to the symmetry involved: t-splitting, for stationary backgrounds; angular-splitting, for spherically-symmetric space-times; and ϕ-splitting, for axisymmetric backgrounds.

We describe very briefly the basic idea of this approach. The field is decomposed in modes in at least the coordinate corresponding to the symmetry present. Also, the points

x and x' are separated along the direction of this coordinate. The counter-terms are expanded in powers of this coordinate separation, and each term is expressed as a mode sum. For example, in t-splitting, the field is decomposed in ω-modes. One separates x and x' in the t direction as $\varepsilon = t' - t$. The counter-terms are expanded in powers of ε, and then written as mode sums, e.g., $\varepsilon^{-2} = -\int_0^\infty \mathrm{d}\omega\, \omega e^{i\omega\varepsilon}$ (these integrals are convergent in a generalized sense, as described below).

Then, the counter-terms may be subtracted from the unrenormalized expression, and the limit $\varepsilon \to 0$ may be taken. However, the remaining mode integral may still be divergent because of undamped oscillations. Thus, it is calculated via a so-called *self-cancellation integral* which renders it convergent in a generalized sense. For more details, see Refs. [29]-[33].

In addition to this method, we also cite the works of Taylor and Breen [34, 35], which appeared during the course of this project. Their approach is similar to the one described in the present thesis, but the calculations are performed in the Euclidean sector. They employed this method in order to renormalize $\langle\hat{\varphi}^2\rangle$ for a scalar field in the Hartle-Hawking state in Schwarzschild-Tangherlini space-time in any dimensions [34, 35].

Chapter 3

Novel method for renormalization

In this chapter, we shall describe the main idea behind our method for renormalization. As seen in section 2.1, the renormalization of $\langle \hat{T}_{\mu\nu} \rangle$ or $\langle \hat{\varphi}^2 \rangle$ is performed by subtracting the singular part of the Hadamard form for the Feynman Green function from the unrenormalized Feynman Green function, according to the point-splitting method, and then taking the coincidence limit after the subtraction.

Explicitly, let $\varphi(x)$ be a scalar field on a space-time manifold $\mathcal{M}$. Let $G_{\mathrm{F}}(x, x')$, with x, $x' \in \mathcal{M}$, be the unrenormalized Feynman Green function for the field equation with the appropriate boundary conditions, according to the quantum state. Also, let $G_{\mathrm{S}}(x, x')$ be the singular part of the Hadamard form for the Feynman Green function. Remember that $G_{\mathrm{S}}(x, x')$ depends on the geometry of the space-time, but not on the quantum state of the field.

After quantization, the field becomes an operator-valued distribution, $\hat{\varphi}(x)$. Accordingly, the expectation value of the square of the field operator, $\langle \psi | \hat{\varphi}^2(x) | \psi \rangle$, in some quantum state $|\psi\rangle$, has to be renormalized. The renormalization is necessary because the product of a distribution with itself on the same point x is not well-defined. The renormalization is carried out according to the point-splitting method,

$$\langle \psi | \hat{\varphi}^2(x) | \psi \rangle_{\mathrm{ren}} = -i \lim_{x' \to x} \Big(G_{\mathrm{F}}(x, x') - G_{\mathrm{S}}(x, x') \Big), \tag{3.1}$$

where $\langle \psi | \hat{\varphi}^2(x) | \psi \rangle_{\mathrm{ren}}$ is the renormalized value of $\langle \psi | \hat{\varphi}^2(x) | \psi \rangle$.

The main difficulty, in general, is in writing the two terms in Eq. (3.1) in a suitable way so as to perform this subtraction in practice. The reason is that $G_{\rm F}$ is usually not known in closed form, but is instead written as a mode decomposition. On the other hand, $G_{\rm S}$ is usually written in a geometrical form – specifically, in terms of Synge's world function $\sigma(x, x')$.

Therefore, the practical implementation requires a suitable handling of these two terms so that they can be conveniently subtracted. In subsection 2.2.1 we saw how it has been achieved in, e.g., Schwarzschild space-time by Euclidianizing the space-time and using the WKB approximation. However, this method is not straightforwardly generalizable to Kerr.

Put simply, the idea of our method is to handle the second term, $G_{\rm S}(x, x')$, by decomposing it in a similar way as the Feynman Green function, i.e., in frequency and multipolar modes. In order to achieve that, we have to point-separate in multiple directions. Note the difference from the method shown in subsection 2.2.1, in which the point-splitting is only along the time coordinate. These ideas are inspired in techniques which arose in the study of the self-force problem in General Relativity. As an example, see Ref. [36].

Then, the subtraction can be performed mode-by-mode and the coincidence limit can be taken (i.e., $x' \to x$). Lastly, we sum over the multipoles and integrate over the frequency. That can be achieved because, after the previous mode-by-mode subtraction, this infinite sum and integral become well-defined.

Let us next clarify these ideas.

3.1 Stationary axisymmetric case

Let $\hat{\varphi}(x)$ be a quantum scalar field. We shall assume that $\hat{\varphi}(x)$ is real, massless and minimally coupled to the Ricci scalar. Also, let us assume a stationary, axisymmetric space-time (e.g., Kerr). We consider a vacuum state $|\psi\rangle$ which is defined by expanding over positive-frequency modes with respect to a "time" coordinate T, on a given Cauchy

surface Σ.[1] We separate[2] the field operator $\hat{\varphi}(x)$ in the time coordinate T, a radial coordinate ξ, and polar and azimuthal angle coordinates, respectively θ and ϕ (with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$).

In these coordinates, a suitable basis for the classical solutions of the field equation is given by the set of functions

$$\varphi^j_{\ell m\omega}(x) = \mathcal{N}^j_{\ell m\omega} e^{-i\omega T} e^{im\phi} S_{\ell m\omega}(\theta)\, R^j_{\ell m\omega}(\xi), \tag{3.2}$$

and their complex conjugate $(\varphi^j_{\ell m\omega}(x))^*$. In Eq. (3.2), ω is a non-negative real number, ℓ is a non-negative integer, and m is an integer from $-\ell$ to ℓ.

The functions $R^j_{\ell m\omega}(\xi)$ are mode solutions of the radial equation. The index j represents the fact that, in order to have a complete set of solutions, we may need more than one type of radial mode for each $\{\ell, m, \omega\}$, satisfying different boundary conditions. E.g., in Schwarzschild space-time, the *in*-modes alone are not sufficient to form a complete set. One must include another set of modes (for instance, the *up*-modes), so that both together form a complete set.

The functions $S_{\ell m\omega}(\theta)$ are mode solutions of the polar angle equation. Note that in general these angular functions may depend not only on ℓ and m, but also on the frequency ω. In Kerr space-time, for instance, these solutions are the *spheroidal harmonics*, which have a dependence on $a\omega$, where a is the angular momentum per unit mass.

Note that we are allowed to decompose in terms of the harmonics $e^{-i\omega T}$ and $e^{im\phi}$ because of the stationarity and the axisymmetry of the space-time.

Finally, $\mathcal{N}^j_{\ell m\omega}$ is a normalization constant.

Given all that, $\{\varphi^j_{\ell m\omega}(x), (\varphi^j_{\ell m\omega}(x))^*\}$, with the appropriate range for the indices ℓ, m, ω and j, form a complete and orthonormal set, with respect to the Klein-Gordon

[1] Strictly, what we need is not necessarily a Cauchy surface, but a characteristic initial data surface – even if it is null. Besides that, such surface may be defined as the union of different subsurfaces. In that case, the "time" coordinate may be different in these subsurfaces. As an example, in Schwarzschild one uses the surface $\mathscr{I}^- \cup \mathscr{H}^-$, i.e., the union of past null infinity and past event horizon. The "time" coordinates, which are actually null, are $u = t - r_*$ along $\mathscr{H}^-$ and $v = t + r_*$ along $\mathscr{I}^-$ (where r_* is the so-called tortoise coordinate).

[2] We assume that the space-time admits separation of the radial (r) and polar (θ) variables. In particular, this is true in the case of Kerr space-time, in which the separation is guaranteed by a hidden symmetry.

inner product. So, we can expand the field in that basis, and quantize by promoting the coefficients to annihilation and creation operators:

$$\hat{\varphi}(x) = \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_j \mathcal{N}^j_{\ell m\omega} e^{-i\omega T} e^{im\phi} S_{\ell m\omega}(\theta)\, R^j_{\ell m\omega}(\xi) \hat{a}^j_{\ell m\omega} + \text{h.c.}. \tag{3.3}$$

Eq. (3.3) allows us to write the Wightman function for the vacuum state $|\psi\rangle$ as a mode decomposition:

$$\begin{aligned} G_+(x,x') &= \langle\psi|\hat{\varphi}(x)\hat{\varphi}(x')|\psi\rangle \\ &= \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_j |\mathcal{N}^j_{\ell m\omega}|^2 e^{-i\omega(T-T')} e^{im(\phi-\phi')} \times \\ &\qquad \times S_{\ell m\omega}(\theta)[S_{\ell m\omega}(\theta')]^* R^j_{\ell m\omega}(\xi)\,[R^j_{\ell m\omega}(\xi')]^*. \end{aligned} \tag{3.4}$$

Defining $f_{\ell m\omega}(\xi,\xi') \equiv \sum_j |\mathcal{N}^j_{\ell m\omega}|^2 R^j_{\ell m\omega}(\xi)\,[R^j_{\ell m\omega}(\xi')]^*$, we can rewrite Eq. (3.4) as

$$G_+(x,x') = \int_0^\infty \mathrm{d}\omega\, e^{-i\omega(T-T')} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} e^{im(\phi-\phi')} S_{\ell m\omega}(\theta)[S_{\ell m\omega}(\theta')]^* f_{\ell m\omega}(\xi,\xi'). \tag{3.5}$$

The Feynman Green function, then, can be obtained from the Wightman function according to

$$G_{\mathrm{F}}(x,x') = i\, G_+(x,x')\,\theta(T-T') + i\, G_-(x,x')\,\theta(T'-T). \tag{3.6}$$

We can choose how the limit $x' \to x$ is to be taken. For simplicity, we henceforth assume $T - T' > 0$. In that case, we have simply

$$G_{\mathrm{F}}(x,x') = i\, G_+(x,x'). \tag{3.7}$$

Next, the idea is to write the singular part of the Hadamard form for the Feynman Green function in a form similar to that in Eq. (3.4).

As we have seen, the Feynman Green function in some state can be expressed analytically, in a normal neighbourhood of the space-time point x, using the so-called Hadamard

form [20]:

$$G_{\rm F}(x,x') = \lim_{\varepsilon\to 0^+} \frac{i}{8\pi^2}\left[\frac{\Delta^{\frac{1}{2}}(x,x')}{\sigma(x,x')+i\varepsilon} + V(x,x')\ln(\sigma(x,x')+i\varepsilon) + W(x,x')\right], \qquad (3.8)$$

where $\Delta(x,x')$ is the Van Vleck-Morette determinant and $\sigma(x,x')$ is the world function (one-half of the square of the geodesic distance between x and x'). The quantities $V(x,x')$ and $W(x,x')$ are regular biscalars. The term '$i\varepsilon$' represents the Feynman prescription. That means that the divergence structure of the Feynman Green function as $x' \to x$ is contained in the first two terms, which we call the *singular part* of the Hadamard form, $G_{\rm S}(x,x')$. The singular part is purely geometrical, i.e., it does not depend on the quantum state or on boundary conditions. On the other hand, $W(x,x')$ is not uniquely determined by the geometry. It is uniquely defined only after we set the quantum state.

Firstly, remembering that we are point-splitting in multiple directions, we expand $G_{\rm S}(x,x')$ in terms of the coordinate separations. Let δ be the typical coordinate separation (i.e., the time, radial, and angular separations are all of order δ). From Eq. (3.8), it is clear that the leading order as $\delta \to 0$ will be δ^{-2}, since the world function must be of order δ^2. We can thus expand $G_{\rm S}$ as

$$G_{\rm S}(x,x') = \frac{G_{\rm S}^{(-2)}(x,x')}{\delta^2} + \frac{G_{\rm S}^{(-1)}(x,x')}{\delta} + G_{\rm S}^{(\log)}(x,x') + G_{\rm S}^{(0)}(x,x') + G_{\rm S}^{(1)}(x,x')\,\delta + \ldots, \qquad (3.9)$$

where the biscalars $G_{\rm S}^{(n)}(x,x')$ are the coefficients of the expansion. In particular, $G_{\rm S}^{(\log)}(x,x')$ is a term that contains the logarithmic dependences on δ. Note that $G_{\rm S}^{(\log)}(x,x')$ is not used in the calculations of Chapter 4. The reason is that, in Ricci-flat space-times, $V(x,x')$ is zero to the order that is necessary for the calculation of $\langle\hat{\varphi}^2\rangle$ (i.e., it does not contribute to this renormalization). That is not the case in the renormalization of $\langle\hat{T}_{\mu\nu}\rangle$, since higher order terms may be needed. In that case, $G_{\rm S}^{(\log)}(x,x')$ will contribute.

In practice, this expansion is obtained as follows: first, we write $G_{\rm S}(x,x')$ in terms of

the coordinate separations[3] by expressing the coordinates of the point x' as

$$T' = T - \delta\Delta T, \qquad \xi' = \xi - \delta\Delta\xi, \qquad \theta' = \theta - \delta\Delta\theta, \qquad \phi' = \phi - \delta\Delta\phi. \tag{3.10}$$

We also set $\varepsilon = \mathcal{O}(\delta^2)$. This is done because the '$i\varepsilon$' term in Eq. (3.8) is added to $\sigma(x, x')$ according to the Feynman prescription, but $\sigma(x, x') = \mathcal{O}(\delta^2)$. Hence we want to keep $i\varepsilon$ as if it were of order δ^2 in the expansion, otherwise we lose the meaning of the Feynman prescription.

Then, we expand $G_{\rm S}(x, x')$ about $\delta = 0$. The coefficients of the series will be the functions $G_{\rm S}^{(n)}(x, x')$ and $G_{\rm S}^{(\log)}$, according to (3.9). Note that δ is just an auxiliary variable to keep track of the order of the expansion; after we find the coefficients, we shall set $\delta = 1$.

Now, we decompose each of the $G_{\rm S}^{(n)}(x, x')$ and $G_{\rm S}^{(\log)}$ in frequency and multipole modes:

$$\begin{aligned} G_{\rm S}^{(n),(\log)}(x, x') =& i \int_{-\infty}^{\infty} {\rm d}\omega \, e^{-i\omega(T-T')} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} e^{im(\phi-\phi')} \times \\ & \times S_{\ell m\omega}(\theta)[S_{\ell m\omega}(\theta')]^* \, g_{\ell m\omega}^{(n),(\log)}(\xi, \xi'). \end{aligned} \tag{3.11}$$

We include the factor 'i' for convenience; the reason is to compensate for the extra factor 'i' that comes from Eq. (3.7). Eq. (3.11) defines the radial modes $g_{\ell m\omega}^{(n)}(\xi, \xi')$. However, in order to perform the subtraction in Eq. (3.1) mode-by-mode, we need an explicit expression for these radial modes. This is achieved by inverting Eq. (3.11):

$$\begin{aligned} g_{\ell m\omega}^{(n),(\log)}(\xi, \xi') = -i \int_{-\infty}^{\infty} {\rm d}T \, e^{i\omega(T-T')} \int_{S^2} {\rm d}^2\Omega \int_{S^2} {\rm d}^2\Omega' \, e^{-im(\phi-\phi')} \times \\ \times [S_{\ell m\omega}(\theta)]^* S_{\ell m\omega}(\theta') \, G_{\rm S}^{(n),(\log)}(x, x'), \end{aligned} \tag{3.12}$$

where ${\rm d}^2\Omega = \sin\theta {\rm d}\theta {\rm d}\phi$ is the surface element on the 2-sphere.

However, $G_{\rm S}^{(n)}$ and $G_{\rm S}^{(\log)}$ are written in Eq. (3.11) as a Fourier integral over the frequency from $-\infty$ to ∞, while we want to subtract it from G_+, which is written as an

[3] In the spherically symmetric case, instead of $\Delta\theta$ and $\Delta\phi$, we find it useful to expand in terms of the new variable $Q = \sqrt{1 - \cos\gamma}$, where γ is the angular separation given by Eq. (3.18) below. This follows Ref. [36].

integral over the frequency from 0 to ∞, in Eq. (3.5). For that reason, we fold the integral in Eq. (3.11):

$$G_{\rm S}^{(n),(\log)} = i\int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{im(\phi-\phi')} S_{\ell m\omega}(\theta)[S_{\ell m\omega}(\theta')]^* \times \\ \times \left(e^{-i\omega(T-T')} g^{(n)(\log)}_{\ell m\omega}(\xi,\xi') + e^{i\omega(T-T')} g^{(n)}_{\ell m,-\omega}(\xi,\xi')\right). \tag{3.13}$$

Now, we can put Eqs. (3.5) and (3.13) (together with (3.7) and (3.9)) back in Eq. (3.1) and take the coincidence limit, $x' \to x$:

$$\langle\psi|\hat{\varphi}^2(x)|\psi\rangle_{\rm ren} = \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} |S_{\ell m\omega}(\theta)|^2 \times \\ \times \left\{ f_{\ell m\omega}(\xi,\xi) - \sum_{n=-2}^{0}\left(g^{(n)}_{\ell m\omega}(\xi,\xi) + g^{(n)}_{\ell m,-\omega}(\xi,\xi)\right) - \right. \\ \left. - \left(g^{(\log)}_{\ell m\omega}(\xi,\xi) + g^{(\log)}_{\ell m,-\omega}(\xi,\xi)\right)\right\}. \tag{3.14}$$

It is worth noticing that, after the coincidence limit is taken inside the integral and the sum in Eq. (3.14), only the terms with $n = -2$, $n = -1$ and $n = 0$ contribute to the renormalization of $\langle\hat{\varphi}^2\rangle$. The reason is that, for $n > 0$, $G_{\rm S}^{(n)}$ is of order δ or greater, as can be seen from Eq. (3.9). Consequently, the terms that come from $G_{\rm S}^{(n)}$ with $n > 0$ do not contribute after the coincidence limit is taken.

Nevertheless, keeping some of the $n > 0$ terms might speed up the convergence of the ℓ-sum and ω-integral in the numerical implementation. Still, that is not needed in our case, since the convergence is rapid enough with only $g^{(n)}_{\ell m\omega}$ for $n = -2, -1, 0$, as we shall see in section 4.2.

Also note that in the renormalization of the stress-energy tensor one would have to keep some of the higher order terms, because the expression of the stress-energy tensor involves derivatives with respect to x and x'.

All of the calculations in this section will be performed analytically, except for the last step, i.e., the ω-integral and the ℓ-sum in Eq. (3.14). Of course, the field equation for the modes also may have to be solved numerically, depending on the space-time.

On the other hand, as mentioned earlier, *all* the calculations can be done analytically

in Bertotti-Robinson, even the infinite sum and integral in Eq. (3.14). We perform them numerically because in this work we want to check the consistency of our method, to use it in the future in more involved space-times, such as Schwarzschild.

3.2 Spherically-symmetric case

We described the method for a stationary axisymmetric space-time, but throughout this work we will apply it only to Minkowski and Bertotti-Robinson space-times, which are spherically symmetric. We will now particularize the method described above to the spherically-symmetric case.

In this case, the angular eigenfunctions are the standard spherical harmonics $Y_\ell^m(\theta,\phi)$ (i.e., the $S_{\ell m\omega}(\theta)$ are proportional to the associated Legendre functions, $P_\ell^m(\cos\theta)$). The decomposition (3.3) becomes

$$\hat{\varphi}(x) = \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_j \mathcal{N}^j_{\ell\omega}\, e^{-i\omega T} Y_\ell^m(\theta,\phi)\, R^j_{\ell\omega}(\xi)\hat{a}^j_{\ell m\omega} + \text{h.c.} \tag{3.15}$$

Note that the radial modes $R_{\ell\omega}(\xi)$ and the normalization constant $\mathcal{N}^j_{\ell\omega}$ now do not depend on m, because of the spherical symmetry. Then, Eq. (3.4) becomes

$$\begin{aligned} G_+(x,x') = \int_0^\infty \mathrm{d}\omega\, e^{-i\omega(T-T')} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)[Y_\ell^m(\theta',\phi')]^* \times \\ \times \sum_j |\mathcal{N}^j_{\ell\omega}|^2 R^j_{\ell\omega}(\xi)\,[R^j_{\ell\omega}(\xi')]^*. \end{aligned} \tag{3.16}$$

We can sum over m using the well-known addition formula for the spherical harmonics (cf. Eq. 12.43 in ref. [37]):

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi) Y_\ell^{m*}(\theta',\phi'), \tag{3.17}$$

where P_ℓ is the Legendre polynomial of order ℓ, and γ is the angular separation in the

2-sphere, namely

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{3.18}$$

We thus obtain

$$\begin{aligned} G_+(x,x') &= \int_0^\infty \mathrm{d}\omega\, e^{-i\omega(T-T')} \sum_{\ell=0}^\infty \frac{2\ell+1}{4\pi} P_\ell(\cos\gamma) \sum_j |\mathcal{N}^j_{\ell\omega}|^2 R^j_{\ell\omega}(\xi)\,[R^j_{\ell\omega}(\xi')]^* \\ &= \int_0^\infty \mathrm{d}\omega\, e^{-i\omega(T-T')} \sum_{\ell=0}^\infty (2\ell+1) P_\ell(\cos\gamma)\, f_{\ell\omega}(\xi,\xi'), \end{aligned} \tag{3.19}$$

where

$$f_{\ell\omega}(\xi,\xi') \equiv \sum_j \frac{|\mathcal{N}^j_{\ell\omega}|^2}{4\pi} R^j_{\ell\omega}(\xi)\,[R^j_{\ell\omega}(\xi')]^*. \tag{3.20}$$

Now, we want to decompose $G^{(n)}_{\mathrm{S}}(x,x')$ and $G^{(\log)}_{\mathrm{S}}$ in a form similar to Eq. (3.19). Hence, in place of Eq. (3.11) we write:

$$G^{(n),(\log)}_{\mathrm{S}}(x,x') = i\int_{-\infty}^\infty \mathrm{d}\omega\, e^{-i\omega(T-T')} \sum_{\ell=0}^\infty (2\ell+1) P_\ell(\cos\gamma)\, g^{(n),(\log)}_{\ell\omega}(\xi,\xi'). \tag{3.21}$$

Solving for $g^{(n)}_{\ell\omega}(\xi,\xi')$ and $g^{(\log)}_{\ell\omega}$, we find

$$g^{(n),(\log)}_{\ell\omega}(\xi,\xi') = -\frac{i}{4\pi}\int_0^\pi \mathrm{d}\gamma \sin\gamma P_\ell(\cos\gamma) \int_{-\infty}^\infty \mathrm{d}T\, e^{i\omega(T-T')} G^{(n),(\log)}_{\mathrm{S}}(x,x'). \tag{3.22}$$

Again, we fold the integral over ω in Eq. (3.21), similarly to (3.13). However, we can make use of the fact that, if $G^{(n)}_{\mathrm{S}}$ and $G^{(\log)}_{\mathrm{S}}$ do not depend on the sign of $\Delta T = T-T'$, then $g^{(n)}_{\ell\omega}$ and $g^{(\log)}_{\ell\omega}$ will not depend on the sign of ω. That can easily be seen from Eq. (3.22), by changing ΔT to $-\Delta T$. Indeed, it is the case for both Minkowski and Bertotti-Robinson space-times that $G^{(n)}_{\mathrm{S}}$ and $G^{(\log)}_{\mathrm{S}}$ do not depend on the sign of ΔT. Then, (3.21) simplifies to

$$G^{(n),(\log)}_{\mathrm{S}}(x,x') = 2i\int_0^\infty \mathrm{d}\omega\, \cos(\omega\Delta T) \sum_{\ell=0}^\infty (2\ell+1) P_\ell(\cos\gamma)\, g^{(n),(\log)}_{\ell\omega}(\xi,\xi'). \tag{3.23}$$

Finally, this allows us to rewrite Eq. (3.14) in the spherically symmetric case:

$$\langle\psi|\hat{\varphi}^2(x)|\psi\rangle_{\rm ren} = \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}(2\ell+1)\left\{f_{\ell\omega}(\xi,\xi) - 2\sum_{n=-2}^{0} g^{(n)}_{\ell\omega}(\xi,\xi) - 2g^{(\log)}_{\ell\omega}(\xi,\xi)\right\}. \quad (3.24)$$

Chapter 4

Application of the proposed method

In this chapter, we apply the method described above to the renormalization of the expectation value of the square of a quantum scalar field $\hat{\varphi}(x)$ in two physical settings. Namely, *(i)* when the field is in the Minkowski vacuum in flat space-time; and *(ii)* when the field is in the Boulware vacuum in Bertotti-Robinson space-time.

4.1 Application to flat space-time

First, we obtain the renormalized value of $\langle M|\hat{\varphi}^2(x)|M\rangle$ in flat space-time, for a massless scalar field $\hat{\varphi}(x)$ in the Minkowski vacuum $|M\rangle$. This is a notoriously simple case, but elucidates the application of the proposed method. In particular, we will see that many of the integrals and series in this section are useful in Bertotti-Robinson space-time as well.

4.1.1 Wightman function in the Minkowski vacuum

Let us start with a massless scalar field $\varphi(x)$. In flat space-time, the field equation for such a field is simply the Klein-Gordon equation,

$$\Box\varphi = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\varphi(x) = 0. \tag{4.1}$$

As usual, the Minkowski vacuum is defined by expanding the field in terms of modes which are positive-frequency with respect to the time coordinate t. Accordingly, the field

is separated in a basis

$$\varphi_{\ell m\omega}(x) = \mathcal{N}_{\ell\omega} e^{-i\omega t} Y_\ell^m(\theta,\phi)\, R_{\ell\omega}(r), \tag{4.2}$$

where (r,θ,ϕ) are the spherical coordinates in $\mathbb{R}^3$, and $\mathcal{N}_{\ell\omega}$ is a normalization constant. Note that, with the boundary conditions stated below, $\{\varphi_{\ell m\omega}, (\varphi_{\ell m\omega})^* \mid \omega \in \mathbb{R}_+,\ \ell \in \mathbb{N},\ m \in \mathbb{Z}, -\ell \leq m \leq \ell\}$, forms a complete set of solutions by itself. Consequently, there is no need for the index j, which was included in Eq. (3.15) considering a more general case.

With these modes, the radial part of the field equation (4.1) becomes:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) - \ell(\ell+1) + \omega^2 r^2\right) R_{\ell\omega}(r) = 0. \tag{4.3}$$

This is a spherical Bessel equation, admitting as independent solutions the spherical Bessel functions of the first kind $j_\ell(\omega r)$ and of the second kind $y_\ell(\omega r)$. The appropriate boundary conditions for the Wightman function are set by requiring that the field be regular at $r = 0$ and at infinity, which excludes $y_\ell(\omega r)$. After normalizing according to the Klein-Gordon inner product, we find

$$\mathcal{N}_{\ell\omega} R_{\ell\omega}(r) = \sqrt{\frac{\omega}{\pi}}\, j_\ell(\omega r). \tag{4.4}$$

This solution allows us to write the Wightman function in the Minkowski vacuum as in Eq. (3.19),

$$G_+(x,x') = \int_0^\infty \mathrm{d}\omega\, e^{-i\omega\Delta t} \sum_{\ell=0}^\infty \frac{2\ell+1}{4\pi} P_\ell(\cos\gamma) \frac{\omega}{\pi} j_\ell(\omega r)\, j_\ell(\omega r'), \tag{4.5}$$

from which we can read off

$$f_{\ell\omega}(r,r') = \frac{\omega}{4\pi^2} j_\ell(\omega r)\, j_\ell(\omega r'). \tag{4.6}$$

From Eq. (4.5), one can find the closed form of the Feynman Green function. The

ℓ-sum can be performed using (cf. Eq. (10.1.45) from ref. [38])

$$\frac{\sin\omega R}{\omega R} = \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)j_\ell(\omega r')j_\ell(\omega r), \tag{4.7}$$

where

$$R \equiv \sqrt{r^2 + r'^2 - 2rr'\cos\gamma} \tag{4.8}$$

is the spatial distance in $\mathbb{R}^3$. For later use, we also exhibit the following relationship (cf. Eq. (10.1.46) from ref. [38]):

$$-\frac{\cos\omega R}{\omega R} = \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)j_\ell(\omega r')y_\ell(\omega r), \quad \text{for } r' < r. \tag{4.9}$$

We substitute Eq. (4.7) into Eq. (4.5). However, in order for the ω-integral to be convergent, we have to add a small imaginary part '$i\varepsilon$' to the exponent, according to the Feynman prescription. Then, Eq. (4.5) becomes

$$\begin{aligned} G_+(x,x') &= \frac{1}{4\pi^2}\lim_{\varepsilon\to 0^+}\int_0^\infty \mathrm{d}\omega\, e^{-i\omega(\Delta t - i\varepsilon)}\,\frac{\sin(\omega R)}{R} \\ &= -\frac{i}{8\pi^2 R}\lim_{\varepsilon\to 0^+}\int_0^\infty \mathrm{d}\omega\, \left(e^{-i\omega(\Delta t - i\varepsilon - R)} - e^{-i\omega(\Delta t - i\varepsilon + R)}\right) \\ &= -\frac{1}{8\pi^2 R}\lim_{\varepsilon\to 0^+}\left(\frac{1}{\Delta t - i\varepsilon - R} - \frac{1}{\Delta t - i\varepsilon + R)}\right) \\ &= -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{(\Delta t - i\varepsilon)^2 - R^2}, \end{aligned} \tag{4.10}$$

From this Wightman function, we obtain the Feynman Green function using Eq. (3.6):

$$G_{\mathrm{F}}(x,x') = -\frac{i}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{(|\Delta t| - i\varepsilon)^2 - R^2}, \tag{4.11}$$

which, with an appropriate reparameterization of ε, can be rewritten in the well-known form of the Feynman Green function for a massless scalar in flat space-time:

$$G_{\mathrm{F}}(x,x') = \frac{i}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{-(\Delta t)^2 + R^2 + i\varepsilon}. \tag{4.12}$$

Note that the denominator in Eq. (4.12) is related to the geodesic distance in spherical

coordinates. Indeed, in flat space-time, Synge's world function is clearly given by

$$\begin{aligned} 2\sigma(x,x') &= -(t-t')^2 + r^2 + r'^2 - 2rr'\cos\gamma \\ &= -(\Delta t)^2 + R^2, \end{aligned} \tag{4.13}$$

so that Eq. (4.12) becomes (again, with a redefiniton of ε)

$$G_{\rm F}(x,x') = \frac{i}{8\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{\sigma + i\varepsilon}. \tag{4.14}$$

In Eq. (4.14), the expression $\frac{1}{\sigma+i\varepsilon}$ should be understood as the distributional relation

$$\lim_{\varepsilon\to 0^+}\frac{1}{z \pm i\varepsilon} = \mathcal{PV}\left(\frac{1}{z}\right) \mp i\pi\delta(z), \tag{4.15}$$

for $z \in \mathbb{C}$, where $\mathcal{PV}$ means Cauchy's principal value.

4.1.2 Singular part of the Hadamard form in flat space-time

Now, it is clear from Eq. (4.14) that the Feynman Green function in the Minkowski vacuum is written in the form of Eq. (3.8). Thus, we can readily read off the singular part of the Hadamard form,

$$G_{\rm S}(x,x') = \frac{i}{8\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{\sigma + i\varepsilon}. \tag{4.16}$$

We can see that the Hadamard form is actually equal to the Feynman Green function in the Minkowski vacuum. That immediately shows, from Eq. (3.1), that $\langle M|\hat{\varphi}^2(x)|M\rangle_{\rm ren} = 0$. However, we carry on with the proposed method, in order to illustrate it and make sure that it works.

The next step is to expand the singular part of the Hadamard form for x' near x. This is achieved by writing Eq. (4.16) in terms of the coordinate separations $\Delta t \equiv t - t'$, $\Delta r \equiv r - r'$, and γ, given by Eq. (3.18). In fact, we find it useful to expand in terms of the new variable $Q = \sqrt{1-\cos\gamma}$ rather than γ, following reference [36]. From now on,

we occasionally omit $\lim_{\varepsilon\to 0^+}$, but remember that eventually this limit shall be taken.

$$\begin{aligned} G_{\rm S}(x,x') &= \frac{i}{4\pi^2}\frac{1}{-(\Delta t)^2 + r^2 + r'^2 - 2rr'\cos\gamma + i\varepsilon} \\ &= \frac{i}{4\pi^2}\frac{1}{-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 - 2r\Delta rQ^2 + i\varepsilon} \end{aligned} \tag{4.17}$$

Then, we multiply Δt, Δr and Q by the auxiliary parameter δ, and multiply ε by δ^2, as explained before. Next, we expand $G_{\rm S}(x,x')$ about $\delta = 0$:

$$\begin{aligned} G_{\rm S}(x,x') &= \frac{i}{4\pi^2}\frac{1}{\delta^2\left(-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 - 2r\Delta rQ^2\delta + i\varepsilon\right)} \\ &= \frac{i}{4\pi^2}\frac{1}{-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 + i\varepsilon}\sum_{n=-2}^{\infty}\left(\frac{2r\Delta rQ^2}{-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 + i\varepsilon}\right)^{n+2}\delta^n, \end{aligned} \tag{4.18}$$

where we have used $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

Thus, we have written $G_{\rm S}$ in the form of Eq. (3.9), from which we can read off

$$G_{\rm S}^{(n)}(x,x') = \frac{i}{4\pi^2}\frac{(2r\Delta rQ^2)^{n+2}}{(-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 + i\varepsilon)^{n+3}}, \qquad n = -2,\,-1,\,0\,,\,\ldots \tag{4.19}$$

Mote that in flat space-time there is no contribution from the logarithmic terms $G_{\rm S}^{(\log)}$, which is a consequence from the fact that $V(x,x') = 0$ in flat space-time. Now we proceed to obtain $g_{\ell\omega}^{(n)}(r,r')$, according to Eq. (3.22). We start with $g_{\ell\omega}^{(-2)}$:

$$\begin{aligned} g_{\ell\omega}^{(-2)}(r,r') &= -\frac{i}{4\pi}\int_0^{\pi}{\rm d}\gamma\,\sin\gamma P_\ell(\cos\gamma)G_\omega^{(-2)} \\ G_\omega^{(-2)} &= \int_{-\infty}^{\infty}{\rm d}t\,e^{i\omega\Delta t}\frac{i}{4\pi^2}\,\frac{1}{-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 + i\varepsilon}. \end{aligned} \tag{4.20}$$

For the t-integral, one can use the residue theorem, closing a semicircular path in the

upper t-complex plane for $\omega > 0$ and in the lower plane for $\omega < 0$:

$$\begin{aligned} G_\omega^{(-2)}(r, \Delta r, Q) &= \frac{i}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{e^{i\omega\Delta t}}{-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 + i\varepsilon} \\ &= -\frac{i}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{e^{i\omega\Delta t}}{(\Delta t - \alpha)(\Delta t + \alpha)} \\ &= \frac{1}{4\pi\alpha} \begin{cases} e^{i\omega\alpha}, & \text{for } \omega > 0 \\ e^{-i\omega\alpha}, & \text{for } \omega < 0 \end{cases} \\ &= \frac{e^{i|\omega|\alpha}}{4\pi\alpha} \\ &= \frac{e^{i|\omega|\sqrt{2r^2Q^2 + (\Delta r)^2 + i\varepsilon}}}{4\pi\sqrt{2r^2Q^2 + (\Delta r)^2 + i\varepsilon}}, \end{aligned} \tag{4.21}$$

in which $\alpha^2 \equiv 2r^2Q^2 + (\Delta r)^2 + i\varepsilon$ gives the pole, a complex number with small positive imaginary part. In Eq. (4.21), the square root is understood to mean the root with positive real part, i.e., for positive real x, $\lim\limits_{\varepsilon \to 0^+} \sqrt{x^2 + i\varepsilon} = x$.

Now, we perform the angular integration. For the t-integral, the small imaginary part '$i\varepsilon$' given by the Feynman prescription was crucial for the convergence. However, after the integration it is not needed anymore, and we can set $\varepsilon = 0$. A more detailed proof, in terms of distributions, is given in Appendix A.1. Hence, we find

$$\begin{aligned} g_{\ell\omega}^{(-2)}(r, r') &= -\frac{i}{4\pi} \int_0^\pi \mathrm{d}\gamma \sin\gamma \, P_\ell(\cos\gamma) \; G_\omega^{(-2)}(r, \Delta r, Q)\big|_{\varepsilon=0} \\ &= -\frac{i}{16\pi^2 r\sqrt{2}} \int_0^\pi \mathrm{d}\gamma \sin\gamma \, P_\ell(\cos\gamma) \, \frac{e^{i|\omega|r\sqrt{2}\sqrt{\frac{(\Delta r)^2}{2r^2} + 1 - \cos\gamma}}}{\sqrt{\frac{(\Delta r)^2}{2r^2} + 1 - \cos\gamma}} \\ &= -\frac{i}{16\pi^2 r\sqrt{2}} (-1)^\ell \int_{-1}^1 \mathrm{d}x \, P_\ell(x) \, \frac{e^{i|\omega|r\sqrt{2}\sqrt{s+1+x}}}{\sqrt{s+1+x}} \\ &= \frac{1}{16\pi r} J_{\ell+\frac{1}{2}} \left(|\omega| r \left(1 + s - \sqrt{s(2+s)} \right)^{\frac{1}{2}} \right) H_{\ell+\frac{1}{2}}^{(1)} \left(|\omega| r \left(1 + s + \sqrt{s(2+s)} \right)^{\frac{1}{2}} \right) \\ &= \frac{|\omega|}{8\pi^2} (1 + s - \sqrt{s(2+s)})^{\frac{1}{4}} (1 + s + \sqrt{s(2+s)})^{\frac{1}{4}} \times \\ &\quad \times j_\ell \left(|\omega| r \left(1 + s - \sqrt{s(2+s)} \right)^{\frac{1}{2}} \right) h_\ell^{(1)} \left(|\omega| r \left(1 + s + \sqrt{s(2+s)} \right)^{\frac{1}{2}} \right) \\ &= \frac{|\omega|}{8\pi^2} j_\ell \left(|\omega| r \left(1 + s - \sqrt{s(2+s)} \right)^{\frac{1}{2}} \right) h_\ell^{(1)} \left(|\omega| r \left(1 + s + \sqrt{s(2+s)} \right)^{\frac{1}{2}} \right), \end{aligned} \tag{4.22}$$

where we defined $s \equiv \frac{(\Delta r)^2}{2r^2}$ and used the definition of $Q^2 \equiv 1 - \cos\gamma$. The integral in the third line was calculated using Eq. 2.17.7.6 from reference [39][1]:

$$\int_{-a}^{a} \frac{1}{\sqrt{x+z}} \left\{ \begin{matrix} \sin(b\sqrt{x+z}) \\ \cos(b\sqrt{x+z}) \end{matrix} \right\} P_\ell\left(\frac{x}{a}\right) \mathrm{d}x = \\ (-1)^\ell \pi \sqrt{2a}\, J_{\ell+1/2}(by_-) \left\{ \begin{matrix} J_{\ell+1/2}(by_+) \\ -Y_{\ell+1/2}(by_+) \end{matrix} \right\}, \tag{4.23}$$

(where $\sqrt{2}y_\pm \equiv (z \pm \sqrt{z^2-a^2})^{1/2}$), which is valid for positive a, $z > a$, real b and integer ℓ. The $J_\ell(x)$ and $Y_\ell(x)$ are the Bessel functions of the first and second kind, respectively. In our case in Eq. (4.22), we use this formula with $a = 1$, $b = |\omega| r\sqrt{2}$, and $z = 1 + s$. We also used the well-known relationship between the Bessel functions and the Hankel function of the first kind, $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ for any $\nu \in \mathbb{C}$, and finally the relationship between these functions and their spherical counterparts, $j_\ell(x)$ and $h_\ell^{(1)}(x)$. In Appendix A.2 we prove that the ℓ-sum and ω-integral of the modes $g_{\ell\omega}^{(-2)}(r, r')$ recover the original function $G_{\rm S}^{(-2)}(x, x')$, as expected.

For $\Delta r = 0$ (in the radial coincidence limit), we have

$$g_{\ell\omega}^{(-2)}(r, r) = \frac{|\omega|}{8\pi^2} j_\ell(|\omega| r) h_\ell^{(1)}(|\omega| r). \tag{4.24}$$

Now we turn to $g_{\ell\omega}^{(n)}$, $n > -2$. In order to obtain it, we first notice the following relationship, which can be seen from Eq. (4.19):

$$G_{\rm S}^{(n-2)}(x, x') = \frac{(r\Delta r)^n (-1)^n}{2^n n!} \left(\frac{\partial}{\partial r} \frac{1}{r}\right)^n G_{\rm S}^{(-2)}(x, x'), \qquad n = 1, 2, \ldots \tag{4.25}$$

The modes $g_{\ell\omega}^{(n)}(r, r')$ can be obtained from this relationship by using Eq. (3.22). We note that the r-derivative commutes with the t- and γ-integrals, so the equivalent relationship

[1] We have noticed a minor error in the formula presented in Ref. [39], which is corrected in here: a negative sign before the Bessel function of the second kind $Y_{\ell+1/2}(by_+)$. We checked this correction both numerically and via inverting the formula (i.e., solving for the integrand) using Eqs. (4.7) and (4.9).

to Eq. (4.25) must hold for $g^{(n)}_{\ell\omega}(r,r')$:

$$g^{(n-2)}_{\ell\omega}(r,r') = \frac{(r\Delta r)^n(-1)^n}{2^n n!}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)^n g^{(-2)}_{\ell\omega}(r,r'), \qquad n = 1,\, 2,\, \ldots \tag{4.26}$$

Let us carry out Eq. (4.26) explicitly for $n = 1$. Defining

$$q_\pm \equiv r(1 + s \pm \sqrt{s(2+s)})^{\frac{1}{2}}, \tag{4.27}$$

we have that

$$\begin{aligned} g^{(-1)}_{\ell\omega}(r,r') &= -\frac{r\Delta r}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}g^{(-2)}_{\ell\omega}(r,r')\right) \\ &= -\frac{|\omega| r\Delta r}{16\pi^2}\frac{\partial}{\partial r}\left(\frac{1}{r}j_\ell(|\omega|q_-)h^{(1)}_\ell(|\omega|q_+)\right) \\ &= -\frac{|\omega| r\Delta r}{16\pi^2}\left\{-\frac{1}{r^2}j_\ell(|\omega|q_-)h^{(1)}_\ell(|\omega|q_+) + \right. \\ &\qquad \left. + \frac{1}{r}\left(\frac{\mathrm{d}j_\ell(|\omega|q_-)}{\mathrm{d}q_-}h^{(1)}_\ell(|\omega|q_+)\frac{\partial q_-}{\partial r} + j_\ell(|\omega|q_-)\frac{\mathrm{d}h^{(1)}_\ell(|\omega|q_+)}{\mathrm{d}q_+}\frac{\partial q_+}{\partial r}\right)\right\}. \end{aligned} \tag{4.28}$$

We can see that

$$\frac{\partial q_\pm}{\partial r} = \frac{q_\pm}{r} - \frac{r}{q_\pm}\left(s \pm \frac{\sqrt{s}(1+s)}{\sqrt{2+s}}\right). \tag{4.29}$$

In the radial coincidence limit ($\Delta r \to 0$), we have that $s \to 0$, $q_\pm \to r$, and, from Eq. (4.29), $\frac{\partial q_\pm}{\partial r} \to 1$. Thus, substituting into Eq. (4.28), we find that, in the radial coincidence limit,

$$g^{(-1)}_{\ell\omega}(r,r) = 0. \tag{4.30}$$

In fact, it is easy to check that the same is true for all $g^{(n)}_{\ell\omega}$, for $n \geq -1$, i.e., $g^{(n)}_{\ell\omega}(r,r) = 0$. Therefore, we see that only $g^{(-2)}_{\ell\omega}$ contributes to the renormalization of $\langle M|\hat{\varphi}^2(x)|M\rangle$, in Eq. (3.24).

4.1.3 Renormalization of $\langle\hat{\varphi}^2(x)\rangle$ in flat space-time

Now that we have obtained $f_{\ell\omega}$ and $g^{(n)}_{\ell\omega}$, the next step is to carry out the ℓ-sum and ω-integral in Eq. (3.24). Substituting Eqs. (4.6) and (4.22) into Eq. (3.24), we find that

$$\begin{aligned}\langle M|\hat{\varphi}^2(x)|M\rangle_{\rm ren} &= \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}(2\ell+1)\left\{f_{\ell\omega}(r,r) - 2\sum_{n=-2}^{\infty} g^{(n)}_{\ell\omega}(r,r)\right\}. \\ &= \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}(2\ell+1)\,\omega\left(j_\ell(\omega r)j_\ell(\omega r) - j_\ell(\omega r)h^{(1)}_\ell(\omega r)\right).\end{aligned} \tag{4.31}$$

(We remind the reader that, in the radial coincidence limit, $g^{(n)}_{\ell\omega}(r,r) = 0$ for $n \geq -1$). It is instructive to separate the real and imaginary parts of Eq. (4.31) – remember that $h^{(1)}_\ell(\omega r) \equiv j_\ell(\omega r) + iy_\ell(\omega r)$, and that j_ℓ and y_ℓ are real (for integer ℓ and real argument):

$$\mathrm{Re}\left(\langle M|\hat{\varphi}^2(x)|M\rangle_{\rm ren}\right) = \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}(2\ell+1)\,\omega\Big(j_\ell(\omega r)j_\ell(\omega r) - j_\ell(\omega r)j_\ell(\omega r)\Big), \tag{4.32}$$

$$\mathrm{Im}\left(\langle M|\hat{\varphi}^2(x)|M\rangle_{\rm ren}\right) = -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}(2\ell+1)\,\omega\, j_\ell(\omega r)y_\ell(\omega r). \tag{4.33}$$

The ℓ-sum in Eq. (4.33) is clearly divergent, as can be seen from Eq. (4.9) with $\gamma = 0$ and $|r-r'| \to 0^+$. Let us investigate the source of this divergence. Consider the Feynman Green function $G_F(x,x')$ given by Eq. (4.12). Note that, as we remarked before, in flat space-time $G_F(x,x')$ is equal to $G_{\rm S}(x,x')$. Via integrating over t and γ, similarly to Eqs.(4.21) and (4.22), one may show that $G_F(x,x')$ is decomposed in frequency and multipole modes according to

$$\begin{aligned}G_F(x,x') &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}\omega\, e^{-\varepsilon|\omega|}\, e^{-i\omega\Delta t}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)\frac{i|\omega|}{8\pi^2}j_\ell(|\omega|r_<)\, h^{(1)}_\ell(|\omega|r_>) \\ &= \lim_{\varepsilon\to 0^+}\int_0^{\infty}\mathrm{d}\omega\, e^{-\varepsilon\omega}\,\cos\omega\Delta t\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)\frac{i\omega}{4\pi^2}j_\ell(\omega r_<)\, h^{(1)}_\ell(\omega r_>),\end{aligned} \tag{4.34}$$

where $r_< \equiv \min(r,r')$ and $r_> \equiv \max(r,r')$. Note that we have already included the '$i\varepsilon$' prescription by introducing a factor $e^{-\varepsilon|\omega|}$.

In the derivation of Eq. (3.24), we used Eq. (3.7), i.e., we assumed that $G_+(x,x') =$

$-iG_F(x,x')$ since we consider $\Delta t > 0$. We shall now see that the spurious divergence in Eq. (4.33) is an artifact of that assumption. Indeed, consider the Wightman function $G_+(x,x')$ given by Eq. (4.10), which may be decomposed in frequency and multipole modes according to Eq. (4.5). Introducing the '$i\varepsilon$' prescription, that may be expressed as:

$$G_+(x,x') = \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\omega\, e^{-\varepsilon\omega}\, e^{-i\omega\Delta t} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\gamma) \frac{\omega}{4\pi^2} j_\ell(\omega r)\, j_\ell(\omega r'). \tag{4.35}$$

Let us calculate $(G_+ + iG_F)$ explicitly. The ℓ-sum may be performed using Eqs.(4.7) and (4.9). The real part yields:

$$\begin{aligned} \mathrm{Re}(G_+(x,x') + iG_F(x,x')) &= \frac{1}{4\pi^2 R} \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\omega\, e^{-\varepsilon\omega} \Big(\cos(\omega\Delta t)\sin(\omega R) - \\ &\qquad - \cos(\omega\Delta t)\sin(\omega R) \Big) \\ &= 0, \end{aligned} \tag{4.36}$$

where R is defined by Eq. (4.8). The imaginary part yields:

$$\begin{aligned} \mathrm{Im}(G_+(x,x') + iG_F(x,x')) &= \frac{1}{4\pi^2 R} \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\omega\, e^{-\varepsilon\omega} \Big(\cos(\omega\Delta t)\cos(\omega R) - \\ &\qquad - \sin(\omega\Delta t)\sin(\omega R) \Big) \\ &= \frac{1}{4\pi^2 R} \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\omega\, \left(e^{i\omega(\Delta t + R + i\varepsilon)} + e^{i\omega(-\Delta t - R + i\varepsilon)} \right) \\ &= \frac{i}{4\pi^2 R} \lim_{\varepsilon\to 0^+} \left(\frac{1}{\Delta t + R + i\varepsilon} + \frac{1}{-\Delta t - R + i\varepsilon} \right) \\ &= \frac{1}{2\pi R} \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi\left[(\Delta t + R)^2 + \varepsilon^2\right]} \\ &= \frac{1}{2\pi R}\, \delta(\Delta t + R), \end{aligned} \tag{4.37}$$

where we have used the Dirac delta representation [40]:

$$\delta(x) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}. \tag{4.38}$$

Now, we can clearly see that $\delta(\Delta t + R)$ is zero for $\Delta t > 0$ (since R is always positive),

which shows that $\mathrm{Im}(G_+(x,x') + iG_F(x,x')) = 0$ for $\Delta t > 0$. The problem arises when one takes $\Delta t = R = 0$, in which case the Dirac delta distribution diverges. However, as one takes the coincidence limit $x' \to x$, we find $\lim_{x'\to x} \delta(\Delta t + R) = 0$. Moreover, one sees that the last line of Eq. (4.37) is equal to the advanced Green function $G_{\mathrm{adv}}(x,x')$ (apart from a factor of 'i'). This is expected, since we may show from Eqs. (1.26), (1.23) and (1.24) that

$$\begin{aligned} iG_F(x,x') &= -G_+(x,x')\theta(\Delta t) - G_-(x,x')\theta(-\Delta t) \\ G_+(x,x') &= G_+(x,x')(\theta(\Delta t) + \theta(-\Delta t)) \\ \Rightarrow G_+(x,x') + iG_F(x,x') &= (G_+(x,x') - G_-(x,x'))\,\theta(-\Delta t) \\ &= i\,G(x,x')\theta(-\Delta t) \\ &= i\,G_{\mathrm{adv}}(x,x'). \end{aligned} \tag{4.39}$$

Note that $G_{\mathrm{adv}}(x,x')$ is zero for $\Delta t > 0$. Similarly, if we had started with the assumption $\Delta t < 0$, we would have found that $(G_- + iG_F)$ is proportional to the retarded Green function, which is zero for $\Delta t < 0$.

This discussion shows that we should take extra care before taking the coincidence limit ($x' \to x$) inside the ω-integral and ℓ-sum. In order to understand this, let us recapitulate the steps that were performed in order to arrive at Eq. (4.33). Starting from Eq. (3.1), we assumed $\Delta t > 0$, so that $G_F = iG_+$, and

$$\begin{aligned} \langle\psi|\hat{\varphi}^2(x)|\psi\rangle_{\mathrm{ren}} &= -i \lim_{x'\to x} \Big(G_F(x,x') - G_{\mathrm{S}}(x,x')\Big) \\ &= \lim_{x'\to x} \Big(G_+(x,x') + iG_{\mathrm{S}}(x,x')\Big) \quad \text{for } \Delta t > 0. \end{aligned} \tag{4.40}$$

Following the steps described in chapter 3, from Eq. (4.40) we then arrive at

$$\begin{aligned}\mathrm{Im}\left(\langle M|\hat{\varphi}^2(x)|M\rangle_{\mathrm{ren}}\right) &= \lim_{\substack{x'\to x\\ \varepsilon\to 0^+}} \mathrm{Im}\left\{\int_0^\infty \mathrm{d}\omega e^{-\varepsilon\omega}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)\times\right.\\ &\qquad\left.\times\left(e^{-i\omega\Delta t}f_{\ell\omega}(r,r') - 2\cos(\omega\Delta t)\sum_{n=-2}^{\infty} g^{(n)}_{\ell\omega}(r,r')\right)\right\} \qquad (4.41)\\ &= \frac{1}{4\pi^2 R}\lim_{\substack{x'\to x\\ \varepsilon\to 0^+}}\int_0^\infty \mathrm{d}\omega\, e^{-\varepsilon\omega}\Big(\cos(\omega\Delta t)\cos(\omega R) -\\ &\qquad - \sin(\omega\Delta t)\sin(\omega R)\Big).\end{aligned}$$

From Eq. (4.37) we see that the result of this is

$$\mathrm{Im}\left(\langle M|\hat{\varphi}^2(x)|M\rangle_{\mathrm{ren}}\right) = \lim_{x'\to x}\frac{\delta(\Delta t+R)}{2\pi R} = 0. \qquad (4.42)$$

The step which was mistaken in the derivation of Eq. (4.33) was taking the limit inside the integral and sum in Eq. (4.41). Indeed, by doing this we recover Eq. (4.33), which is divergent. Taking the limit inside the integral and sum is equivalent to setting $\Delta t = R = 0$ directly, which is of course wrong, since we assumed $\Delta t > 0$ in the beginning in order to state that $G_F = iG_+$. In that sense, this divergence may be seen, schematically, as $\frac{\delta(\Delta t+R)}{2\pi R}\Big|_{\Delta t=R=0}$. This means that, for the imaginary part, we were not allowed to take the limit inside the integral and sum; if we do not do that, we would then obtain the correct result given by Eq. (4.42), not Eq. (4.33). This result that the imaginary part of $\langle M|\hat{\varphi}^2(x)|M\rangle_{\mathrm{ren}}$ is null is expected, since every expectation value of any Hermitian operator is real.

The above discussion applies to the imaginary part. For the real part no problem arises. This can be understood from Eq. (4.39): the difference between G_+ and $-iG_F$ is purely imaginary (since the advanced Green function is always real), thus for the real part we can change between G_+ and $-iG_F$ seamlessly. Note that this statement is not specific to flat space-time, since Eq. (4.39) holds in general space-times[2].

Finally, now that we have seen that the imaginary part of $\langle M|\hat{\varphi}^2(x)|M\rangle_{\mathrm{ren}}$ is zero, we

[2]Provided basic assumptions in order to formulate a field theory and define Green functions, such as global hyperbolicity (for more details, see standard textbooks, e.g. Ref. [18]).

are left with the real part. But Eq. (4.32) clearly shows that the real part is also zero. That immediately yields the final result:

$$\langle M|\hat{\varphi}^2(x)|M\rangle_{\rm ren} = 0, \tag{4.43}$$

as we have already noted in the beginning of subsection 4.1.2.

4.2 Application to Bertotti-Robinson space-time

Bertotti-Robinson space-time is a particularly useful space-time as a second check of our method because most of the calculations in it can be done in closed form. Indeed, the renormalization of the stress-energy tensor in this space-time has already been carried out analytically by Ottewill and Taylor [41], which means that we can compare our results against previous literature. In other words, we already know the end results before the numerical implementation itself. We would not have this guidance if we tested our method directly in Schwarzschild space-time, for instance, since the unrenormalized modes are not known in closed form in that case.

4.2.1 The Bertotti-Robinson space-time

Let us first describe briefly the properties of Bertotti-Robinson space-time. We start from Einstein's equation[3] coupled to the electromagnetic stress-energy tensor,

$$R_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{4.44}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor satisfying the source-free Maxwell's equations in curved space-time,

$$\nabla_\mu F^{\mu\nu} = 0; \qquad \partial_{[\mu}F_{\nu\rho]} = 0, \tag{4.45}$$

[3]Note that the electromagnetic stress-energy tensor is traceless. As a consequence, the curvature scalar is zero in electrovacuum solutions, and the left-hand side of Einstein's equation is given just by $R_{\mu\nu}$.

and A_μ is the electromagnetic 4-potential. The Bertotti-Robinson space-time [42, 43] is a solution of this system of equations (the so-called Einstein-Maxwell's equations) with a uniform electromagnetic field (i.e., the covariant derivatives of $F_{\mu\nu}$ are null). This space-time has a geometry given by the product space $AdS_2 \times S_2$ (where AdS_2 is two-dimensional anti-de Sitter space-time and S_2 is the two-dimensional sphere). The metric may be expressed as

$$\mathrm{d}s^2 = \frac{e^2}{\tilde{r}^2}\left(-\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\theta^2 + \tilde{r}^2\sin^2\theta\mathrm{d}\phi^2\right), \tag{4.46}$$

for some constant $e > 0$ with units of length, where the coordinates range over $\tilde{r} \in [0, \infty)$, $\tilde{t} \in (-\infty, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. In these coordinates, the components of the electromagnetic field are given by

$$F_{\tilde{t}}^{\ \tilde{r}} = F_{\tilde{r}}^{\ \tilde{t}} = \frac{\sqrt{2}}{e}, \tag{4.47}$$

and all the other components are null. Note that the metric (4.46) is conformally flat. An important consequence of the conformal flatness is that the biscalar $V(x, x')$ in the Hadamard form (3.8) vanishes for a conformal field in Bertotti-Robinson space-time.

Throughout this work, we will use the metric in the form

$$\mathrm{d}s^2 = e^2\left(-(\rho^2-1)\mathrm{d}t^2 + (\rho^2-1)\mathrm{d}\rho^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{4.48}$$

where the coordinates range over $\rho \in [1, \infty)$, $t \in (-\infty, \infty)$. This metric is obtained from Eq. (4.46) after the change of variables

$$\begin{aligned} \tilde{t} &= \frac{(\rho^2-1)^{1/2}\sinh t}{\rho + (\rho^2-1)^{1/2}\cosh t}, \\ \tilde{r} &= \frac{1}{\rho + (\rho^2-1)^{1/2}\cosh t}. \end{aligned} \tag{4.49}$$

Let us now build the Penrose diagram for Bertotti-Robinson space-time. The space

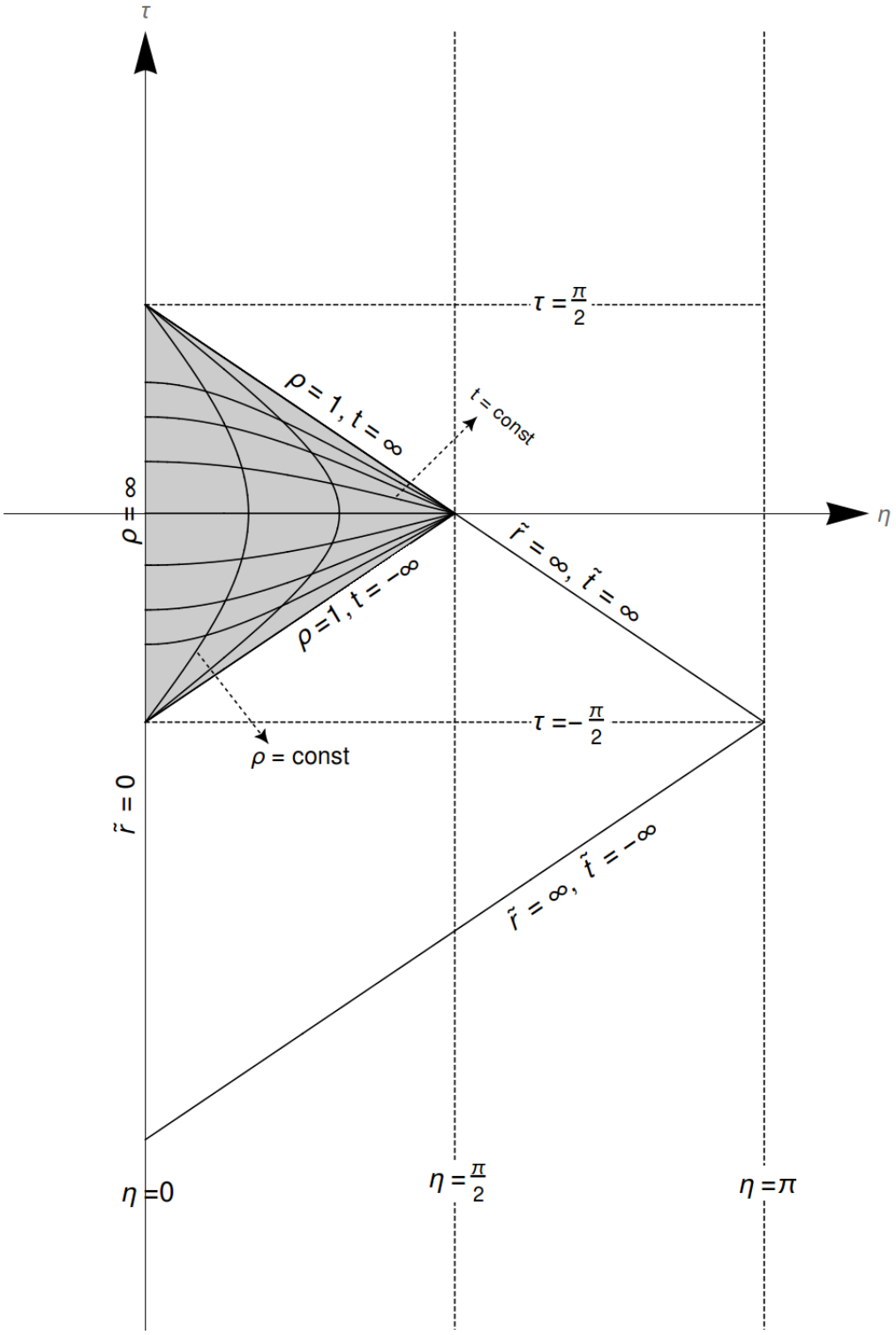

Figure 4.1: Penrose diagram for Bertotti-Robinson space-time. The shaded region is covered by the (ρ, t) coordinates. The curves of constant ρ and t are shown. Note that the shaded region is restricted to $\tau \in [-\pi/2, \pi/2]$.

may be compactified by the change of coordinates

$$\begin{aligned} \sin\tau &= \frac{\sqrt{\rho^2-1}\sinh t}{\sqrt{1+(\rho^2-1)\cosh^2 t}}, \\ \cot\bar{\eta} &= \sqrt{\rho^2-1}\cosh t. \end{aligned} \tag{4.50}$$

Such transformation yields the metric

$$\mathrm{d}s^2 = \frac{e^2}{\sin^2\bar{\eta}}\big(-\mathrm{d}\tau^2 + \mathrm{d}\bar{\eta}^2 + \sin^2\bar{\eta}\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\big), \tag{4.51}$$

which is conformal to the Einstein static universe. The full space-time (i.e., the maximal extension) is covered by $\tau \in (-\infty, \infty)$ and $\bar{\eta} \in [0, \pi]$. Note that this range of the τ coordinate gives the *universal covering* of anti-de Sitter space-time, while the time coordinate in AdS is periodic. The coordinates (ρ, t) and $(\tilde{t}, \tilde{r})$, on the other hand, do not cover the whole space-time.

The Penrose diagram for the extended space-time is shown in Fig. 4.1. The curves of constant ρ and constant t are also plotted in the figure. From Eq. (4.50), it is easy to show that these curves are given by the equations $\sin\tau = \pm\big(1-\rho^2\sin^2\bar{\eta}\big)^{1/2}$ and $\sin\tau = \cos\bar{\eta}\tanh t$, respectively. Also note that outgoing radial null geodesics are given by curves of constant U, and ingoing radial null geodesics are given by curves of constant V, where $U \equiv \tau - \bar{\eta}$ and $V \equiv \tau + \bar{\eta}$ are null coordinates.

In the rest of this section, we shall consider only the region covered by the (ρ, t) coordinates, which is the shaded region in Fig. 4.1.

It is also noteworthy that Bertotti-Robinson space-time may be obtained as the near-horizon limit of an extremal Reissner-Nordström black hole [44].

Now, following the method described in Chapter 3, in order to renormalize $\langle\hat{\varphi}^2(x)\rangle$ we need the Wightman function in a given state (Boulware, in our case) and the Hadamard form for the Feynman Green function.

4.2.2 Wightman function in the Boulware vacuum

In this subsection, we review the Wightman function in the Boulware vacuum,[4] as obtained in Ottewill and Taylor [41].

Let us start with a massless real scalar field $\varphi(x)$. The field equation is simply $\Box\varphi = 0$, with $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, and $g_{\mu\nu}$ is the Bertotti-Robinson metric (4.48) [5] (from now on, we particularize to the case $e = 1$).

Now, the different vacua are distinguished by the decomposition of the field into positive- and negative-frequency modes. In particular, the Boulware vacuum $|B\rangle$ is characterized by a decomposition in modes which are positive-frequency with respect to the time coordinate t. Therefore, we use coordinates t, ρ, θ and ϕ, with the metric given by (4.48). In this coordinate system, the field equation becomes

$$\left\{-\frac{1}{\rho^2-1}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial\rho}\left((\rho^2-1)\frac{\partial}{\partial\rho}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\varphi = 0, \quad (4.52)$$

and the field can be separated – similarly to (3.15)[6] – in a basis given by

$$\varphi_{\ell m\omega}(x) = \mathcal{N}_{\ell\omega}\, e^{-i\omega t} Y_\ell^m(\theta,\phi)\, R_{\ell\omega}(\rho). \quad (4.53)$$

These modes, when subjected to the boundary conditions given below, form a complete set of solutions for the field equation (together with their complex conjugates). Again, there is no need for the index j defined in Eq. (3.15).

Substituting these modes into the field equation, we find an equation for the radial part $R_{\ell\omega}(\rho)$, namely

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\rho}\left((\rho^2-1)\frac{\mathrm{d}}{\mathrm{d}\rho}\right) + \frac{\omega^2}{\rho^2-1} - \ell(\ell+1)\right\} R_{\ell\omega}(\rho) = 0. \quad (4.54)$$

[4] The vacuum state which is defined from modes which are positive-frequency with respect to the coordinate t is called *Boulware vacuum*, in analogy to the case of Schwarzschild space-time. Such state is a zero-temperature state, and the stress-energy tensor diverges on the "horizon" given by $\rho = 1$.

[5] Note that, being a real scalar field (and therefore uncharged), $\varphi(x)$ does not couple directly to the electromagnetic field, despite the electromagnetic field being non-null in the Bertotti-Robinson solution (e.g., Eq. (4.47)).

[6] Remember that Eq. (3.3) refers to the quantized field operator $\hat{\varphi}(x)$. For the classical field $\varphi(x)$, we have an analogous decomposition, but with the operators replaced with complex functions.

This is an associated Legendre equation, whose general solution is a linear combination of $P_\ell^{i\omega}(\rho)$ and $Q_\ell^{i\omega}(\rho)$ (associated Legendre functions of the first and second kind, respectively). The purely imaginary order comes from the positive sign in the second term of the radial equation.

Remember that we are using coordinates in which $\rho > 1$. Accordingly, the associated Legendre functions are defined with a branch cut along $\rho \in (-\infty, 1]$ on the real axis, instead of the traditional $(-\infty, -1] \cup [1, \infty)$. These definitions can be found in [45]. By requiring that the field vanishes at spatial infinity as a boundary condition, the $P_\ell^{i\omega}(\rho)$ is ruled out, so that $R_{\ell\omega}(\rho) \propto Q_\ell^{i\omega}(\rho)$. Finally, we normalize it with respect to the Klein-Gordon inner product, finding

$$\mathcal{N}_{\ell\omega} R_{\ell\omega}(x) = \frac{1}{\pi} e^{\omega\pi} \sqrt{\sinh \omega\pi}\, Q_\ell^{i\omega}(\rho). \tag{4.55}$$

Now we can substitute this result into Eq. (3.19) in order to obtain the Wightman function in the Boulware vacuum:

$$\begin{aligned} G_+(x, x') = \frac{1}{4\pi^3} \int_0^\infty \mathrm{d}\omega\, e^{-i\omega\tau} e^{2\omega\pi} \sinh(\omega\pi) \times \\ \times \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\gamma) Q_\ell^{i\omega}(\rho) \left[Q_\ell^{i\omega}(\rho')\right]^*, \end{aligned} \tag{4.56}$$

where $\tau \equiv t - t'$. One can yet rewrite Eq. (4.56) as

$$\begin{aligned} G_+(x, x') = \frac{i}{8\pi^2} \int_0^\infty \mathrm{d}\omega\, e^{-i\omega\tau} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\gamma) \times \\ \times \left(e^{\omega\pi} P_\ell^{-i\omega}(\rho_<) Q_\ell^{i\omega}(\rho_>) - e^{-\omega\pi} P_\ell^{i\omega}(\rho_<) Q_\ell^{-i\omega}(\rho_>) \right), \end{aligned} \tag{4.57}$$

where $\rho_<$ is the smallest of ρ and ρ', and $\rho_>$ is the greatest. Hence we have written $G_+(x, x')$ in the form of Eq. (3.5), and we can read off

$$f_{\ell\omega}(\rho, \rho') = \frac{i}{8\pi^2} \left(e^{\omega\pi} P_\ell^{-i\omega}(\rho_<) Q_\ell^{i\omega}(\rho_>) - e^{-\omega\pi} P_\ell^{i\omega}(\rho_<) Q_\ell^{-i\omega}(\rho_>) \right). \tag{4.58}$$

This function is purely real when $\rho' = \rho$, as can be seen from Eq. (4.56).

For later use, we now obtain the closed form of the Wightman function in the Boulware vacuum. Ref. [41] derives a non-standard summation formula for the ℓ-sum in Eq. (4.57), namely:

$$\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)e^{-i\mu\pi}P_\ell^{-\mu}(\rho_<)Q_\ell^{\mu}(\rho_>)=\frac{e^{-\mu\beta}}{\widetilde{R}^{\frac{1}{2}}}, \tag{4.59}$$

where $\mu\in\mathbb{C}$ and

$$\begin{aligned}\cosh\beta &\equiv \eta\eta'(\rho\rho'-\cos\gamma)\\ \widetilde{R} &\equiv \rho^2+\rho'^2-2\rho\rho'\cos\gamma-\sin^2\gamma\\ &= \left(\frac{\sinh\beta}{\eta\eta'}\right)^2\end{aligned} \tag{4.60}$$

where we have defined

$$\eta\equiv\frac{1}{\sqrt{\rho^2-1}}, \tag{4.61}$$

and similarly for η'.

Eq. (4.59) allows us to perform the ℓ-sum in Eq. (4.57) and obtain the closed form of the Wightman function in the Boulware vacuum. Note that, according to the Feynman prescription, we add a small imaginary part 'iε' to the exponent, as we have done in Eq. (4.10):

$$\begin{aligned}G_+(x,x') &= \frac{1}{4\pi^2\widetilde{R}^{\frac{1}{2}}}\lim_{\varepsilon\to0^+}\int_0^\infty \mathrm{d}\omega\, e^{-i\omega(\tau-i\varepsilon)}\sin(\omega\beta)\\ &= \frac{1}{4\pi^2\widetilde{R}^{\frac{1}{2}}}\lim_{\varepsilon\to0^+}\frac{\beta}{-(\tau-i\varepsilon)^2+\beta^2}.\end{aligned} \tag{4.62}$$

With the aid of Eq. (3.6), that allows us to write the Feynman Green function in the Boulware vacuum,

$$\begin{aligned}G_{\mathrm{F}}(x,x') &= \frac{i}{4\pi^2\widetilde{R}^{\frac{1}{2}}}\lim_{\varepsilon\to0^+}\frac{\beta}{-(|\tau|-i\varepsilon)^2+\beta^2}\\ &= \frac{i}{4\pi^2\widetilde{R}^{\frac{1}{2}}}\lim_{\varepsilon\to0^+}\frac{\beta}{-\tau^2+\beta^2+i\varepsilon},\end{aligned} \tag{4.63}$$

with a redefinition of ε.

4.2.3 Singular part of the Hadamard form in Bertotti-Robinson space-time

Now we proceed to obtain the singular part of the Hadamard form for the Feynman Green function in Bertotti-Robinson space-time.

We start from Eq. (3.8). For a conformal scalar field in a conformally flat space-time, which, as stated before, is the case of Bertotti-Robinson, one has $V(x,x') \equiv 0$. Moreover, since it has a structure of $AdS_2 \times S^2$, the world function can be written as a sum of the world function for these two subspaces [47], namely

$$\sigma(x,x') = \frac{\lambda^2}{2} + \frac{\gamma^2}{2}, \tag{4.64}$$

where λ is the geodesic distance on AdS_2, given by

$$\cosh\lambda = \rho\rho' - (\rho^2-1)^{\frac{1}{2}}(\rho'^2-1)^{\frac{1}{2}}\cosh(t'-t) \tag{4.65}$$

and γ is the geodesic distance on the sphere S^2, given by Eq. (3.18). Similarly, the van Vleck-Morette determinant can be written as a product of the determinants for the AdS_2 and S^2 parts, so that, in the end, the singular part of the Hadamard form is

$$G_{\rm S}(x,x') = \lim_{\varepsilon\to 0^+}\left\{\frac{i}{4\pi^2}\left(\frac{\lambda}{\sinh\lambda}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\sin\gamma}\right)^{\frac{1}{2}}\frac{1}{\lambda^2+\gamma^2+i\varepsilon}\right\}. \tag{4.66}$$

Following the proposed method, the next step is to expand the singular part of the Hadamard form in terms of the coordinate separations, $\tau \equiv t-t'$, $\Delta \equiv \rho-\rho'$, and γ. As in the case of flat space-time, we find it useful to expand in terms of $Q = \sqrt{1-\cos\gamma}$ rather than γ. This expansion is performed in the following way: first, we rewrite λ as a function of τ and Δ in Eq. (4.65),

$$\lambda = \operatorname{arccosh}\left(\rho(\rho+\Delta) - (\rho^2-1)^{\frac{1}{2}}[(\rho+\Delta)^2-1]^{\frac{1}{2}}\cosh\tau\right), \tag{4.67}$$

and write γ in terms of Q,

$$\gamma = \arccos(1 - Q^2). \tag{4.68}$$

Then, as described in Chapter 3, we multiply τ, Δ and Q by an auxiliary parameter δ (and multiply ε by δ^2). Next, we expand for small δ. Remember that we only need to keep the terms up to order 0 in δ. Each one of the coefficients of this series is collected as one of the functions $G_{\mathrm{S}}^{(n)}$, as in Eq. (3.9). The result is the following:

$$G_{\mathrm{S}}^{(-2)}(\tau,\eta,\Delta,Q) = \frac{i\eta^2}{4\pi^2}\,\frac{1}{-\tau^2+\eta^2\left(\Delta^2\eta^2+2Q^2+i\varepsilon\right)}, \tag{4.69}$$

$$G_{\mathrm{S}}^{(-1)}(\tau,\eta,\Delta,Q) = \frac{i\eta^3}{4\pi^2}\,\frac{\Delta\sqrt{\eta^2+1}\left(\Delta^2\eta^4+\tau^2\right)}{\left(-\tau^2+\eta^2\left(\Delta^2\eta^2+2Q^2+i\varepsilon\right)\right)^2}, \tag{4.70}$$

$$\begin{aligned} G_{\mathrm{S}}^{(0)}(\tau,\eta,\Delta,Q) = {} & \frac{1}{48\pi^2\left(-\tau^2+\eta^2\left(\Delta^2\eta^2+2Q^2+i\varepsilon\right)\right)^3}\times \\ & \Big\{\eta^2\left(\tau^4\left(i\eta^2\left(\Delta^2\left(19\eta^2+12\right)+2Q^2\right)+\varepsilon(1-\eta^2)\right)- \\ & i\eta^2\tau^2\left(-33\Delta^4\eta^6-2i\Delta^2\eta^2\varepsilon+6\Delta^2\eta^4\left(-6\Delta^2+2Q^2+\right.\right. \\ & \left.i\varepsilon\right)+\varepsilon^2\Big)+\eta^4\left(-3i\Delta^6\eta^8+\Delta^4\left(15\eta^2+13\right)\eta^4\varepsilon+\right. \\ & \left.i\Delta^2\eta^2\varepsilon^2-4Q^4\varepsilon-2iQ^2\left(3\Delta^4\left(5\eta^2+4\right)\eta^4+\varepsilon^2\right)\right)-i\tau^6\Big)\Big\}, \end{aligned} \tag{4.71}$$

$$\begin{aligned} G_{\mathrm{S}}^{(1)}(\tau,\eta,\Delta,Q) = {} & \frac{1}{48\pi^2\left(-\tau^2+\eta^2\left(\Delta^2\eta^2+2Q^2+i\varepsilon\right)\right)^4}\times \\ & \Big(\Delta\eta^3\sqrt{\eta^2+1}\left(\varepsilon^3\eta^4\left(\Delta^2\eta^4+\tau^2\right)-i\varepsilon^2\eta^4\left(3\Delta^4\left(7\eta^2+4\right)\eta^6+\right.\right. \\ & 6\Delta^2\eta^4\tau^2+\tau^4\Big)+\varepsilon\left(\Delta^6\left(\eta^{12}-12\eta^{14}\right)+\Delta^4\left(72\eta^2+47\right)\eta^8\tau^2+\right. \\ & \Delta^2\left(20\eta^2-1\right)\eta^4\tau^4+\tau^6\Big)+i\left(3\Delta^8\eta^{16}-12\Delta^6\left(\eta^2-1\right)\eta^{10}\tau^2+\right. \\ & 6\Delta^4\left(13\eta^2+12\right)\eta^6\tau^4+4\Delta^2\left(7\eta^2+3\right)\eta^2\tau^6-\tau^8\Big)+ \\ & 4i\eta^4Q^4\left(3i\varepsilon\left(\Delta^2\eta^4+\tau^2\right)+3\Delta^4\left(7\eta^2+4\right)\eta^6+6\Delta^2\eta^4\tau^2+\right. \\ & \tau^4\Big)+4\eta^2Q^2\left(-2i\varepsilon^2\eta^2\left(\Delta^2\eta^4+\tau^2\right)-\varepsilon\left(\Delta^4\left(21\eta^2+13\right)\eta^8+\right. \\ & 6\Delta^2\eta^6\tau^2+\left(\eta^2-1\right)\tau^4\Big)+2i\Delta^2\eta^4\left(3\Delta^4\eta^8-\right. \\ & \left.\left.\left.6\Delta^2\left(3\eta^2+2\right)\eta^2\tau^2-5\tau^4\right)\right)\right), \end{aligned} \tag{4.72}$$

where η is given by Eq. (4.61). Again, as in the case of flat space-time, note that in Bertotti-Robinson space-time there is no contribution from the logarithmic terms $G_S^{(\log)}$.

Before we continue, we remark that in the radial coincidence limit ($\Delta \to 0$) $g_{\ell\omega}^{(-1)}(\rho,\rho) = 0$ and $g_{\ell\omega}^{(1)}(\rho,\rho) = 0$. That can be seen by explicitly calculating the τ-integral of Eqs. (4.70) and (4.72) (the results are not given here). Consequently, $G_{\rm S}^{(-1)}$ and $G_{\rm S}^{(1)}$ do not contribute to the renormalization. Note that a similar thing happens in flat space-time, in which we have shown that $G_{\rm S}^{(n)}$, for $n \geq -1$, do not contribute to the renormalization because $g_{\ell\omega}^{(n)}(r,r) = 0$, $n \geq -1$ in the radial coincidence limit.

Now, let us obtain $g_{\ell\omega}^{(-2)}$, with the aid of Eq. (3.12). As in the case of flat space-time, we use the residue theorem, closing a semicircular path in the upper τ-complex plane for $\omega > 0$ and in the lower plane for $\omega > 0$:

$$\begin{aligned} G_{\omega}^{(-2)}(\eta,\Delta,Q) &= \frac{i\eta^2}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \frac{e^{i\omega\tau}}{-\tau^2+\eta^2\left(\Delta^2\eta^2+2Q^2+i\varepsilon\right)} \\ &= -\frac{i\eta^2}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \frac{e^{i\omega\tau}}{(\tau-\widetilde{\alpha})(\tau+\widetilde{\alpha})} \\ &= \frac{\eta^2 e^{i|\omega|\widetilde{\alpha}}}{4\pi\widetilde{\alpha}} \\ &= \frac{\eta}{4\pi}\,\frac{e^{i|\omega|\eta\sqrt{\Delta^2\eta^2+2Q^2+i\varepsilon}}}{\sqrt{\Delta^2\eta^2+2Q^2+i\varepsilon}}, \end{aligned} \tag{4.73}$$

in which $\widetilde{\alpha}^2 \equiv \eta^2\left(\Delta^2\eta^2+2Q^2+i\varepsilon\right)$.

The angular integration is also similar to the case of flat space-time. Again, we set $\varepsilon = 0$ (see Appendix A.1) and employ Eq. (4.23):

$$\begin{aligned} g_{\ell\omega}^{(-2)}(\rho,\rho') &= -\frac{i}{4\pi}\int_0^{\pi} \mathrm{d}\gamma \sin\gamma\, P_\ell(\cos\gamma)\, G_{\omega}^{(-2)}(\eta,\Delta,Q) \\ &= -\frac{i\eta}{16\pi^2\sqrt{2}}\int_0^{\pi} \mathrm{d}\gamma \sin\gamma\, P_\ell(\cos\gamma)\, \frac{e^{i\eta|\omega|\sqrt{2}\sqrt{\frac{\Delta^2\eta^2}{2}+1-\cos\gamma}}}{\sqrt{\frac{\Delta^2\eta^2}{2}+1-\cos\gamma}} \\ &= \frac{\eta}{16\pi} J_{\ell+\frac{1}{2}}\left(\eta|\omega|\left(1+\tilde{s}-\sqrt{\tilde{s}\,(2+\tilde{s})}\right)^{\frac{1}{2}}\right)\times \\ &\qquad \times H_{\ell+\frac{1}{2}}^{(1)}\left(\eta|\omega|\left(1+\tilde{s}+\sqrt{\tilde{s}\,(2+\tilde{s})}\right)^{\frac{1}{2}}\right) \\ &= \frac{\eta^2|\omega|}{8\pi^2} j_\ell\left(|\omega|\tilde{q}_-\right) h_\ell^{(1)}\left(|\omega|\tilde{q}_+\right), \end{aligned} \tag{4.74}$$

where we defined $\tilde{s} \equiv \frac{\Delta^2\eta^2}{2}$ and $\tilde{q}_\pm \equiv \eta\left(1+\tilde{s}\pm\sqrt{\tilde{s}\,(2+\tilde{s})}\right)$. In the radial coincidence limit ($\Delta = 0$), Eq. (4.74) gives

$$g^{(-2)}_{\ell\omega}(\rho,\rho) = \frac{\eta^2|\omega|}{8\pi^2} j_\ell(|\omega|\eta) h^{(1)}_\ell(|\omega|\eta). \tag{4.75}$$

Now we turn to $g^{(0)}_{\ell\omega}$. We first have to take extra care of the τ-integral. If we naively try to integrate $e^{i\omega\tau}G^{(0)}_{\rm S}$ as given in Eq. (4.71), we find that the integral diverges. The reason for this divergence can be understood by analysing the behaviour of $G^{(0)}_{\rm S}$ as $|\tau|\to\infty$:

$$G^{(0)}_{\rm S}(\tau,\eta,\Delta,Q) = \frac{i\eta^2}{48\pi^2} + \mathcal{O}(\tau^{-2}). \tag{4.76}$$

The first term in this series is the term responsible for the divergence, as the integrand does not decay for $|\tau|\to\infty$. Indeed, it is independent of τ, hence when substituted into Eq. (3.22) it would give rise to a term proportional to $\delta(\omega)$, which is divergent for $\omega = 0$. We solve this in the following way: first, we define

$$\widetilde{G}^{(0)}_{\rm S}(\tau,\eta,\Delta,Q) \equiv G^{(0)}_{\rm S}(\tau,\eta,\Delta,Q) - iA, \tag{4.77}$$

where $A \equiv \frac{\eta^2}{48\pi^2}$. Now $\widetilde{G}^{(0)}_{\rm S}$ can be substituted into Eq. (3.22). We then define

$$\widetilde{g}^{(0)}_{\ell\omega}(\rho,\rho') \equiv -\frac{i}{4\pi}\int_0^\pi \mathrm{d}\gamma \sin\gamma P_\ell(\cos\gamma)\int_{-\infty}^{\infty}\mathrm{d}\tau\, e^{i\omega\tau}\widetilde{G}^{(0)}_{\rm S}(x,x'). \tag{4.78}$$

With these definitions, Eq. (3.24) becomes

$$\langle B|\hat{\varphi}^2(x)|B\rangle_{\rm ren} = -A + \int_0^\infty \mathrm{d}\omega \sum_{\ell=0}^{\infty}(2\ell+1)\left\{f_{\ell\omega}(\rho,\rho) - 2g^{(-2)}_{\ell\omega}(\rho,\rho) - 2\widetilde{g}^{(0)}_{\ell\omega}(\rho,\rho)\right\} \tag{4.79}$$

(remember that we do not consider $g^{(n)}_{\ell\omega}$ for $n=-1$ and $n>0$ since they do not contribute to the renormalization of $\langle\psi|\hat{\varphi}^2(x)|\psi\rangle$).

Now, the τ-integral in Eq. (4.78) can be performed:

$$\widetilde{G}^{(0)}_{\omega}(\eta,\Delta,Q) \equiv \int_{-\infty}^{\infty}\mathrm{d}\tau\, e^{i\omega\tau}\,\widetilde{G}^{(0)}_{\rm S}(\tau,\eta,\Delta,Q). \tag{4.80}$$

The full result is too large to be shown here. However, we can find a simpler expression by setting $\varepsilon = 0$. Indeed, while the Feynman prescription was needed for the convergence of Eq. (4.80), it is not needed anymore. Besides, since in Eq. (4.79) we only need $\widetilde{g}^{(0)}_{\ell\omega}$ in the radial coincidence limit, we can set $\Delta = 0$. Then, we find

$$\widetilde{G}^{(0)}_{\omega}(\eta, \Delta = 0, Q)\Big|_{\varepsilon=0} = -\frac{\eta^3 Q}{96\pi}\, e^{i\sqrt{2}\eta Q\omega}\left(3\sqrt{2} + 2i\eta Q\omega\right). \tag{4.81}$$

Now, we notice a relationship between $\widetilde{G}^{(0)}_{\omega}$ and $G^{(-2)}_{\omega}$:

$$\widetilde{G}^{(0)}_{\omega}(\eta, \Delta = 0, Q)\Big|_{\varepsilon=0} = \frac{\eta^3}{24\omega^2}\frac{\mathrm{d}^3}{\mathrm{d}\eta^3}\left[G^{(-2)}_{\omega}(\eta, \Delta = 0, Q)\big|_{\varepsilon=0}\right], \tag{4.82}$$

i.e., $\widetilde{G}^{(0)}_{\omega}$ is proportional to the third derivative of $G^{(-2)}_{\omega}$ after taking $\Delta = 0$ and $\varepsilon = 0$. Such relation can be explicitly seen from Eqs. (4.73) and (4.81). That also ties in with the case of flat space-time, in which we have seen that $G^{(n)}_{\mathrm{S}}$, $n \geq -1$, are related to $G^{(-2)}_{\mathrm{S}}$ through Eq. (4.25).

The ρ-derivative commutes with the γ-integral, hence the same relation must hold for $\widetilde{g}^{(0)}_{\ell\omega}$ and $g^{(-2)}_{\ell\omega}$:

$$\widetilde{g}^{(0)}_{\ell\omega}(\rho, \rho) = \frac{\eta^3}{24\omega^2}\frac{\mathrm{d}^3}{\mathrm{d}\eta^3}\left(g^{(-2)}_{\ell\omega}(\rho, \rho)\right), \tag{4.83}$$

where $g^{(-2)}_{\ell\omega}(\rho, \rho)$ is given by (4.75). This allows us to obtain the angular integral of $\widetilde{G}^{(0)}_{\omega}(\eta, \Delta = 0, Q)\Big|_{\varepsilon=0}$, even if we do not know how to calculate it directly.

4.2.4 Asymptotic behaviour of the modes

Before we continue with the renormalization, it is interesting to briefly analyse the asymptotic behaviour of the modes obtained, given by Eqs. (4.58) and (4.75).

We start with $f_{\ell\omega}$ in Eq. (4.58). The asymptotic behaviour of the associated Legendre functions as $\ell \to \infty$ is given by (see [45], Eqs. 14.15.13 and 14.15.14, together with 10.40.1

and 10.25.3):[7]

$$
\begin{aligned}
P_\ell^{-i\omega}(\cosh\xi) &\sim \frac{1}{\ell^{i\omega}}\frac{1}{\sqrt{\sinh\xi}}\frac{e^{\left(\ell+\frac{1}{2}\right)\xi}}{\sqrt{\pi\,(2\ell+1)}} \\
Q_\ell^{i\omega}(\cosh\xi) &\sim \ell^{i\omega}e^{-\omega\pi}\frac{1}{\sqrt{\sinh\xi}}\sqrt{\frac{\pi}{2\ell+1}}e^{-\left(\ell+\frac{1}{2}\right)\xi},
\end{aligned}
\tag{4.84}
$$

uniformly for $\xi \in (0,\infty)$. Thus, we find that, for large ℓ, and in the radial coincidence limit ($\rho' = \rho$),

$$
\begin{aligned}
e^{\omega\pi}P_\ell^{-i\omega}(\rho)Q_\ell^{i\omega}(\rho) &\sim \frac{\eta}{2\ell+1} \\
e^{-\omega\pi}P_\ell^{i\omega}(\rho)Q_\ell^{-i\omega}(\rho) &\sim \frac{\eta}{2\ell+1}.
\end{aligned}
\tag{4.85}
$$

However, we have that $f_{\ell\omega}(\rho,\rho) = \frac{i}{8\pi^2}\left(e^{\omega\pi}P_\ell^{-i\omega}(\rho)Q_\ell^{i\omega}(\rho) - e^{-\omega\pi}P_\ell^{i\omega}(\rho)Q_\ell^{-i\omega}(\rho)\right)$. Eq. (4.85) shows that, in the asymptotic limit, the two terms $e^{\omega\pi}P_\ell^{-i\omega}(\rho)Q_\ell^{i\omega}(\rho)$ and $e^{\omega\pi}P_\ell^{i\omega}(\rho)Q_\ell^{-i\omega}(\rho)$ cancel out, at order $(2\ell+1)^{-1}$. In fact, we can see from Eq. (4.56) that $f_{\ell\omega}$ can be rewritten as

$$
f_{\ell\omega}(\rho,\rho') = \frac{1}{4\pi^3}e^{2\omega\pi}\sinh(\omega\pi)Q_\ell^{i\omega}(\rho)\,[Q_\ell^{i\omega}(\rho')]^*. \tag{4.86}
$$

Then, Eq. (4.84) shows that, for large ℓ,

$$
\begin{aligned}
f_{\ell\omega}(\rho,\rho) &\sim \frac{1}{4\pi^2}\sinh(\omega\pi)\frac{\eta}{2\ell+1}e^{-(2\ell+1)\,\mathrm{arccosh}\,\rho} \\
&\sim \frac{1}{4\pi^2}\sinh(\omega\pi)\frac{\eta}{2\ell+1}\left(\rho+\frac{1}{\eta}\right)^{-(2\ell+1)}.
\end{aligned}
\tag{4.87}
$$

We see that $f_{\ell\omega}$ exhibits an exponential decay with ℓ for $\ell \to \infty$. That means that the two terms in Eq. (4.85) must cancel each other in the asymptotic limit to all powers in ℓ.

As for $g_{\ell\omega}^{(-2)}$, we analyse the real and imaginary part separately. The asymptotic behaviour of the spherical Bessel functions as $\ell \to \infty$ is given by (see [45], Eqs. 10.19.1

[7] Note that there are two definitions for the associated Legendre function of the second kind in Ref. [45], Q_ν^μ and $\mathbf{Q}_\nu^\mu$. These are related by $Q_\nu^\mu = \Gamma(\nu+\mu+1)e^{i\pi\mu}\mathbf{Q}_\nu^\mu$. Eq. 14.15.14 in [45] refers to $\mathbf{Q}_\nu^\mu$, but in here we refer to Q_ν^μ in Eq. (4.84). Also note that the asymptotic behaviour of the functions $P_\nu^{-\mu}$ and $\mathbf{Q}_\nu^\mu$, as $\nu \to \infty$, are given by Eqs. 14.15.13 and 14.15.14 in Ref. [45] only for real values of the order μ. However, we checked numerically that it also holds for imaginary values of μ.

and 10.19.2)

$$
\begin{aligned}
j_\ell(\eta|\omega|) &\sim \frac{1}{\sqrt{2\eta|\omega|}}\frac{1}{\sqrt{2\ell+1}}\left(\frac{e\eta|\omega|}{2\ell+1}\right)^{\ell+\frac{1}{2}} \\
y_\ell(\eta|\omega|) &\sim -2\frac{1}{\sqrt{2\eta|\omega|}}\frac{1}{\sqrt{2\ell+1}}\left(\frac{e\eta|\omega|}{2\ell+1}\right)^{-\left(\ell+\frac{1}{2}\right)}.
\end{aligned}
\tag{4.88}
$$

With $g^{(-2)}_{\ell\omega}$ given by Eq. (4.75), we find that, for large ℓ,

$$
\begin{aligned}
\mathrm{Re}(g^{(-2)}_{\ell\omega}(\rho,\rho)) &\sim \frac{\eta}{16\pi^2}\frac{1}{2\ell+1}\left(\frac{e\eta|\omega|}{2\ell+1}\right)^{2\ell+1} \\
\mathrm{Im}(g^{(-2)}_{\ell\omega}(\rho,\rho)) &\sim \frac{\eta}{8\pi^2}\frac{1}{2\ell+1}.
\end{aligned}
\tag{4.89}
$$

Note that both $f_{\ell\omega}$ and the real part of $g^{(-2)}_{\ell\omega}$ decay exponentially for large ℓ. Consequently, we can expect that the real part of the ℓ-sum in Eq. (4.79) is amenable to a numerical calculation, with a finite upper limit for the sum given a certain desired accuracy. On the other hand, the asymptotic behaviour of the imaginary part of $g^{(-2)}_{\ell\omega}(\rho,\rho)$ means that the imaginary part of the ℓ-sum in Eq. (4.79) diverges. However, as in the case of flat space-time (described in subsection 4.1.3), this divergence is inocuous, and in fact the imaginary part of $\langle B|\hat{\varphi}^2(x)|B\rangle_{\mathrm{ren}}$ is zero. Because of this, we include only the real part of the modes in the numerical calculation.

4.2.5 Renormalization of $\langle\hat{\varphi}^2(x)\rangle$ in Bertotti-Robinson space-time

Before we implement the ℓ-sum and ω-integral in Eq. (4.79) numerically, let us first analytically obtain $\langle B|\hat{\varphi}^2(x)|B\rangle_{\mathrm{ren}}$. It is important to have this result beforehand, so that we can check the output of the numerical implementation.

Note that, in our case, it is trivial to obtain $\langle B|\hat{\varphi}^2(x)|B\rangle_{\mathrm{ren}}$ from Eqs. (3.1) and (3.9). That is because we already have the closed form of the Feynman Green function and the singular part of the Hadamard form, given by Eqs.(4.63) and (4.69-4.71), respectively.

We can choose how the coincidence limit $x' \to x$ is to be taken. For simplicity, we

choose time separation, so that $\Delta = 0$ and $\gamma = 0$, but $\tau \neq 0$. Then, we find that

$$\begin{aligned}
&\lim_{\Delta\to 0,\,\gamma\to 0} G_{\rm F}(x,x') = -\frac{i\eta^2}{4\pi^2\tau^2},\\
&\lim_{\Delta\to 0,\,\gamma\to 0} G_{\rm S}^{(-2)}(\tau,\eta,\Delta,Q) = -\frac{i\eta^2}{4\pi^2\tau^2},\\
&\lim_{\Delta\to 0,\,\gamma\to 0} G_{\rm S}^{(-1)}(\tau,\eta,\Delta,Q) = 0,\\
&\lim_{\Delta\to 0,\,\gamma\to 0} G_{\rm S}^{(0)}(\tau,\eta,\Delta,Q) = \frac{i\eta^2}{48\pi^2}.
\end{aligned} \tag{4.90}$$

In particular, we have already set $\varepsilon = 0$. Hence, we have

$$\begin{aligned}
\langle B|\hat{\varphi}^2(x)|B\rangle_{\rm ren} &= -i\lim_{x'\to x}\left(G_{\rm F}(x,x') - \sum_{n=-2}^{0} G_{\rm S}^{(n)}(x,x')\right)\\
&= -i\lim_{\tau\to 0}\left(-\frac{i\eta^2}{4\pi^2\tau^2} + \frac{i\eta^2}{4\pi^2\tau^2} - \frac{i\eta^2}{48\pi^2}\right)\\
&= -\frac{\eta^2}{48\pi^2}.
\end{aligned} \tag{4.91}$$

This is the same result given in Ottewill and Taylor [41], where it was obtained differently.

It is interesting to obtain the same result via a radial point separation, so that $\tau = 0$, $\gamma = 0$, but $\Delta \equiv \rho - \rho' \neq 0$. In that case, we find

$$\begin{aligned}
&\lim_{\tau\to 0,\,\gamma\to 0} G_{\rm F}(x,x') = \frac{i}{4\pi^2\Delta^2\eta^2} + \frac{i\rho}{4\pi^2\Delta} - \frac{i\eta^2}{12\pi^2},\\
&\lim_{\tau\to 0,\,\gamma\to 0} G_{\rm S}^{(-2)}(\tau,\eta,\Delta,Q) = \frac{i}{4\pi^2\Delta^2\eta^2},\\
&\lim_{\tau\to 0,\,\gamma\to 0} G_{\rm S}^{(-1)}(\tau,\eta,\Delta,Q) = \frac{i\rho}{4\pi^2\Delta},\\
&\lim_{\tau\to 0,\,\gamma\to 0} G_{\rm S}^{(0)}(\tau,\eta,\Delta,Q) = -\frac{i\eta^2}{16\pi^2},
\end{aligned} \tag{4.92}$$

and then

$$\begin{aligned}
\langle B|\hat{\varphi}^2(x)|B\rangle_{\rm ren} &= -i\lim_{\Delta\to 0}\left(\frac{i}{4\pi^2\Delta^2\eta^2} + \frac{i\rho}{4\pi^2\Delta} - \frac{i\eta^2}{12\pi^2} - \frac{i}{4\pi^2\Delta^2\eta^2} - \frac{i\rho}{4\pi^2\Delta} + \frac{i\eta^2}{16\pi^2}\right)\\
&= -\frac{\eta^2}{48\pi^2}.
\end{aligned} \tag{4.93}$$

It is particularly interesting that, in this case, the divergences in $G_{\rm F}(x,x')$ are canceled

both by $G_{\rm S}^{(-2)}$ and $G_{\rm S}^{(-1)}$, while the term which is $\mathcal{O}(1)$ in $G_{\rm F}(x,x')$ adds to $G_{\rm S}^{(0)}$ to give the final result.

Note that this final result, $\langle B|\hat{\varphi}^2(x)|B\rangle_{\rm ren} = -\frac{\eta^2}{48\pi^2}$, is exactly equal to $-A$, with $A = \frac{\eta^2}{48\pi^2}$ as defined in the paragraph after Eq. (4.77). Thus, we see that the sum and integral in Eq. (4.79) must give zero. This is the result that will be checked numerically.

We now describe our numerical implementation of the sum and integral in Eq. (4.79) in Mathematica. Because of what is explained above, we only include the real part of the modes. We first implement the ℓ-sum for a fixed value of ω and ρ, with $f_{\ell\omega}$, $g_{\ell\omega}^{(-2)}$ and $\widetilde{g}_{\ell\omega}^{(0)}$ given respectively by Eqs. (4.58), (4.75) and (4.83). The ℓ-sum is computed by summing each term in the series from $\ell = 0$ until a value of ℓ such that the relative error is smaller then a given desired precision. We should mention that, because the sum converges to zero, this implementation is not straightforward. If S_n is the nth partial sum, a_{n+1} is the next term in the series, and p is the desired precision, we want to check if $\frac{|a_{n+1}|}{|S_n|} < 10^{-p}$ in order to stop the sum. However, if the sum converges to zero, $\frac{|a_{n+1}|}{|S_n|}$ diverges, and the check always fails. The solution is to use the slightly modified expression $\frac{|a_{n+1}|}{|S_n|+10^{-p}} < 10^{-p}$. Fig. 4.2 shows the maximum value of ℓ for various values of ω and $\rho = 1.5$.

We find that the computation of the ℓ-sum is more intensive as one approaches $\rho = 1$. As an example: in our running of the code, for the values of $\rho = 1.1$ and $\omega = 1.5$, the calculation takes about one second with a precision of $p = 20$; for $\rho = 1.001$ with the same value of ω and the same precision, the calculation takes about forty seconds. The value of ω has a smaller impact: for $\rho = 1.1$ and $\omega = 10.5$, the calculation takes only about a second and a half. For these computations, we use one core of an Intel i7 processor at 1.80GHz.

Then, for a given value of ρ, the ℓ-sum is computed for ω between $\omega_{\rm min}$ and $\omega_{\rm max}$, in steps of $\Delta\omega$. The points obtained from this calculation are interpolated. Finally, the integration of the interpolated integrand is performed. In our tests, we used the values $\omega_{\rm min} = 0$, $\omega_{\rm max} = 10$, and $\Delta\omega = 0.01$. We tested this implementation for different values of ρ, in each case finding zero up to the precision specified, as expected from our analytical considerations. Fig. 4.3 shows the total computing time for various values of ρ.

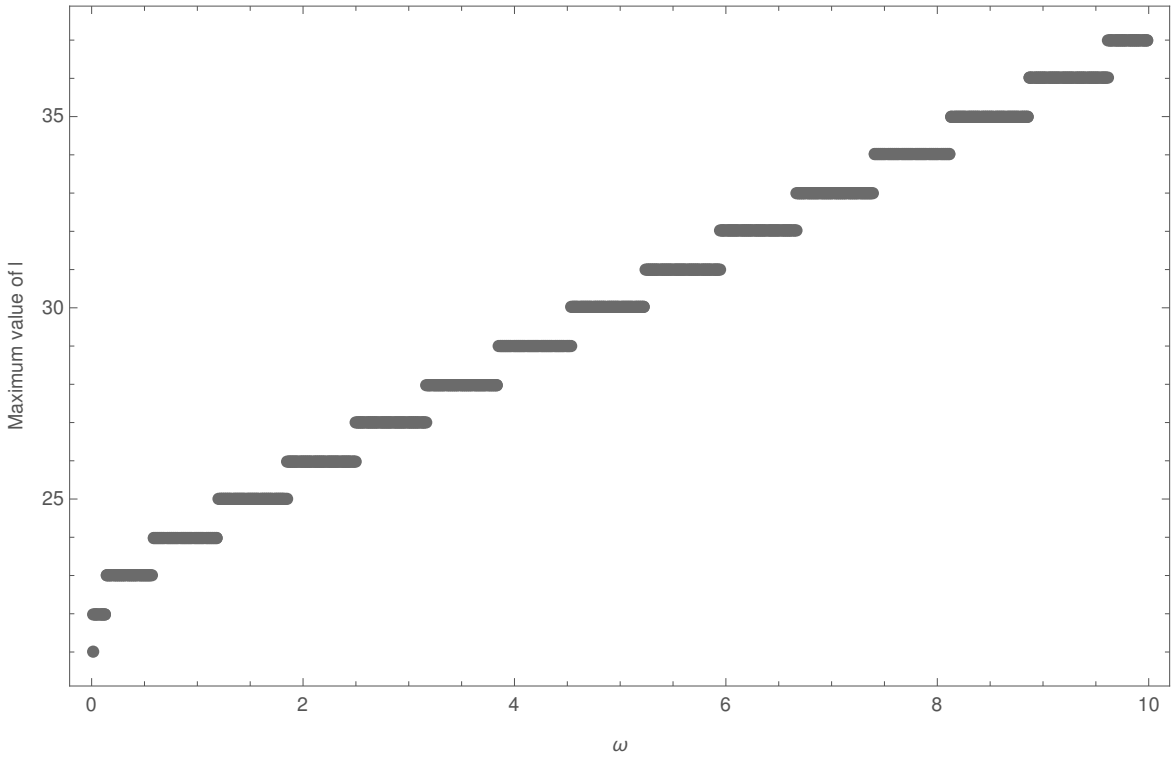

Figure 4.2: Maximum value of ℓ for which the ℓ-sum is computed, as a function of ω, for the value of $\rho = 1.5$. The precision is set to $p = 10$.

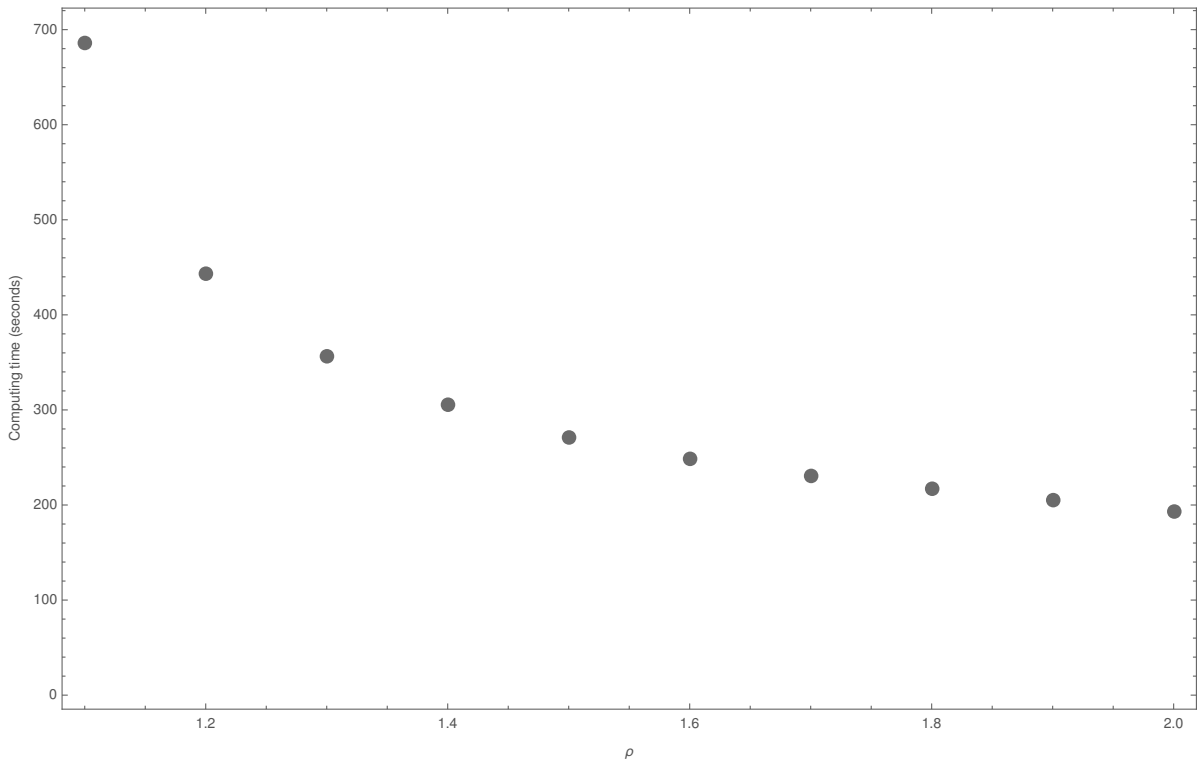

Figure 4.3: Total computing time in seconds for the ℓ-sum and ω-integration, as a function of ρ, with precision $p = 10$.

Conclusion

In this work, we have presented a novel method for the renormalization of $\langle\hat{\varphi}^2\rangle$ in the quantum theory of a scalar field in curved space-times using the Hadamard form of the Feynman Green function. It is a practical implementation of the point-splitting procedure via a mode-by-mode subtraction of the counter-terms. It does not make use of the WKB approximation, and does not require an Euclidean sector in the space-time. Therefore, it may be used in space-times more general than the static, spherically-symmetric case described in Refs. [7]-[12].

We have detailed the implementation of this method in order to renormalize $\langle\hat{\varphi}^2\rangle$ for a scalar field in flat space-time and in Bertotti-Robinson space-time. These examples were taken for their simplicity, as a first test of our approach. In both cases, the method works and our results agree with previous results in the literature.

In the future, this method may be used to calculate the renormalization of the expectation value of the stress-energy tensor $\langle\hat{T}_{\mu\nu}\rangle$. Also, a future step is to apply it to an example of a space-time which is not static and spherically-symmetric, such as Kerr space-time.

Appendix A

Proofs using distributions

A.1 Fourier transform of the Hadamard form in flat space-time

In subsection 4.1.2, we calculate the Fourier transform of $G_{\rm S}^{(-2)}(x,x')$ directly from the integral which defines the Fourier transform of a function. However, the Hadamard form is in fact a distribution, and thus its Fourier transform should be treated in a more rigorous way.

Let us first define the notation that shall be used in this section. A distribution D is defined as a bounded linear functional on the space of test functions (i.e., smooth functions of compact support) [40]. The action of a distribution D on a test function f shall be denoted $\langle D, f\rangle$ throughout. Any locally integrable function $g(x)$ defines a distribution via

$$\langle g, f\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, g(x) f(x) \tag{A.1}$$

for any test function f, and as an abuse of notation we use the same letter g to represent the distribution. The Dirac delta distribution δ is defined according to

$$\langle \delta, f\rangle \equiv f(0) \tag{A.2}$$

for any test function f. We also use the notation $\delta(x)$ inside an integral similar to Eq. (A.1)

to represent the action of the Dirac delta distribution, although $\delta(x)$ is *not* a function. The principal value distribution $\mathcal{PV}\left(\frac{1}{x}\right)$, which will be used below, is defined as

$$\left\langle \mathcal{PV}\left(\frac{1}{x}\right), f \right\rangle \equiv \mathcal{PV}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{f(x)}{x} \equiv \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \mathrm{d}x\, \frac{f(x)}{x} \tag{A.3}$$

for any test function f. Finally, the Fourier transform $\mathcal{F}\{D\}$ of a distribution[1] D is defined according to:

$$\langle \mathcal{F}\{D\}, f\rangle \equiv \langle D, \mathcal{F}\{f\}\rangle, \tag{A.4}$$

where $\mathcal{F}\{f\}$ is the Fourier transform of the test function f:

$$\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{ikx} f(x), \tag{A.5}$$

which is always well-defined since f is smooth and has compact support[2]. Note that the Fourier transform is a function of the k variable (i.e., it is defined in the Fourier space), while the original function is defined on the space of positions (x variable). When we need to make these dependences explicit, we shall denote the Fourier transform as $\mathcal{F}\{D(x)\}(k)$, as we did in Eq. (A.5) for the Fourier transform of f.

In a distributional sense, Eq. (4.19) (with $n=-2$) for $G_{\mathrm{S}}^{(-2)}(x,x')$ in flat space-time should be understood, according to Eq. (4.15), as:

$$G_{\mathrm{S}}^{(-2)}(x,x') = \frac{i}{4\pi^2}\left\{\mathcal{PV}\left(\frac{1}{-t^2+\zeta^2}\right) - i\pi\,\delta\left(-t^2+\zeta^2\right)\right\}, \tag{A.6}$$

where we have defined $\zeta \equiv 2r^2Q^2 + \Delta r^2$. In order to simplify the notation, we also take $t'=0$, so that $\Delta t = t$. Given these considerations, let us calculate the Fourier transform of $G_{\mathrm{S}}^{(-2)}$, denoted by $G_{\omega}^{(-2)} \equiv \mathcal{F}\left\{G_{\mathrm{S}}^{(-2)}(x,x')\right\}(\omega)$, via the action on an arbitrary test

[1] In fact, the Fourier transform of a distribution is defined for tempered distributions, which are functionals on the space of test functions of rapid decay (briefly: test functions which, along with their derivatives of all orders, decay at infinity faster than the reciprocal of any polynomial). However, we shall not get into these details in this discussion.

[2] However, one cannot guarantee that $\mathcal{F}\{f\}$ is a test function on its own, because it may not have compact support. In that case, Eq. (A.4) would not be well defined. This is the reason why the space of test functions must be enlarged to include test functions of rapid decay, as described in the previous footnote.

function f, using Eq. (A.4):

$$\begin{aligned}\left\langle G_\omega^{(-2)}, f\right\rangle &= \left\langle G_{\mathrm{S}}^{(-2)}, \mathcal{F}\{f\}\right\rangle \\ &= \frac{i}{4\pi^2}\left\{\mathcal{PV}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\mathcal{F}\{f\}(t)}{-t^2+\zeta^2} - i\pi\int_{-\infty}^{\infty}\mathrm{d}t\,\mathcal{F}\{f\}(t)\delta(-t^2+\zeta^2)\right\} \\ &= \frac{i}{8\pi^2\zeta}\left\{-\mathcal{PV}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\mathcal{F}\{f\}(t)}{t-\zeta} + \mathcal{PV}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\mathcal{F}\{f\}(t)}{t+\zeta} - \right. \\ &\qquad \left. -i\pi\int_{-\infty}^{\infty}\mathrm{d}t\,\mathcal{F}\{f\}(t)\delta(t-\zeta) - i\pi\int_{-\infty}^{\infty}\mathrm{d}t\,\mathcal{F}\{f\}(t)\delta(t+\zeta)\right\},\end{aligned} \tag{A.7}$$

where we have used the well-known result

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|} \tag{A.8}$$

for an arbitrary function $g(x)$, where x_i runs over the simple zeros of $g(x)$, and $g'(x)$ denotes the derivative of $g(x)$. Therefore:

$$\begin{aligned}G_\omega^{(-2)} = \frac{i}{8\pi^2\zeta}\Bigg[&-\mathcal{F}\left\{\mathcal{PV}\left(\frac{1}{t-\zeta}\right)\right\}(\omega) + \mathcal{F}\left\{\mathcal{PV}\left(\frac{1}{t+\zeta}\right)\right\}(\omega) - \\ &- i\pi\mathcal{F}\left\{\delta(t-\zeta)\right\}(\omega) - i\pi\mathcal{F}\left\{\delta(t+\zeta)\right\}(\omega)\Bigg].\end{aligned} \tag{A.9}$$

Now, using the translation theorem for Fourier transforms,

$$\mathcal{F}\left\{D(x-a)\right\}(k) = e^{ika}\,\mathcal{F}\left\{D(x)\right\}(k), \tag{A.10}$$

we find that

$$\begin{aligned}G_\omega^{(-2)} = \frac{i}{8\pi^2\zeta}\Bigg[&-e^{i\omega\zeta}\,\mathcal{F}\left\{\mathcal{PV}\left(\frac{1}{t}\right)\right\}(\omega) + e^{-i\omega\zeta}\,\mathcal{F}\left\{\mathcal{PV}\left(\frac{1}{t}\right)\right\}(\omega) - \\ &- i\pi\,e^{i\omega\zeta}\,\mathcal{F}\left\{\delta(t)\right\}(\omega) - i\pi\,e^{-i\omega\zeta}\,\mathcal{F}\left\{\delta(t)\right\}(\omega)\Bigg].\end{aligned} \tag{A.11}$$

It is proven in Ref. [40] that

$$\mathcal{F}\left\{\mathcal{PV}\left(\frac{1}{t}\right)\right\}(\omega) = i\pi\,\mathrm{sgn}(\omega), \qquad \mathcal{F}\left\{\delta(t)\right\}(\omega) = 1. \tag{A.12}$$

Thus, we have finally

$$\begin{aligned} G_{\omega}^{(-2)} &= \frac{1}{8\pi\zeta}\left(e^{i\omega\zeta}\ \mathrm{sgn}(\omega) - e^{-i\omega\zeta}\ \mathrm{sgn}(\omega) + e^{i\omega\zeta} + e^{-i\omega\zeta}\right) \\ &= \frac{1}{4\pi\zeta}\left(i\ \sin(\omega\zeta)\,\mathrm{sgn}(\omega) + \cos(\omega\zeta)\right) \\ &= \frac{e^{i|\omega|\zeta}}{4\pi\zeta}, \end{aligned} \tag{A.13}$$

which is equal to the result in Eq. (4.21) with $\varepsilon = 0$, as claimed in the text.

A.2 Sum of the $g_{\ell\omega}^{(-2)}$ modes in flat space-time

In this Appendix we shall prove that the $g_{\ell\omega}^{(-2)}$ modes calculated in Eq. (4.22) for flat space-time recover the original function $G_{\rm S}^{(-2)}(x,x')$, after the ℓ-sum and ω-integral. Remember that $G_{\rm S}^{(-2)}(x,x')$ is expressed as

$$G_{\rm S}^{(-2)}(x,x') = \frac{i}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{-(\Delta t)^2 + 2r^2Q^2 + (\Delta r)^2 + i\varepsilon}, \tag{A.14}$$

as given by Eq. (4.19) for $n=-2$. We start from the result

$$g_{\ell\omega}^{(-2)}(r,r') = \frac{|\omega|}{8\pi^2} j_\ell\left(|\omega|q_-\right) h_\ell^{(1)}\left(|\omega|q_+\right), \tag{A.15}$$

with $q_\pm$ defined by Eq. (4.27). We want to check whether

$$G_{\rm S}^{(-2)}(x,x') = i\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}\omega\, e^{-\varepsilon|\omega|}e^{-i\omega\Delta t}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)g_{\ell\omega}^{(-2)}(r,r') \tag{A.16}$$

holds. Note that we have already included the '$i\varepsilon$' prescription by introducing the factor $e^{-\varepsilon|\omega|}$. The ℓ-sum may be performed with the aid of Eqs. (4.7) and (4.9), yielding:

$$\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)\,\frac{|\omega|}{8\pi^2} j_\ell\left(|\omega|q_-\right) h_\ell^{(1)}\left(|\omega|q_+\right) = -\frac{i}{8\pi^2}\,\frac{e^{i|\omega|\zeta}}{\zeta}, \tag{A.17}$$

where

$$\begin{aligned}
\zeta^2 &= q_+^2 + q_-^2 - 2q_+q_-\cos\gamma \\
&= r^2\left(1+s+\sqrt{s(2+s)}\right) + r^2\left(1+s-\sqrt{s(2+s)}\right) - \\
&\quad - 2r^2\left(1+s+\sqrt{s(2+s)}\right)^{\frac{1}{2}}\left(1+s-\sqrt{s(2+s)}\right)^{\frac{1}{2}}\cos\gamma \\
&= 2r^2(1+s) - 2r^2\cos\gamma \\
&= 2r^2Q^2 + (\Delta r)^2,
\end{aligned} \tag{A.18}$$

where we have used the definitions of s and Q given in the text. We see that the ℓ-sum recovers the Fourier modes obtained in Eq. (4.21) (with $\varepsilon = 0$) and checked in Appendix A.1. Now, we take the inverse Fourier transform:

$$\begin{aligned}
G_{\rm S}^{(-2)}(x,x') &= \frac{1}{8\pi^2}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}\omega\, e^{-\varepsilon|\omega|}\, e^{-i\omega\Delta t}\,\frac{e^{i|\omega|\zeta}}{\zeta} \\
&= \frac{1}{8\pi^2\zeta}\lim_{\varepsilon\to 0^+}\int_0^{\infty}\mathrm{d}\omega\,\left\{e^{i\omega(\Delta t+\zeta+i\varepsilon)} + e^{i\omega(-\Delta t+\zeta+i\varepsilon)}\right\} \\
&= \frac{i}{8\pi^2\zeta}\lim_{\varepsilon\to 0^+}\left\{\frac{1}{\Delta t+\zeta+i\varepsilon} + \frac{1}{-\Delta t+\zeta+i\varepsilon}\right\} \\
&= \frac{i}{4\pi^2\zeta}\lim_{\varepsilon\to 0^+}\frac{\zeta+i\varepsilon}{-(\Delta t)^2+(\zeta+i\varepsilon)^2}.
\end{aligned} \tag{A.19}$$

We shall now prove that the distribution $D \equiv \lim_{\varepsilon\to 0^+}\left(\frac{\varepsilon}{-x^2+(a+i\varepsilon)^2}\right)$ is the zero distribution. Indeed, applying it to a test function f, we have that:

$$\begin{aligned}
\langle D, f\rangle &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{\varepsilon}{-x^2+(a+i\varepsilon)^2}f(x) \\
&= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{\varepsilon}{2(a+i\varepsilon)}\left\{\frac{1}{-x+a+i\varepsilon} + \frac{1}{x+a+i\varepsilon}\right\}f(x).
\end{aligned} \tag{A.20}$$

Let us focus on the terms inside the brackets separately, defining the distributions $D_\pm \equiv \lim_{\varepsilon\to 0^+}\left(\frac{\varepsilon}{2(a+i\varepsilon)}\frac{1}{\pm x+a+i\varepsilon}\right)$. Let $A_\pm \in \mathbb{R}$ be a large real number so that $f(x) = 0$ for $|x \pm a| > A_\pm$. Such a number always exists, since $f(x)$ has compact support. Then, transforming

$x \mapsto x \pm a$, we have that

$$\begin{aligned}\langle D_\pm, f\rangle &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\varepsilon}{2(a+i\varepsilon)}\,\frac{f(x)}{\pm x + a + i\varepsilon} \\ &= \lim_{\varepsilon\to 0^+}\int_{-A_\pm}^{A_\pm} \mathrm{d}x\, \frac{\varepsilon}{2(a+i\varepsilon)}\,\frac{f(x\mp a)}{\pm x + i\varepsilon}.\end{aligned} \tag{A.21}$$

We may use Taylor's theorem to state that there exists a smooth function $g(x)$ such that $f(x\mp a) = f(\mp a) + x g(x)$. Then:

$$\langle D_\pm, f\rangle = \lim_{\varepsilon\to 0^+}\int_{-A_\pm}^{A_\pm} \mathrm{d}x\, \frac{\varepsilon}{2(a+i\varepsilon)}\,\frac{f(\mp a)}{\pm x + i\varepsilon} + \lim_{\varepsilon\to 0^+}\int_{-A_\pm}^{A_\pm} \mathrm{d}x\, \frac{\varepsilon}{2(a+i\varepsilon)}\,\frac{x\, g(x)}{\pm x + i\varepsilon}. \tag{A.22}$$

Because the integrand on the second integral is smooth as $\varepsilon \to 0^+$, we may take the limit inside the integral, which shows that the second integral is zero. As for the first integral, we calculate:

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{-A_\pm}^{A_\pm} \mathrm{d}x\, \frac{\varepsilon}{2(a+i\varepsilon)}\,\frac{f(\mp a)}{\pm x + i\varepsilon} &= \lim_{\varepsilon\to 0^+}\left\{\frac{\varepsilon f(\mp a)}{a+i\varepsilon}\arctan\left(\frac{A_\pm}{\varepsilon}\right)\right\} \\ &= 0.\end{aligned} \tag{A.23}$$

Thus, we see that $\langle D_\pm, f\rangle = 0$. Consequently, $\langle D, f\rangle = 0$ for any test function f, which proves our proposition that D is the zero distribution.

Back to Eq. (A.19), we use what we have just proven in order to simplify the expression:

$$\begin{aligned}G_{\mathrm{S}}^{(-2)}(x,x') &= \frac{i}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1+i\varepsilon/\zeta}{-(\Delta t)^2 + (\zeta+i\varepsilon)^2} \\ &= \frac{i}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{-(\Delta t)^2 + (\zeta+i\varepsilon)^2} \\ &= \frac{i}{4\pi^2}\lim_{\varepsilon\to 0^+}\frac{1}{-(\Delta t)^2 + \zeta^2 + i\varepsilon},\end{aligned} \tag{A.24}$$

where we suitably redefined ε in the last line. This is the same original expression for $G_{\mathrm{S}}^{(-2)}(x,x')$, given by Eq. (A.14), which was to be proven.

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