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Disorder in $d = 0$ field theory

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Disorder in Quantum Field Theory

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Abstract

We use a new method for treating disordered systems in field theory, the distributional zeta function approach, to analyze the perturbative expansion of the random field Ising model and of the random mass Ising model.

Resumo

Utilizamos um novo método para tratar sistemas desordenados em teoria de campos para analisar a expansão perturbativa para o modelo de Ising com campo magnético aleatório e para o modelo de Ising com massa aleatória.

Palavras-chave: **desordem, replica** .

Contents

Chapter 1

Introduction

The study of disordered systems finds application everywhere in physics, from gravitation, where one can use randomness to study analog models for quantum gravity [1–5], to statistical mechanics and condensed matter physics, where disorder is used in the treatment of impure metals and semiconductors, spin glasses, surface growth and directed polymers.

A straightforward way to introduce disorder is to begin by considering the well-known Ising model, describing a d-dimensional lattice of spins $\sigma_i = \pm 1$ which interact with nearest neighbors with a coupling J_{ij}. There are then two ways to introduce disorder into this model. First, one can couple this system to an external random magnetic field, giving the random field Ising model [6]. Another possibility is to let the coupling between spins, J_{ij}, be given by a random distribution. This leads to the vast field of spin glasses [7–9].

The random field Ising model (RFIM) can be used to model diluted frustrated magnets and binary liquids in porous media. Despite being a simple model and being intensively researched the RFIM is still not completely solved. For example, although the upper and lower critical dimensions are known the dimensional reduction breaks at the perturbative level [10].

A possible framework to study this questions is to use the methods of Statistical Field Theory [11–13], since the behavior of the system near the critical point can be described by the Landau-Ginzburg scalar field model where the order parameter is a continuous field $\phi(x)$, coupled with a random field $h(x)$. The presence of disorder causes the ground-state configurations to depend explicitly on specific realizations of the random field. This leads to the existence of several local minima for the field, which makes the implementation of perturbation theory. To solve this problem one usually introduces the replica trick [14] to average the free energy over the disorder field.

This replica method to solve disordered systems has had great success [15] but some authors still consider that a more mathematically rigorous derivation to support this procedure is still necessary [16–19]. This was done in the recent work [20].

As a testing ground for this new approach to disordered system we will consider a $\lambda\phi^4$ scalar field theory in dimension $d = 0$.

The choice of this model is motivated by the fact that it is possible to find an analytic solution in the pure (disorder-free) case [21] and by the existing literature studying the analytic properties of the free energy after the addition of disorder into the model [22–24].

Although this model is an extreme simplification it is still physically interesting, in part because this simplicity makes it very useful for comparison of different perturbative and resummation techniques, such as the ϵ-expansion and the large-N expansion. The zero-dimensional quantum field theory is also relevant in the context of random matrix theory and quantum dots.

This dissertation is organized as follows. In the next chapter we give a brief review of standard Quantum Field Theory, setting the stage for chapter three, where we give an introduction to the methods to treat disorder in QFT. In chapter four we perform some calculations in the $d = 0$ model and compare with the results in the literature.

Chapter 2

Basic Quantum Field Theory

In this chapter we review the basic functional methods in Quantum Field Theory [25–27]. We start with the generating functional and go through the usual perturbative approaches to $\lambda\phi^4$. Then we perform the Wick rotation to arrive at a Euclidean Field Theory (EQFT). This sets the stage for the next chapter where we will introduce disorder and compare with the usual results.

In the subject of quantum field theory there are two approaches. The oldest and main approach is concerned with the direct connection to particle physics experiments, in which the physical quantities measured are transition probabilities. This connection between theory and experimental results is made through the S-matrix, using the LSZ (Lehmann, Symanzik and Zimmermann) reduction formula. In this approach one calculate the Green functions of the theory, G_n, and the translates these quantities to scattering amplitudes, which can be compared with experimental results.

The other approach is to fully utilize the strength of the path integral formulation of quantum field theory to obtain a new set of observables. Some examples are: calculation of critical exponents, symmetry breaking, metastable state decay rates and topological solutions.

2.1 General Functional methods

Non-relativistic Quantum mechanics is constructed from a set of axioms. One of the axioms establishes that the time evolution of the state of a system is given by the Schrödinger equation, which is linear in the time derivative and quadratic in the space derivatives and thus is not compatible with special relativity. One can then attempt to generalize to a relativistic time evolution equation, such as Klein-Gordon and Dirac equations, but there are still some problems. The correct way to proceed is then to go to a quantum theory of fields.

In quantum mechanics one usually starts from a classical Hamiltonian and, through the processes of canonical quantization, arrives at the quantum theory. In QFT we begin with the same process, the difference being that the classical theory used as a starting point is a classical theory of fields.

For example, we can begin by considering the classical scalar field ϕ in d dimensions. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi), \tag{2.1}$$

where $V(\phi)$ describes the field self-interaction.

The Hamiltonian density is then obtained by

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L}, \tag{2.2}$$

where the conjugate field π is defined as

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}. \tag{2.3}$$

Canonical quantization then proceeds in the same way as usual quantum mechanics.

The dynamics are obtained from the Heisenberg equations of motion

$$\partial_0\phi(x) = i[H,\phi] \tag{2.4}$$

$$\partial_0\pi = i[H,pi]. \tag{2.5}$$

We also have the equal-time commutation relations

$$[\phi(x^0,\mathbf{x}),\pi(y^0,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \tag{2.6}$$

$$[\phi(x^0,\mathbf{x}),\phi(y^0,\mathbf{y})] = [\pi(x^0,\mathbf{x}),\pi(y^0,\mathbf{y})] = 0 \tag{2.7}$$

Then, eqs. (2.4) can be rewritten as

$$\partial_0\phi = \frac{\partial\mathcal{H}}{\partial\pi} \tag{2.8}$$

$$\partial_0\pi = -\frac{\partial\mathcal{H}}{\partial\phi} \tag{2.9}$$

Combining this equations leads to the operator-valued Euler-Lagrange equation

$$(\Box_x + m^2)\phi(x) + V'(\phi) = 0 \tag{2.10}$$

which is obtained from a variation of the classical action functional

$$S[\phi] = \int \mathrm{d}x\mathcal{L}(\partial_\mu\phi,\phi) \tag{2.11}$$

The connection with experimental results is made through the Lehmann-Symanzik-Zimmermann (LSZ) formula, which expresses scattering cross-sections in terms of the Green functions of the theory, defined as

$$G_n(x_1,\cdots,x_n) = \langle 0|\, T[\phi(x_1)\cdots\phi(x_n)]\, |0\rangle\,. \tag{2.12}$$

A convenient way to obtain a solution for the set of Green functions is to begin with a definition of a functional generator,

$$Z[j] = \langle 0| \, T \left[\mathrm{e}^{i \int \mathrm{d}^d x j(x)\phi(x)} \right] |0\rangle \, . \tag{2.13}$$

Notice that here the field ϕ is an operator valued function.

The Green functions are then obtained by functional derivatives,

$$(-i)^n \left. \frac{\delta^n Z[j]}{\delta j(x_1) \cdots \delta j(x_n)} \right|_{j=0} = \langle 0| \, T[\phi(x_1) \cdots \phi(x_n)] \, |0\rangle \, . \tag{2.14}$$

This translates the problem of obtaining the set of all Green functions to the problem of obtaining $Z[j]$. To solve the latter we follow the approach of Symanzik, where one finds a differential equation for $Z[j]$.

We begin by defining $E(x'^0, x^0)$

$$E(x'^0, x^0) = T \left[\exp\left(i \int_{x^0}^{x'^0} \mathrm{d}y^0 \int \mathrm{d}\mathbf{y} j(y^0, \mathbf{y}) \phi(y^0, \mathbf{y}) \right) \right] . \tag{2.15}$$

This allows us to rewrite the generating functional $Z[j]$ as

$$Z[j] = \langle 0| \, E(\infty, -\infty) \, |0\rangle = \langle 0| \, E(\infty, x^0) E(x^0, -\infty) \, |0\rangle \, . \tag{2.16}$$

Let us now insert the Heisenberg equations of motion (2.10) inside this vacuum persistence amplitude to obtain

$$\langle 0| \, E(\infty, x^0) \left[\partial^2 \phi(x) + V'(\phi) \right] E(x^0, -\infty) \, |0\rangle = 0. \tag{2.17}$$

Now, using that

$$\begin{aligned} \partial_0^2 \langle 0| \, E(\infty, x^0) \phi(x) E(x^0, -\infty) \, |0\rangle = & j(x) \langle 0| \, E(\infty, x^0) E(x^0, -\infty) \, |0\rangle + \\ & + \langle 0| \, E(\infty, x^0) \partial_0^2 \phi(x) E(x^0, -\infty) \, |0\rangle \, . \end{aligned} \tag{2.18}$$

We can rewrite eq. (2.17) as

$$\left[\partial^2\left(-i\frac{\delta}{\delta j(x)}\right)+V'\left(-i\frac{\delta}{\delta j(x)}\right)-j(x)\right]Z[j]=0. \tag{2.19}$$

This is the Schwinger-Dyson equation, obtained by the Symanzik's construction. This equation defines the linear functional differential operator, which we denote by $SD[-i\frac{\delta}{\delta j(x)}]$ and the Schwinger-Dyson equation can be written as $SD[-i\frac{\delta}{\delta j(x)}]Z[j]=0$.

Having obtained this differential equation for $Z[j]$ we now proceed on an attempt to solve it. Let us write $Z[j]$ as the functional Fourier transform of another functional, $\tilde{Z}[\varphi]$, which depends on a classical function φ [28]

$$Z[j]=\int\mathcal{D}\varphi\tilde{Z}[\varphi]\mathrm{e}^{i\int \mathrm{d}^dx j(x)\varphi(x)}. \tag{2.20}$$

Applying this on (2.19) we obtain that $\tilde{Z}[\varphi]=\mathrm{e}^{i\int \mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-V(\varphi)\right)}$, and thus the generating functional for Green functions can be written as the functional integral

$$Z[j]=\int\mathcal{D}\varphi\mathrm{e}^{i\int \mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-V(\varphi)+j(x)\varphi(x)\right)}. \tag{2.21}$$

This is the formal solution to the SD equations, up to a normalization constant, in form of a functional integral. It is convenient to normalize the generating functional as $Z[0]=1$, which fixes the normalization $\mathcal{N}$ as

$$Z[0]=\mathcal{N}\int\mathcal{D}\varphi\mathrm{e}^{i\int \mathrm{d}^dx\left(\frac{1}{2}(\partial\varphi)^2-V(\varphi)\right)}=1 \tag{2.22}$$

$$\tag{2.23}$$

Then we rewrite (2.21) as

$$Z[j] = \frac{\int \mathcal{D}\varphi \mathrm{e}^{i\int \mathrm{d}^d x\left(\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + j(x)\varphi(x)\right)}}{\int \mathcal{D}\varphi \mathrm{e}^{i\int \mathrm{d}^d x\left(\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right)}} \tag{2.24}$$

This is our final desired form for the functional generator of Green functions. However, since we are interest in the Statistical Field theory let us perform a Wick rotation to write the Euclidean form of this functional. Making the transformation $t \mapsto -it$ we get

$$Z[j] = \frac{\int \mathcal{D}\varphi \mathrm{e}^{-\int \mathrm{d}^d x\left(\frac{1}{2}\varphi\partial_E^2\varphi + V(\varphi) - j(x)\varphi(x)\right)}}{\int \mathcal{D}\varphi \mathrm{e}^{-\int \mathrm{d}^d x\left(\frac{1}{2}\varphi\partial_E^2\varphi + V(\varphi)\right)}}. \tag{2.25}$$

This is called Euclidean field theory because now the metric is simply `diag`$(+1)$ and we have a d-dimensional Euclidean space.

In the case of a free field ($V(\varphi) = \frac{1}{2}m^2\varphi^2$) the integral for $Z_0[j]$ is Gaussian and can be solved directly,

$$Z_0[j] = \left[\det(-\partial^2 + m^2)\right]^{-\frac{1}{2}} \mathrm{e}^{\int \mathrm{d}^d x\, \mathrm{d}^d y j(x) D(x-y) j(y)} \tag{2.26}$$

where the normalization $Z_0[j] = 1$ fixes $\mathcal{N}_0 = 1$ and $D(x-y)$ is the Euclidean propagator, defined as

$$D(x-y) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{e}^{-ip(x-y)}}{p^2 + m^2}. \tag{2.27}$$

For the more general theory it is not possible to solve the integral directly so we must use perturbation theory. This amounts to writing the interaction term of the Lagrangian in terms of a small interaction parameter, which we will use to expand the exponentials in powers of this parameter.

For example, for the usual $\lambda\phi^4$ theory we have

To organize this series expansion we introduce Feynman diagrams, where a line segment corresponds to for a propagator $-iD_F(x-y)$, a filled circle at the end of a line corresponds to a source term $i\int \mathrm{d}^d x j(x)$ and a vertex connecting four lines corresponds to $i\lambda \int \mathrm{d}^d x$.

In the expansion (**??**) many terms are algebraically identical. In terms of diagrams this means that the same diagram appears many times in the expression. So we can simply add them all, where the overall counting factor can be calculated by noticing that in each diagram the number of lines is P and the number of vertices is V. We can rearrange the functional derivatives from a particular vertex without altering the diagrams, so we have a factor 4! for each vertex. We can also rearrange the vertex themselves, which yields a factor $V!$. We can also rearrange the two sources at the ends of a particular propagator, which yields a factor 2! for each propagator. We can rearrange the propagators, giving $P!$. All together these counting factors cancel the numbers from the dual Taylor expansion in (**??**).

This still leads to overcounting. The factor by which we have overcounted is called the symmetry factor. This is of combinatonic origin.

Now the generating functional is given by the sum of all diagrams D.

$$Z[j] = \mathcal{N} \sum_{\{n_I\}} D \tag{2.28}$$

$$Z[j] = \mathcal{N} \sum_{\{n_I\}} \prod_I \frac{1}{C_I}^{n_I} \tag{2.29}$$

$$Z[j] = \mathcal{N} \prod_I \sum_{n_I=0}^{\infty} \frac{1}{C_I}^{n_I} \tag{2.30}$$

$$Z[j] = \mathcal{N} \prod_I \exp\left[C_I\right] \tag{2.31}$$

$$Z[j] = \mathcal{N} \exp\left[\sum_I C_I\right] \tag{2.32}$$

This expresses that the generating functional $Z[j]$ is given by the exponential of the sum of connected diagrams. If we omit the vacuum diagrams it already fixes the normalization $Z[0] = 1$. Then

$$Z[j] = \exp\left[iW[j]\right] \tag{2.33}$$

where $W[j=0]=0$.

2.2 Pure Zero-Dimensional Field Theory

We will work with the real scalar field in zero dimensions as our toy model. This means that our space-time is a simply connected zero-dimensional manifold M and the fields are maps $\phi : M \mapsto \mathbb{R}$. Since M is just a point the fields ϕ can be treated as ordinary real numbers and thus the functional integration measure $\mathcal{D}\phi$ is just the usual Lebesgue measure over the real line, $\mathrm{d}\phi$. The partition function is then simply

$$Z = \int_{\mathbb{R}} \mathrm{d}\phi \mathrm{e}^{-S(\phi)}. \tag{2.34}$$

Since there are no spatial directions to define differentiation and no notion of integration over the zero-dimensional manifold the most general action is a polynomial,

$$S(\phi) = \sum_{n=0}^{\infty} \frac{g_n}{n!}\phi^n. \tag{2.35}$$

In particular, if we set all couplings $g_n = 0$ except for $g_2 = m_0^2 > 0$ we recover the free theory and denote this actions by $S_0 = \frac{1}{2}m_0^2\phi^2$. In this case the partition function can be trivially solved, using the well-known Gaussian integration formula, to obtain $Z_0(m_0^2) = \sqrt{\frac{2\pi}{m_0^2}}$ and we can absorb this normalization constant to set $Z_0 = 1$.

If we also consider as non-zero the coupling $g_4 = \lambda$ we recover the self-interacting scalar field. The action is this case is given by

$$S(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{2.36}$$

And the partition function is given by

$$Z(m_0^2, \lambda) = \int_{\mathbb{R}} \mathrm{d}\phi \mathrm{e}^{-\frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{4!}\phi^4}. \tag{2.37}$$

In this case the partition function can be solved exactly, for $\mathsf{Re}\lambda > 0$, in terms of the confluent hypergeometric function of the second kind, $U(a,b,z)$ [21, 29, 30],

$$Z(m_0^2,\lambda) = \sqrt{2\pi}\left(\frac{3}{2\lambda}\right)^{\frac{3}{4}} m_0^4 U\left(\frac{3}{4},\frac{3}{2},\frac{3m_0^4}{2\lambda}\right). \tag{2.38}$$

On the other hand, we can also use perturbative techniques to treat the partition function. For this we perform an asymptotic expansion in powers of λ

$$\begin{aligned} Z(m_0^2,\lambda) &= \sum_{n=0}^{\infty}\left(\frac{-\lambda}{4!}\right)\frac{1}{n!}\int_{\mathbb{R}} \mathrm{d}\phi\phi^4 \mathrm{e}^{-\frac{1}{2}m_0^2\phi^2} \\ &= \left[1 - \frac{1}{8}\frac{\lambda}{m_0^4} + \frac{35}{384}\frac{\lambda^2}{m_0^8} + \cdots\right] Z[m_0^2, 0], \end{aligned} \tag{2.39}$$

where $Z[m_0^2, 0]$ corresponds to the free case. This series zero radius of convergence, as expected, since the integral (2.37) is not defined for $\mathsf{Re}\lambda < 0$, and thus is not defined on an open disk around the origin. This means that perturbation theory is not capable of recovering all the information present in the analytical non-perturbative solution (2.38). However, if we apply a Borel transformation we are able to perform a resummation and, if the Borel transformed sum converges, recover the exact result.

Since this $\lambda\phi^4$ action has a $\mathbb{Z}_2$ symmetry under the the transformation $\phi \mapsto -\phi$ all n-point Green functions, with n odd, vanish. Thus, for this zero-dimensional field theory, all Green functions can be obtained by successive derivatives of the partition function (2.38) with respect to the mass parameter m_0^2,

$$G^{(n)} = \frac{1}{Z}\frac{\partial^n}{\partial m_0^{2^n}} Z(m_0^2,\lambda). \tag{2.40}$$

However, since this feature is not easily generalized to other QFTs it is more instructive to follow the usual procedure of introducing an external source current $J\phi$ and taking derivatives with respect to J. In this case we have

$$Z[J] = \int_{\mathbb{R}} \mathrm{d}\phi \mathrm{e}^{-\frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi} \qquad G^{(n)} = \frac{1}{Z}\frac{\partial^n}{\partial J^n} Z[J]\bigg|_{J=0}. \tag{2.41}$$

Using the exact result from (2.38) we can thus obtain all Green functions analytically [31].

This contrasts with the usual perturbative approach where one would expand (2.41) around $\lambda = 0$ and then obtain the Green functions as an asymptotic series in λ [32]. To recover the analytical results one has to perform a resummation of the perturbative expansion.

Chapter 3

Disorder in Quantum Field Theory

In the context of the renormalization group the action can be written in the general form

$$S = \int \mathrm{d}^d x \sum_i g^i(x)\mathcal{O}_i(x). \tag{3.1}$$

Here $g^i(x)$ are the couplings parameters, which are allowed to depend on space, and $\mathcal{O}_i$ are general field operators that can be any combination of field monomials and derivative terms such that the action satisfies the symmetries of the theory. In particular, for theories with $O(N)$ symmetry we can have a general function $V(\rho)$ of $\rho = \frac{1}{2}\phi^a\phi_a$.

The introduction of disorder in QFT now amounts to letting one of the couplings be given by a probability distribution. There are two ways to do this. The first and simplest one is to have this probability distribution be independent of the quantum fields. This case is called ”quenched” disorder and represents the case where the time-scale of fluctuations for the external disorder field is much larger than the time-scale of quantum fluctuations of the fields. The other option would be to let the probability distribution depend on specific realizations of the quantum field, called ”annealed” disorder, the random field and the quantum field have same orders of time-scale fluctuations.

In this work we are interested in the quenched case. It is conceptually more simple but leads to more complicated calculations.

Without loss of generality one can set $g^0(x) = h(x)$ [1] to be the coupling associated with the disorder in the action (3.1). Now, to solve tis theory one could in principle proceed as usual by calculating the partition function

$$Z[h, g^i] = \int \mathcal{D}\phi \mathrm{e}^{-\int \mathrm{d}^d x \left(h(x)\mathcal{O}_0(x) + \sum_i g^i(x)\mathcal{O}_i(x)\right)}, \tag{3.2}$$

where we have separated the path integral measure for the disorder field.

Here one can already see a problem with this approach arising. The partition function, and thus all observables of the theory, depends on the specific realization of the disorder $h(x)$. If one was trying to model physical systems with impurities this means that one would have to solve the theory for each possibility of the disorder. But of course we are interested instead in developing a general theory of disordered systems, not one theory for each sample. In other words, we need to develop a theory based on quantities that do not depend directly on $h(x)$. This quantities are called self-averaging, in the sense in large enough systems this quantities average out in a way that the result is independent of specific realizations of the disorder.

If we describe by N the size of the system, then a functional $A[g^i]$ is self-averaging if they are disorder-independent in the limit that the size of the system goes to infinity,

$$\lim_{N\to\infty} A_N[h, g^i] = A[g^i]. \tag{3.3}$$

Now if the probability distribution of the random coupling is $P(h)$ we can define the quenched average of a functional $A[h, g^i]$ as [2]

$$A_q[g^i] \equiv \overline{A[h, g^i]} = \int \mathcal{D}h P(h) A[h, g^i]. \tag{3.4}$$

It should be clear that the quenched average of a self-averaging quantity is equal to

[1] We include the possibility that the operator $\mathcal{O}_0$ can have an additional, non-random, coupling g^0 besides h.

[2] We denote by an overline the average over the disorder and by the usual brakets the average over quantum fluctuations

its h-independent value.

The free energy is extensive while the partition function is not. This means that F is, in general, self-averaging, while Z is not. This is why we work primarily with F.

The natural step now would be to obtain the quenched partition function, given by

$$Z_q[g^i] = \int \mathcal{D}hP(h) \int \mathcal{D}\phi \mathrm{e}^{-\int \mathrm{d}^d x \left(h(x)\mathcal{O}_0(x) + \sum_i g^i(x)\mathcal{O}_i(x)\right)}. \tag{3.5}$$

But this correspond to the case of annealed disorder, since this treats the fields h and ϕ in equal footing. Since we are interested in quenched disorder we must instead consider the quenched free energy

$$\begin{aligned} F_q[g^i] &= \int \mathcal{D}hP(h)F[h, g^i] \\ &= -\int \mathcal{D}hP(h) \log(Z[h, g^i]). \end{aligned} \tag{3.6}$$

Here we see yet another difficulty because we want to integrate the logarithm of the partition function. There are several ways to address this issue. The most common one is the replica trick, which we will now briefly review. An alternative is to use the zeta distributional method, that we will develop shortly. And then there are the supersymmetry and Keldysh approaches, that we will mention.

3.1 Replica Trick

The idea of the replica trick originates ([14]) from the simple identity

$$\log Z = \lim_{k\to 0} \frac{Z^k - 1}{k}, \tag{3.7}$$

where, of course, $k \in reals$.

Let us now consider an integer power of the partition function,

$$Z[h, g^i]^k = \int \left[\prod_{a=1}^{k} \mathcal{D}\phi_a \right] \mathrm{e}^{-\sum_{a=1}^{k} \int \mathrm{d}^d x \left(h(x)\mathcal{O}_{0,a}(x) + \sum_i g^i(x)\mathcal{O}_{i,a}(x) \right)}, \tag{3.8}$$

and interpret this to be the partition function of a new systems, made of k statistically independent identical replicas of the original system.

Taking the disorder average we can define of this new partition function we write

$$Z_k[g^i] = \int \mathcal{D}h P(h) Z[h, g^i]^k. \tag{3.9}$$

This leads to the definition of

$$F_k[g^i] = -\frac{1}{k} \log Z_k[g^i]. \tag{3.10}$$

Suppose now that the above expression, as a function of the index $k \in \mathbb{Z}$, can be analytically extended to a function with $k \in \mathbb{R}$. Then, from the identity (3.7), it follows that the original quenched free energy (3.6) can be written as

$$F_q[g^i] = \lim_{k \to 0} F_k[g^i]. \tag{3.11}$$

While the replica trick greatly simplifies the problem of obtaining the quenched free energy by reducing it to the problem of calculating powers of the partition function it is severely harmed by some mathematical problems.

The first and most immediate one is the fact that the whole trick hinges on the possibility of performing the analytical extension of a function $F : \mathbb{Z} \mapsto \mathbb{C}$ to a function $F : \mathbb{R} \mapsto \mathbb{C}$. Since it is not possible to analytically extend a function defined only on the integers([33]), this has to be broken into two steps: First, we must find an analytic function,f that coincides with F_n on the integers and then perform the analytical continuation of this new function to the real line on the complex plane. The problem then is that it is not possible to guarantee that such function f exists.

The second problem is that in eq. (3.8) we arrive at a effective action that has a symmetry under the permutation group of k elements, S_k. The problem is then to understand how this symmetry group behaves in the limit $k \to 0$. This can also be seen if we consider the set of all replicas to form a multiplet of fields, $\mathbf{\Phi} = (\phi_1, \phi_2, \cdots, \phi_k)$. When taking the limit $k \to 0$ we must understand the physical implications of a field theory for a multiplet of zero fields.

For all this problems that it is called the replica "*trick*" instead of replica *theory*.

Despite this problems the replica trick generally gives sensible results [3] and thus is still widely used in the statistical physics and related fields.

As an example of these ideas we briefly go over the two important examples, the random field Ising model (RFIM) and the random energy model.

The random field Ising model:

The action for the RFIM is

$$S[h, \phi] = \int \mathrm{d}^d\, x \left(\frac{1}{2} \left(\partial\phi(x)\right)^2 + \frac{1}{2} m_0^2 \phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 - h(x)\phi(x) \right). \tag{3.12}$$

To proceed using the replica trick let us now construct the replicated partition function,

$$Z^k[h] = \int \left[\prod_{a=1}^{k} \mathcal{D}\phi_a \right] \mathrm{e}^{-\sum_{a=1}^{k} S[h,\phi_a]}. \tag{3.13}$$

Now suppose the probability distribution for the disorder field h is Gaussian, i.e., given by

$$P(h) = \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{\frac{-h(x)^2}{2\sigma}}, \tag{3.14}$$

[3] one notable exception is the thermodynamics of the Sherrington-Kirkpatrick model

then the disorder average of the replica partition function is given by

$$Z_k = \int \mathcal{D}hP(h)Z^k[h] \tag{3.15}$$

$$Z_k = \int \left[\prod_{a=1}^{k} \mathcal{D}\phi_a\right] \mathrm{e}^{-S_{\mathtt{eff}}[k,\phi_a]}, \tag{3.16}$$

where the effective action $S_{\mathtt{eff}}[k, \phi_a]$ is given by

$$S_{\mathtt{eff}}[k,\phi_a] = \frac{1}{2}\sum_{a,b=1}^{k} \int \mathrm{d}^d\, x\phi_a(x) \left[\left(-\Delta + m_0^2\right)\delta_{ab} - \sigma\right]\phi_b(x) + \frac{\lambda}{4!}\sum_{a=1}^{k} \int \mathrm{d}^d\, x\phi_a(x)^4. \tag{3.17}$$

Here we can see that the processes of averaging over the disorder for the system of k independent replicas introduces an off-diagonal term in the propagator, connecting different replicas and thus giving rise to a non-trivial effective field theory.

In order to study the perturbative series for this RFIM we introduce the external sources $j_a(x)$ which are used to generate the Green functions and begin with the generating functional for the theory with λ interaction turned off,

$$Z_{0k} = \int \left[\prod_{a=1}^{k} \mathcal{D}\phi_a\right] \mathrm{e}^{-\frac{1}{2}\sum_{a,b=1}^{k} \int \mathrm{d}^d\, x\phi_a(x)\left[\left(-\Delta+m_0^2\right)\delta_{ab}-\sigma\right]\phi_b(x)}. \tag{3.18}$$

Then it follows immediately from the well-knows results for Gaussian integrals that

$$Z_{0k}[j_i] = \mathrm{e}^{\frac{1}{2}\sum_{a,b=0}^{k} \int \mathrm{d}^d\, x \int \mathrm{d}^d\, y j_a(x) G^{-1}_{0,ab}(m_0,\sigma;x-y) j_b(y)}. \tag{3.19}$$

The propagator in momentum space is, at tree-level,

$$G_{0,ab}(m_0,\sigma;p) = \frac{1}{\left(p^2 + m_0^2\right)\delta_{ab} - \sigma} \tag{3.20}$$

To invert equation we define the projector $P_{ab} = \frac{1}{k}$ and its complement $Q_{ab} = \delta_{ab} - P_{ab}$

and use the relation

$$(AQ_{ab}+BP_{ab})^{-1}=\frac{1}{A}Q_{ab}+\frac{1}{B}P_{ab}, \tag{3.21}$$

to obtain

$$G_{0,ab}(m_0,\sigma;p)=\frac{\delta_{ab}}{p^2+m_0^2}+\frac{\sigma}{(p^2+m_0^2)p^2+m_0^2-k\sigma} \tag{3.22}$$

The first term is the usual propagator. The second term contains the corrections to the bare propagator due to the averaging over disorder h.

The random mass Ising model:

In this model the disorder couples to ϕ^2, thus acting as an random correction to the field mass.

$$S=\int \mathrm{d}^d\, x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}m_0^2\phi^2+h(x)\phi^2+\frac{\lambda}{4!}\phi^4\right) \tag{3.23}$$

If, as above, we construct the replicated partition function and take the disorder average we again obtain $Z_k=\int\mathcal{D}\phi\exp\left(-S_{\texttt{eff}}\right)$, where now the effective action is given by

$$S_{\texttt{eff}}[k,\phi_a]=\int \mathrm{d}^d\, x\sum_{a=1}^{k}\phi_a(x)\frac{1}{2}\left(-\Delta+m_0^2\right)\phi_a(x)+\frac{\sigma}{2}\int \mathrm{d}^d\, x\left(\sum_{a=1}^{k}\phi_a(x)^2\right)^2+$$
$$+\int \mathrm{d}^d\, x\frac{\lambda}{4!}\sum_{a=1}^{k}\int \mathrm{d}^d\, x\phi_a(x)^4. \tag{3.24}$$

Here we see that the disorder averaging generates an effective interaction between the replicas

3.2 Distributional Zeta Function

In this section we summarize the results obtained in [20] for an alternative method for calculating the quenched free energy of a disordered systems, without use of replicas.

In this approach we define the distributional zeta-function $\mathbf{\Phi}(s)$, for $s \in \mathbb{C}$, as

$$\mathbf{\Phi}(s) = \int \mathrm{d}[h] P(h) \frac{1}{Z(h)^s}, \tag{3.25}$$

where $P(h)$ is the probability distribution for the random variable h and $Z(h)$ is the h-dependent partition function

$$Z[h] = \int \mathcal{D}\phi \mathrm{e}^{-S[h,\phi]}. \tag{3.26}$$

It is possible to show that this function $\mathbf{\Phi}(s)$ is well defined in the half complex plane $\mathtt{Re}(s) \geq 0$.

Now, since we can write $Z(h)^{-s} = \mathrm{e}^{-s\log(Z(h))}$, it follows that

$$-\frac{d}{ds} Z(h)^{-s}\bigg|_{s=0^+} = \log Z(h), \tag{3.27}$$

$$\tag{3.28}$$

and recalling , we have

$$F_q = -\int \mathcal{D}h P(h) \, -\frac{d}{ds} Z(h)^{-s}\bigg|_{s=0^+} = -\frac{d}{ds}\mathbf{\Phi}(s)\bigg|_{s=0^+}. \tag{3.29}$$

We have thus obtained an analytic expression for the quenched free energy that does not involve the integer moments of the partition function and instead depends only on the calculation of the function $\mathbf{\Phi}(s)$. It is possible to show that with this approach the quenched free energy can be written as

$$F_q = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!k} \overline{Z^k} - \gamma + R(1), \quad |R(1)| \leq \frac{1}{Z(0)} \mathrm{e}^{-Z(0)}, \tag{3.30}$$

where $\gamma = 0.577\dots$ is Euler's constant and $R(1)$ is just some constant.

Thus, defining a distributional zeta-function it is possible to show that all moments of the partition function contribute to the quenched free energy, as opposed to the replica trick, where one has to take the limit $k \to 0$.

Chapter 4

Zero-dimensional Model

We now wish to introduce disorder into the zero-dimensional field theory. As we have seen in chapter 3 there are two main possibilities considered in the literature. The random field Ising model, where the disorder h couples to ϕ in the action, and the random energy model, where the disorder couples to ϕ^2. In either case we are interested in the quenched free-energy.

4.1 Random Mass Ising Model

First we consider the case where the disorder couples as a mass term. In this case the action is given by

$$S(h, m_0^2, \lambda) = \frac{1}{2}\left(m_0^2 + h\right)\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{4.1}$$

Thus the quenched free energy is given by

$$Z[h] = \int \mathcal{D}\phi \mathrm{e}^{-\frac{1}{2}\left(m_0^2+h\right)\phi^2 - \frac{\lambda}{4!}\phi^4}, \qquad -F[h] = \int \frac{\mathrm{d}h}{\sqrt{2\pi\sigma}} \mathrm{e}^{\frac{-h^2}{2\sigma}} \log Z[h]. \tag{4.2}$$

In this case, the usual perturbative expansion of the free energy will be a double expansion, in terms of the coupling interaction λ and the disorder parameter σ. To simplify and set us up for more straight forward comparisons with the literature [22–24] we fix the ratio $\frac{\sigma}{\lambda} = \gamma$, which allows us to expand the free energy in terms only of powers of λ,

$$-F = \sum_{k=1}^{\infty} A_k(\lambda)\lambda^k. \tag{4.3}$$

We are now interested in the question of determining the behavior of the coefficients $A_K(\lambda)$ in the asymptotic limit $k \to \infty$ and whether the Borel transform is analytic, which would allow us to obtain some non-perturbative information from the system.

The first step in this calculation is to obtain the disorder average of the replica partition function

$$Z[h]^k = \int \left(\prod_{a=1}^{k} \mathrm{d}\phi_a \right) \mathrm{e}^{\sum_{a=1}^{k} \left(-\frac{1}{2}(m_0^2+h)\phi_a^2 - \frac{\lambda}{4!}\phi_a^4 \right)}. \tag{4.4}$$

Assuming the disorder has a Gaussian probability distribution and defining $\mathrm{d}\phi \equiv \prod_{a=1}^{k} \mathrm{d}\phi_a$, it follows that

$$\begin{aligned}
\mathbb{E}(Z^k) &= \int \mathcal{D}h\, \mathrm{d}\phi \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{h^2}{2\sigma}} \mathrm{e}^{\sum_{a=1}^{k} \left(-\frac{1}{2}(m_0^2+h)\phi_a^2 - \frac{\lambda}{4!}\phi_a^4 \right)} \\
\mathbb{E}(Z^k) &= \int \mathrm{d}\phi \mathrm{e}^{\sum_{a=1}^{k} \left(-\frac{1}{2}m_0^2\phi_a^2 - \frac{\lambda}{4!}\phi_a^4 \right)} \int \mathcal{D}h \mathrm{e}^{-\frac{h^2}{2\sigma} - \frac{1}{2}h\left(\sum_{a=1}^{k} \phi_a^2 \right)} \\
\mathbb{E}(Z^k) &= \int \mathrm{d}\phi \mathrm{e}^{\sum_{a=1}^{k} \left(-\frac{1}{2}m_0^2\phi_a^2 - \frac{\lambda}{4!}\phi_a^4 \right)} \mathrm{e}^{\frac{\sigma}{2}\left(\frac{1}{2}\sum_{a=1}^{k} \phi_a^2 \right)^2} \\
\mathbb{E}(Z^k) &= \int \mathrm{d}\phi \mathrm{e}^{-S_{\mathsf{eff}}(k,\phi_a)},
\end{aligned} \tag{4.5}$$

where the effective action is given by

$$S_{\mathsf{eff}}(k,\phi_a) = \frac{1}{2}m_0^2 \sum_{a=1}^{k} \phi_a^2 + \frac{\lambda}{4!} \sum_{a=1}^{k} \phi_a^4 - \frac{\sigma}{8} \left(\sum_{a=1}^{k} \phi_a^2 \right)^2. \tag{4.6}$$

Here we note how the act of averaging over the disorder induces a $O(N)$ type interaction and thus leaves the Gaussian part of the action the same as in the disorder-free case.

Going back to the averaged partition function and fixing the ratio $\frac{\sigma}{\lambda} = \gamma$ we can write

$$\mathbb{E}(Z^k) = \int \mathrm{d}\phi \mathrm{e}^{-\lambda \sum_{a=1}^{k} \left(\frac{1}{4!}\phi_a^4 - \frac{\gamma}{8}\left(\sum_{a=1}^{k} \phi_a^2\right)^2 \right)} \mathrm{e}^{-\sum_{a=1}^{k} \frac{1}{2} m_0^2 \phi_a^2}$$
$$\mathbb{E}(Z^k) = \int \mathrm{d}\phi \left[\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(-\frac{1}{4!} \sum_{a=1}^{k} \phi_a^4 + \frac{\gamma}{8} \left(\sum_{a=1}^{k} \phi_a^2 \right)^2 \right)^n \right] \mathrm{e}^{-\frac{m_0^2}{2} \sum_{a=1}^{k} \phi_a^2} \tag{4.7}$$

Thus we have the replica partition function in the form of an expansion $\mathbb{E}(Z^k) = \sum_{n=0}^{\infty} A_{k,n}(\lambda)\lambda^n$, where the coefficients are given by

$$A_{k,n}(\lambda) = \frac{(-1)^n}{n!4!^n} \int \mathrm{d}\phi \left(\sum_{a=1}^{k} \phi_a^4 - 3\gamma \left(\sum_{a=1}^{k} \phi_a^2 \right)^2 \right)^n \mathrm{e}^{-\frac{m_0^2}{2} \sum_{a=1}^{k} \phi_a^2} \tag{4.8}$$

Let us now perform a rescaling of the fields $\phi_a \mapsto \sqrt{n}\phi_a$. The above expression then becomes

$$A_{k,n}(\lambda) = \frac{(-1)^n}{n!4!^n} \int \mathrm{d}\phi \left(\sum_{a=1}^{k} \phi_a^4 - 3\gamma \left(\sum_{a=1}^{k} \phi_a^2 \right)^2 \right)^n \mathrm{e}^{-\frac{m_0^2}{2} \sum_{a=1}^{k} \phi_a^2}$$
$$A_{k,n}(\lambda) = \frac{(-1)^n}{n!4!^n} n^{\frac{k}{2}} \mathrm{e}^{2n \log n} \int \mathrm{d}\phi \mathrm{e}^{nG[k,\phi_a]}, \tag{4.9}$$

where the functional $G[k, \phi_a]$ is defined by

$$G[k, \phi_a] = -\frac{1}{2} m_0^2 \sum_{a=1}^{k} \phi_a^2 + \log \left[\sum_{a=1}^{k} \phi_a^4 - 3\gamma \left(\sum_{a=1}^{k} \phi_a^2 \right)^2 \right]. \tag{4.10}$$

Recalling the expression for the free energy in the distributional zeta-function formalism as a sum over all moments of the replica partition function (3.30) we can write the free energy as

$$F_q = \sum_{n=0}^{\infty} B_n(\lambda)\lambda^n, \tag{4.11}$$

where the expansion coefficients are given by

$$B_n(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^{k+n+1}}{k!kn!4!^n} \mathrm{e}^{\left(2n+\frac{k}{2}\right)\log n} \int \mathrm{d}\phi \mathrm{e}^{nG[k,\phi_a]}. \tag{4.12}$$

To study the behavior of these coefficients in large orders of perturbation theory we perform a saddle-point analysis of the integral above. The extrema are then found by the equation

$$\frac{\partial G[k,\phi_a]}{\partial \phi_a} = \phi_a \left(-m_0^2 + 4\frac{\phi_a^2 - 3\gamma \sum_{b=1}^{k} \phi_b^2}{\sum_{b=1}^{k} \phi_b^4 - 3\gamma \left(\sum_{b=1}^{k} \phi_b^2\right)^2} \right) = 0. \tag{4.13}$$

To proceed let us assume the replica symmetric ansatz, where all replica fields are equal, $\phi_a = \varphi$. The solution for the saddle-point equation above is then

$$\varphi^2 = \frac{4}{km_0^2}. \tag{4.14}$$

The value of the G functional in this saddle-point is given by

$$G[k,\phi_a = \varphi] = -\frac{2}{m_0^2} + \log\left[\frac{16}{m_0^4}\right] + \log\left[\frac{1}{k} - 3\gamma\right] \tag{4.15}$$

Here we find an upper bound for the number of replicas, $k > \frac{1}{3\gamma}$, since the logarithm has a discontinuity at this point.

For the fluctuations around this saddle-point the Hessian matrix is

$$\begin{aligned} \frac{\partial G[k,\phi_a]}{\partial \phi_a \partial \phi_b} &= \delta_{ab} \left(-m_0^2 + 4\frac{\phi_a^2 - 3\gamma \sum_{c=1}^{k} \phi_c^2}{\sum_{c=1}^{k} \phi_c^4 - 3\gamma \left(\sum_{c=1}^{k} \phi_c^2\right)^2} \right) \\ &+ \phi_a \left(4\frac{2\phi_a\delta_{ab} - 3\gamma 2\phi_b}{\sum_{c=1}^{k} \phi_c^4 - 3\gamma \left(\sum_{c=1}^{k} \phi_c^2\right)^2} + 4\frac{\phi_a^2 - 3\gamma \sum_{c=1}^{k} \phi_c^2}{\left[\sum_{c=1}^{k} \phi_c^4 - 3\gamma \left(\sum_{c=1}^{k} \phi_c^2\right)^2\right]^2} \left[4\phi_b^3 - 3\gamma 2 \left(\sum_{c=1}^{k} \phi_c^2\right) 2\phi_b\right] \right) \end{aligned} \tag{4.16}$$

Substituting the saddle-point value of the field φ we obtain

$$\frac{\partial G[k,\phi_a]}{\partial \phi_a \partial \phi_b} = m_0^2 \left(\frac{2\delta_{ab}}{1 - 3\gamma k} - \frac{6\gamma}{1 - 3\gamma k} - \frac{4}{k} \right) \tag{4.17}$$

The eigenvalues of this matrix are

$$\Lambda_1 = -2m_0^2$$
$$\Lambda_2 = m_0^2 \frac{2}{1-3\gamma k},$$

with degeneracies 1 and $k-1$, respectively. For this saddle-point to be a true minimum this eigenvalues should be negative. But for Λ_2 to be negative we must have $1-3\gamma k < 0$, which contradicts the requirement from (4.1), thus we need to break the replica symmetry. Let us ignore this for now and proceed.

Then

$$\int \mathrm{d}\phi \mathrm{e}^{nG[s,\phi_a]} = \mathrm{e}^{nG[k,\varphi]} \left(\frac{2\pi}{n}\right)^{\frac{k}{2}} (-\Lambda_1)^{-\frac{1}{2}} (-\Lambda_2)^{-\frac{k-1}{2}} \tag{4.18}$$

With this we can write

$$B_n(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^{k+n+1}}{k!kn!4!^n} \mathrm{e}^{\left(2n+\frac{k}{2}\right)\log n} \mathrm{e}^{nG[k,\varphi]} \left(\frac{2\pi}{n}\right)^{\frac{k}{2}} (-\Lambda_1)^{-\frac{1}{2}} (-\Lambda_2)^{-\frac{k-1}{2}}$$
$$B_n(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^{k+n+1}}{k!kn!4!^n} \left(\frac{2\pi}{n}\right)^{\frac{k}{2}} \mathrm{e}^{\left(2n+\frac{k}{2}\right)\log n} \mathrm{e}^{-\frac{2n}{m_0^2}+n\log\left[\frac{16}{m_0^4}\right]+n\log\left[\frac{1}{k}-3\gamma\right]} \left(2m_0^2\right)^{-\frac{1}{2}} \left(-m_0^2\frac{2}{1-3\gamma k}\right)^{-\frac{k-1}{2}}$$
$$B_n(\lambda) = \left(2m_0^2\right)^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^{n+k+1}(4\pi)^k}{k!k^{n+1}n!4!^n} \mathrm{e}^{2n\log n} \mathrm{e}^{-\frac{2n}{m_0^2}+n\log\left[\frac{16}{m_0^4}\right]} [1-3\gamma k]^n \left(-m_0^2\frac{2}{1-3\gamma k}\right)^{-\frac{k-1}{2}} \tag{4.19}$$

For large orders of perturbation theory, i.e. for $n \to \infty$, this diverges. The expansion in the replica number is also not possible to be summed. This suggests that we must break the replica symmetry.

This can be done in a way that $\phi_a = \varphi$ for $a < s$ and $\phi_a = 0$ for $a > 0$. This leads to $\varphi^2 = \frac{4}{sm_0^2}$ and there are $2^s\binom{k}{s}$ different solutions.

The value of the G functional in this saddle-point is given by

$$G[s, \phi_a = \varphi] = -\frac{2}{m_0^2} + \log\left[\frac{16}{m_0^4}\right] + \log\left[\frac{1}{s} - 3\gamma\right] \tag{4.20}$$

For the fluctuations around the saddle-point the Hessian matrix is

$$\frac{\partial G[s,\phi_a]}{\partial\phi_a\partial\phi_b} = \delta_{ab}\left(-m_0^2 + 4\frac{\phi_a^2 - 3\gamma\sum_{c=1}^k\phi_c^2}{\sum_{c=1}^k\phi_c^4 - 3\gamma\left(\sum_{c=1}^k\phi_c^2\right)^2}\right)$$
$$+\phi_a\left(4\frac{2\phi_a\delta_{ab} - 3\gamma 2\phi_b}{\sum_{c=1}^k\phi_c^4 - 3\gamma\left(\sum_{c=1}^k\phi_c^2\right)^2} + 4\frac{\phi_a^2 - 3\gamma\sum_{c=1}^k\phi_c^2}{\left[\sum_{c=1}^k\phi_c^4 - 3\gamma\left(\sum_{c=1}^k\phi_c^2\right)^2\right]^2}\left[4\phi_b^3 - 3\gamma 2\left(\sum_{c=1}^k\phi_c^2\right)2\phi_b\right]\right) \tag{4.21}$$

At the saddle point $\sum_{c=1}^k\phi_c^2 = \frac{4}{m_0^2}$. If $a, b > s$

$$\frac{\partial G[s,\phi_a]}{\partial\phi_a\partial\phi_b} = -\frac{m_0^2\delta_{ab}}{1-3\gamma s}. \tag{4.22}$$

If $a \le s, b > s$

$$\frac{\partial G[s,\phi_a]}{\partial\phi_a\partial\phi_b} = 0. \tag{4.23}$$

If $a, b \le s$

$$\frac{\partial G[s,\phi_a]}{\partial\phi_a\partial\phi_b} = m_0^2\left(\frac{2\delta_{ab}}{1-3\gamma s} - \frac{6\gamma}{1-3\gamma s} - \frac{4}{s}\right) \tag{4.24}$$

The eigenvalues of this Hessian matrix are

$$\Lambda_1 = -2m_0^2, \quad \Lambda_2 = m_0^2\frac{2}{1-3\gamma s}, \quad \Lambda_3 = -\frac{1}{1-3\gamma s} \tag{4.25}$$

with degeneracies 1,$s-1$ and $k-s$, respectively. The true minimum is when all eigenvalues are negative, which is only possible for $s = 1$ and $\gamma < \frac{1}{3}$.

For multiple saddle-points

$$\int \mathrm{d}\phi\,\mathrm{e}^{nG[s,\phi_a]} = \sum_{s=1}^k 2^s\binom{n}{s}\mathrm{e}^{nG[s,\varphi]}\left(\frac{2\pi}{n}\right)^{\frac{k}{2}}(-\Lambda_1)^{-\frac{1}{2}}(-\Lambda_2)^{-\frac{s-1}{2}}(-\Lambda_3)^{-\frac{k-s}{2}} \tag{4.26}$$

For the case with $s = 1$

$$\int \mathrm{d}\phi \mathrm{e}^{nG[s,\phi_a]} = 2n\mathrm{e}^{nG[s,\varphi]} \left(\frac{2\pi}{n}\right)^{\frac{k}{2}} \left(2m_0^2\right)^{-\frac{1}{2}} \left(\frac{1}{1-3\gamma}\right)^{-\frac{k-1}{2}} \tag{4.27}$$

Then

$$B_n(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^{k+n+1}}{k!kn!4!^n} \mathrm{e}^{\left(2n+\frac{k}{2}\right)\log n} 2n\mathrm{e}^{n\left(-\frac{2}{m_0^2}+\log\left[\frac{16}{m_0^4}\right]+\log[1-3\gamma]\right)} \left(\frac{2\pi}{n}\right)^{\frac{k}{2}} \left(2m_0^2\right)^{-\frac{1}{2}} \left(\frac{1}{1-3\gamma}\right)^{-\frac{k-1}{2}} \tag{4.28}$$

We can evaluate this series and find that it converges to

$$B_n(\lambda) = \frac{1}{\sqrt{m^2}n!}\sqrt{\frac{2}{1-3\gamma}} \left(-\frac{1}{4!}\right)^n n^{2n+1} \mathrm{e}^{nG[1,\varphi]} \left(\log\left(\sqrt{2\pi(1-3\gamma)}\right) + \Gamma\left(\sqrt{2\pi(1-3\gamma)}\right) + \gamma_E\right). \tag{4.29}$$

The possibility to sum the replica expansion is already very interesting, since it points to the existence of physical effects due to the contributions of all replica numbers and such effects would not be present in the usual replica trick calculation since one would lose all this information by taking the limit where the number of replicas goes to zero. But despite this interesting progress the result we have found for perturbative expansion of the free energy is still not Borel summable, due to the existence of the n^n factor in (4.29). This factor is not present in the analysis in [22], although it is not clear how they removed it. If this factor were not present the series would indeed be summable, yielding an analytic result.

4.2 Random Field Ising Model

We now perform the same analysis for the model where the disorder couples as an external random source. Returning to the action of the random field model we have that the partition function in the presence of a random source h is given by

$$Z[h] = \int \mathrm{d}\phi \mathrm{e}^{-\left[\frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4 - h\phi\right]}. \tag{4.30}$$

In order to calculate the quenched free energy using the distributional zeta-function approach we need to construct Z^k

$$Z^k[h] = \int \left(\prod_{a=0}^{k} \mathrm{d}\phi_a\right) \mathrm{e}^{-\sum\limits_{a=0}^{k}\left[\frac{1}{2}m_0^2\phi_a^2 + \frac{\lambda}{4!}\phi_a^4 - h\phi_a\right]}. \tag{4.31}$$

We need now the expected value of this partition function with regard to the external disorder field, which is necessary to calculate the quenched free energy.

We choose a Gaussian distribution with mean zero and variance σ for the disorder,

$$P(h) = \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{\frac{-h^2}{2\sigma}}. \tag{4.32}$$

Then, the expectation value is

$$\mathbb{E}(Z^k) = \int \mathrm{d}h P(h) Z^k[h]$$

$$\mathbb{E}(Z^k) = \int \mathrm{d}h P(h) \int \left(\prod_{a=0}^{k} \mathrm{d}\phi_a\right) \mathrm{e}^{-\sum\limits_{a=0}^{k}\left[\frac{1}{2}m_0^2\phi_a^2 + \frac{\lambda}{4!}\phi_a^4 - h\phi_a\right]}$$

$$\mathbb{E}(Z^k) = \int \mathrm{d}h \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{\frac{-h^2}{2\sigma}} \int \left(\prod_{a=0}^{k} \mathrm{d}\phi_a\right) \mathrm{e}^{-\sum\limits_{a=0}^{k}\left[\frac{1}{2}m_0^2\phi_a^2 + \frac{\lambda}{4!}\phi_a^4 - h\phi_a\right]}$$

$$\mathbb{E}(Z^k) = \int \left(\prod_{a=0}^{k} \mathrm{d}\phi_a\right) \mathrm{e}^{-\sum\limits_{a=0}^{k}\left[\frac{1}{2}m_0^2\phi_a^2 + \frac{\lambda}{4!}\phi_a^4\right]} \int \mathrm{d}h \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{\frac{-h^2}{2\sigma} + h\sum\limits_{i=0}^{k}\phi_a}$$

$$\mathbb{E}(Z^k) = \int \left(\prod_{a=0}^{k} \mathrm{d}\phi_a\right) \mathrm{e}^{-\sum\limits_{a=0}^{k}\left[\frac{1}{2}m_0^2\phi_a^2 + \frac{\lambda}{4!}\phi_a^4\right]} \mathrm{e}^{\sum\limits_{a,b=0}^{k}\phi_a \frac{\sigma}{2}\phi_b}$$

$$\mathbb{E}(Z^k) = \int \left(\prod_{a=0}^{k} \mathrm{d}\phi_a\right) \exp\left[-\sum_{a,b=0}^{k} \frac{1}{2}\left(\delta_{ab}m_0^2 - \sigma\right)\phi_a\phi_b - \frac{\lambda}{4!}\sum_{a=1}^{k}\phi_a^4\right] \tag{4.33}$$

So the effective action is given by

$$S_{\text{eff}}(\phi_i, k) = \frac{1}{2}\sum_{a,b=1}^{k}\phi_a\left(\delta_{ab}m_0^2 - \sigma\right)\phi_b + \frac{\lambda}{4!}\sum_{a=1}^{k}\phi_b^4 \tag{4.34}$$

Utilizing the replica symmetric ansatz ($\phi_i = \phi$) we can calculate the saddle-point for this effective action.

$$\left(m_0^2 - k\sigma\right)\phi = -\frac{k\lambda}{3!}\phi^3. \tag{4.35}$$

Since the mass term must be positive definite we have that $m_0^2 - k\sigma$ must be positive and thus we have an upper bound for the replica number, $k_m = m_0^2/\sigma$.

Note that this action is not diagonal in the space of replicas - the term with σ connects different replicas. To solve this model we need to diagonalize the action by some similarity transformation for the $k \times k$ matrix operator defined by $A_{k,ij} = \delta_{ij}m_0^2 - \sigma^2 J_k$, where J_k is a $k \times k$ matrix where every element is equal to one.

Since A is a real symmetric matrix the spectral theorem says that it can be diagonalized by an orthogonal matrix. The eigenvalues of A_k are $\frac{m_0^2}{2}$, with degenerescence $k-1$, and $\frac{1}{2}\left(m_0^2 - k\sigma\right)$.

Performing a perturbative expansion we have

$$\mathbb{E}(Z^k) = \int\left(\prod_{a=0}^{k}\mathrm{d}\phi_a\right)\exp\left[-\sum_{a,b=0}^{k}\frac{1}{2}\left(\delta_{ab}m_0^2 - \sigma\right)\phi_a\phi_b - \frac{\lambda}{4!}\sum_{a=1}^{k}\phi_a^4\right] \tag{4.36}$$

$$\mathbb{E}(Z^k) = \int\left(\prod_{a=0}^{k}\mathrm{d}\phi_a\right)\left[\sum_{n=0}\frac{(-1)^n\lambda^n}{n!4!^n}\left(\sum_{a=1}^{k}\phi_a^4\right)^n\right]\mathrm{e}^{-\sum_{a,b=0}^{k}\frac{1}{2}\left(\delta_{ab}m_0^2-\sigma\right)\phi_a\phi_b} \tag{4.37}$$

We will analyze in full some low order terms. For $k = 1$ we have that the effective action (4.34) is just the original action of the zero dimensional $\lambda\phi^4$ with a mass correction due to the disorder.

$$S_{\text{eff}}(\phi_i, k) = \frac{1}{2}\left(m_0^2 - \sigma\right)\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{4.38}$$

It follows that, writing $m_1^2 = m_0^2 - \sigma$ the expected value can be calculated just as in the pure model:

$$\mathbb{E}(Z) = \int \mathrm{d}\phi \exp\left[-\frac{1}{2}m_1^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] \tag{4.39}$$

$$\mathbb{E}(Z) = \sqrt{2\pi}\left(\frac{3}{2\lambda}\right)^{\frac{3}{4}} m_1^2 U\left(\frac{3}{4}, \frac{3}{2}, \frac{3m_1^2}{2\lambda}\right) \tag{4.40}$$

For $k = 2$ we can define the doublet

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{4.41}$$

Then, the effective action can be written as

$$S_{\text{eff}}(\phi_i, 2) = \boldsymbol{\phi}^{\intercal} A_2 \boldsymbol{\phi} + \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right), \tag{4.42}$$

where the matrix A_2 is given by

$$A_2 = \begin{pmatrix} \frac{1}{2}\left(m_0^2 - \sigma\right) & -\frac{1}{2}\sigma \\ -\frac{1}{2}\sigma & \frac{1}{2}\left(m_0^2 - \sigma\right) \end{pmatrix}. \tag{4.43}$$

This matrix can be diagonalized by the orthogonal matrix P_2:

$$P_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \tag{4.44}$$

,

and introducing this similarity transformation in the effective action for $k = 2$ gives:

$$S_{\text{eff}}(\phi_i, 2) = \boldsymbol{\phi}^{\intercal} A_2 \boldsymbol{\phi} + \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) \tag{4.45}$$

$$S_{\text{eff}}(\phi_i, 2) = \boldsymbol{\phi}^{\intercal} P_2 P_2^{\intercal} A_2 P_2 P_2^{\intercal} \boldsymbol{\phi} + \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) \tag{4.46}$$

$$S_{\text{eff}}(\phi_i, 2) = (P_2^{\intercal}\boldsymbol{\phi})^{\intercal}\, D\, (P_2^{\intercal}\boldsymbol{\phi}) + \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) \tag{4.47}$$

where D_2 is the diagonal matrix with the eigenvalues of A_2

$$D_2 = \begin{pmatrix} \frac{1}{2}m_0^2 & 0 \\ 0 & \frac{1}{2}\left(m_0^2 - 2\sigma\right) \end{pmatrix} \tag{4.48}$$

Defining a new double, given by $\tilde{\boldsymbol{\phi}} = P_2^{\intercal}\boldsymbol{\phi}$ and substituting in the effective action gives

$$S_{\text{eff}}(\tilde{\phi}_i, 2) = \tilde{\boldsymbol{\phi}}^{\intercal} D \tilde{\boldsymbol{\phi}} + \frac{\lambda}{4!4}\left(\left(-\tilde{\phi}_1 + \tilde{\phi}_2\right)^4 + \left(\tilde{\phi}_1 + \tilde{\phi}_2\right)^4\right) \tag{4.49}$$

$$S_{\text{eff}}(\tilde{\phi}_i, 2) = \frac{1}{2}m_0^2\tilde{\phi}_1^2 + \frac{1}{2}\left(m_0^2 - 2\sigma\right)\tilde{\phi}_2^2 + \frac{\lambda}{4!2}\left(\tilde{\phi}_1^4 + \tilde{\phi}_2^4 + 6\tilde{\phi}_1^2\tilde{\phi}_2^2\right) \tag{4.50}$$

The effect of diagonalizing the operator A by a rotation in the space of fields is introducing a new interaction term.

The path integral for this action can be solved by weak perturbation expansion. Introducing two source fields, g_1 and g_2, we have:

$$\mathbb{E}_{g_1 g_2}(Z^2) = \int \mathrm{d}\tilde{\phi}_1\, \mathrm{d}\tilde{\phi}_2 \exp\Bigg[-\frac{1}{2}m_0^2\tilde{\phi}_1^2 - \frac{1}{2}\left(m_0^2 - 2\sigma\right)\tilde{\phi}_2^2 + \\ -\frac{\lambda}{4!2}\left(\tilde{\phi}_1^4 + \tilde{\phi}_2^4 + 6\tilde{\phi}_1^2\tilde{\phi}_2^2\right) + g_1\tilde{\phi}_1 + g_2\tilde{\phi}_2\Bigg] \tag{4.51}$$

The original expectation value is recovered in the limit where the sources go to zero. Considering the interaction as a perturbation we can rewrite the above as

$$\mathbb{E}_{g_1 g_2}(Z^2) = \mathrm{e}^{\frac{\lambda}{4!2}\left(\frac{\partial^4}{\partial g_1^4}+\frac{\partial^4}{\partial g_2^4}+6\frac{\partial^2}{\partial g_1^2}\frac{\partial^2}{\partial g_1^2}\right)} \int \mathrm{d}\tilde{\phi}_1 \, \mathrm{d}\tilde{\phi}_2 \mathrm{e}^{\frac{1}{2}m_0^2\tilde{\phi}_1^2+\frac{1}{2}\left(m_0^2-2\sigma\right)\tilde{\phi}_2^2+g_1\tilde{\phi}_1+g_2\tilde{\phi}_2} \tag{4.52}$$

$$\mathbb{E}_{g_1 g_2}(Z^2) = \frac{2\pi}{m_0\sqrt{m_0^2-2\sigma^2}} \mathrm{e}^{\frac{\lambda}{4!2}\left(\frac{\partial^4}{\partial g_1^4}+\frac{\partial^4}{\partial g_2^4}+6\frac{\partial^2}{\partial g_1^2}\frac{\partial^2}{\partial g_2^2}\right)} \mathrm{e}^{\frac{g_1^2}{2m_0^2}+\frac{g_2^2}{2m_0^2-4\sigma^2}} \tag{4.53}$$

$$\mathbb{E}(Z^2) = \frac{2\pi}{m_0\sqrt{m_0^2-2\sigma^2}} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4!2}\right)^n \left(\frac{\partial^4}{\partial g_1^4}+\frac{\partial^4}{\partial g_2^4}+6\frac{\partial^2}{\partial g_1^2}\frac{\partial^2}{\partial g_2^2}\right)^n \mathrm{e}^{\frac{g_1^2}{2m_0^2}+\frac{g_2^2}{2m_0^2-4\sigma^2}} \tag{4.54}$$

We can do this same calculation for every value of k, up to the maximum value obtained from the saddle-point. Since this is effectively a truncation of the free energy expansion obtained from the distributional zeta functional approach this gives an summable series. That is, since the term with $k = 1$ is analytical, just as in the non-disordered case [21] and the terms with $k \geq 2$ are just a perturbative expansion of k interacting fields, where one has the original bare mass and the $k - 1$ other ones have a mass correction from disorder, the free energy in this model is Borel summable.

Chapter 5

Conclusions

Motivated by the mathematical problems in replica trick calculations of disordered systems, Svaiter [20] introduced a new way to obtain the quenched free energy of a disordered system. While in the usual replica trick one must take the limit where the number of replicas goes to zero in the end of calculations, in this new method, using the distributional zeta function, one takes in consideration the contributions from all replica numbers and thus all moments of the partition function contribute to the quenched free energy.

As a way to test the validity of this new method of calculating the quenched free energy of a disordered system we analyze the Borel summability of the perturbative expansion of two zero-dimensional models, the random field and the random mass models. For the random mass model we are able to compare our results with the ones present in the literature. In [22] it is found that the series expansion for the free energy is not Borel summable. They note that the lack of summability occurs because of the finite probability that the total mass term is negative and thus the perturbative expansion is meaningless since the expansion would be around a false vacuum. They also note that the calculations made in that work using the replica trick are very questionable and one would probably need to do away with them in order to obtain a full analytical result. This was attempted in [23], where they identify the types of non-analytical behavior stemming from a exact calculation of the partition function and are able to recover the results of Bray et al. On the other hand, in [24] some alternative summation methods are addressed and they show that the perturbative expansion for the free energy is in fact summable if one perform the summation in a sequential method: first sum the coupling expansion and then sum the disorder expansion, but they leave as an open problem the solution to this problem using the replica trick.

In our work we make use of the distributional zeta function to shed new light on the contributions of the different replicas to the summability of the free energy. In section 4.1 we find that the free energy is not Borel summable. This is a sign that although the distributional zeta function offers new insight into the physics of the replicas, since we do not take the limit where the number of replicas goes to zero, there is still some effects that might contribute to the summability of the perturbative expansion. One possibility is to better explore the spontaneous breaking of the replica symmetry.

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