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"TWO PATHS BEYOND THE STANDARD MODEL: SUPERSYMMETRY AND LORENTZ SYMMETRY VIOLATION MEET IN THE GAUGE-BOSON/GAUGINO TRANSITION"

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Abstract

In this Dissertation, we aim to give a general contextualization of the activity on Lorentz Symmetry Violation (LSV) and its relation with Supersymmetry (SUSY). We adopt the superfield approach to SUSY in $(1+3)$ dimensions and use it to formulate an supersymmetric version of the Yang-Mills-Carroll-Field-Jackiw model, with the Lorentz-symmetry violating background introduced by means of a superfield. This LSV supermultiplet induces a mixing term between the gauge-boson and the gaugino in the component-field action. An introductory inspection of this mixing mechanism, induced by background fermionic condensates and analogue to the axion/photon Primakoff effect, is carried out as a closure of this Dissertation.

Keyword: Supersymmetry, Lorentz Symmetry Violation, Carroll-Field-Jackiw models, Breaking of Supersymmetry, Primakoff effect, Mixing of particles.

Resumo

Nesta Dissertação, objetivamos dar uma contextualização geral das atividades em Violação da Simetria de Lorentz (LSV) e em sua relaçao com a Supersimetria (SUSY). Nós adotamos a abordagem de supercampos para a SUSY em $(1 + 3)$ dimensões e à usamos para formular uma versão supersimétrica do modelo de Yang-Mills-Carroll-Field-Jackiw, com o fundo violador da simetria de Lorentz introduzido por meio de um supercampo. Este supermultipleto LSV induz um termo de mixing entre o boson de gauge e o gaugino na ação em termos dos campos componentes. Uma inspeção introdutória à este mecanismo de mixing, induzido pelos condensados do fundo fermiônico e analógo ao efeito Primakoff áxion/photon, é realizada como um fechamento desta Dissertação.

Palavras-Chave: Supersimetria, Violação da Simetria de Lorentz, Modelo de Carroll-Field-Jackiw, Quebra da Supersimetria, Efeito Primakoff, Mixing de partículas.

Contents

1 Introduction and General Considerations on the Work

The Standard Model of Particle Physics (SM) is an cornerstone in modern science. Nowadays, it's predictions are verified with even greater precision and it stands as the fundamental pillar of our present description of nature. However, even with so many conquests, there are questions unanswered by the SM and they still persist ever since new physics has being proposed. New physical scenarios, that we may collect in a general framework we refer to as Physics Beyond the Standard Model, are then expected and most welcome.

One of this new approaches is Supersymmetry (SUSY). First introduced in the former Soviet Union by [21] by Gol'fand and Likhtman (1971) and some time later on in the west by Abdus Salam, J. Strathdee [34] (1974) and J. Wess, B. Zumino [39] (1973), it is centered in the idea of extending the Minkowski space-time by adding grassmannian variables, and therefore proposing that physical space has a "fermionic nature" implicit to it. Assuming that the fundamental particles of physics comes from the irreducible representations of the group that preserves the isometry of the space (an idea also present in the SM), the so called supermultiplet carried by the superfields contains the Lorentz non-unitary irreducible representation fields and it's supersymmetric partners. Today, SUSY has gave many contributions to contemporary physics, from the low-energy condensaded matter physics to the highest energy scales physics like Superstring theory, despite the fact it has not being experimentally verified yet.

Another key aspect linked to new physics beyond the SM is related to the conservation of symmetries. From the perspective of the theories that constitute the SM, for instance Quantum Field Theory, it's expected the conservation of its fundamental symmetries. But, experience shows the other way around. CP-symmetry, for instance, can be violated in the context of Weak interactions, as was experimentally shown by

C.S. Wu [41]. Since the finding of CP-violation, many physics have prospected, from different points of view, the possibility that more general symmetries could also be violated, for instance CPT and Lorentz symmetry. In the context of high energy physics, it has being proposed that at the Planck scale Lorentz symmetry could be violated [26].

Ever since, many theories have arise to describe effects from the violation of Lorentz symmetry (LSV) and a model of this kind, reviewed in this dissertation, is the so called Carrol-Field-Jackiw (CFJ) Electrodynamics [11]. In this theoretical framework, the electromagnetic field couples to a background vector in which its origins comes from a more fundamental physics. Since LSV occurs near the Planck scale 10^{19} GeV, H. Belich et al. [5] observed the fact that, when LSV happens SUSY is still preserved because it's breaking scale is expected to be much lower (10^{12} GeV). Therefore, they proposed that the LSV must be included in a supersymmetric scenario and introduced it via a LSV supermultiplet. With this approach, the vectorial background not only gains a more fundamental description, but also a fermionic background superpartner. In recently works [38], the supersymmetric non-abelian version of an CFJ-type model was developed, where the possibility of a gauge-boson/gaugino mixing mechanism is open, analogously to the Primakoff effect, induced by the Majorana fermionic background.

In this dissertation, we will adopt the perspective present by H. Belich et al [5]. We start with a brief review of the historical context (subsection 1.1) in which LSV arises and it's interception with Supersymmetry (subsection 1.2). After that, the physical perspective assumed here (subsection 1.3), the CFJ-Electrodynamics, and its supersymmetric version, are reviewed in more details. The Primakoff effect, and the mixing induced by it, is algo present.

In section (2), we do a recapitulation of the Poincaré algebra and introduce the so called Van-der Warden 2-component spinor notation, which we use in the entire text to facilitate the construction of SUSY. Following that, in section (3) we construct the SUSY algebra and its representation, and demonstrate the so called supermultiplets.

Some aspects of Supersymmetry, like the preservation of the symmetry between bosonic and fermionic degress of freedom, are also discussed. The procedure to construct the superspace is also demonstrated, with a brief discussion of Grassmann variables and its algebra. Afterward, actions invariant under supersymmetric transformation were constructed and a SUSY covariant derivative was also introduced. In the same section, the chiral/antichiral superfields were presented as the first representations of SUSY and with it the procedure to derive Lorentz non-unitary representation matter fields terms from it.

In section (4), we demonstrate how the framework of SUSY can also supply us with gauge fields and gauge couplings. Higher representations of SUSY were explored, and the abelian and non-abelian versions of Super-Gauge theories, with its gauge couplings, were also developed. We ended the section by giving an example of an extended supersymmetric theory, i.e. $N = 2$ Super-Yang-Mills theory, and discussed its properties.

With all the theoretical framework set in place, we derived from first principles the supersymmetric version of the non-abelian CFJ model in section (5). A superfield which induces the LSV is proposed, with its physical consequences pointed out. In sequence, we prospected possible paths towards a gauge-boson/gaugino mixing , inspired in the Primakoff effect, in section (6). After the conclusions in section (7), we presented in the appendix the dictionary to translate from the 2-components notation to the 4-components spinor notation. The procedure of the Fierz rearrangement is also demonstrated.

1.1 Historical Contextualization of Lorentz Symmetry Violation

Historically, the discussion about the breaking of Lorentz symmetry started with Dirac [16], Bondi and Gold [10] in 1951. Dirac proposed a new Electrodynamics with the presence of a privileged direction in space. The idea was to understand the origins of the divergences present in QED. Later on, Bondi and Gold showed that Dirac's Electrodynamics, with spatial anisotropy, has cosmological implications.

Since the 60's were the decade of the CP violation in the weak interactions, with Lee's and Yang's theoretical work on parity conservation [27] and the experimental verification of CP violation by C.S. Wu [41], the question of whatever or not others symmetries could also be violated, for instance CPT and Lorentz symmetry, was "in the air" during that time. In 1963, J.D.Bjorken [8] constructed a model of 4 fermions (an analog to Heisemberg's model) which constitutes vacuum expectation values characterized by vectorial billinears. The point was to try to find an emerging Electromagnetism with composed photons in the context of a spontaneous breaking of Lorentz symmetry. A background vector which breaks the Lorentz symmetry also appears when we consider the graviton as a Goldstone boson, as proposed by P.R. Phillips [32] in 1966. T.G. Pavlopoulos also investigated J.D. Bjorken's notion of Lorentz symmetry breaking in a more deepen way [31]. In a more phenomenological perspective, L.B. Rédei (1967) proposed space-time anisotrophys smaller than 10^{-16} cm (Electroweak scale) and explored the implications of Lorentz symmetry violation on muon's $(g-2)$ factor. Later on, in 1978, H.B. Nielsen and M. Ninomyia studied the β-function of a non-covariant Yang-Mills theory and showed that the fixed point is in the region where Lorentz symmetry is restored.

In early 80's, the epoch of the Grand Unification, J. Ellis, M.K. Gaillard, D. Nanopoulos, S. Rudaz [17] (1980) and A.Zee [42] (1982) discussed the possibility that the proton decay, predicted by the Grand Unified Theories, could also happen with the presence of small violations of the Lorentz symmetry. On the other hand, at the same time, Mandelstrom demonstrated that the $N = 4$ Super-Yang-Mills theories are finite in all orders of perturbation theory, but the demonstration was not explicitly Lorentz covariant. There was a need at the time to understand the limits of Lorentz symmetry. With this context in view, H.B. Nielsen et al. (1982-3) investigated the phenomenological limits in which Lorentz symmetry could be really preserved [29, 12, 30].

After the String's Revolution (1984), a new phase on the understanding of LSV is inaugurated by V.A. Kostelecký and S. Samuel in the context of String theory [26, 25] (1989). It was shown that there exists tensorial modes in open strings which could take non-trivial vacuum expectation values, and in turn could break Lorentz symmetry. Also, phenomenological bounds on string theory and Lorentz symmetry violation were discussed. In the early 90's, a topological model due to background vector, which breaks Lorentz symmetry, was proposed by S.M. Carroll, G.B. Field and R. Jackiw in 1990 [11]. The idea, initially, was to use this model with the assistance of astrophysical data to propose bounds and estimates on the parameters that violates Lorentz symmetry. Some time later, at the Indiana University Center for Space-times Symmetries (IUCSS), directed by V.A. Kostelecký, summer school of 1995, D. Colladay presented "the minimal Standard Model Extension (SME)". But, it was only in (1997-8) that V.A. Kostelecký and D. Colladay full presented what is nowadays know as the Standard Model Extended (SME). [14, 15]. Today, this theoretical framework is the most popular one used to investigates Lorentz symmetry violation phenomenologically.

However, the SME was not the only scheme at display. N. Seiberg and E. Witten proposed in 1999 another framework know as non-commutative field theory [36]. They proposed a non-commutability in the space-time coordinates, which in turn induces the LSV. The fields associated with this theory are low energy limits of String theory. This theoretical framework is not so popular as the SME, but is a very good approach to Lorentz symmetry breaking.

At the same time, S. Coleman and S.L. Glashow [13] introduced a new approach based on the hypotheses that Lorentz symmetry should be violated at very high energies, i.e. at the Planck scale. They proposed to test special relativity at scenarios with extreme high energies, like ultra high cosmic rays with energies to the order of 10^{11} GeV.

In (2002-3), G. Amelino-Camelia [3] and J. Magueijo [28] introduced a new method for Lorentz symmetry violation; the so called Double-Special Relativity. A theory with two base invariants: the speed of light and an energy scale.

Another possible path to study LSV phenomenologically is to consider high frequency photons with expanded or modified dispersion relations. Supported by astrophysical high energy photons measurements, it is possible to relate deviations from the photon dispersion relation with violation of Lorentz symmetry or even from effects due to Quantum Gravity, as discussed by G. Amelino-Camelia, et al. [2, 18, 4] in (1997-2000). Even dispersion relations arising from Loop Quantum Gravity effects were considered by R. Gambini and J. Pullin [20] (1999) and J. Alfaro, et al. [1] (2002).

1.2 The Relation Between Supersymmetry and Lorentz Symmetry Violation

In today's understanding, Supersymmetry is a very good symmetry at high energies. Despite the fact that there is no experimental verification of SUSY, we can prospect this condition from a very solid theoretical point of view. The most general symmetry that preserves the light cone space-time structure is the conformal symmetry ($SO(2, 4)$). The representations of the conformal group must have zero mass or a continuous spectrum of mass. Therefore, in the ultra-relativistic limit, the conformal symmetry is a good symmetry. We can show that the conformal transformations can be derived from a set of transformations given by Majorana spinors which obey a killing vector type equation. Since the conformal transformations are related to this set of transformations by fermionic billinears, we can show that the conformal group is the square of the local SUSY group (SUGRA), i.e. $SO(2, 4) \simeq (SUGRA)^2$.

From the perspective of high energy physics, (String theory[24] [25][26] or even Loop-Quantum Gravity [20]) Lorentz symmetry is expected to be violated near Planck scale. Therefore, a relationship between SUSY and LSV can be prospected. The first work in this direction was done by V.A. Kostelecký and M.S. Berger [7] in 2002. The idea was to alter the Supersymmetry algebra by introducing a coupling with a background vector. Later on, M.S. Berger studied the superfields realizations of this superalgebra [6] in 2003. Following a different path, A.L.M.A. Nogueira, et al. [5, 35] proposed in (2003-4) that, since Lorentz symmetry is expected to be violated at Planck scale (10^{19} GeV) and Supersymmetry is only broken at 10^{12} GeV, the primer should happen in a scenario where the latter still holds. With this perspective in mind, they argued that the Supersymmetry algebra should be preserved and that the LSV must be introduced via a superfield. In this way, a supersymmetric fermionic background would also appear and a microscopic origin for the Lorentz symmetry breaking could be studied.

In this spirit, P.A. Bolokhov, et. al [9] constructed a Supersymmetric QED with Lorentz violation in 2005. They studied QED's vacuum birefringence due to Lorentz breaking. Also, the relation between supersymmetry breaking and LSV is explored by A.Katz and Y. Shadmi [23] (2006) in the context of the splitting of the superpartner's masses. In modern days, A.Yu.Petrov, et al. [19] (2012) followed the same line of introducing Lorentz breaking in the supersymmetry algebra.

1.3 The Supersymmetric Carroll-Field-Jackiw Model

Now, we would like to give some contextualization to the line of work followed by this dissertation. Here, we consider the perspective of [5] in which the LSV happens in a supersymmetric scenario. The Lorentz violating background now has a supersymmetric nature, i.e. is introduced via a superfield. In this LSV multiplet, it's present not only the bosonic background but also it fermionic superpartner background. This also allows us to bring forth a possible fundamental origin to the LSV background.

The so called Carroll-Field-Jackiw Electrodynamics,

$$\mathcal{L}_{\mathrm{CFJ}} = \frac{1}{4}\varepsilon^{\mu\nu\lambda\kappa}\upsilon_\mu A_\nu F_{\lambda\kappa},$$

proposes a topological term in which, in order to preserve gauge symmetry, imposes the following condition on the bosonic background vector:

$$\partial_\mu v_\nu - \partial_\nu v_\mu = 0. \tag{1.1}$$

In section 5, we constructed the supersymmetric $N = 1$ non-abelian version of this CFJ-term [37]. A Wess-Zumino chiral supermultiplet $S = \{s, \Psi, f\}$, which breaks the Lorentz symmetry,

$$S(x, \theta) = e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}(s(x) + \sqrt{2}\theta\psi(x) - \theta^2 f(x)), \tag{1.2}$$

is introduced via a supersymmetric interaction term present in the superaction. It contains a complex scalar boson s(x), a Majorana fermion $\Psi(x)$ and an auxiliary field f(x) (which will be eliminated by it's fields equations). This superfield is neutral under the gauge group, and imposing the right conditions to recover the CFJ-term, we obtain the bosonic background in the form

$$v_\mu \propto \partial_\mu \,\mathrm{Im}\{s\}, \tag{1.3}$$

i.e as a 4-gradient of a real scalar, which naturally has the condition to preserve gauge symmetry (1.1). Since the fermionic background is a Majorana one, it can contribute with the following billinears:

$$\begin{cases} \bar{\Psi}\Psi \rightarrow \text{scalar}, \\ \bar{\Psi}\gamma_5\Psi \rightarrow \text{pseudo-scalar}, \\ \bar{\Psi}\gamma_5\gamma_\mu\Psi \rightarrow \text{pseudo-vector}. \end{cases} \tag{1.4}$$

In the supersymmetric gauge sector $V^a = \{A^a_\mu, \Lambda^a, D^a\}$, these fermionic billinears contributes to the gaugino's dispersion relations with the following terms:

$$\begin{cases} (\bar{\Psi}\Psi)(\bar{\Lambda}^a\Lambda^a), \\ (\bar{\Psi}\gamma_5\Psi)(\bar{\Lambda}^a\gamma_5\Lambda^a), \\ (\bar{\Psi}\gamma_5\gamma_\mu\Psi)(\bar{\Lambda}^a\gamma_5\gamma^\mu\Lambda^a), \\ (\bar{\Psi}\Sigma^{\mu\nu}\gamma_5\Psi)(\bar{\Lambda}^a\Sigma_{\mu\nu}\gamma_5\Lambda^a). \end{cases} \tag{1.5}$$

It was also shown in [38] that the gauge boson and the gaugino, in this supersymmetric LSV scenario, gains a mass:

$$\begin{aligned} m_A &= 2(\bar{\Psi}\Psi) + \frac{|\vec{v}|}{2}, \\ m_\Lambda &= |\vec{v}|, \end{aligned} \tag{1.6}$$

of the order of the background vector. Since Supersymmetry requires that particles of the same supermultiplet to have the same mass, we can easily see that Supersymmetry is broken. The last billinear in (1.5), after Fierz rearrangement, generates a mixing term intermediated by the fermionic background:

$$\mathcal{L}_{\text{Mixing}} = -i\sqrt{2}\bar{\Psi}\Sigma^{\mu\nu}\gamma_5\Lambda^a F^a_{\mu\nu}. \tag{1.7}$$

This term allows for a possible gauge-boson/gaugino conversion mechanism, an analogue to the Primakoff effect, induced here by the Majorana fermionic background.

The introduction of this supersymmetric background turned possible a microscopic explanation for the generalized Lorentz violating Dirac equation proposed by V.A. Kostelecký and R. Lehnert [24] in the context of the SME model. Therefore, the perspective assumed in this thesis allows for a more fundamental explanation for the phenomenological nature of the violation of Lorentz symmetry.

1.4 The Primakoff Effect

Now, we would like to discuss more in details the Primakoff effect. In the latter, a axion-photon mixing is possible in the context of the Axion-Electrodynamics [40]:

$$\mathcal{L}_{\text{Axion-ED}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{g}{4}aF_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}(\partial_\mu a)^2 - \frac{m}{2}a^2, \tag{1.8}$$

where A_μ is the 4-potential photon field, a the axion field and $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$ the electromagnetic field dual tensor. Considering a constant external magnetic field [33], and the Lorenz gauge ($\partial_\mu A^\mu = 0$), the field equations becomes:

$$\begin{cases} \Box\phi = g\boldsymbol{\nabla} a \cdot \vec{B}, \\ \Box\vec{A} = g\partial_t a\vec{B} - g\boldsymbol{\nabla} a \times \vec{E}, \\ (\Box + m_a^2)a = g\partial_t\vec{A}\cdot\vec{B} + g\boldsymbol{\nabla}\phi\cdot\vec{B}. \end{cases} \tag{1.9}$$

with $\Box = \partial_t^2 - \boldsymbol{\nabla}^2$. If the external magnetic field is transverse to the momentum of the waves, i.e. $\vec{k}\cdot\vec{B}_T = 0$, then the gradient terms does not contribute to the field equations, i.e. $\boldsymbol{\nabla} a\cdot\vec{B}_T = \boldsymbol{\nabla}\phi\cdot\vec{B}_T = 0$. This can be seen by considering these terms in momentum space. Also, we make the assumption that $\vec{E} << \vec{B}_T$. The field equations becomes:

$$\begin{cases} \Box\phi = 0, \\ \Box\vec{A} = g\partial_t a\vec{B}_T, \\ (\Box + m_a^2)a = g\partial_t\vec{A}\cdot\vec{B}_T. \end{cases} \tag{1.10}$$

Now, we consider the photon and axion dispersion relations as

$$\omega^2 = |\vec{k}|^2 + \tilde{\omega}^2, \tag{1.11}$$

where $\tilde{\omega}$ is the contribution from some external effect in the presence of a magnetic field, like the Cotton-Mouton effect or some solar plasma medium [22]. Assuming plane waves solutions with $\vec{\mathbf{k}} = k\hat{\mathbf{e}}_z$, we can write the field equations in the form:

$$\left[(\omega^2+\partial_z^2)\mathbb{1}_3 - 2\omega^2\begin{pmatrix}\Delta_\perp/\omega & 0 & 0\\ 0 & \Delta_{||}/\omega & \Delta_M/\omega\\ 0 & \Delta_M/\omega & \Delta_a/\omega\end{pmatrix}\right]\begin{pmatrix}A_\perp\\ A_{||}\\ a\end{pmatrix} = 0, \tag{1.12}$$

with the defined quantities:

$$\Delta_\perp = \frac{\tilde{\omega}_\perp^2}{2\omega},\quad \Delta_{||} = \frac{\tilde{\omega}_{||}^2}{2\omega},\quad \Delta_a = \frac{m_a^2}{2\omega},\quad \Delta_M = \frac{igB_T}{2} \tag{1.13}$$

and $A_{||}$ the photon's polarization component parallel to the magnetic field and $A_\perp$ the photon's polarization component perpendicular to the magnetic field. In the ultra-relativistic limit, i.e. $\omega >> m_a$ and $\omega >> \tilde{\omega}$, we can linearize the operator:

$$(\omega^2+\partial_z^2) = (\omega + \underbrace{i\partial_z}_{=k=\omega})(\omega - i\partial_z) = 2\omega(\omega - i\partial_z). \tag{1.14}$$

The linearized field equations becomes:

$$\left[(\omega - i\partial_z)\mathbb{1}_3 - \begin{pmatrix}\Delta_\perp & 0 & 0\\ 0 & \Delta_{||} & \Delta_M\\ 0 & \Delta_M & \Delta_a\end{pmatrix}\right]\begin{pmatrix}A_\perp\\ A_{||}\\ a\end{pmatrix} = 0, \tag{1.15}$$

From the undiagonal sector of the field equations, we can identify the Schrödinger like equation:

$$-i\partial_z\begin{pmatrix}A_{||}\\ a\end{pmatrix} = \underbrace{\begin{pmatrix}\Delta_{||} & \Delta_M\\ \Delta_M & \Delta_a\end{pmatrix}}_{H_I}\begin{pmatrix}A_{||}\\ a\end{pmatrix}. \tag{1.16}$$

Since the H_I matrix is symmetric, the states can be diagonalized by a SO(2) matrix:

$$\begin{pmatrix}A'_{||}\\ a'\end{pmatrix} = \underbrace{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}}_{P}\begin{pmatrix}A_{||}\\ a\end{pmatrix}, \tag{1.17}$$

with the eigenvalues,

$$\lambda_\pm = \frac{\Delta_{||}+\Delta_a}{2} \pm \sqrt{\frac{(\Delta_{||}-\Delta_a)^2}{4} + \Delta_M^2}, \tag{1.18}$$

and the mixing angle

$$\tan(2\theta) = \frac{2\Delta_M}{\Delta_{||} - \Delta_a}. \tag{1.19}$$

Consider now a beam of free particles (axions and photons) traveling in the presence of a constant external magnetic field transverse to the momentum of the beam. We can study the mixing process by analyzing the difference in phase between the diagonal modes and the undiagonal parallel photon mode after a distance z is travalled (we consider momentum of the modes in the z direction):

$$\begin{pmatrix} A'(z) \\ a'(z) \end{pmatrix} = \begin{pmatrix} e^{-i(\lambda_+ - \Delta_{||})z} & 0 \\ 0 & e^{-i(\lambda_- - \Delta_{||})z} \end{pmatrix} \begin{pmatrix} A'(0) \\ a'(0) \end{pmatrix}.$$

In the undiagonal basis:

$$\begin{pmatrix} A(z) \\ a(z) \end{pmatrix} = P^{-1} M(z) P \begin{pmatrix} A(0) \\ a(0) \end{pmatrix}, \tag{1.20}$$

with

$$P^{-1}M(z)P = e^{-i\Delta_{||}z} \begin{pmatrix} \cos^2(\theta)e^{-i\lambda_+ z} + \sin^2(\theta)e^{-i\lambda_- z}, & \sin(\theta)\cos(\theta)(e^{-i\lambda_+ z} - e^{-i\lambda_- z}) \\ \sin(\theta)\cos(\theta)(e^{-i\lambda_+ z} - e^{-i\lambda_- z}), & \cos^2(\theta)e^{-i\lambda_- z} + \sin^2(\theta)e^{-i\lambda_+ z} \end{pmatrix}. \tag{1.21}$$

The modulus squared of the off-diagonal elements gives us the photon-axion transition probability:

$$P_{a\to\gamma}(z) = \sin^2(\theta)\sin^2(\Delta_{\rm osc} z), \quad \text{with} \quad \Delta_{\rm osc} = \frac{\lambda_+ - \lambda_-}{2},$$

which is the probability that a free axion particle travelling in a region with constant transverse magnetic field can oscillate into a photon after a distance z. Since the mixing matrix is symmetric, the probability is the same for the photon oscillating into an axion. The parameter $\Delta_{\rm osc}$ has dimensions of length^{-1}, therefore it can be used to estimate the scale in which the oscillations can be observed, i.e. $l_{\rm osc} = 2\pi/\Delta_{\rm osc}$.

In section (6) we showed that it is possible to generate an similar mechanism for the mixing of gauge-boson/gaugino particles, induced not by an external magnetic field, but now by a fermionic background. We developed the field equations of the model in linearized form and discussed some implications of it.

2 Spinors as Non-Unitary Representations of the Lorentz Group

Here, we make a brief review of spinors as non-unitary irreducible representations of the Lorentz Group in (1+3) dimensional Minkowski space-time. The space-time metric is chosen to be $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, with $\mu, \nu, ... = 0, 1, 2, 3$ space-time indices and $i, j, ... = 1, 2, 3$ space indices only.

2.1 Arbitrary Non-Unitary Representations of The Lorentz Group

In covariant form, the Lorentz algebra, given by the antisymmetric generators $M_{\mu\nu}$, has the form:

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\rho} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho}, \tag{2.1}$$

with $M_{oi} = K_i$ and $M_{ij} = \varepsilon_{ijk}J_k$ the boost and rotations generators, respectively. In terms of these generators, the algebra can be cast as:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}K_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k. \tag{2.2}$$

To study it's representations, we can write the generators as:

$$J_i^{\pm} = \frac{1}{2}(J_i \pm K_i), \tag{2.3}$$

in terms of which the algebra separates into two commuting $su(2)$ algebras:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm}, \qquad [J_i^{\pm}, J_j^{\mp}] = 0. \tag{2.4}$$

From these non-hermitian generators, we see that the Lorentz Group is a complexified version of $SU(2) \times SU(2)$, which is the group $SL(2, \mathbb{C})$ (actually, this group is the universal cover of the Lorentz group). Since the Lorentz group is composed of two commuting $su(2)$ algebras, the space of the representations of this group is a tensor product of the spaces of representations of two $SU(2)$ groups. Therefore, we can write the generators as:

$$J_i^{(+)} = J_i^{(A)} \otimes \mathbb{1}, \qquad J_i^{(-)} = \mathbb{1} \otimes J_i^{(B)}, \tag{2.5}$$

where J_i^A are the generators of the su(2) algebra in the representation labeled by the highest weight $A = 0, 1/2, 1, 3/2, ...$ In this way, we can label the irreducible representations of the group $SL(2,\mathbb{C})$ by the pair (A,B). Note that this representations are field representations, i.e. they are non-unitary. Therefore, the Lorentz transformations can be written as:

$$\Lambda^{(A,B)}(a,b) = \exp\bigl(ia^i J_i + ib^j K_j\bigr) = \exp\Bigl(i(a^i - ib^i)J_i^{+(A)} + i(a^i + ib^i)J_i^{-(B)}\Bigr). \tag{2.6}$$

Since the group $SO(3) \subset SL(2,\mathbb{C})$, we can decompose the irreducible representations of the Lorentz group into irreducible representations $S \in SO(3)$ as:

$$(A,B) = \bigoplus_{S=|A-B|}^{(A+B)} S. \tag{2.7}$$

2.2 Poincaré Algebra

The Poincaré group contains, in addition to the Lorentz group, the space-time translations. The generators of space-time translations are the 4-momentum P_μ. It has the following commutation relations with the Lorentz group generators:

$$[P_\mu, P_\nu] = 0, \quad [J_i, P_j] = i\epsilon_{ijk}P_k, \quad [J_i, P_0] = 0, \quad [K_i, P_\mu] = iP_\mu, \tag{2.8}$$

which contains the facts that the 4-translations commutes among themselves, that the P_i and P_0 are a vector and a scalar under 3-space rotations, and how P_μ is mixed under a boost. In terms of the covariant Lorentz generators $M_{[\mu\nu]}$, the full Poincaré algebra can be read as:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho}, \qquad (2.9)\\
[M_{\mu\nu}, P_\rho] &= -ig_{\rho\mu}P_\nu + ig_{\rho\nu}P_\mu.
\end{aligned}$$

Any element of the Poincaré group can be written in terms of it's generators as:

$$P(\omega, a) = \exp\left(\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} - ia^\mu P_\mu\right). \qquad (2.10)$$

2.3 Two-Component Spinors

We construct here spinors as objects which its elements transforms under the fundamental representations of $SL(2,\mathbb{C})$. Since, in the fundamental representation, the elements of the group $SL(2,\mathbb{C})$ are 2×2 matrices with complex entries M_β^α, then a spinor is a two component object $\psi_\alpha \in \mathbb{C}$, $\alpha, \beta = 1, 2$, transforming under the elements of $SL(2,\mathbb{C})$ as,

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta. \qquad (2.11)$$

The representations of $SL(2,\mathbb{C})$, namely M, are not equivalent to their complex conjugate M^*. Therefore, we denote the two component objects transforming under M^* as $\bar{\psi}$, i.e.:

$$\bar{\psi}'_{\dot{\beta}} = (M^*)_{\dot{\beta}}{}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}. \qquad (2.12)$$

This spinor is called dotted spinor and the spinor ψ is called an undotted one. We use the dotted notation because the representations of the two spinor are not equivalent, therefore we need to differentiate both indexes. If we complex conjugate the relation (2.11), we can identify $\bar{\psi}_{\dot{\alpha}}$ with ψ^*_α.

The irreducible representation carried by ψ (i.e. M) is the $(1/2, 0)$ representation of the Lorentz group and the one carried by $\bar{\psi}$ (i.e. M^*) is the $(0, 1/2)$. They can be written as:

$$\begin{aligned}\Lambda^{(1/2,0)} &= M = \exp\bigg(i(a^i - ib^i)\frac{\sigma^i}{2}\bigg),\\ \Lambda^{(0,1/2)} &= M^* = \exp\bigg(i(a^i + ib^i)\frac{\sigma^i}{2}\bigg),\end{aligned} \tag{2.13}$$

where the σ^i, $i = 1, 2, 3$, are the Pauli matrices, i.e. half of then are the generators of the $su(2)$ algebra in the $(1/2)$ irreducible representation.

Now, we introduce the antisymmetric matrices ϵ that lower and raise indices in the spinorial space. They have the form:

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.14}$$

with $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^{\alpha}{}_{\beta}$ and $\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}$. They lower and raise the indices as

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}, \quad \psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}. \tag{2.15}$$

Using the notation defined in (2.15), we find that the spinors with raised indices transforms as:

$$\begin{aligned}(\psi')^{\alpha} &= \psi^{\beta}(M^{-1})_{\beta}{}^{\alpha},\\ (\bar{\psi}')^{\dot{\beta}} &= \bar{\psi}^{\dot{\alpha}}(M^{*-1})_{\dot{\alpha}}{}^{\dot{\beta}}.\end{aligned} \tag{2.16}$$

Now, we introduce the matrices σ^{μ} which has naturally dotted and undotted index:

$$(\sigma^{\mu})_{\alpha\dot{\alpha}} = (\mathbb{1}, -\sigma^i)_{\alpha\dot{\alpha}}. \tag{2.17}$$

Raising the indices with the antisymmetric tensors ϵ yields:

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^{\mu})_{\beta\dot{\beta}} = (\mathbb{1}, +\sigma^i)^{\dot{\alpha}\alpha}. \tag{2.18}$$

Since spinors are anti-commuting variables, i.e. $\psi_1\chi_2 = -\chi_2\psi_1$, the definition (2.15) allows us to define the scalar product with the following summing convention:

$$\begin{aligned}\psi\chi &= \psi^{\alpha}\chi_{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}\chi_{\alpha} = -\psi_{\beta}\epsilon^{\beta\alpha}\chi_{\alpha} = -\psi_{\beta}\chi^{\beta},\\ \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}.\end{aligned} \tag{2.19}$$

Using this scalar product convention, we can construct the following vectorial quantities:

$$\psi\sigma^\mu\bar{\chi} = \psi^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}}, \qquad \bar{\psi}\bar{\sigma}^\mu\chi = \bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\chi_\beta. \tag{2.20}$$

One can use this summing convention and prove the following identities:

$$\begin{aligned}
\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi \quad &, \quad (\chi\sigma^\mu\bar{\psi})^\dagger = \psi\sigma^\mu\bar{\chi} \\
\chi\sigma^\mu\bar{\sigma}^\nu\psi = \psi\sigma^\nu\bar{\sigma}^\mu\chi \quad &, \quad (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger = \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi} \\
\psi\chi = \chi\psi, \quad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi} \quad &, (\psi\chi)^\dagger = \bar{\psi}\bar{\chi}.
\end{aligned} \tag{2.21}$$

2.4 Dirac, Weyl and Majorana Spinors in the Weyl Representation

In the Weyl representation, the Dirac matrices are given by:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}. \tag{2.22}$$

A 4-component Dirac spinor carries a reducible representation $(1/2,0)\oplus(0,1/2)$ of the Lorentz group. It's composed of a dotted and a undotted spinors as:

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.23}$$

The Weyl (left/right) spinors are defined as as the Dirac spinors which are eigenstates of the chirality matrix γ_5, i.e. $\gamma_5\psi_D = \pm\psi_D$. The Dirac spinors with eigenvalue $(+1)$ are called Weyl-Left and the ones with (-1) Weyl-Right. In the Weyl representations of the gamma matrices, the Weyl-Left spinors are given by the undotted spinors and the Weyl-Right by the dotted ones, i.e.:

$$\Psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad \Psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.24}$$

.

We define the Majorana spinor as a Dirac spinor that is equal to it's charge conjugate, i.e. $\Psi_D = \Psi_D^c = C\bar{\Psi}_D^t$. In the Weyl representation of Dirac matrices, the Majorana spinors has the form:

$$\Psi_{\text{Maj.}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.25}$$

In terms of the gamma matrices γ^μ, the Lorentz generators can be cast as:

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} = i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \tag{2.26}$$

where we can see that the undotted and dotted spinors transforms separately, the generators being $i\sigma^{\mu\nu}$ for ψ_α and $i\bar{\sigma}^{\mu\nu}$ for $\bar{\psi}^{\dot{\alpha}}$ with

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha^{\ \beta} &= \frac{1}{4}(\sigma^\mu_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu,\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\bar{\sigma}^{\mu,\dot{\gamma}\beta}), \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} &= \frac{1}{4}(\bar{\sigma}^{\mu,\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - \bar{\sigma}^{\nu,\dot{\alpha}\gamma}\sigma^\mu_{\gamma\dot{\beta}}). \end{aligned} \tag{2.27}$$

We finalize this section by saying that in the appendix A there is a dictionary to pass from the 2-component notation to the 4-component one. From what follows, we will consider the 2-component spinors notation and mention when the notation is changed.

3 Superspace and Superfields

In the following section, we construct the superspace, which is the natural space to study the emergence of supersymmetric theories. The idea is to expand the physical space (x^μ) by adding grassmannian coordinates (θ_α) that preserves the isometry of the space.

3.1 Superspace in (1+3)D

We expand the Minkowski (1+3)D space-time (x^μ) with grassmannian coordinates. The coordinates θ_α are chosen to be Majorana spinors and the superspace it's described by the coordinates $(x^\mu; \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$, with $\alpha, \dot{\alpha} = 1, 2$.

Since the θ's are anti-commuting coordinates, i.e. $\{\theta_\alpha, \theta_\beta\} = 0$, then any product of more than two θ's or more than $\bar{\theta}$'s vanishes. For grassmannian variables, the integration is defined as $\int d\theta^1 \theta^1 = 1$ and $\int d\theta^1 = 0$. Since $\theta\theta = 2\theta^2\theta^1$ and $\bar{\theta}\bar{\theta} = \bar{\theta}^1\bar{\theta}^2$, we define $d^2\theta = \frac{1}{2} d\theta^1 d\theta^2$ and $d^2\bar{\theta} = \frac{1}{2} d\bar{\theta}^2 d\bar{\theta}^1$ such that

$$\int d^2\theta \ \theta\theta = \int d^2\bar{\theta} \ \bar{\theta}\bar{\theta} = 1 \longrightarrow \int d^2\theta d^2\bar{\theta} \ \theta\theta\bar{\theta}\bar{\theta} = 1. \tag{3.1}$$

Note also that

$$\begin{gathered} 1 = \frac{\partial\theta^1}{\partial\theta^1}\frac{\partial\theta^2}{\partial\theta^2} = \frac{1}{2}\left(\frac{\partial\theta^1}{\partial\theta^1}\frac{\partial\theta^2}{\partial\theta^2} - \frac{\partial\theta^2}{\partial\theta^2}\frac{\partial\theta^1}{\partial\theta^1}\right) = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta \ \theta\theta \\ \therefore \int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta, \quad \int d^2\bar{\theta} = \frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}} \end{gathered} \tag{3.2}$$

where we defined the grassmannian derivatives as $\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$ and $\bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$. With the above definitions, and the fact that $\theta_\alpha^\dagger = \bar{\theta}_{\dot{\alpha}}$, we can easily see the hermicity property

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^\dagger = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \tag{3.3}$$

with $\alpha = \dot{\alpha}$.

Using the summing convention (2.19), we can easily prove the following relations with the grassmannian coordinates:

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \quad , \quad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \\ \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \quad , \quad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \\ \theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}g^{\mu\nu} \quad , \quad \theta\psi\theta\chi = -\frac{1}{2}\theta\theta\psi\chi. \end{aligned} \tag{3.4}$$

A superfield is a continuous function of the coordinates of the superspace that carries non-unitary irreducible representations of the Lorentz group. Due to the nature of the grassmannian variables, an arbitrary scalar field can be expanded as:

$$\begin{aligned} F(x,\theta,\bar{\theta}) &= s(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}m(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &\quad + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\Gamma(x) + \theta\theta\bar{\theta}\bar{\theta}D(x), \end{aligned} \tag{3.5}$$

where we can see the Lorentz representations carried by the components of the superfield.

Now, we introduce the Supersymmetry (SUSY) transformations as translations in superspace:

$$\theta'_\alpha = \theta_\alpha + \epsilon_\alpha, \quad \bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}, \quad (x^\mu)' = x^\mu + i\theta\sigma^\mu\bar{\epsilon} + i\epsilon\sigma^\mu\bar{\theta}. \tag{3.6}$$

Since the translations are a continuous symmetry, we can find the representation of it's generators in the coordinate space. Considering infinitesimal translations on the θ coordinates only:

$$\begin{aligned} &(1+i\epsilon Q)F(x^\mu,\theta,\bar{\theta}) = F(x^\mu + i\epsilon\sigma^\mu\bar{\theta}, \theta+\epsilon, \bar{\theta}) \\ &\rightarrow i\epsilon^\alpha Q_\alpha F(x^\mu,\theta,\bar{\theta}) = \epsilon^\alpha(\partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu)F(x^\mu,\theta,\bar{\theta}). \end{aligned} \tag{3.7}$$

We can see that the generators of Supersymmetry are given by:

$$Q_\alpha = -i\partial_\alpha + \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \tag{3.8}$$

and the generators of translations on the $\bar{\theta}$ coordinates are it's hermitian conjugate, i.e.

$$\bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + \theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\partial_{\mu}. \tag{3.9}$$

The generators satisfy the algebra

$$\begin{aligned}\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} &= \{-i\partial_{\alpha} + \sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{\mu}, i\bar{\partial}_{\dot{\alpha}} + \theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\partial_{\mu}\} \\ &= -2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} = 2\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu},\end{aligned} \tag{3.10}$$

where we identified the generator of space-time translations as $P_{\mu} = -i\partial_{\mu}$. Therefore, the SUSY variation of a Superfield is given by:

$$\delta F = (i\epsilon Q + i\bar{\epsilon}\bar{Q})F. \tag{3.11}$$

Now, we are interested in objects that transforms in the same way as a superfield. Since the generators are differential operators, they obey the Libniz's derivative rule and with it a product of superfields transforms as a superfield. The space-time derivative ∂_{μ} commutes with the fermionic derivatives ∂_{α} and $\bar{\partial}_{\dot{\alpha}}$, which implies that $\delta\partial_{\mu}F = \partial_{\mu}\delta F$, i.e. a space-time derivative of a superfield transforms as a superfield. On the other hand, a fermionic derivative has a non-trivial commutator with the generators:

$$\begin{aligned}\partial_{\alpha}\delta F &= \partial_{\alpha}(i\epsilon Q + i\bar{\epsilon}\bar{Q})F \\ &= i(\epsilon^{\beta}[-i\partial_{\beta} + \sigma^{\mu}_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\partial_{\mu}]\partial_{\beta} + \bar{\epsilon}^{\dot{\beta}}[i\bar{\partial}_{\dot{\beta}} + \theta^{\beta}\sigma^{\mu}_{\beta\dot{\beta}}\partial_{\mu} - \delta^{\beta}_{\alpha}\sigma^{\mu}_{\beta\dot{\beta}}\partial_{\mu}]) \\ &= \delta(\partial_{\alpha}F) - i\bar{\epsilon}^{\dot{\beta}}(\sigma^{\mu}_{\alpha\dot{\beta}}\partial_{\mu}),\end{aligned} \tag{3.12}$$

and therefore $\partial_{\alpha}F$ does not transform as a superfield. In order to preserve the tensorial covariancy, we need to introduce a fermionic covariant derivative $\partial_{\alpha} \to D_{\alpha}$, such that a covariant derivative of a superfield transforms as a superfield, i.e. $D_{\alpha}\delta F = \delta D_{\alpha}F$. In order to obtain such covariant derivative, we need that the generators of SUSY symmetry anti-commute with the covariant derivatives, i.e. $\{D_{\alpha}, Q_{\beta}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$, due to the nature of the fermionic parameters of the transformations $\epsilon, \bar{\epsilon}$. We find that:

$$
\begin{aligned}
D_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\
\bar{D}_{\dot{\alpha}} &= \bar{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu,
\end{aligned}
\tag{3.13}
$$

where $\bar{D}_{\dot{\alpha}} = (D_\alpha)^\dagger$ and $(\partial_\mu)^\dagger = -\partial_\mu$. The covariant derivatives obey the anti-commutation relations:

$$
\begin{aligned}
&\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \quad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\beta}}, \bar{D}_{\dot{\alpha}}\} = 0 \\
&\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.
\end{aligned}
\tag{3.14}
$$

3.2 The SUSY Algebra and its Representations

We constructed the superspace as an extension of the ordinary space-time. Therefore, its natural to view the SUSY algebra, which is an algebra of the superspace, as an extension of the Poincaré algebra (the space-time algebra). In the following section, we develop the SUSY algebra from this point of view.

We enlarge the ordinary Poincaré algebra by generators that transform as dotted Q^I_α and undotted spinors $\bar{Q}^I_{\dot{\alpha}}$ under the Lorentz group, and that commute with space-time translations. We denote by $I = 1, ..., N$ the different pairs of SUSY generators that we add to the Poincaré algebra. Therefore, the SUSY algebra is given by

$$
\begin{aligned}
[P_\mu, Q^I_\alpha] &= [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0, \\
[M_{\mu\nu}, Q^I_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta, \\
[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\dot{Q}^{\dot{\beta},I}, \\
\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu\delta^{IJ}.
\end{aligned}
\tag{3.15}
$$

Note that $M_{12} = J_3$ and thus $[J_3, Q^I_1] = \frac{1}{2}Q^I_1$ and $[J_3, Q^I_2] = -\frac{1}{2}Q^I_2$. Since $\bar{Q}^{1,I} = (Q^I_2)^\dagger$ and $\bar{Q}^{I,2} = -(Q^I_1)^\dagger$, we also have $[J_3, (Q^I_2)^\dagger] = \frac{1}{2}(Q^I_2)^\dagger$ and $[J_3, (Q^I_1)^\dagger] = -\frac{1}{2}(Q^I_1)^\dagger$. Therefore, the generators Q^I_1 and $(Q^I_2)^\dagger$ rise the spin (helicity) by 1/2, while Q^I_2 and $(Q^I_1)^\dagger$ lower it by 1/2.

We also have:

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{[IJ]}, \qquad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{[IJ]})^*, \tag{3.16}$$

where the $Z^{[IJ]}$ are central charges, which means they commute with all generators of the full SUSY algebra. In the unextended SUSY algebra (N=1) there is no possibility of central charges. For $N > 1$, we one talks about extended susy algebra. From the algebric point of view there is no limit to N, but with increasing N the theories must also contain particles with higher spin (helicity), and there seem to be no consistent quantum field theory with spin (helicity) higher than 1 without gravity, or larger than 2 if we have gravity.

Since the full SUSY algebra contains the Poincaré algebra as a sub-algebra, any representation of the full SUSY algebra also contains a representation of the Poincaré algebra, although in general a reducible one. Since each irreducible representation of the Poincaré algebra corresponds to a particle, an irreducible representation of the SUSY algebra in general correspond to several particles. The states of the representations are related to each other by Q^I_α and $\bar{Q}^J_{\dot{\beta}}$, and thus have spin (helicity) differing by units of half. They form what is called a supermultiplet. Here, we will call an irreducible representation of the SUSY algebra simply by supermultiplet. Using the spin-statistics theorem, we can show that the Q and $\bar{Q}$ change bosons into fermions and vice versa.

An irreducible representations of the SUSY algebra has the proprieties:

- Since P^2 is a Casimir operator of the SUSY algebra, all particles belonging to the same supermultiplet have the same mass;

- For any state of a free SUSY theory $|\Psi\rangle$, the positivity of the Hilbert space:

$$0 \leq ||Q^I_\alpha |\Psi\rangle ||^2 + ||(Q^I_\alpha)^\dagger |\Psi\rangle ||^2 = \langle\Psi| \{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\} |\Psi\rangle = 2\sigma^\mu_{\alpha\dot{\alpha}} \langle\Psi| P^\mu |\Psi\rangle \,. \tag{3.17}$$

  Taking the trace with respect to the spinorial indices, and using $\mathrm{Tr}(\sigma^\mu) = 2\delta^{\mu 0}$, yields:

$$0 \leq \langle\Psi| P^0 |\Psi\rangle \,, \tag{3.18}$$

i.e. the energy P^0 in a supersymmetric free theory is always positive.

- A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom.

3.2.1 Massless Supermultiplets

Since for a massless supermultiplet $P^2 = 0$, we choose the reference frame where $P^\mu = E(1,0,0,1)$, so that

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = \begin{bmatrix} 0 & 0 \\ 0 & 4E \end{bmatrix} \delta^{IJ}. \tag{3.19}$$

Note that $\{Q^I_1, \bar{Q}^J_{\dot{1}}\} = 0$. Therefore, due to the positivity of Hilbert space, we must set $Q^I_1 = \bar{Q}^I_{\dot{1}} = 0$. We are left with only the N generators Q^I_2 and $\bar{Q}^J_{\dot{2}}$. Due to the relations (3.16), we see that, for massless supermultiplets, the central charges Z^{IJ} must vanish.

We define

$$a_I = \frac{1}{\sqrt{4E}} Q^I_2 \quad , \quad a_I^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot{2}}. \tag{3.20}$$

The a_I and $a_I^\dagger$ are anti-commuting creation and annihilation operators:

$$\{a_I, a_J^\dagger\} = \delta_{IJ}, \quad \{a_I, a_J\} = 0. \tag{3.21}$$

Then, we choose a vacuum state, i.e. a state annihilated by all a_I. Such a state will carry some irreducible representation of the Poincaré algebra, i.e. it will be characterized by $m = 0$ and some helicity λ. We denote this state by $|\lambda\rangle$. From the commutators of Q^I_2 and $\bar{Q}^J_{\dot{2}}$ with the helicity operator, one sees that Q^I_2 lowers the helicity by half and $\bar{Q}^J_{\dot{2}}$ rises it by half. Then the supermultiplets have the form $a_1^\dagger....a_N^\dagger\,|\lambda\rangle = |\lambda + N/2\rangle$. In general, such supermultiplets will not have the helicities symmetrically distributed about zero. Such supermultiplets cannot be invariant under CPT, since CPT flips the sign of the helicity. To satisfy CPT, one needs to add to these multiplets its CPT conjugate multiplets with opposite helicities and opposite quantum numbers.

For $N = 1$ SUSY, the supermultiplets only contains 2 states: $|\lambda\rangle$ and $|\lambda + 1/2\rangle$. We denote then by $(\lambda, \lambda + 1/2)$. We need to double then so they can be invariant under CPT. We get the following supermultiplets:

- The chiral multiplet $(0, 1/2) \oplus (-1/2, 0)$, corresponding to a Weyl fermion and a complex scalar;

- The vector multiplet $(1/2, 1) \oplus (-1, -1/2)$, corresponding to a gauge boson and a Weyl fermion (gaugino), both necessarily in the adjoint representation of the gauge group;

- The gravitino multiplet $(1, 3/2) \oplus (-3/2, -1)$, corresponding to a gravitino and a gauge boson;

- The graviton multiplet $(3/2, 2) \oplus (-2, -3/2)$, corresponding to a graviton and a gravitino.

Since the gravitino should only be present in a theory with gravity, so if $N = 1$ it can only occur once and in the graviton multiplet. For helicities up to 2, we need to stop here in the $N = 1$ supermultiplets.

A $N = 2$ supermultiplet has the form $(\lambda, \lambda + 1/2, \lambda + 1/2, \lambda + 1)$. Considering helicities not exceeding one, we get the following multiplets:

- The $N = 2$ vector multiplet $(0, 1/2, 1/2, 1) \oplus (-1, -1/2, -1/2, 0)$, corresponding to a gauge boson, 2 Weyl fermions and a complex scalar, all in the adjoint representation of the gauge group;

- The hypermultiplet: if $\lambda = -1/2$ we get $(-1/2, 0, 0, 1/2)$. This may or may not be CPT self conjugate. If it is, we call it a half-hypermultiplet. If its not, we have to add its CPT conjugate $(-1/2, 0, 0, 1/2)$.

For $N = 4$, and restricting again to helicities not exceeding one, there is a single $N = 4$ multiplet $(-1, 4 \times (-1/2), 6 \times 0, 4 \times 1/2, 1)$. It contains a gauge boson, 4 Weyl fermions and 3 complex scalars, all transforming under the adjoint representation of the gauge group.

3.2.2 Massive Supermultiplets

Considering now the massive case $P^2 > 0$, in the rest frame $P_\mu = (m, 0, 0, 0)$, the SUSY algebra becomes:

$$\{Q^I_\alpha, (Q^J_\beta)^\dagger\} = 2m\delta_{\alpha\beta}\delta^{IJ}, \quad \{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ}, \quad \{(Q^I_\alpha)^\dagger, (Q^J_\beta)^\dagger\} = \epsilon_{\alpha\beta}(Z^{IJ})^*. \tag{3.22}$$

By use of an appropriate $U(N)$ rotation among the generators Q^I, we can put the matrix representation the central charges Z^{IJ} in the diagonal form:

$$Z^{IJ} = \begin{bmatrix} 0 & q_1 & 0 & 0 & \\ -q_1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & q_2 & \\ 0 & 0 & -q_2 & 0 & \\ & & \vdots & & \end{bmatrix} \tag{3.23}$$

with all $q_n \leq 0$, $n = 1, ..., N/2$. We assume N even, since otherwise there is an extra zero eigenvalue of the Z-matrix which can be handled trivially.

We define the operators:

$$\begin{aligned} a^1_\alpha &= \frac{1}{\sqrt{2}}(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger) \\ b^1_\alpha &= \frac{1}{\sqrt{2}}(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger) \\ a^2_\alpha &= \frac{1}{\sqrt{2}}(Q^3_\alpha + \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger) \\ b^2_\alpha &= \frac{1}{\sqrt{2}}(Q^3_\alpha - \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger) \\ &\vdots \end{aligned} \tag{3.24}$$

The operators a^r_α and b^r_α, $r = 1, ..., N/2$, and their hermitian conjugate satisfy the commutation relations

$$
\begin{aligned}
\{a^r_\alpha, (a^s_\beta)^\dagger\} &= (2m - q_r)\delta_{rs}\delta_{\alpha\beta}, \\
\{b^r_\alpha, (b^s_\beta)^\dagger\} &= (2m + q_r)\delta_{rs}\delta_{\alpha\beta}, \qquad (3.25) \\
\{a^r_\alpha, (b^s_\beta)^\dagger\} &= \{a^r_\alpha, a^s_\beta\} = ... = 0.
\end{aligned}
$$

The positivity of Hilbert space requires that $2m \geq |q_n|$. If any of the q_n saturate the bound, i.e. $|q_n| = 2m$, then the corresponding operators must be set equal to zero, as we did in the massless case. Clearly, in the massless case the bound becomes $0 \geq |q_n|$, thus $q_n = 0$ always. Again, we see that there cannot be central charges in the massless case and the bound is always saturated, with only half non-vanishing fermionic generators.

In the more general massive case, i.e $|q_n| < 2m$ for all n, then we can construct the states starting from an vacuum state with a minimal spin S, which is annihilated by all a^n_α and b^n_β.

For $N = 1$, we got four states denoted by $\left(S, S+\frac{1}{2}, S+\frac{1}{2}, S+1\right)$. The $(-1/2, 0, 0, 1/2)$ multiplet, which is CPT self conjugate, or $(-1, -1/2, -1/2, 0) \oplus (1, 1/2, 1/2, 0)$. The latter has the same states as a massless vector plus a massless chiral multiplet and can be obtained from then via Higgs Mechanism. It corresponds to massive vector, a Dirac spinor and a single real scalar field.

For $N = 2$, we have 16 states with spins ranging from $[-1, 1]$. This massive multiplet can be seen as the sum of a massless $N = 2$ vector and hypermultiplets. A massive $N = 4$ multiplet contains $2^8 = 256$ states, including at least a ± 2 spin. Such a theory must include a spin-2 massive particle, which is not possible in quantum field theory.

The multiplets where some of the q_n are equal to 2m, then we have whats is called short or BPS multiplets. If all the q_n equal to $2m$, then we have the so called shortest multiplets.

3.3 Chiral/Anti-Chiral Superfields

A chiral superfield, i.e. the chiral multiplet $(0,1/2)\oplus(-1/2,0)$, is defined by the superfield equations:

$$\bar{D}_{\dot{\alpha}}\phi = 0, \tag{3.26}$$

and a anti-chiral one by

$$D_{\alpha}\bar{\phi} = 0. \tag{3.27}$$

Translating the space-time coordinates by $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and $\bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$, we can see that

$$D_\alpha\bar{\theta} = \bar{D}_{\dot{\alpha}}\theta = D_\alpha\bar{y}^\mu = \bar{D}_{\dot{\alpha}}y^\mu = 0, \tag{3.28}$$

hence $\phi = \phi(y^\mu,\theta)$ and $\bar{\phi} = \bar{\phi}(\bar{y}^\mu,\bar{\theta})$.

The chiral superfield ϕ has the component expansion:

$$\phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y), \tag{3.29}$$

or Taylor expanding in x^μ:

$$\phi(x,\theta) = z(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu z(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2 z(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \theta\theta f(x). \tag{3.30}$$

Such a chiral superfied is composed of a complex scalar z and a Weyl fermion ψ, i.e a $(0,1/2)$ representation. For the anti-chiral superfield:

$$\bar{\phi}(x,\theta) = z^\dagger(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu z^\dagger(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2 z^\dagger(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) - \bar{\theta}\bar{\theta}\bar{f}(x). \tag{3.31}$$

Now, let us analysis the SUSY variation of the chiral superfield. The SUSY generators in the $(y^\mu,\theta,\bar{\theta})$ variables are given by:

$$Q_\alpha = -i\partial_\alpha \quad , \quad \bar{Q}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\frac{\partial}{\partial y^\mu}. \tag{3.32}$$

Then,

$$\delta\phi(y,\theta) = (i\epsilon Q + i\bar{\epsilon}\bar{Q})\phi(y,\theta) = \left(\epsilon^{\alpha}\partial_{\alpha} + 2i\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\frac{\partial}{\partial y^{\mu}}\right)\phi(y,\theta)$$
$$= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(\sqrt{2}i\sigma^{\mu}\bar{\epsilon}\partial_{\mu}z - \sqrt{2}\epsilon f) - \theta\theta(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^{\mu}\partial_{\mu}\psi). \tag{3.33}$$

In terms of field components, the SUSY variations can be read as:

$$\begin{aligned} \delta z &= \sqrt{2}\epsilon\psi, \\ \delta\psi &= \sqrt{2}i\partial_{\mu}z\sigma^{\mu}\bar{\epsilon} - \sqrt{2}f\epsilon, \\ \delta f &= \sqrt{2}i\partial_{\mu}\psi\sigma^{\mu}\bar{\epsilon}. \end{aligned} \tag{3.34}$$

3.4 SUSY Actions

Now, let's consider supersymmetric actions. Note that the product of chiral (anti-chiral) superfields are still chiral (anti-chiral) superfields. Typically, one will have chiral superpotentials $W(\phi)$, which may depend on several chiral superfields ϕ_i. We can Taylor expand, in terms of the variables $(y,\theta,\bar{\theta})$,:

$$W(\phi) = W(z(y)) + \sqrt{2}\frac{\partial W}{\partial z_i}\theta\psi_i(y) - \theta\theta\left(\frac{\partial W}{\partial z_i}f_i(y) + \frac{1}{2}\frac{\partial^2 W}{\partial z_i \partial z_j}\psi_i(y)\psi_j(y)\right), \tag{3.35}$$

where it's understood that the derivatives are taken with respect to $z(y)$. Now, since supersymmetric theories are considered here as Representations of a superspace, we introduce the supersymmetric action with a supersymmetric measure on integration $(d^2\theta d^2\bar{\theta} d^4x)$:

$$S = \int d^4x \left\{ d^2\theta d^2\bar{\theta}\ F(x,\theta,\bar{\theta}) + \int d^2\theta\ W[\phi] + \int d^2\bar{\theta}\ (W[\phi])^{\dagger} \right\}. \tag{3.36}$$

Actions of this form are automatically SUSY invariant, i.e. they transform at most by a total derivative which in turn does not contribute to the field equations. This can be seen by the SUSY variation of the superfields:

$$\delta F = (i\epsilon Q + i\bar{\epsilon}\bar{Q})F = \partial_{\alpha}(-\epsilon^{\alpha}F) + \bar{\partial}_{\dot{\alpha}}(-\bar{\epsilon}^{\dot{\alpha}}F) + \partial_{\mu}(-i(\epsilon\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\epsilon})F). \tag{3.37}$$

Since the ϵ and $\bar{\epsilon}$ are constant spinors, the integration on the grassmannian coordinates $\int d^2\theta d^2\bar{\theta}$ only leaves the last term, which is a total derivative as claimed before. For the superpotential $W[\phi]$, we can consider the SUSY variation on chiral superfields. In the $(y, \theta, \bar{\theta})$ variables:

$$\delta\phi = \partial_\alpha(-\epsilon^\alpha \phi(i, \theta)) + \frac{\partial}{\partial y^\mu}[-i(\epsilon\sigma^\mu\bar{\theta} - \theta\sigma^\mu\bar{\epsilon})]\phi(y, \theta) \tag{3.38}$$

Again, integration on $\int d^2\theta$ only leaves the last term which is a total derivative. The analogous argument holds for anti-chiral superpotentials $(W[\phi])^\dagger$ and integration on $\int d^2\bar{\theta}$. This proves the Supersymmetry of (3.36).

From the nature of the expansion of the superfields and superpotentials, the only term that will survive the integration in the grassmannian coordinates are the $\theta\theta\bar{\theta}\bar{\theta}$ terms, called D-terms. For the chiral/anti-chiral superfield, only the $\theta\theta/\bar{\theta}\bar{\theta}$ terms, which are called F-terms.

3.5 N=1 Matter Lagrangian

Since $W[\phi]$ and $(W[\phi])^\dagger$ are superpotentials, the kinetic terms must be developed from $F(x, \theta, \bar{\theta})$. A natural, and simple proposal, is to choose $F = \phi^\dagger\phi$. In terms of the $(y, \theta, \bar{\theta})$ variables, the D-term is given by:

$$\phi_i^\dagger\phi_i\big|_{\theta\theta\bar{\theta}\bar{\theta}} = \partial_\mu z_i^\dagger \partial^\mu z_i - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i + f_i^\dagger f_i + \text{Total Derivatives.} \tag{3.39}$$

Therefore, the action assume the form:

$$\begin{aligned} S &= \int d^4x d^2\theta d^2\bar{\theta}\ \phi_i^\dagger\phi_i + \int d^2\theta\ W[\phi_i] + h.c \\ &= \int d^4x \left(|\partial_\mu z_i|^2 - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i + f_i^\dagger f_i - \frac{\partial W}{\partial z_i} f_i\bigg|_{\theta=0} + h.c. - \frac{1}{2}\frac{\partial^2 W}{\partial z_i \partial z_j}\psi_i\psi_j\bigg|_{\theta=0} + h.c.\right). \end{aligned} \tag{3.40}$$

Note that the auxiliary fields f_i have no kinetic term. They can be eliminated by their field equations:

$$f_i = \frac{\partial W}{\partial z_i^\dagger}, \quad f_i^\dagger = \frac{\partial W}{\partial z_i}. \tag{3.41}$$

Substituting back into the action and putting in 4-components notation:

$$S = \int d^4x \left(|\partial_\mu z_i|^2 - i\bar{\Psi}_{L,i}\gamma^\mu\partial_\mu\Psi_{L,i} - \left|\frac{\partial W}{\partial z_i}\right|^2_{\theta=0} - \frac{1}{2}\frac{\partial^2 W}{\partial z_i \partial z_j}\Psi_{L,i}\Psi_{L,j}\bigg|_{\theta=0} + h.c. \right), \tag{3.42}$$

where we can see that the scalar potential is given by

$$V = \sum_i \left|\frac{\partial W}{\partial z_i}\right|^2 . \tag{3.43}$$

This kinetic term gives us several complex scalars z_i and Weyl left $\psi_{L,i}$ fields. It also contains the Weyl right fermions. We can see this by rewriting the spinor kinetic term with the use of the relations (2.21) :

$$i\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\psi_\alpha = -i\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \underbrace{i\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}}}_{i\bar{\Psi}_R\gamma^\mu\partial_\mu\Psi_R} + \text{Total Derivative} \tag{3.44}$$

This proposal of kinetic term can also describe Majora fermions kinetic terms. Again, we can rewrite the 2-components Weyl-left term as

$$\begin{aligned} i\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\psi_\alpha &= \frac{i}{2}(\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\psi_\alpha + \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}}) + \text{Total Derivative} \\ &= \frac{i}{2}\bar{\Psi}_{\text{Maj.}}\gamma^\mu\partial_\mu\Psi_{\text{Maj.}} + \text{Total Derivative.} \end{aligned} \tag{3.45}$$

Now, for the Majorana and the scalar mass terms we can propose a superpotential in the form:

$$W[\phi] = \frac{m}{2}\phi_i^2, \quad \frac{\partial W}{\partial z_i} = m\phi_i, \quad \frac{\partial^2 W}{\partial z_i \partial z_j} = m\delta_{ij} \tag{3.46}$$

where the parameter m has canonical dimension 1 and the derivatives of the superpotential with respect to z_i are taken with $\theta = \bar{\theta} = 0$. Substituting this superpotential in the action (3.42), with the spinors written as Majorana's ones in 4-component notation, we obtain

$$S = \int d^4x \left[|\partial_\mu z_i|^2 - \frac{i}{2}\bar{\Psi}_{M,i}\gamma^\mu\partial_\mu\Psi_{M,i} - m^2|z_i|^2 - \frac{m}{2}\underbrace{(\psi^\alpha\psi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}})}_{\bar{\Psi}_{M,i}\Psi_{M,i}} \right] \tag{3.47}$$

For the Dirac fermions, we need that the right and left components are unrelated. To archive this, we need another chiral superfield Ξ to construct the right-independent part:

$$\Xi(y) = s(y) + \sqrt{2}\theta\chi(y) - \theta^2 h. \tag{3.48}$$

Therefore, the Dirac kinetic terms can be derived from:

$$\begin{aligned}
\phi_i^\dagger\phi_i\Big|_{\theta^2\bar{\theta}^2} + \Xi_i^\dagger\Xi_i\Big|_{\theta^2\bar{\theta}^2} &= |\partial_\mu s_i|^2 + |\partial_\mu z_i|^2 - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i - i\chi_i\sigma^\mu\partial_\mu\bar{\chi}_i + f_i^\dagger f_i + h_i^\dagger h_i + \text{T.D.} \\
&= |\partial_\mu s_i|^2 + |\partial_\mu z_i|^2 - i\bar{\Psi}_{D,i}\gamma^\mu\partial_\mu\Psi_{D,i} + f_i^\dagger f_i + h_i^\dagger h_i + \text{T.D.}
\end{aligned} \tag{3.49}$$

To generate the Dirac's mass terms, we write the superpotential as

$$\begin{aligned}
&W[\phi,\Xi] = m(\phi\ \Xi), \qquad \frac{\partial W}{\partial z_i} = m\Xi_i, \qquad \frac{\partial W}{\partial s_i} = m\phi_i, \\
&\frac{\partial^2 W}{\partial z_i\partial s_j} = m\delta_{ij}, \quad \frac{\partial^2 W}{\partial s_i\partial s_j} = \frac{\partial^2 W}{\partial z_i\partial z_j} = 0,
\end{aligned} \tag{3.50}$$

where again we consider that the derivatives with respect to the scalar fields are taken with $\theta = \bar{\theta} = 0$. Adding the superpotential terms, as in (3.42), to the Dirac's kinetics terms, we obtain the action:

$$S = \int d^4x \left[|\partial_\mu s_i|^2 + |\partial_\mu z_i|^2 - i\bar{\Psi}_{D,i}\gamma^\mu\partial_\mu\Psi_{D,i} - m^2|z_i|^2 - m^2|s_i|^2 - m(\underbrace{\chi^\alpha\psi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}}_{\bar{\Psi}_D\Psi_D})\right] \tag{3.51}$$

We end this section by demonstrating how to construct Yukawa interactions and scalar self-interactions. Considering the action of the Weyl-left fermions (3.42), we construct a superpotential in the form:

$$W[\phi_i] = \sum_i \frac{g}{3!}\phi_i^3, \quad \frac{\partial W}{\partial z_i} = \frac{g}{2}\phi_i^2, \quad \frac{\partial^2 W}{\partial z_i\partial z_j} = g\phi_i\delta_{ij}, \tag{3.52}$$

where again we consider the derivatives in the context of $\theta = \bar{\theta} = 0$. Substituting back in (3.42):

$$S = \sum_i \int d^4x \left[|\partial_\mu z_i|^2 - i\bar{\Psi}_{L,i}\gamma^\mu\partial_\mu\Psi_{L,i} - g^2|z_i|^4 + g(z_i\psi_i^\alpha\psi_{\alpha,i} + z_i^\dagger\bar{\psi}_{\dot{\alpha},i}\bar{\psi}_i^{\dot{\alpha}})\right]. \tag{3.53}$$

4 Super-Gauge Theories

In this section, we introduce a supersymmetric scenario where Gauge theories can naturally emerge, i.e. the Super-Gauge theories, and the supersymmetric generalization of the gauge transformations. We will focus on the $N = 1$ Supersymmetry first, but in the last subsection, we will construct SUSY's with higher N.

4.1 Pure N=1 Abelian Super-Gauge Theory

The next $N = 1$ supermultiplet of higher spin is the vector multiplet $(1, 1/2) \oplus (-1, -1/2)$, which is in the adjoint representation of the gauge group. The corresponding superfield $V(x, \theta, \bar{\theta})$ is real and has the expansion:

$$\begin{aligned} V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi - i\bar{\theta}\bar{\chi} + \theta\sigma^{\mu}\bar{\theta}A_{\mu} + \frac{i}{2}\theta\theta B - \frac{i}{2}\bar{\theta}\bar{\theta}B^{\dagger} + i\theta\theta\bar{\theta}\left(\bar{\lambda} + \frac{i}{2}\bar{\partial\!\!\!/}\chi\right) \\ -i\bar{\theta}\bar{\theta}\theta\left(\bar{\lambda} - \frac{i}{2}\partial\!\!\!/\bar{\chi}\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D - \frac{1}{2}\partial^2 C\right). \end{aligned} \tag{4.1}$$

where $\partial\!\!\!/ \equiv \sigma^{\mu}\partial_{\mu}$ and $\bar{\partial\!\!\!/} = \bar{\sigma}^{\mu}\partial_{\mu}$.

We can eliminate many of the components of this multiplet by making use of a supersymmetric generalization of a gauge transformation. Note that the transformation:

$$\begin{aligned} V' &= V + \phi + \phi^{\dagger} \\ &= C + z + z^{\dagger} + \theta(i\chi + \sqrt{2}\psi) + \bar{\theta}(-i\bar{\chi} + \sqrt{2}\bar{\psi}) + \frac{i}{2}\theta\theta(B + 2if) \\ &- \frac{i}{2}(B^{\dagger} - 2if^{\dagger}) + \theta\theta\bar{\theta}\left(i\bar{\lambda} - \frac{1}{2}\bar{\partial\!\!\!/}\chi + \frac{i}{\sqrt{2}}\bar{\partial\!\!\!/}\psi\right) + \bar{\theta}\bar{\theta}\theta\left(-i\lambda - \frac{1}{2}\partial\!\!\!/\bar{\chi} - \frac{i}{\sqrt{2}}\partial\!\!\!/\bar{\psi}\right) \\ &+ \theta\theta\bar{\theta}\bar{\theta}\left(\frac{D}{2} - \frac{\partial^2 C}{4} - \frac{\partial^2 z}{4}\right) + \theta\sigma^{\mu}\bar{\theta}(A_{\mu} + i\partial_{\mu}z - i\partial_{\mu}z^{\dagger}), \end{aligned} \tag{4.2}$$

with ϕ a chiral superfield, implies that the vector field component A_{μ} must transforms as an abelian gauge field,

$$A'_{\mu} = A_{\mu} + \partial_{\mu}(2\,\mathrm{Im}(z)). \tag{4.3}$$

Therefore, (4.2) is the supersymmetric generalization of an abelian gauge transformation. We can use this supersymmetric gauge freedom to eliminate the components $B, B^\dagger, \chi, C$. This choice of "super gauge" is called the Wess-Zumino gauge, and it reduces the vector multiplet to the form,

$$V_{\rm WZ} = \theta\sigma^\mu\bar\theta A_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x). \tag{4.4}$$

Since the expansion of (4.4) contains terms with at least one θ and one $\bar\theta$, the only non-vanishing power of $V_{\rm WZ}$ is

$$V_{\rm WZ}^2 = \theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta A_\mu A_\nu = \frac{1}{2}\theta\theta\bar\theta\bar\theta A_\mu A^\mu, \tag{4.5}$$

with $V_{\rm WZ}^n = 0$, for $n \geq 3$.

Since the product of $V_{\rm WZ}$ does not generate kinetic terms for A_μ, we need to act with covariant derivatives on V to generate this terms. We define the chiral and anti-chiral quantities:

$$W_\alpha = -\frac{1}{4}\bar D\bar D D_\alpha V, \qquad \bar W_{\dot\alpha} = -\frac{1}{4}DD\bar D_{\dot\alpha}V. \tag{4.6}$$

The condition $D^3 = \bar D^3 = 0$ reflects the fact that W_α and $\bar W_{\dot\alpha}$ are chiral and anti-chiral.

Note that, if we transform the vector multiplet by the super gauge transformations (4.2), the W superfields remain invariant:

$$\begin{aligned} W'_\alpha &= W_\alpha - \frac{1}{4}\bar D\bar D D_\alpha(\phi + \phi^\dagger) = W_\alpha + \frac{1}{4}\bar D^{\dot\alpha}\bar D_{\dot\alpha}D_\alpha\phi \\ &= W_\alpha + \frac{1}{4}\bar D^{\dot\alpha}\{\bar D_{\dot\alpha}, D_\alpha\}\phi = W_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\cancelto{0}{(\bar D^{\dot\alpha}\phi)} = W_\alpha, \end{aligned} \tag{4.7}$$

due to the condition that defines the chiral superfields, i.e. $\bar D\phi = D\phi^\dagger = 0$. To calculate W_α, we can use the Wess-Zumino gauge and the covariant derivatives in the $(y, \theta, \bar\theta)$,

$$D_\alpha = \partial_\alpha + 2i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial y^\mu}, \qquad \bar D_{\dot\alpha} = \bar\partial_{\dot\alpha}. \tag{4.8}$$

We can write the vector multiplet as:

$$V_{\rm WZ} = \theta\sigma^\mu\bar\theta A_\mu(y) + i\theta\theta\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\theta\lambda(y) + \frac{1}{2}\theta\theta\bar\theta\bar\theta(D(y) - i\partial_\mu A^\mu(y)). \tag{4.9}$$

Then,

$$D_\alpha V_{\rm WZ} = (\sigma^\mu\theta)_\alpha A_\mu + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta\bar\theta\lambda_\alpha + \theta_\alpha\bar\theta\bar\theta D + i\bar\theta\bar\theta((\sigma^\mu\bar\sigma^\nu)_\alpha{}^\gamma - g^{\mu\nu}\delta_\alpha{}^\gamma)\theta_\gamma\partial_\mu A_\nu$$
$$+ \theta\theta\bar\theta\bar\theta(\partial\!\!\!/\bar\lambda)_\alpha. \tag{4.10}$$

Using the identity $\sigma^\mu\bar\sigma^\nu - g^{\mu\nu} = 2\sigma^{\mu\nu}$, we find that

$$\begin{aligned} D_\alpha V_{\rm WZ} &= (\sigma^\mu\theta)_\alpha A_\mu + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta\bar\theta\lambda_\alpha + \theta_\alpha\bar\theta\bar\theta D \\ &\quad + 2i\bar\theta\bar\theta(\sigma^{\mu\nu}\theta)_\alpha\partial_\mu A_\nu + \theta\theta\bar\theta\bar\theta(\partial\!\!\!/\bar\lambda)_\alpha. \end{aligned} \tag{4.11}$$

After noting that $\bar D\bar D\bar\theta\bar\theta = -4$, we can compute the explicit form of the W's superfields:

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta(\partial\!\!\!/\bar\lambda)_\alpha(y), \tag{4.12}$$

where we used the antisymmetry of $\sigma^{\mu\nu}$ contracted with $\partial_\mu A_\nu$ to write the field strength tensor associated with A_μ,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.13}$$

Since W_α is chiral, $\int d^2\theta\ W^\alpha W_\alpha$ is a SUSY invariant Lagrangian. The F-term associated with $W^\alpha W_\alpha$ is given by

$$\begin{aligned} W^\alpha W_\alpha\big|_{\theta\theta} &= -2i\lambda\partial\!\!\!/\bar\lambda + D^2 - \frac{1}{2}(\sigma^{\mu\nu}\bar\sigma^{\rho\sigma})^\alpha{}_\alpha F_{\mu\nu}F_{\rho\sigma} \\ &= -2i\lambda\partial\!\!\!/\bar\lambda + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}F_{\mu\nu}\tilde F^{\mu\nu}, \end{aligned} \tag{4.14}$$

where we used the fact that

$$(\sigma^{\mu\nu}\bar\sigma^{\rho\sigma})^\alpha_{\ \alpha} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}), \tag{4.15}$$

and we defined the dual field strength as $\tilde F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. To obtain the full supersymmetric Maxwell Lagrangian, we need to add the complex conjugate of (4.14). With the adequate normalization, we can write the supersymmetric $N = 1$ Maxwell theory as

$$\begin{aligned} S^{\rm SUSY}_{\rm Maxwell} &= \frac{1}{4}\int d^4x\left(\int d^2\theta\ W^\alpha W_\alpha + \int d^2\bar\theta\bar W_{\dot\alpha}\bar W^{\dot\alpha}\right) \\ &= \int d^4x\left(-\frac{i}{2}\Lambda\partial\!\!\!/\bar\Lambda + \frac{1}{2}D^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right), \end{aligned} \tag{4.16}$$

with the spinors,

$$\Lambda = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}, \tag{4.17}$$

in the 4-component notation.

4.2 Pure N=1 Non-Abelian Super-Gauge Theory

Here, we are interested in generalizing the theory (3.42) for any SU(N) gauge group. Consider the vector multiplet (4.1) in the adjoint representation of the gauge group G, i.e. $V \equiv V_a T^a$, $a = 1, 2, ..., \dim$ (G), where the T^a are the generators of the gauge group in the adjoint representation. Therefore, by the exponential mapping, we can see that the more fundamental object is actually e^V. The non-abelian supersymmetric gauge transformation now becomes:

$$e^V \to e^{\phi^\dagger} e^V e^\phi, \qquad e^{-V} \to e^{-\phi} e^{-V} e^{-\phi^\dagger}, \tag{4.18}$$

where we can see that to first order in ϕ it reproduces (4.2). We will construct a susy action that is a local symmetry of this transformation. This transformation (4.18) can again be used to put the vector multiplet (4.1) in the WZ-gauge expression (4.4). From what follows, we assume all quantities on this WZ-gauge. Note that one simply has:

$$e^V = 1 + V + \frac{V^2}{2}. \tag{4.19}$$

The non-abelian generalization of the superfield strength tensor is now defined as:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}(e^{-V} D_\alpha e^V), \qquad \bar{W}_{\dot{\alpha}} = +\frac{1}{4} DD(e^V \bar{D}_{\dot{\alpha}} e^{-V}), \tag{4.20}$$

which to first order in V reproduces the abelian case (4.12). Under the the transformations (4.18), the superfield strength tensor:

$$
\begin{aligned}
W'_\alpha &= -\frac{1}{4}\bar{D}\bar{D}(e^{-\phi}e^{-V}e^{-\phi^\dagger}D_\alpha[e^{\phi^\dagger}e^{V}e^{\phi}]) \\
&= -\frac{1}{4}\bar{D}\bar{D}(e^{-\phi}e^{-V}[(D_\alpha e^{V})e^{\phi} + e^{V}(D_\alpha e^{\phi})]) \\
&= e^{-\phi}\left\{ -\frac{1}{4}\bar{D}\bar{D}(e^{-V}D_\alpha e^{V}) - \cancelto{0}{\frac{1}{4}\bar{D}\bar{D}D_\alpha\phi}\right\}e^{\phi} = e^{-\phi}W_\alpha e^{\phi},
\end{aligned}
\tag{4.21}
$$

i.e. it transforms covariantly under(4.18). Analogously, we can show that

$$
\bar{W}'_{\dot\alpha} = e^{\phi^\dagger}\bar{W}_{\dot\alpha}e^{-\phi^\dagger}. \tag{4.22}
$$

Using the expansion (4.19), we can find the explicit form of (4.20):

$$
\begin{aligned}
W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\left(\left[1 - V + \frac{V^2}{2}\right]D_\alpha\left[1 + V + \frac{V^2}{2}\right]\right) \\
&= -\frac{1}{4}\bar{D}\bar{D}\left(\left[1 - V + \frac{V^2}{2}\right]\left[D_\alpha V + \frac{D_\alpha V.V + V.D_\alpha V}{2}\right]\right) \\
&= -\frac{1}{4}\bar{D}\bar{D}\left(D_\alpha V + \frac{D_\alpha V.V}{2} + \frac{V.D_\alpha V}{2} - V.D_\alpha V\right) \\
&= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V],
\end{aligned}
\tag{4.23}
$$

which the first term is the same as in the abelian case (4.12). The second term, therefore, is the non-abelian contribution of the gauge group. The commutator has the form:

$$
\begin{aligned}
[V, D_\alpha V] = \Bigg[&\left(\theta\sigma^\mu\bar\theta A_\mu(y) + i\theta\theta\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\theta\lambda(y) + \frac{1}{2}\theta\theta\bar\theta\bar\theta(D(y) - i\partial_\mu A^\mu(y))\right), \\
&(\sigma^\mu\bar\theta)_\alpha A_\mu + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta\bar\theta\lambda_\alpha + \theta_\alpha\bar\theta\bar\theta D + i\bar\theta\bar\theta((\sigma^\mu\bar\sigma^\nu)_\alpha{}^\gamma - g^{\mu\nu}\delta_\alpha{}^\gamma)\theta_\gamma\partial_\mu A_\nu + \theta\theta\bar\theta\bar\theta(\partial\!\!\!/\bar\lambda)_\alpha\Bigg] \\
&= \bar\theta\bar\theta(\sigma^{\nu\mu}\theta)_\alpha[A_\mu, A_\nu] + i\theta\theta\bar\theta\bar\theta\sigma^\mu_{\alpha\dot\beta}[A_\mu, \bar\lambda^{\dot\beta}],
\end{aligned}
\tag{4.24}
$$

and using again $\bar{D}\bar{D}\bar\theta\bar\theta = -4$, then

$$
\frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] = \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha[A_\mu, A_\nu] - \frac{i}{2}\theta\theta\sigma^\mu_{\alpha\dot\beta}[A_\mu, \bar\lambda^{\dot\beta}]. \tag{4.25}
$$

Adding this to (4.12), we finally obtain the generalization of the super non-abelian field strength tensor:

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta(\not{D}\bar{\lambda})_\alpha(y), \tag{4.26}$$

where now the field strength $F_{\mu\nu}$ and the covariant derivatives contains non-abelian terms:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i}{2}[A_\mu, A_\nu], \quad D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - \frac{i}{2}[A_\mu, \bar{\lambda}]. \tag{4.27}$$

Now, we rescale the vector multiplet V by a factor of $2g$, to express the non-abelian coupling constant g,

$$V \to 2g\ V \Leftrightarrow A_\mu \to 2g\ A_\mu, \quad \lambda \to 2g\ \lambda, \quad D \to 2g\ D. \tag{4.28}$$

The field strength tensor and the covariant derivatives now assume the form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \quad D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - ig[A_\mu, \bar{\lambda}]. \tag{4.29}$$

Finally, the F-term associated with $W^\alpha W_\alpha$ is given by:

$$W^\alpha W_\alpha\big|_{\theta\theta} = -2g^2(F_{\mu\nu}F^{\mu\nu} + 4i\lambda\not{D}\bar{\lambda} - 2D^2 - iF_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{4.30}$$

where we defined the dual of $F^{\mu\nu}$ in the same way as in the abelian case, i.e.:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}. \tag{4.31}$$

Now, the vector multiplet (4.1) is considered in some representation of the gauge group. Its generators satisfy the commutations relations:

$$[T^a, T^b] = if^{[abc]}T^c, \tag{4.32}$$

with real structure constants $f^{[abc]}$. The field strength tensor can be cast in the form:

$$\begin{aligned} F^a_{\mu\nu}T^a &= \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a - igA^b_\mu A^c_\nu[T^b, T^c] \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{bca}A^b_\mu A^c_\nu)T^a \\ \therefore\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \end{aligned} \tag{4.33}$$

and the covariant derivative,

$$\begin{aligned} D_\mu \bar{\lambda}^a T^a &= \partial_\mu \bar{\lambda}^a T^a - igA^b_\mu \bar{\lambda}^c [T^b, T^c] = (\partial_\mu \bar{\lambda}^a + gf^{bca} A^b_\mu \bar{\lambda}^c) T^a \\ \therefore \quad D_\mu \bar{\lambda}^a &= \partial_\mu \bar{\lambda}^a + gf^{abc} A^b_\mu \bar{\lambda}^c. \end{aligned} \tag{4.34}$$

To construct the supersymmetric non-abelian Lagrangian, we need to take the trace with respect to the adjoint representation of the gauge group and introduce the coupling constant:

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{4.35}$$

Therefore, the N=1 gauge generalized SUSY action becomes:

$$\begin{aligned} S^{\text{N=1}}_{\text{Non-Abe.Gauge}} &= \frac{1}{32\pi} \int d^4x \, \mathrm{Im} \left[\tau \int d^2\theta \, \mathrm{Tr}(W^\alpha W_\alpha) \right] \\ &= \frac{1}{32\pi} \int d^4x \, \mathrm{Im} \left[\left(\frac{\Theta}{2\pi} + \frac{4\pi i}{g^2} \right) (-2g^2) \, \mathrm{Tr} \left[(F_\mu F^{\mu\nu} + 4i\lambda \not{D} \bar{\lambda} - 2D^2 - iF_{\mu\nu} \tilde{F}^{\mu\nu}) \right] \right] \\ &= \int d^4x \left\{ \mathrm{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \bar{\Lambda} \not{D} \Lambda + \frac{1}{2} D^2 \right) + \frac{\Theta}{32\pi^2} g^2 \, \mathrm{Tr} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \right\}, \end{aligned} \tag{4.36}$$

with the photino field Λ in 4-components notation. Note that the SUSY Lagrangian $\mathrm{Tr}(W^\alpha W_\alpha)$, properly normalized, produced the conventionally normalizes gauge kinetic terms $\mathrm{Tr}\left(F^2_{\mu\nu}\right)$ and the instanton density $\frac{g^2}{32\pi^2} \mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$ associated with the Θ term.

4.3 N=1 Super-Gauge-Matter Theory

To include the matter Lagrangian, we consider now the chiral multiplets ϕ^i in some representation of the gauge group R where the generators are given by the matrices $(T^a_R)^i{}_j$. They transform under the gauge group:

$$\phi \to e^{i\Lambda}\phi, \quad \phi^\dagger \to \phi^\dagger e^{-i\Lambda^\dagger}, \tag{4.37}$$

with $\Lambda = \Lambda^a T^a_R$ a chiral superfield. Then, the gauge generalization of the matter kinetic term is given by

$$\phi^\dagger e^{V^a T^a} \phi = \phi^\dagger_i (e^V)^i{}_j \phi^j. \tag{4.38}$$

To calculate the D-terms associated with the kinetic terms (4.38), we must evaluate the terms of the expansion,

$$\phi^\dagger e^V \phi = \phi^\dagger\phi + \phi^\dagger V\phi + \frac{\phi^\dagger V^2\phi}{2}, \tag{4.39}$$

where the first term was calculated in (3.39). The second one is given by

$$\phi^\dagger V\phi = \frac{i}{2}z^\dagger A^\mu\partial_\mu z - \frac{i}{2}\partial_\mu z^\dagger A^\mu z - \frac{1}{2}\bar{\psi}\bar{\sigma}^\mu A_\mu\psi + \frac{i}{\sqrt{2}}z^\dagger\lambda\psi - \frac{i}{\sqrt{2}}\bar{\psi}\bar{\lambda}z + \frac{1}{2}z^\dagger Dz, \tag{4.40}$$

and the third one by

$$\frac{1}{2}\phi^\dagger V^2\phi\big|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{4}z^\dagger A^\mu A_\mu z. \tag{4.41}$$

Rescaling the vector multiplet V as in (4.28), we find that the kinetic matter terms are given by:

$$\begin{aligned}\phi^\dagger e^{2gV}\phi\big|_{\theta\theta\bar{\theta}\bar{\theta}} = (D_\mu z)^\dagger D^\mu z - i\bar{\psi}\bar{\not{D}}\psi + f^\dagger f + g^2 z^\dagger A^\mu A_\mu z + i\sqrt{2}gz^\dagger\lambda\psi \\ - i\sqrt{2}g\bar{\psi}\bar{\lambda}z + gz^\dagger Dz + \text{Total Derivatives},\end{aligned} \tag{4.42}$$

with the covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T_R^a$.

Now, there is a last type of term that can appear if the gauge group is U(1) or has U(1) factors. If there is at least one extra U(1) factor, we have several coupling constants, i.e the so called Fayet-Ilioupoulos terms. Consider V^A, the vector multiplet, in the abelian case, or the component corresponding to an abelian factor. It transform under (4.2). From the component expansion of the chiral (3.30) and anti-chiral (3.31), one sees that the D-term transform as a $D^A \to D^A + \text{Total Derivative}$. Being a D-term, it also transforms as a total derivative under SUSY. Therefore, the Lagrangian associated with those abelian terms:

$$\mathcal{L}_{FI} = \sum_{A\in\text{Abelian Factors}} \chi^A \int d^2\theta d^2\bar{\theta}\; V^A = \sum_{A\in\text{Abelian Factors}} \frac{1}{2}\chi^A D^A. \tag{4.43}$$

is SUSY and gauge invariant under the action.

Finally, we can write the full N=1 SUSY non-abelian matter theory as:

$$\begin{aligned}
S^{N=1}_{\text{Non-Abe Gau/Matter}} &= \int d^4x \Big\{ \int d^2\theta d^2\bar{\theta}\ \phi^\dagger e^{2gV}\phi + \int d^2\theta\ W(\phi) + h.c. \\
&+ \frac{1}{32\pi}\,\mathrm{Im}\left[\tau \int d^2\theta\,\mathrm{Tr}\,(W^\alpha W_\alpha)\right] + 2g\sum_A \chi^A \int d^2\theta d^2\bar{\theta}\ V^A. \Big\} \\
&= (D_\mu z)^\dagger D^\mu z - i\bar{\psi}\bar{\not{D}}\psi + f^\dagger f + g^2 z^\dagger A^\mu A_\mu z + i\sqrt{2}g z^\dagger \lambda\psi \\
&- i\sqrt{2}g\bar{\psi}\bar{\lambda}z + g z^\dagger D z - \frac{1}{2}w^i w^j \psi_i\psi_j + h.c. \\
&+ \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\not{D}\bar{\lambda} + \frac{1}{2}D^2\right) + \frac{\Theta}{32\pi^2}g^2\,\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \\
&+ g\sum_A \chi^A D^A,
\end{aligned} \tag{4.44}$$

where we defined the derivatives of the superpotential $W(z_i, z_i^\dagger)$ as

$$w^i \equiv \frac{\partial W}{\partial z_i}, \quad w_j \equiv \frac{\partial W}{\partial z_j^\dagger}, \tag{4.45}$$

and we left the spinors in 2-component notation in order to emphasize the couplings of the fermions with the superpotential.

From the action, we see that the auxiliary field equations are

$$f_i^\dagger = \frac{\partial W}{\partial z_i} = w^i, \quad D^a = -g z^\dagger T^a z - g\chi^a, \tag{4.46}$$

where it is understood that $\chi^a = 0$ represents the absence of abelian gauge factors.

Substituting back into the action, we finally get:

$$
\begin{aligned}
S^{N=1}_{\text{Non-Abe Gau/Matter}} &= \int d^4x \Big\{ \int d^2\theta d^2\bar{\theta}\ \phi^\dagger e^{2gV}\phi + \int d^2\theta\ W(\phi) + h.c. \\
&+ \frac{1}{32\pi}\,\mathrm{Im}\left[\tau \int d^2\theta\,\mathrm{Tr}\ (W^\alpha W_\alpha)\right] + 2g\sum_A \chi^A \int d^2\theta d^2\bar{\theta}\ V^A. \Big\} \\
&= (D_\mu z)^\dagger D^\mu z - i\bar{\psi}\bar{\not{D}}\psi + g^2 z^\dagger A^\mu A_\mu z + i\sqrt{2}gz^\dagger\lambda\psi \\
&- i\sqrt{2}g\bar{\psi}\bar{\lambda}z - \frac{1}{2}w^i w^j \psi_i\psi_j + h.c. + \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\not{D}\bar{\lambda}\right) \\
&+ \frac{\Theta}{32\pi^2}g^2\,\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) - V(z,z^\dagger),
\end{aligned}
\tag{4.47}
$$

where the scalar potential $V(z,z^\dagger)$ is given by

$$
V(z,z^\dagger) = f^\dagger f + \frac{1}{2}D^2 = \sum_i |w^i|^2 + \frac{g^2}{2}\sum_a |z^\dagger T^a z + \chi^a|^2. \tag{4.48}
$$

In order to recover the abelian case, we just need to make $f^{abc} = 0$. The covariant derivative would reduce to the usual abelian derivative, i.e. with the dependence on the electric charge. Also, there would be no trace under the gauge fields and their superpartners since the generator of the abelian group is proportional to the identity. In addition, the gaugino field would have no gauge coupling, since it's a Majorana fermion and the latter has no charge. The latter is a consequence of the Majorana's condition.

4.4 Super Yang-Mills N=2 Theory

The N=2 multiplets with helicities (spins) not exceeding one are the massless $N = 2$ vector multiplet and the hypermultiplet. The latter can be massless or be a short (BPS) massive multiplet. In this section, we will concentrate on $N = 2$ vector multiplets to construct $N = 2$ super yang-mills theories.

We can decompose the $N = 2$ vector multiplet into an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet, but now all fields must be in the adjoint representation of the gauge group. Therefore, we must take the trace of the kinetic gauge matter term (4.38) with respect to the indices of the gauge group representation. The Lagrangian for this term becomes:

$$\mathcal{L}^{N=1}_{\text{Matter}} = \int d^2\theta d^2\bar{\theta}\,\text{Tr}\,(\phi^\dagger e^{2gV}\phi), \tag{4.49}$$

but now with

$$z = z^a T^a, \quad \psi = \psi^a T^a, \quad f = f^a T^a, \ldots.., \tag{4.50}$$

i.e. all fields in the adjoint representation of the gauge group.

In the adjoint representation of the gauge group, the generators are given by the structure constants:

$$(T^a)_{bc} = -if_{abc}, \tag{4.51}$$

with the generators normalized by

$$\text{Tr}\,(T^a T^b) = \delta^{ab}. \tag{4.52}$$

Therefore, we can write terms of the form $z^\dagger\lambda\psi$ as

$$\begin{aligned} z^\dagger\lambda\psi \to z_b^\dagger \lambda^a (T^a)_{bc}\psi^c &= -iz_b^\dagger\lambda^a f_{abc}\psi^c = iz_b^\dagger f_{bac}\lambda^a\psi^c = z_b^\dagger i f_{bac}\delta^{ae}\lambda^e\psi^c \\ &= z_b^\dagger\,\text{Tr}\,(if_{bac}T^aT^e)\lambda^e\psi^c = z_b^\dagger\,\text{Tr}\,([T^c,T^b]T^e)\lambda^e\psi^c \\ &= z_b^\dagger\lambda^e\psi^c\,\text{Tr}\,(T^b[T^e,T^c]) = \text{Tr}\,(z^\dagger[\lambda,\psi]). \end{aligned} \tag{4.53}$$

Then, in terms of field components, the Lagrangian (4.49) becomes:

$$\mathcal{L}^{N=1}_{\text{Matter}} = \text{Tr}\,\left[(D_\mu z)^\dagger D^\mu z \ - i\bar{\psi}\bar{\not{D}}\psi + f^\dagger f + i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\bar{\psi},\bar{\lambda}\}z + gD[z,z^\dagger]\right]. \tag{4.54}$$

Adding the Lagrangian of the theory (4.36) to (4.49), we obtain

$$\begin{aligned} \mathcal{L}^{N=2}_{\text{S-YM}} &= \int d^2\theta d^2\bar{\theta}\ \text{Tr}\,(\phi^\dagger e^{2gV}\phi) + \frac{1}{32\pi}\,\text{Im}\left[\tau\int d^2\theta\,\text{Tr}\,(W^\alpha W_\alpha)\right] \\ &= \text{Tr}\,\Bigg[(D_\mu z)^\dagger D^\mu z - i\bar{\psi}\bar{\not{D}}\psi + f^\dagger f + i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\bar{\psi},\bar{\lambda}\}z + gD[z,z^\dagger] \\ &\quad - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\not{D}\bar{\lambda} + \frac{1}{2}D^2 + \frac{\Theta}{32\pi^2}g^2F_{\mu\nu}\tilde{F}^{\mu\nu}\Bigg] \end{aligned} \tag{4.55}$$

A sufficient condition, and a necessary one, for $N = 2$ SUSY is the existence of an $SU_R(2)$ symmetry between the generators Q^I_α. Note that, since the $N = 2$ vector multiplet contains the two Weyl fermions ψ and λ, this symmetry must also act on then. The coefficients in (4.55) were also chosen in such a way as to have this $SU_R(2)$ symmetry. Note also that the terms with anti-commutators in (4.55) also preserves this symmetry.

We have not added a superpotential because a term of this type (unless linear in ϕ) would break the $SU_R(2)$ symmetry and not lead to a $N = 2$ SUSY theory.

The auxiliary fields equations:

$$\begin{aligned} f^a &= 0 \\ D^a &= -g[z, z^\dagger]^a, \end{aligned} \tag{4.56}$$

leading to a scalar potential

$$V(z, z^\dagger) = \frac{1}{2} g^2 \,\mathrm{Tr}\, ([z, z^\dagger])^2, \tag{4.57}$$

which is due only to the auxiliary D fields of the $N = 1$ gauge multiplet.

With the ending of this section, we concluded the construction the Super-Gauge theories. We stopped at the N=2 Super-Yang-Mills theories, but its possible to construct theories with higher N. In the next section, we are interested in apply the developed formalism to the study of Lorentz violating models.

5 Lorentz Symmetry Violation in Supersymmetric Scenarios

In this section, we develop a supersymmetric version of the non-abelian-Carrol-Field-Jackiew model [38]. Following, we focus on the abelian case to study a possible photon-photino mixing analogous to the Primakoff effect [33]. This mixing is possible due to the fermionic sector of the supersymmetric multiplet which breaks the Lorentz symmetry.

5.1 Derivation of the Supersymmetric Non-Abelian Carrol-Field-Jackiw Model

We wish to build up the supersymmetric version of the Yang-Mills-Carroll-Field-Jackiw Lorentz symmetry breaking type-term:

$$\mathcal{L} = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}v_\mu A^a_\nu \partial_\lambda A^a_\kappa + \frac{g}{3!}\epsilon^{\mu\nu\lambda\kappa} f^{abc} v_\mu A^a_\nu A^b_\lambda A^c_\kappa, \tag{5.1}$$

with $F^{\mu\nu}$ defined in the context of some gauge group as in (4.29) and (4.33). The supersymmetric version of the CFJ-term is given by:

$$S^{\mathrm{CFJ}}_{\mathrm{SUSY}} = \int d^4x d^4\theta \{W^{\alpha,a}(D_\alpha V^a)S + c.c.\}, \tag{5.2}$$

where c.c. denotes the complex conjugate, $W^{\alpha,a}$ and $D_\alpha V^a_{\mathrm{WZ}}$ are in the adjoint representation of the gauge group, as was given in (4.26) for the former and the latter is defined as:

$$D_\alpha V^a = (\sigma^\mu\bar{\theta})_\alpha A^a_\mu + 2i\theta_\alpha\bar{\theta}\bar{\lambda}^a - i\bar{\theta}\bar{\theta}\lambda^a_\alpha + \theta_\alpha\bar{\theta}\bar{\theta}D^a + i\bar{\theta}\bar{\theta}(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu} + \theta\theta\bar{\theta}\bar{\theta}(\not{D}^{ab}\bar{\lambda}^b)_\alpha. \tag{5.3}$$

The chiral superfield S(x) is responsible for the breaking of the Lorentz symmetry:

$$S(x) = s(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu s(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 s(x) + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\bar{\theta}^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) - \theta^2 f(x), \tag{5.4}$$

with $\bar{D}_{\dot{\alpha}}S(x) = 0$. This superfield is neutral under the gauge group and its canonical mass dimension is 0.

The F-term of $W^{\alpha,a}(D_\alpha V^a)S$:

$$\begin{aligned} W^{\alpha,a}(D_\alpha V^a)S\Big|_{\theta^2\bar{\theta}^2} &= W^{\alpha,a}(D_\alpha V^a)\bigg(s(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu s(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 s(x) + \sqrt{2}\theta\psi(x) \\ &\quad - \frac{i}{\sqrt{2}}\theta^2(\sigma^\mu\bar{\theta})\partial_\mu\psi(x) - \theta^2 f(x)\bigg)\bigg|_{\theta^2\bar{\theta}^2} \\ &= -i2s\lambda^a \not{D}^{ab}\bar{\lambda}^b + s(D^a)^2 - \frac{s}{2}([F^a_{\mu\nu}]^2 - iF^a_{\mu\nu}\tilde{F}^{\mu\nu,a}) + \sqrt{2}(\lambda^a\sigma^{\mu\nu}\psi)F^a_{\mu\nu} \\ &\quad + (\lambda^a)^2 f + i\sqrt{2}(\lambda^a\psi)D^a, \end{aligned} \tag{5.5}$$

and the complex conjugate

$$\begin{aligned} \bar{W}^a_{\dot{\alpha}}(\bar{D}^{\dot{\alpha}}V^a)S\Big|_{\theta^2\bar{\theta}^2} &= -i2s^\dagger\bar{\lambda}^a\bar{\not{D}}^{ab}\lambda^b + s^\dagger(D^a)^2 - \frac{s^\dagger}{2}([F^a_{\mu\nu}]^2 + iF^a_{\mu\nu}\tilde{F}^{\mu\nu,a}) \\ &\quad - \sqrt{2}(\bar{\lambda}^a\bar{\sigma}^{\mu\nu}\bar{\psi})F^a_{\mu\nu} + (\bar{\lambda}^a)^2 f^\dagger - i\sqrt{2}(\bar{\lambda}^a\bar{\psi})D^a. \end{aligned} \tag{5.6}$$

The super-action becomes:

$$\begin{aligned} S^{\text{CFJ}}_{\text{SUSY}} &= \int d^4x \bigg\{ -i2s\lambda^a\not{D}^{ab}\bar{\lambda}^b - i2s^\dagger\bar{\lambda}^a\bar{\not{D}}^{ab}\lambda^b + (s+s^\dagger)(D^a)^2 - \frac{s+s^\dagger}{2}(F^a)^2_{\mu\nu} \\ &\quad + i\frac{s-s^\dagger}{2}F^a_{\mu\nu}\tilde{F}^{\mu\nu,a} + \sqrt{2}(\lambda^a\sigma^{\mu\nu}\psi - \bar{\lambda}^a\bar{\sigma}^{\mu\nu}\bar{\psi})F^a_{\mu\nu} + i\sqrt{2}(\lambda^a\psi - \bar{\lambda}^a\bar{\psi})D^a \\ &\quad + (\lambda^a)^2 f + (\bar{\lambda}^a)^2 f^\dagger \bigg\} \\ &= \int d^4x \bigg\{ -2i(\partial_\mu s)\bar{\lambda}^a\bar{\sigma}^\mu\lambda^a - 4i\,\mathrm{Re}\{s\}\bar{\lambda}^a\bar{\sigma}^\mu\partial_\mu\lambda^a - 4i\,\mathrm{Re}\{s\}gf^{abc}\lambda^a\sigma^\mu\bar{\lambda}^b A^c_\mu \\ &\quad 2\,\mathrm{Re}\{s\}(D^a)^2 - \mathrm{Re}\{s\}(F^a_{\mu\nu})^2 + 2\epsilon^{\mu\nu\lambda\kappa}(\partial_\mu\,\mathrm{Im}(s))A^a_\nu\partial_\lambda A^a_\kappa + \frac{2g}{3}\epsilon^{\mu\nu\lambda\kappa}(\partial_\mu\,\mathrm{Im}(s))f^{abc}A^a_\nu A^b_\lambda A^c_\kappa \\ &\quad + \sqrt{2}(\lambda^a\sigma^{\mu\nu}\psi - \bar{\lambda}^a\bar{\sigma}^{\mu\nu}\bar{\psi})F^a_{\mu\nu} + i\sqrt{2}(\lambda^a\psi - \bar{\lambda}^a\bar{\psi})D^a + (\lambda^a)^2 f + (\bar{\lambda}^a)^2 f^\dagger \bigg\} \end{aligned} \tag{5.7}$$

To obtain the CFJ-term, we impose under S(x) the following boundaries conditions:

$$\mathrm{Re}\{s\} = \frac{s+s^\dagger}{2} = 0 \to s = -s^\dagger \quad \text{and} \quad \mathrm{Im}\{s\} = \varphi = \frac{v_\mu x^\mu}{4}, \tag{5.8}$$

such that,

$$\partial_\mu \operatorname{Im}\{s\} = \partial_\mu \varphi = \frac{v_\mu}{4}. \tag{5.9}$$

Note also that:

$$f\lambda^2 + f^\dagger \bar{\lambda}^2 = \operatorname{Re}\{f\}(\lambda^2 + \bar{\lambda}^2) + i\operatorname{Im}\{f\}(\lambda^2 - \bar{\lambda}^2). \tag{5.10}$$

Reorganizing the terms, we find

$$\begin{aligned} S^{\text{CFJ}}_{\text{SUSY}} = \int d^4x \Big\{ & \frac{v_\mu}{4}(2\bar{\lambda}^a \bar{\sigma}^\mu \lambda^a) + \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa} v_\mu A^a_\nu \partial_\lambda A^a_\kappa + \frac{g}{6}\epsilon^{\mu\nu\lambda\kappa} v_\mu f^{abc} A^a_\nu A^b_\lambda A^c_\kappa \\ & + \sqrt{2}(\lambda^a \sigma^{\mu\nu}\psi - \bar{\lambda}^a \bar{\sigma}^{\mu\nu}\bar{\psi})F^a_{\mu\nu} + i\sqrt{2}(\lambda^a\psi - \bar{\lambda}^a\bar{\psi})D^a + \operatorname{Re}\{f\}([\lambda^a]^2 + [\bar{\lambda}^a]^2) \\ & + i\operatorname{Im}\{f\}([\lambda^a]^2 - [\bar{\lambda}^a]^2) \Big\} \end{aligned} \tag{5.11}$$

Note that SUSY already provides the 4-curl condition that preserves the gauge symmetry of the theory

$$v_\mu \propto \partial_\mu \varphi \quad \rightarrow \quad \partial_\mu v_\nu - \partial_\nu v_\mu = 0. \tag{5.12}$$

Translating to 4-components, we find

$$\begin{aligned} S^{\text{CFJ}}_{\text{SUSY}} = \int d^4x \Big\{ & \frac{v_\mu}{4}\bar{\Lambda}^a \gamma^\mu \gamma_5 \Lambda^a + \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa} v_\mu A^a_\nu \partial_\lambda A^a_\kappa + \frac{g}{6}\epsilon^{\mu\nu\lambda\kappa} v_\mu f^{abc} A^a_\nu A^b_\lambda A^c_\kappa - i\sqrt{2}(\bar{\Lambda}^a \Sigma^{\mu\nu}\gamma_5 \Psi)F^a_{\mu\nu} \\ & + i\sqrt{2}(\bar{\Lambda}^a \gamma_5 \Psi)D^a + \operatorname{Re}\{f\}(\bar{\Lambda}^a \Lambda^a) + i\operatorname{Im}\{f\}\bar{\Lambda}^a \gamma_5 \Lambda^a \Big\}. \end{aligned} \tag{5.13}$$

Adding (4.36) (with $\Theta = 0$):

$$\begin{aligned} S^{\text{CFJ}}_{\text{SUSY}} + S^{\text{Gauge}}_{\text{SUSY}} = \int d^4x \Big\{ & -\frac{(F^a_{\mu\nu})^2}{4} + \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa} v_\mu A^a_\nu \partial_\lambda A^a_\kappa + \frac{g}{6}\epsilon^{\mu\nu\lambda\kappa} v_\mu f^{abc} A^a_\nu A^b_\lambda A^c_\kappa - \frac{i}{2}\bar{\Lambda}^a \not{D}^{ab}\Lambda^b \\ & + \frac{v_\mu}{4}\bar{\Lambda}^a \gamma^\mu \gamma_5 \Lambda^a - i\sqrt{2}(\bar{\Lambda}^a \Sigma^{\mu\nu}\gamma_5 \Psi)F^a_{\mu\nu} + i\sqrt{2}(\bar{\Lambda}^a \gamma_5 \Psi)D^a + \frac{1}{2}(D^a)^2 \\ & + \operatorname{Re}\{f\}(\bar{\Lambda}^a \Lambda^a) + i\operatorname{Im}\{f\}\bar{\Lambda}^a \gamma_5 \Lambda^a \Big\}. \end{aligned} \tag{5.14}$$

The D field equations are $D^a = -i\sqrt{2}\bar{\Lambda}^a\gamma_5\Psi$. Substituting back these field equations, and after Fiertz rearrangement (B), we finally obtain:

$$S^{\text{CFJ}}_{\text{SUSY}} + S^{\text{Gauge}}_{\text{SUSY}} = \int d^4x \Big\{ \underbrace{-\frac{(F^a_{\mu\nu})^2}{4}}_{\text{Gauge term}} + \underbrace{\frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}v_\mu A^a_\nu \partial_\lambda A^a_\kappa + \frac{g}{6}\epsilon^{\mu\nu\lambda\kappa}v_\mu f^{abc}A^a_\nu A^b_\lambda A^c_\kappa}_{\text{YM-CFJ type-term}} - \frac{i}{2}\bar{\Lambda}^a \not{D}^{ab}\Lambda^b$$
$$+ \frac{R_\mu}{4}\bar{\Lambda}^a\gamma^\mu\gamma_5\Lambda^a - i\sqrt{2}(\bar{\Lambda}^a\Sigma^{\mu\nu}\gamma_5\Psi)F^a_{\mu\nu} + M_1\bar{\Lambda}^a\Lambda^a + iM_2\bar{\Lambda}^a\gamma_5\Lambda^a\Big\}. \tag{5.15}$$

where we defined the quantities:

$$R_\mu = v_\mu - \bar{\Psi}\gamma_\mu\gamma_5\Psi, \quad M_1 = \text{Re}\{f\} - \frac{\bar{\Psi}\Psi}{4}, \quad M_2 = \text{Im}\{f\} + \frac{i}{4}\bar{\Psi}\gamma_5\Psi. \tag{5.16}$$

As can be seen, we were able to generate the YM-CFJ model with many contributions from supersymmetry. The gaugino field Λ, which is the supersymmetric partner of the gauge-boson A^μ, in contrast to the conventional Super-Yang-Mills model has a much more rich Lagrangian. In particular, the presence of the fermionic background induces a mass term for the gaugino, which shows that SUSY is explicitly broken. Also, a bilinear term in the gauge-boson and the gaugino field intermediated by the fermionic background is present. This term allows for a possible gauge-boson/gaugino mixing mechanism, analogue to the Primakoff effect, but here induced by a fermionic background.

6 Future Perspectives

In this section, we analyze the perspective of a possible Gauge-boson/Gaugino mixing due to the fermionic background induce by the SUSY LSV Non-Abelian Carroll-Field-Jackiw model. Here, we focus on the abelian case, i.e. $f^{abc} = 0$, just for the simplicity.

For a quantum mechanical mixing approach, analogue to the Primakoff effect, we need to show the possibility to generate a mixing matrix from the equations of motion. The field equations of motion are given by:

$$\begin{cases} \partial_\mu F^{\mu\nu} + v_\mu \tilde{F}^{\mu\nu} = \; -i2\sqrt{2}\bar{\Psi}\Sigma^{\mu\nu}\gamma_5\partial_\mu\Lambda, \\ \left(i\partial\!\!\!/ - \frac{R\!\!\!/}{2}\gamma_5 - 2(iM_2\gamma_5 + M_1)\right)\Lambda = -i2\sqrt{2}\Sigma^{\mu\nu}\gamma_5\Psi\partial_\mu A_\nu = -i\sqrt{2}\Sigma^{\mu\nu}\gamma_5\Psi F_{\mu\nu}, \\ \partial_\mu \tilde{F}^{\mu\nu} = 0, \end{cases} \tag{6.1}$$

with the field strength (and its dual) components given by:

$$\begin{cases} F^{i0} = E^i \to \vec{E}, \quad F_{ij} = -\epsilon_{ijk}B^k, \\ \tilde{F}^{i0} = B^i \to \vec{B}, \quad \tilde{F}_{ij} = \epsilon_{ijk}E^k. \end{cases} \tag{6.2}$$

The Lorentz generators (with the γ_5 matrix) are given by:

$$\Sigma^{\mu\nu}\gamma_5 = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\gamma_5 = \frac{1}{2}\left(\begin{array}{c|c} 0 & i\Sigma^i \\ \hline -i\Sigma^i & \epsilon^{ijk}\Sigma^k\gamma_5 \end{array}\right)_{4\times4}, \quad \text{with} \quad \Sigma^i = \mathbf{1}_2 \otimes \sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{6.3}$$

and σ^i the Pauli's matrices. Since we are interested in the fermionic background's effects only, for simplicity we disconsider effects from the vector background , and other billinears different from $\bar{\Psi}\Psi$, by considering $v_\mu = R_\mu = M_2 = 0$. In the Lorenz gauge, i.e $\partial_\mu A^\mu = 0$, the field equations becomes:

$$\begin{cases} \Box A^\nu = -i2\sqrt{2}\bar{\Psi}\Sigma^{\mu\nu}\gamma_5\partial_\mu\Lambda, \\ (\Box + 4M_1^2)\Lambda = -(-i\partial\!\!\!/ + 2M_1)(-i2\sqrt{2}\Sigma^{\mu\nu}\gamma_5\Psi\partial_\mu A_\nu). \end{cases} \tag{6.4}$$

Now, we consider plane wave solutions to the equations,

$$A^\nu(x) = A^\nu(k)e^{-ik^\mu x_\mu}, \quad \Lambda(x) = \Lambda(k)e^{-ik^\mu x_\mu}, \tag{6.5}$$

with the 3-momentum $\vec{k} = k\hat{z}$ and the dispersion relation $k^\mu k_\mu = 0$. We linearize the D'Alembertian operator as

$$\Box = (\partial_t^2 - \partial_z^2) = -(\omega + i\partial_z)(\omega - \underbrace{i\partial_z}_{-k}) = -2\omega(\omega + i\partial_z). \tag{6.6}$$

Therefore, in 3-momentum space, we can write the field equations as

$$\begin{cases} -2\omega(\omega + i\partial_z)A^\nu = -2\sqrt{2}\bar{\Psi}\Sigma^{\mu\nu}\gamma_5 k_\mu \Lambda, \\ -2\omega(\omega + i\partial_z)\Lambda + 4M_1^2\Lambda = (\not{k} + 2M_1)(2\sqrt{2}\Sigma^{\mu\nu}\gamma_5\Psi k_\mu A_\nu). \end{cases} \tag{6.7}$$

Dividing by -2ω, we have

$$\begin{cases} (\omega + i\partial_z)A^\nu = \frac{\sqrt{2}}{\omega}\bar{\Psi}\Sigma^{\mu\nu}\gamma_5 k_\mu \Lambda, \\ \left(\omega + i\partial_z - \frac{2M_1^2}{\omega}\right)\Lambda = (\not{k} + 2M_1)\left(-\frac{\sqrt{2}}{\omega}\Sigma^{\mu\nu}\gamma_5\Psi k_\mu A_\nu\right). \end{cases} \tag{6.8}$$

As we can see from the symmetry of the coupling, its possible to build a mixing mass matrix from these field equations, but there are some details that must be deal with first. One of then is the difference in canonical dimension between the photon's field A^μ and the photino's field Λ. The primer has canonical dimension 1 and the latter canonical dimension 3/2. If we try to write down a mass matrix from these equations, it would have a undefined mass dimension, which in turn would lead to problems in the diagonalization process necessary to obtain the matrix which relates the mixed fields. Other question necessary to be addressed is how to relate the different nature of the fields. The photino field is a spinor, and therefore different from the vectorial nature of the photon field. To relate then, one must be projected into the space of the other. This must also be considered in the mixing process. These points are going to be considered in future research before considering a mixing matrix.

7 Conclusions

In this Master Dissertation, we have developed the full formalism of Supersymmetry and derived its superfield as irreducible non-unitary representations of SUSY algebra. In the process, we have shown how the superfields could generate, in the context of the superactions, all the irreducible non-unitary Lorentz fields (all the gauge and matter fields present in the Standard Model of Particle Physics and it's supersymmetric partners) kinetic, potential and interaction terms.

With this context in mind, we would like to, first of all, emphasize here the full power and richness of Supersymmetry. It not only allows us to construct an formalism able to board in the physics of the Standard Model, but also proposes new particles, i.e. the supersymmetric partners, that can be candidates to solve some of the challenges in physics. Furthermore, there are many contributions done by SUSY to different aspects and areas of Physics, namely Non-Renormalizability, implications on String theory, on gravitational interaction, i.e. named Supergravity, it's applications to condensed matter theory, the neutralino as a dark matter candidate, and so on, that are so profound and vast that cannot be contained in this dissertation.

Second, we would like to highlight the incredible capacity of SUSY to provide a theoretical framework for a Unified Fundamental Physics. As we have shown over this dissertation, Supersymmetry allows us to derive all the Standard Model like matter field, with its self interactions and Yukawa couplings, and also the Gauge fields with it's respectively gauge couplings. Therefore, if we assume that Supersymmetry is a fundamental symmetry of Nature, as once envisioned by Abdus Salam and many others, it allows us to unify all the fundamental aspects present in the Standard Model in one unique fundamental principle which is, in the point of view of this Master's student, a desirable feature.

As a third, we desire to shed light upon the great contributions given by Supersymmetry to the context of the Lorentz symmetry violation. In the construction of the supersymmetric Yang-Mills-CFJ model, we showed how adopting the perspective described in section (1.3) allowed for a more fundamental explanation of the LSV. We demonstrated that, not only the vector background gained a more fundamental description of it's origin, but also the Supersymmetry allowed it to have a Majorana fermionic background superpartner. The latter gave many contributions to the model. Both the gauge-boson and the gaugino gained a mass in the model, with a splitting of it. This showed that bringing the LSV to a supersymmetric scenario caused also the breaking of SUSY. Also, the model allowed for a mixing term between the gauge-boson and the gaugino, induce by the fermionic background. As we have discussed in section (6), it's possible to derive a mixing process similar to the Primakoff effect. We emphasized, however, that some points must be deal with before this mixing process could be done. These points center deeply in the difference of both fields as mathematical and physical objects. They belong to different spaces and has different canonical mass dimensions. A new approach is needed to deal with this aspects. Nevertheless, we would like to stress that the mixing process is still possible and will be topic of future research. It's an important, and great conclusion of this dissertation, to highlight the perspective of this Primakoff like gauge-boson/gaugino mixing induced by the Majorana background.

Finally, it is the most honest and truthful aim of this Dissertation to argue for, and to defend, the Supersymmetry not only as a theory, but also a fundamental aspect and symmetry of Nature. In our Research Group, we adopt the viewpoint that the superspace is not a merely mathematical tool, but a representation of physical space, with the grassmannian coordinates reflecting the fermionic nature intrinsic to space-time. Although we cannot measure these coordinates, they have physical implications to it's non-unitary representations and therefore to the physics derived from it. Despite the fact Supersymmetry was not experimentally verified yet, the implications of it to Elementary Particle Physics gives sufficient pro arguments to believe on it, in the opinion of this author. Also, it's important to point out that the most important playrole of theoretical physics is not to just give physical sciences mathematical models, but well ornamented principles and visions of the physical world. As once quoted by Einstein, "Physics is not the merely act of cataloging experimental data, but first of all a vision of the world" and that "Science is Man's truthful wish to create a simple and clear vision of the world". What gives Science consistency, clarity, and depth meaning is harmony, beauty and simplicity. We, as scientists, should aim at these aspects in order to build up knowledge based on truth and humility before the diverse challenges of Nature. The author attributes these qualities to Supersymmetry, and therefore as an extraordinary path to understand Nature.

A Translation of the 2-components notation to the 4-components notation

Here, we show how to translate the 4-components spinors to the 2-components dotted/undotted spinors notation. The γ-matrices are given in the Weyl representation as discussed in section 2.4.

A.1 Weyl Spinors

In terms of the 2-component spinors, the Weyl left Ψ_L and the Weyl right Ψ_R spinors are given by:

$$\Psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad \Psi_R = \begin{pmatrix} 0 \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.1}$$

The Dirac's conjugate of then:

$$\bar{\Psi}_L = \Psi_L^\dagger \gamma_0 = \begin{pmatrix} 0 & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi}_R = \Psi_R^\dagger \gamma_0 = \begin{pmatrix} \psi^\alpha & 0 \end{pmatrix}. \tag{A.2}$$

The Weyl's left kinetic terms:

$$\begin{aligned} i\bar{\Psi}_L \gamma^\mu \partial_\mu \Psi_L &= i \begin{pmatrix} 0 & \bar{\psi}_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \\ &= i\bar{\psi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha, \end{aligned} \tag{A.3}$$

and the Weyl right ones,

$$\begin{aligned} i\bar{\Psi}_R \gamma^\mu \partial_\mu \Psi_R &= i \begin{pmatrix} \psi^\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \\ &= i\psi^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \bar{\psi}^{\dot{\alpha}}. \end{aligned} \tag{A.4}$$

A.2 Dirac Spinors

We can think of Dirac spinors as spinors with left and right components unrelated, i.e. independent. Therefore:

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}}, \end{pmatrix} \tag{A.5}$$

with the Dirac's conjugate,

$$\bar{\Psi}_D = \Psi_D^\dagger \gamma^0 = \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}. \tag{A.6}$$

The Dirac spinor kinetic term can be written as:

$$\begin{aligned} i\bar{\Psi}_D \gamma^\mu \partial_\mu \Psi_D &= i \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}}, \end{pmatrix} \\ &= i\bar{\psi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha + i\chi^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \bar{\chi}^{\dot{\alpha}}, \end{aligned} \tag{A.7}$$

and the Dirac spinor mass term as:

$$m\bar{\Psi}_D \Psi_D = m\chi^\alpha \psi_\alpha + m\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}. \tag{A.8}$$

A.3 Majorana Spinors

From the Majorana condition, i.e. $\Psi = \Psi^c$, we find that, in terms of the 2-components spinors, the Majorana spinors have the form:

$$\Psi_{\text{Maj.}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.9}$$

with the left and right components related by $\psi_R = \varepsilon \psi_L^*$ and the antisymmetric matrix given by

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.10}$$

The Dirac's conjugate:

$$\bar{\Psi}_{\text{Maj.}} = \Psi_{\text{Maj.}}^\dagger \gamma^0 = \begin{pmatrix} \psi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}. \tag{A.11}$$

The Majorana's kinetic terms can be expressed as:

$$\begin{aligned} \frac{i}{2}\bar{\Psi}_{\text{Maj.}}\gamma^{\mu}\partial_{\mu}\Psi_{\text{Maj.}} &= \frac{i}{2}\begin{pmatrix}\psi^{\alpha} & \bar{\psi}_{\dot{\alpha}}\end{pmatrix}\begin{pmatrix}0 & (\sigma^{\mu})_{\alpha\dot{\alpha}}\partial_{\mu} \\ (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\partial_{\mu} & 0\end{pmatrix}\begin{pmatrix}\psi_{\alpha} \\ \bar{\psi}^{\dot{\alpha}}\end{pmatrix} \\ &= \frac{i}{2}\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\partial_{\mu}\psi_{\alpha} + \frac{i}{2}\psi^{\alpha}(\sigma^{\mu})_{\alpha\dot{\alpha}}\partial_{\mu}\bar{\psi}^{\dot{\alpha}} \end{aligned} \tag{A.12}$$

The Majorana's mass term:

$$m\bar{\Psi}_{\text{Maj.}}\Psi_{\text{Maj.}} = m\psi^{\alpha}\psi_{\alpha} + m\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}. \tag{A.13}$$

For use in section 5, we also demonstrate the following Majorana's billinears:

$$\begin{aligned} i\bar{\Lambda}\gamma_5\Lambda &= \begin{pmatrix}\lambda^{\alpha} & \bar{\lambda}_{\dot{\alpha}}\end{pmatrix}\begin{pmatrix}\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2\end{pmatrix}\begin{pmatrix}\lambda_{\alpha} \\ \bar{\lambda}^{\dot{\alpha}}\end{pmatrix} = i\lambda^{\alpha}\lambda_{\alpha} - i\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}, \\ \bar{\Lambda}\gamma^{\mu}\gamma_5\Lambda &= \begin{pmatrix}\lambda^{\alpha} & \bar{\lambda}_{\dot{\alpha}}\end{pmatrix}\begin{pmatrix}0 & (\sigma^{\mu})_{\alpha\dot{\alpha}}\partial_{\mu} \\ (\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\partial_{\mu} & 0\end{pmatrix}\begin{pmatrix}\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2\end{pmatrix}\begin{pmatrix}\lambda_{\alpha} \\ \bar{\lambda}^{\dot{\alpha}}\end{pmatrix} \\ &= \bar{\lambda}_{\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\lambda_{\alpha} - \lambda^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} = 2\bar{\lambda}_{\dot{\alpha}}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\lambda_{\alpha}, \\ \bar{\Lambda}\Sigma^{\mu\nu}\gamma_5\Psi &= \begin{pmatrix}\lambda^{\alpha} & \bar{\lambda}_{\dot{\alpha}}\end{pmatrix}\begin{pmatrix}i(\sigma^{\mu\nu})_{\alpha}^{\ \beta} & 0 \\ 0 & i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}\end{pmatrix}\begin{pmatrix}\mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2\end{pmatrix}\begin{pmatrix}\psi_{\beta} \\ \bar{\psi}^{\dot{\beta}}\end{pmatrix} \\ &= i\lambda^{\alpha}(\sigma^{\mu\nu})_{\alpha}^{\ \beta}\psi_{\beta} - i\bar{\lambda}_{\dot{\alpha}}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\psi}^{\dot{\beta}}. \end{aligned} \tag{A.14}$$

B Fierz Rearrangement

In this appendix, we demonstrate how to do the so called Fierz Rearrangement. The set of matrices:

$$\Gamma^A = \{\mathbb{1}_4, \gamma^\mu, \gamma_5, \gamma^\mu\gamma_5, \Sigma^{\mu\nu}\}, \tag{B.1}$$

forms a basis for the space of the complex 4×4 matrices. Therefore, any matrix that belongs to this space can be written as

$$M_{\alpha\beta} = \sum_A c_A \Gamma^A_{\alpha\beta}. \tag{B.2}$$

With the inverse matrices of the basis, i.e.

$$(\Gamma^A)^{-1} = \Gamma_A = \{\mathbb{1}_4, \gamma_\mu, \gamma_5, -\gamma_\mu\gamma_5, \Sigma_{\mu\nu}\}, \tag{B.3}$$

we can calculate the expansion coefficients:

$$\begin{aligned} M\Gamma_B =& \sum_A c_A \Gamma^A \Gamma_B \\ \mathrm{Tr}\{M\Gamma_B\} =& \sum_A \mathrm{Tr}\big\{c_A \Gamma^A \Gamma_B\big\} \\ \mathrm{Tr}\{M\Gamma_B\} =& \sum_A c_A \,\mathrm{Tr}\big\{\delta^A_B \mathbb{1}_4\big\} \\ \mathrm{Tr}\{M\Gamma_B\} =& 4\ c_B. \end{aligned} \tag{B.4}$$

Therefore, we can cast the 4×4 matrices as:

$$M_{\alpha\beta} = \sum_A \left(\frac{\mathrm{Tr}\{M\Gamma_A\}}{4}\right) \Gamma^A_{\alpha\beta} \tag{B.5}$$

As an example, we will calculate an rearrangement needed in section (5). If Λ and Ψ are Majorana spinors, then we can rewrite the billinear:

$$\bar{\Lambda}\gamma_5\Psi = (\bar{\Lambda}\gamma_5\Psi)^t = -\Psi^t\gamma_5^t\bar{\Lambda}^t = \bar{\Psi}\underbrace{C\gamma_5^t C^{-1}}_{\gamma_5}\Lambda = \bar{\Psi}\gamma_5\Lambda. \tag{B.6}$$

Also, the following quantities vanishes:

$$\begin{aligned}
\bar{\Lambda}\gamma_\mu\Lambda &= (\bar{\Lambda}\gamma_\mu\Lambda)^t = -\Lambda^t\gamma_\mu^t\bar{\Lambda}^t = \bar{\Lambda}C\gamma_\mu^t C^{-1}\Lambda = -\bar{\Lambda}\gamma_\mu\Lambda = 0 \\
\bar{\Lambda}\Sigma_{\mu\nu}\Lambda &= (\bar{\Lambda}\Sigma_{\mu\nu}\Lambda)^t = -\Lambda^t\Sigma_{\mu\nu}^t\bar{\Lambda}^t = \bar{\Lambda}C\Sigma_{\mu\nu}^t C^{-1}\Lambda = -\bar{\Lambda}\Sigma_{\mu\nu}\Lambda = 0.
\end{aligned} \tag{B.7}$$

With these billinears in mind, we construct the Fierz rearrangement of the quantity:

$$\begin{aligned}
(\bar{\Lambda}\gamma_5\Psi)(\bar{\Lambda}\gamma_5\Psi) &= \bar{\Lambda}_\alpha(\gamma_5)_{\alpha\beta}\Psi_\beta\bar{\Psi}_\sigma(\gamma_5)_{\sigma\rho}\Lambda_\rho \\
&= (\Psi_\beta\bar{\Psi}_\sigma)\bar{\Lambda}_\alpha(\gamma_5)_{\alpha\beta}(\gamma_5)_{\sigma\rho}\Lambda_\rho \\
&= \frac{1}{4}(\Psi_\beta\bar{\Psi}_\sigma\Gamma_{A,\sigma\beta})\Gamma^A_{\beta\sigma}\bar{\Lambda}_\alpha(\gamma_5)_{\alpha\beta}(\gamma_5)_{\sigma\rho}\Lambda_\rho \\
&= -\frac{1}{4}(\bar{\Psi}_\sigma\Gamma_{A,\sigma\beta}\Psi_\beta)\bar{\Lambda}_\alpha(\gamma_5)_{\alpha\beta}\Gamma^A_{\beta\sigma}(\gamma_5)_{\sigma\rho}\Lambda_\rho \\
&= -\frac{1}{4}\left[\bar{\Psi}\Psi\bar{\Lambda}\Lambda + \bar{\Psi}\gamma_5\Psi\bar{\Lambda}\gamma_5\Lambda + \bar{\Psi}\gamma_\mu\gamma_5\Psi\bar{\Lambda}\gamma^\mu\gamma_5\Lambda\right],
\end{aligned} \tag{B.8}$$

where we used the expansion:

$$\Psi_\beta\bar{\Psi}_\sigma = \frac{1}{4}\,\mathrm{Tr}\big\{\Psi\bar{\Psi}\Gamma_A\big\}\Gamma^A_{\beta\sigma}. \tag{B.9}$$

C Bibliography

[1] Jorge Alfaro, Hugo A Morales-Tecotl, and Luis F Urrutia. Loop quantum gravity and light propagation. *Physical Review D*, 65(10):103509, 2002.

[2] G. Amelino-Camelia, John R. Ellis, N. E. Mavromatos, and Dimitri V. Nanopoulos. Distance measurement and wave dispersion in a Liouville string approach to quantum gravity. *Int. J. Mod. Phys. A*, 12:607–624, 1997.

[3] Giovanni Amelino-Camelia. Doubly-special relativity: first results and key open problems. *International Journal of Modern Physics D*, 11(10):1643–1669, 2002.

[4] Giovanni Amelino-Camelia and Shahn Majid. Waves on noncommutative space–time and gamma-ray bursts. *International Journal of Modern Physics A*, 15(27):4301–4323, 2000.

[5] H Belich, JL Boldo, LP Colatto, JA Helayel-Neto, and ALMA Nogueira. Supersymmetric extension of the lorentz-and cpt-violating maxwell-chern-simons model. *Physical Review D*, 68(6):065030, 2003.

[6] MS Berger. Superfield realizations of lorentz and cpt violation. *Physical Review D*, 68(11):115005, 2003.

[7] MS Berger and V Alan Kostelecký. Supersymmetry and lorentz violation. *Physical Review D*, 65(9):091701, 2002.

[8] James D Bjorken. A dynamical origin for the electromagnetic field. *Ann. Phys.*, 24(CERN-TH-241):174–187, 1963.

[9] Pavel A Bolokhov, Stefan Groot Nibbelink, and Maxim Pospelov. Lorentz violating supersymmetric quantum electrodynamics. *Physical Review D*, 72(1):015013, 2005.

[10] H_ BONDI and T Gold. Is there an æther? *Nature*, 169:146–146, 1952.

[11] Sean M Carroll, George B Field, and Roman Jackiw. Limits on a lorentz-and parity-violating modification of electrodynamics. *Physical Review D*, 41(4):1231, 1990.

[12] Sudhir Chadha and Holger Bech Nielsen. Lorentz invariance as a low energy phenomenon. *Nuclear Physics B*, 217(1):125–144, 1983.

[13] Sidney Coleman and Sheldon L Glashow. High-energy tests of lorentz invariance. *Physical Review D*, 59(11):116008, 1999.

[14] Don Colladay and V Alan Kosteleckỳ. Cpt violation and the standard model. *Physical Review D*, 55(11):6760, 1997.

[15] Don Colladay and V Alan Kosteleckỳ. Lorentz-violating extension of the standard model. *Physical Review D*, 58(11):116002, 1998.

[16] Paul Adrien Maurice Dirac. Is there an aether? *Nature*, 168(4282):906–907, 1951.

[17] John Ellis, Mary K Gaillard, Dimitri V Nanopoulos, and Serge Rudaz. Uncertainties in the proton lifetime. *Nuclear Physics B*, 176(1):61–99, 1980.

[18] John R Ellis, NE Mavromatos, and DIMITRI V Nanopoulos. Probing models of quantum spacetime foam. Technical report, 1999.

[19] C. F. Farias, A. C. Lehum, J. R. Nascimento, and A. Yu. Petrov. On the superfield supersymmetric aether-like Lorentz-breaking models. *Phys. Rev. D*, 86:065035, 2012.

[20] Rodolfo Gambini and Jorge Pullin. Nonstandard optics from quantum space-time. *Phys. Rev. D*, 59, 05 1999.

[21] Yuri A Gol'fand and Evgeny P Likhtman. Extension of the algebra of poincaré group generators and violation of p invariance. In *Supergravities in diverse dimensions. Volume 1*. 1989.

[22] Ersilia Guarini, Pierluca Carenza, Javier Galán, Maurizio Giannotti, and Alessandro Mirizzi. Production of axionlike particles from photon conversions in large-scale solar magnetic fields. *Physical Review D*, 102(12):123024, 2020.

[23] Andrey Katz and Yael Shadmi. Lorentz violation and superpartner masses. *Physical Review D*, 74(11):115021, 2006.

[24] V Alan Kosteleckỳ and Ralf Lehnert. Stability, causality, and lorentz and cpt violation. *Physical Review D*, 63(6):065008, 2001.

[25] V. Alan Kostelecký and Stuart Samuel. Phenomenological gravitational constraints on strings and higher-dimensional theories. *Phys. Rev. Lett.*, 63:224–227, Jul 1989.

[26] V. Alan Kostelecký and Stuart Samuel. Spontaneous breaking of lorentz symmetry in string theory. *Phys. Rev. D*, 39:683–685, Jan 1989.

[27] T. D. Lee and C. N. Yang. Question of parity conservation in weak interactions. *Phys. Rev.*, 104:254–258, Oct 1956.

[28] Joao Magueijo. New varying speed of light theories. *Reports on Progress in Physics*, 66(11):2025, 2003.

[29] Holger Bech Nielsen and I Picek. The rédei-like model and testing lorentz invariance. *Physics Letters B*, 114(2-3):141–146, 1982.

[30] Holger Bech Nielsen and I Picek. Lorentz non-invariance. *Nuclear Physics B*, 211(2):269–296, 1983.

[31] TG Pavlopoulos. Breakdown of lorentz invariance. *Physical Review*, 159(5):1106, 1967.

[32] Peter R Phillips. Is the graviton a goldstone boson? *Physical Review*, 146(4):966, 1966.

[33] Georg Raffelt and Leo Stodolsky. Mixing of the photon with low-mass particles. *Physical Review D*, 37(5):1237, 1988.

[34] Abdus Salam and J Strathdee. Super-gauge transformations. *Selected Papers of Abdus Salam (with Commentary). Edited by ALI A ET AL. Published by World Scientific Publishing Co. Pte. Ltd*, pages 404–409, 1994.

[35] AP Baeta Scarpelli, H Belich, JL Boldo, LP Colatto, JA Helayel-Neto, and ALMA Nogueira. Remarks on the causality, unitarity and supersymmetric extension of the lorentz and cpt-violating maxwell-chern-simons model. *Nuclear Physics B-Proceedings Supplements*, 127:105–109, 2004.

[36] Nathan Seiberg and Edward Witten. String theory and noncommutative geometry. *JHEP*, 09:032, 1999.

[37] R. C. Terin, W. Spalenza, H. Belich, and J. A. Helayël-Neto. Aspects of the gauge boson-gaugino mixing in a supersymmetric scenario with Lorentz-symmetry violation. *Phys. Rev. D*, 105(11):115006, 2022.

[38] RC Terin, W Spalenza, H Belich, and JA Helayël-Neto. Aspects of the gauge boson-gaugino mixing in a supersymmetric scenario with lorentz-symmetry violation. *Physical Review D*, 105(11):115006, 2022.

[39] Julius Wess and Bruno Zumino. Supergauge transformations in four dimensions. *Nuclear Physics B*, 70(1):39–50, 1974.

[40] Frank Wilczek. Two applications of axion electrodynamics. *Phys. Rev. Lett.*, 58:1799–1802, May 1987.

[41] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson. Experimental test of parity conservation in beta decay. *Phys. Rev.*, 105:1413–1415, Feb 1957.

[42] A Zee. Perhaps proton decay violates lorentz invariance. *Physical Review D*, 25(7):1864, 1982.