Techniques for Cayley-Dickson Algebras and Graded Lie (Super)Algebras for Physics

Isaque Porto de Freitas

Master degree thesis

Advisor:
Prof. Francesco Toppan

Rio de Janeiro, RJ
2023

CBPF Centro Brasileiro de Pesquisas Físicas

MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E INOVAÇÃO

GOVERNO FEDERAL BRASIL UNIÃO E RECONSTRUÇÃO

"TECHNIQUES FOR CAYLEY-DICKSON ALGEBRAS AND GRADED LIE (SUPER) ALGEBRAS FOR PHYSICS"

ISAQUE PORTO DE FREITAS

Dissertação de Mestrado em Física apresentada no Centro Brasileiro de Pesquisas Físicas do Ministério da Ciência Tecnologia e Inovação. Fazendo parte da banca examinadora os seguintes professores:

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Rio de Janeiro, 24 de abril de 2023.

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Thesis submitted to the post-graduate program at the Centro Brasileiro de Pesquisas Físicas in partial fulfillment of the requirements for the degree of the Master in Physics.

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Resumo

Esta dissertação de mestrado tem como intuito principal apresentar novas técnicas matemáticas em álgebras de Cayley-Dickson e (super)álgebras graduadas de Lie $\mathbb{Z}_2^n$. As principais aplicações destas novas ideias são as teorias de partículas e parapartículas, eletromagnestismo de dyons e Teoria-M octoniônica. No contexto das álgebras de Cayley-Dickson, procurou-se apresentá-las sob a óptica da Construção de Cayley-Dickson, na qual foi criada uma representação matricial que comporta a não-associtividade das álgebras hipercomplexas como octônions $\mathbb{O}$ e sedênions $\mathbb{S}$. Assim, foi possível estabelecer uma relação entre a álgebra da base octoniônica e a álgebra das matrizes de Pauli e Dirac para, então, criar a lagrangiana octoniônica do eletromagnetismo de dyons. Para as álgebras graduadas de Lie, nos concentramos na apresentação de um método de dobração, no qual se obtem uma álgebra $\mathbb{Z}_2^n$ a partir de uma outra de dimensão $n-1$. Além disso, foi demonstrado um cálculo para perturbações em sistemas de álgebras graduadas que permite a análise não só para pequenas perturbações, mas de ordens maiores. Espera-se que este último tenha aplicação em sistemas de partículas e parapartículas que, mediante uma perturbação, passam a interagir de modo não usual entre si.

Palavras chave: Álgebras de Cayley-Dickson; Álgebras graduadas de Lie; Superálgebras graduadas de Lie; Parapartículas; Dyons.

ABSTRACT

This master's dissertation aims to present new mathematical techniques in Cayley-Dickson algebras and graded Lie (super)algebras over $\mathbb{Z}_2^n$. The main applications of these new ideas are in the theories of particles and para-particles, dyon electromagnetism, and octonionic M-theory.
In the context of Cayley-Dickson algebras, the focus was on presenting them from the perspective of the Cayley-Dickson construction, in which a matrix representation was created that accommodates the non-associativity of hypercomplex algebras such as octonions $\mathbb{O}$ and sedenions $\mathbb{S}$. This made it possible to establish a relationship between the algebra of the octonionic base and the algebra of Pauli and Dirac matrices, and then to create the octonionic lagrangian for dyon electromagnetism.
For graded Lie algebras, the focus was on presenting a doubling method, in which an algebra $\mathbb{Z}_2^n$ is obtained from another of dimension $n-1$. In addition, a calculation was demonstrated for perturbations in systems of graded algebras that allows analysis not only for small perturbations but also for higher orders. It is expected that the latter will have applications in systems of particles and para-particles that, through a perturbation, interact in an unusual way.
Keywords: Cayley-Dickson algebras; Graded Lie algebras; Graded super Lie algebras; Para-particles; Dyons.

I dedicate this work to my sister Ester.

Acknowledgements

Firstly, to my parents Luiz and Sebastiana, for their immeasurable efforts and words of encouragement that have always accompanied me on my educational journey.

To my advisor Francesco Toppan, for all the support and immense solicitude in the moments when I needed help. I am also grateful for allowing and encouraging the creation of the new ideas that are present in this work.

To my colleagues during these two years of Master's degree, Matheus Balbino, Rodrigo Rana, and Luiza Miranda, who greatly contributed to this work with discussions, ideas, and constructive criticism. I am very happy to have shared this brief journey with all of you.

To Professor José Helayël-Neto, for the encouragement of new ideas and for the time dedicated to discussing them.

To my girlfriend Fernanda de Oliveira, for her tremendous companionship, help with the subjects, and encouragement since the undergraduate period.

To my old and new friends Felipe Sobrero, Filipe Fernandes and André Lima, for all the help, every word of encouragement, the help with coursework, for discussions on various topics in physics and for moments of fun.

Finally, I would like to thank the CAPES agency for funding this work.

Mathematics, rightly viewed,
possesses not only truth, but
supreme beauty - a beauty cold
and austere, like that of sculpture.

– RUSSEL, BERTRAND. 1912

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Introduction

Mathematical physics is an area of study that combines the theories and methods of physics with advanced mathematics. However, more than just a connection between two fields of knowledge, we can also understand mathematical physics as the area in which physicists create advanced mathematical techniques useful for overcoming mathematical limitations existing in physical theories. Therefore, the purpose of this master's dissertation is to present new techniques created in the context of Cayley-Dickson algebras [1], [2], [3] and graded Lie (super)algebras $\mathbb{Z}_2^n$ [4], [5], [6] hoping they will be useful for particle and para-particle physics theories [7] as well as for Maxwell and dyonic electromagnetism [8].

At first, this work aims to revisit the main Cayley-Dickson algebras - complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), octonions ($\mathbb{O}$), and Cayley-Dickson sedenions ($\mathbb{S}$) - according to the doubling construction. This mathematical formalism is an important tool for verifying how we can build new algebras by doubling algebras of smaller dimensions. In addition, it makes it possible to understand how the properties of algebras relate to those of other algebras used in doubling.

It is known that for complex numbers and quaternions there is a matrix representation that is compatible with their algebras [9], [10]. Therefore, a detailed construction of the matrices that correspond to the algebras $\mathbb{C}$ and $\mathbb{H}$ is also presented. However, this work proposes a matrix construction for algebras of higher dimensions. We sought to develop the matrix representation technique for the construction of Cayley-Dickson algebras, allowing elements of non-associative algebras, such as octonions and sedenions, to be written as matrices. For octonions and split-octonions, this technique is an alternative to Zorn matrices [11],[12] and it is expected to be used, for example, as a possibility of writing Clifford algebra $Cl_O(10,1)$ of octonionic M-theory [13] matrices with real entries and in dimensions of 32×32. In addition, through the matrix representation of the bases of octonionic numbers, it was possible to construct the octonionic version of the dyonic electromagnetism Lagrangian [8]. It was also possible to verify global and local quaternionic symmetries for the $SU(2)$ group, through a simple and already known relationship between quaternionic base elements and Pauli matrices [14].

In the context of $\mathbb{Z}_2^n$-graded Lie algebras and superalgebras, after a brief presentation of $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, and $\mathbb{Z}_2^3$, a new technique was demonstrated that allows for the creation of these algebras from others of the same nature but with smaller dimensions. This practice is called *Doubling Construction for* $\mathbb{Z}_2^n$ *(super)algebras* and was used to create a representation for a $\mathbb{Z}_2^3$ algebra of supersymmetry $\mathcal{N} = 2$.

Also presented was an approach to perturbations in graded (super)algebra systems. In this formalism, the perturbation was inserted into the mapping of a (super)algebra of any dimension, so that the rules of commutation or anticommutation were modified. As a result, we had that the result for the Lie bracket was ambiguous, generating both a commutation and an anticommutation simultaneously for the generators of the algebra in question; unlike the usual case, in which the Lie bracket will be either just a commutation or just an anticommutation. This formalism seems to indicate a new way of constructing Volichenko algebras from graded Lie (super)algebras [15]. It is expected that the results obtained are fully capable of being applied to descriptions of systems where there is symmetry breaking or perturbations capable of modifying the interaction between particles.

Chapter 1

Cayley-Dickson Algebras

In this work, we will call by Cayley-Dickson algebras all the hyper-complex algebras that can be produced from a sequence of doublings starting from real numbers. This sequential procedure of creating new algebras was introduced by Arthur Cayley and Leonard Dickson [1] and such algebras are constantly applied in mathematical physics. Some examples of this application is the use of Quaternion algebra ($\mathbb{H}$), a non-commutative Cayley-Dickson algebra, as a tool for Maxwell electromagnetism [16], [17] and for rotation of rigid bodies [18]. Recently, Giardino also used quaternions to build a constrained four-dimensional quaternion-parametrized conformal field theory [19]. Cohl Furey used division algebras - a special kind of parabolic Cayley-Dickson algebras - to construct standard model particles representations [20]. Octonions ($\mathbb{O}$), a normed non-associative algebra, is a elegant mathematical tool for dyonic electromagnetism and Sedenions ($\mathbb{S}$), a non-alternative and non-division algebra, are useful for introduce gravitation in this model [8].

In this chapter, Cayley-Dickson algebras will be presented and its most important properties, attempting to a particular presentation of elements of algebras coincident Cayley-Dickson doubling construct. So it is not wrong to say that the most important subject of this chapter is not the algebras but the procedure of doubling construction as a powerful tool to understand characteristic aspects of each algebra.

1.1 Preliminary concepts

A Cayley-Dickson algebra $\mathbb{A}$ is a hyper-complex algebra over the Reals ($\mathbb{R}$) with a bilinear mapping $\mathscr{M}$ that is characterized as follows:

$$\begin{aligned} \mathscr{M} : \mathbb{A} \times \mathbb{A} \quad &\rightarrow \quad \mathbb{A} \\ a, b \in \mathbb{A} \quad &| \quad \mathscr{M}(a, b) \in \mathbb{A} \end{aligned}$$

$\mathscr{M}$ is the multiplication and it will be defined shortly afterwards. There are other properties for A and M that are important. They are:

$$1, a \in \mathbb{A} \quad | \quad \mathscr{M}(1,a) = \mathscr{M}(a,1) = a \tag{1.1}$$

that is $1 \in \mathbb{A}$ and 1 is the neutral element of mapping the $\mathscr{M}$.

The Cayley-Dickson doubling construction is a mathematical tool that allows the creation of Cayley-Dickson algebras that have $2N$ base elements, from the doubling of another algebra with N base elements [1]. The most important algebras formed from this process are Complex numbers($\mathbb{C}$), Quaternions ($\mathbb{H}$), Octonions ($\mathbb{O}$) and Sedenions ($\mathbb{S}$). They are obtained through successive doubling of the Reals ($\mathbb{R}$).

Before starting presentation of Cayley-Dickson construction, it is important to mention that, according to the Hurwitz theorem, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only four possible division algebras [2]. A division algebra A over $\mathbb{R}$ is an alternative, finite-dimensional, real vector space with a bi-linear product $\mathscr{M}$, such that:

$$\begin{aligned} \mathscr{M} : A \times A &\rightarrow A \\ (a,b) &\rightarrow ab, \quad a,b \in A \end{aligned}$$

It satisfies the conditions, for any a, b:

$$ab = 0 \Leftrightarrow a = 0 \;\text{ or }\; b = 0 \tag{1.2}$$

and

$$a(ab) = a^2 b \tag{1.3}$$

The relation (1.3) is the alternativity property for division algebras.

There are two main operations that arises from definition of division algebras:

- Conjugation: If $a, b \in \mathbb{A}$, a division algebra, then the conjugation satisfies:

$$\begin{aligned} (a^*)^* &= a \\ (ab)^* &= b^* a^* \end{aligned}$$

- Norm: A division algebra can be normed A if $\mathscr{N}$ is a positive-definite quadratic form, such that:

$$\mathscr{N} : A \rightarrow \mathbb{R}^+$$

and for any $a \in A$, one can have:

$$\mathscr{N}(a) = a^* a, \quad \mathscr{N}(a) \in \mathbb{R}^+.$$

The most important Cayley-Dickson algebras have the following properties:

- $\mathbb{R}$: Commutative and associative.
- $\mathbb{C}$: Commutative, associative and alternative.
- $\mathbb{H}$: Non-commutative, associative and alternative.
- $\mathbb{O}$: Non-commutative, non-associative and alternative.
- $\mathbb{S}$: Non-commutative, non-associative and non-alternative[1].

The Cayley-Dickson doubling construction is a powerful tool to understand how a new property arises when a new algebra is constructed from another. It is, suppose a algebra $\mathbb{A}$ with a certain quantity of properties and now let's use Cayley-Dickson doubling to construct a new algebra $\mathbb{A}^2$ from $\mathbb{A}$. When doubling occurs, it's is possible to verify that there are more properties in $\mathbb{A}^2$. Then, if we doubling $\mathbb{A}^2$ its new property - the one that does not exists in $\mathbb{A}$ - will be directly responsible for the new property of $\mathbb{A}^4$. For example, lately will be shown that the non-associativity of octonions is a consequence of non-commutative of quaternions.

1.2 Cayley-Dickson doubling construction

As mentioned before, Cayley-Dickson doubling construction is a process to produce a new hyper-complex algebras from a given algebra. Furthermore, following this process, one can verify that a 2^n-dimensional Cayley-Dickson algebra can be constructed after n successive doublings from $\mathbb{R}$, so that the elements of a 2^n-dimensional algebra are "vectors" represented by 2^n real components each multiplying an element of the hyper-complex basis.

A Cayley-Dickson algebra $\mathbb{A}^2$ is defined by the following operations [21]:

1. Multiplication: $(x,y)\cdot(z,w) = (xz + \epsilon w^* y, wx + yz^*)$.
2. Conjugation: $(x,y)^* = (x^*, -y)$.
3. "Norm": $\mathcal{N}(x,y) = \mathcal{N}(x) - \epsilon\mathcal{N}(y)$.
4. Multiplication by a real number: $a(x,y) = (ax, ay), \quad a \in \mathbb{R}$.
5. Conjugation of multiplication: $[(x,y)\cdot(z,w)]^* = (z,w)^* \cdot (x,y)^*$

[1] Later, we will see that since Cayley-Dickson Sedenions are formed from a doubling of octonions, thus they cannot be classified as a division algebra, because property (1.2) is not always true.

where $x, y \in \mathbb{A}$. If $\mathbb{A}^2$ is a division algebra, then $\epsilon = -1$. For split-division $\mathbb{A}^2$, $\epsilon = 1$. Operations 4 and 5 are immediate consequence of operations 1 and 2, but is important to explicit it here because it will be useful to next calculations.

For Cayley-Dickson doubling construction we choose $\epsilon = \pm 1$ and a algebra $\mathbb{A}$ to represent elements of other algebra $\mathbb{A}^2$ as the ordered pair (x, y), whose $x, y \in \mathbb{A}$.

Split-division algebras are not a division algebra because the property (1.2) is not always true for its elements.

Operation (3) is only called norm for division algebras and that is why the word was written in quotes.

Now we have all the concepts needed to start building the Cayley-Dickson algebras. But first, we need to know one last detail. By choosing $\epsilon = -1$,

- $\mathbb{C}$ are a doubling of $\mathbb{R}$;
- $\mathbb{H}$ are a doubling of $\mathbb{C}$;
- $\mathbb{O}$ are a doubling of $\mathbb{H}$ and
- $\mathbb{S}$ are a doubling of $\mathbb{O}$.

1.3 Complex numbers

The usual presentation for a complex number z is:

$$z = a + bi, \quad a, b \in \mathbb{R} \tag{1.4}$$

One can also present z as the ordered pair (a, b), such that:

$$z = a(1,0) + b(0,1) \tag{1.5}$$

where:

$$1 \equiv (1,0), \quad i \equiv (0,1). \tag{1.6}$$

If we use the property of multiplication for Cayley-Dickson construction, it will be possible to verify that $i^2 = -1$:

$$i^2 \equiv (0,1) \cdot (0,1) = (-1^* 1, 0) = -(1,0) \tag{1.7}$$

Multiplication between two complex numbers $z = a + bi$ and $w = c + di$ written according to (1.4) is

$$\begin{aligned} z \cdot w &= (a+bi) \cdot (c+di) \\ &= (ac - bd) + (ad + bc)i \end{aligned} \tag{1.8}$$

If we use the Cayley-Dickson's multiplication and representation (1.5),

$$\begin{aligned} z \cdot w &= (a,b)\cdot(c,d) \\ &= (ac - d^* b, da + bc^*) \\ &= (ac - bd, ad + bc) \\ &= (ac - bd)(1,0) + (ad + bc)(0,1) \end{aligned} \tag{1.9}$$

It is easy to verify that the results (1.9) and (1.8) are equivalents. The conjugation z^* of the complex number z is:

$$z^* = a - bi. \tag{1.10}$$

According to Cayley-Dickson's conjugation,

$$z^* = (a, -b) = a(1,0) - b(0,1) \tag{1.11}$$

The norm of a complex number can be calculated by $z^* z$ using the form (3):

$$\begin{aligned} z^* z &= (a+bi)(a-bi) \\ &= a^2 + b^2 \end{aligned} \tag{1.12}$$

Using the Cayley-Dickson's definition of norm,

$$\mathcal{N}(z) = \mathcal{N}(a,b) = \mathcal{N}(a) + \mathcal{N}(b) \tag{1.13}$$

and the property of $\mathcal{N}(a)$, then

$$\mathcal{N}(z) = a^2 + b^2 \tag{1.14}$$

1.4 Quaternions

To create quaternions ($\mathbb{H}$) we will use ordered pairs of complex numbers. A quaternion v can be presented as

$$v = (z, w), \quad z, w \in \mathbb{C} \tag{1.15}$$

The quaternion number v can be represented in by a vector with reals components. For it, just write $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. Then,

$$\begin{aligned} v = (z, w) &= (a + bi, c + di) \\ &= a(1,0) + b(i,0) + c(0,1) + d(0,i) \end{aligned} \tag{1.16}$$

Let's represent the quaternionic base as:

$$\mathbf{e}_0 = (1,0), \quad \mathbf{e}_1 = (i,0), \quad \mathbf{e}_2 = (0,1), \quad \mathbf{e}_3 = (0,i) \tag{1.17}$$

By using the Cayley-Dickson multiplication is possible to establish that:

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_0\eta_{ij} + \varepsilon_{ijk}\mathbf{e}_k \tag{1.18}$$

where η_{ij} is the metric of quaternions and ε_{ijk} is the totally antisymmetric Levi-Civita symbol. It is easy to verify that:

$$\eta_{ij} = \delta_{ij} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.19}$$

Since the quaternions metric is euclidean is not necessary to distinguish to upper and lowed indices. The same occurs for octonions. Split-quaternions and split-octonions have a non-euclidean metric and therefore for both will be necessary to write indices superimposed ou lowered[2].

But now, let's back to the representation of quaternions as ordered pairs of complex numbers. From this presentation it is possible to analyse the origin of the non-commutativity of quaternions. Given two quaternions, $v = (z, w)$ and $u = (x, y)$, where $x, y, z, w \in \mathbb{C}$. We have

$$\begin{aligned} u \cdot v &= (x, y) \cdot (z, w) \\ &= (xz - w^* y, wx + yz^*) \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} v \cdot u &= (z, w) \cdot (x, y) \\ &= (zx - y^* w, yz + wx^*) \end{aligned} \tag{1.21}$$

Results (1.20) and (1.21) are not the same because $y^* \neq y$, $w^* \neq w$, $z^* \neq z$ and $x^* \neq x$. Quaternions are non-commutative because operation of conjugation applied in a complex number generates a different complex number - this not happens to real numbers. That is the first example of how a property of a algebra can be responsible to a new property when it is doubled.

Later will presented a matrix realization for quaternions and complex and how it can be used to construct a matrix representation for Cayley-Dickson doubling construction.

1.5 Octonions

The procedure to represent octonions in terms of the Cayley-Dickson construction is the same as before, remembering that the ordered pair is of quaternions.

[2]This reasoning is similar to when we are working with elements of Minkowski space

Therefore, let's choose quaternions v and u such that $q = (v, u)$, $q \in \mathbb{O}$. To represent octonion q as a "vector" with real components we need at first to do the same to quaternions v and u. Considering

$$v = a\mathbf{e}_0 + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3, \quad a, b, c, d \in \mathbb{R} \tag{1.22}$$

and

$$u = f\mathbf{e}_0 + g\mathbf{e}_1 + h\mathbf{e}_2 + m\mathbf{e}_3, \quad f, g, h, m \in \mathbb{R}, \tag{1.23}$$

then,

$$\begin{aligned} q &= (a\mathbf{e}_0 + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3, f\mathbf{e}_0 + g\mathbf{e}_1 + h\mathbf{e}_2 + m\mathbf{e}_3) \\ &= a(\mathbf{e}_0, 0) + b(\mathbf{e}_1, 0) + c(\mathbf{e}_2, 0) + d(\mathbf{e}_3, 0) + f(0, \mathbf{e}_0) + \\ &\quad + g(0, \mathbf{e}_1) + h(0, \mathbf{e}_2) + m(0, \mathbf{e}_3) \end{aligned} \tag{1.24}$$

One can define:

$$\begin{aligned} &\mathbf{E}_0 = (\mathbf{e}_0, 0), \quad \mathbf{E}_1 = (\mathbf{e}_1, 0), \quad \mathbf{E}_2 = (\mathbf{e}_2, 0), \quad \mathbf{E}_3 = (\mathbf{e}_3, 0) \\ &\mathbf{E}_4 = (0, \mathbf{e}_0), \quad \mathbf{E}_5 = (0, \mathbf{e}_1), \quad \mathbf{E}_6 = (0, \mathbf{e}_2), \quad \mathbf{E}_7 = (0, \mathbf{e}_3) \end{aligned} \tag{1.25}$$

The basis can be divided into two groups: $\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are an associative subalgebra of the base and it's is isomorphic to quaternion basis. $\mathbf{E}_4, \mathbf{E}_5, \mathbf{E}_6, \mathbf{E}_7$ are the non-associative part of the octonionic basis.

The multiplication between two elements of the basis is defined as:

$$\mathbf{E}_i \mathbf{E}_j = -\mu_{ij} \mathbf{E}_0 + C_{ijk} \mathbf{E}_k, \tag{1.26}$$

where C_{ijk} is the structure constant of octonions, it is totally anti-symmetric. μ_{ij} Is the $ij-$component of a metric. It is defined as:

$$\mu_{ij} = \delta_{ij} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.27}$$

The values of C_{ijk} are given in the following table:

$C_{123}=1$	$C_{213}=-1$	$C_{312}=1$
$C_{132}=-1$	$C_{231}=1$	$C_{321}=-1$
$C_{145}=1$	$C_{246}=1$	$C_{347}=1$
$C_{154}=-1$	$C_{257}=1$	$C_{356}=-1$
$C_{167}=-1$	$C_{264}=-1$	$C_{365}=1$
$C_{176}=1$	$C_{275}=-1$	$C_{374}=-1$

$C_{415}=-1$	$C_{514}=1$	$C_{617}=1$	$C_{716}=-1$
$C_{426}=-1$	$C_{527}=-1$	$C_{624}=1$	$C_{725}=1$
$C_{437}=-1$	$C_{536}=1$	$C_{635}=-1$	$C_{734}=1$
$C_{451}=1$	$C_{541}=-1$	$C_{642}=-1$	$C_{743}=-1$
$C_{462}=1$	$C_{563}=-1$	$C_{653}=1$	$C_{752}=-1$
$C_{473}=1$	$C_{572}=1$	$C_{671}=-1$	$C_{761}=1$

Table 1.1: Structure constants of octonions.

One can now consider three different octonions, p, q and r, and apply definition of the associator $[\bullet,\bullet,\bullet]$:

$$[p,q,r]=(pq)r-p(qr) \tag{1.28}$$

For associative algebras, operation (1.28) is null. But for $p,q,r\in\mathbb{O}$ it is not always true. To verify this cause within the scope of the Cayley-Dickson construction, let's write p, q and r as follows:

$$p=(a,b),\quad q=(c,d),\quad r=(g,h) \tag{1.29}$$

where $a,b,c,d,g,h\in\mathbb{H}$.

Therefore,

$$\begin{aligned}(pq)r &= ((a,b)(c,d))(g,h)\\ &= (ac-d^*b, da+bc^*)(g,h)\\ &= (acg-d^*bg-h^*da-h^*bc^*,\quad hac-hd^*b+dag^*+bc^*g^*)\end{aligned} \tag{1.30}$$

and

$$\begin{aligned}p(qr) &= (a,b)((c,d)(g,h))\\ &= (a,b)(cg-h^*d, hc-dg^*)\\ &= (acg-ah^*d-(hc-dg^*)^*b, (hc+dg^*)a+b(cg-h^*d)^*)\\ &= (acg-ah^*d-c^*h^*b-gd^*b,\quad hca+dg^*a+bg^*c^*-bd^*h)\end{aligned} \tag{1.31}$$

Let's now substitute (1.30) and (1.31) in (1.28):

$$\begin{aligned}[p,q,r] = & ([a,h^*d]+[c^*,h^*b]+[g,d^*b],\quad h[a,c]+d[g^*,a]+b[c^*,g^*]\\ & +b[d^*,h]+[b,hd^*])\end{aligned} \tag{1.32}$$

(1.32) Shows that the non-commutativity of quaternions is responsible for the non-associativity of octonions.

One can use the associator to proof the alternativity of octonions. At first, let's back to the table 1.1 and verify the following identity:

$$C_{ijk}C_{mjk} = \delta_{im} \tag{1.33}$$

It is important to say that result (1.33) is not a summation. For a summation, then[3]:

$$\frac{1}{6}C_{ijk}C_{mjk} = \delta_{im}$$

The identity (1.33) will be useful on the next calculation. Now, let's apply the associator on any three of the seven elements of octonion basis:

$$\begin{aligned}
[\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k] &= (\mathbf{E}_i\mathbf{E}_j)\mathbf{E}_k - \mathbf{E}_i(\mathbf{E}_j\mathbf{E}_k) \\
&= (\delta_{ij}\mathbf{E}_0 + C_{ijl}\mathbf{E}_l)\mathbf{E}_k - \mathbf{E}_i(-\delta_{jk}\mathbf{E}_0 + C_{jkm}\mathbf{E}_m) \\
&= -\delta_{ij}\mathbf{E}_k + \delta_{jk}\mathbf{E}_i + C_{ijl}(-\delta_{lk} + C_{lkn}\mathbf{E}_n) - C_{jkm}(-\delta_{im} + C_{imp}\mathbf{E}_p) \\
&= -\delta_{ij}\mathbf{E}_k + \delta_{jk}\mathbf{E}_i - C_{ijk} + C_{jki} + C_{ijl}C_{lkn}\mathbf{E}_n - C_{jkm}C_{imn}\mathbf{E}_n \\
&= -\delta_{ij}\mathbf{E}_k + \delta_{jk}\mathbf{E}_i + (C_{ijl}C_{lkn} + C_{jkm}C_{min})\mathbf{E}_n
\end{aligned} \tag{1.34}$$

So, let's do $i = j$ and $i, j \neq k$:

$$[\mathbf{E}_i, \mathbf{E}_i, \mathbf{E}_k] = -\delta_{ii}\mathbf{E}_k + C_{ikm}C_{min}\mathbf{E}_n \tag{1.35}$$

Using the result (1.33) on (1.35),

$$[\mathbf{E}_i, \mathbf{E}_i, \mathbf{E}_k] = -\mathbf{E}_k + \delta_{kn}\mathbf{E}_n = 0 \tag{1.36}$$

When choosing $j = k$ and $j, k \neq i$, the calculation will be analogous to the immediately previous process.

If one choose $i = k$ and $i, k \neq j$, then:

$$\begin{aligned}
[\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_i] &= (C_{ij}{}^{l}C_{li}{}^{n} + C_{ji}{}^{m}C_{mi}{}^{n})\mathbf{E}_n \\
&= (\delta_j{}^{n} - \delta_j{}^{n})\mathbf{E}_n = 0
\end{aligned} \tag{1.37}$$

So, according to (1.36) and (1.37), one can see that the elements of octonion basis form a alternative basis. Furthermore, it's is possible to state that the operation $[\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k]$ can be written in terms of a totally anti-symmetric tensor T_{ijkn} such that:

$$[\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k] = T_{ijk}{}^{n}\mathbf{E}_n \tag{1.38}$$

[3]The Einstein sum convention is adopted in this case.

Returning to table 1.1, it's is possible to verify that:

$$[\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k] = [\mathbf{E}_k, \mathbf{E}_i, \mathbf{E}_j] = [\mathbf{E}_j, \mathbf{E}_k, \mathbf{E}_i] \tag{1.39}$$

where,

$$[\mathbf{E}_k, \mathbf{E}_i, \mathbf{E}_j] = -\delta_{ki}\mathbf{E}_j + \delta_{ij}\mathbf{E}_k + (C_{kil}C_{ljn} + C_{ijm}C_{mkn})\mathbf{E}_n \tag{1.40}$$

and

$$[\mathbf{E}_j, \mathbf{E}_k, \mathbf{E}_i] = -\delta_{jk}\mathbf{E}_i + \delta_{ki}\mathbf{E}_j + (C_{jkl}C_{lin} + C_{kim}C_{mjn})\mathbf{E}_n \tag{1.41}$$

When adding up (1.34), (1.40) and (1.41), we have:

$$[\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k] = \frac{2}{3}(C_{ijl}C_{lkn} + C_{jkm}C_{min} + C_{kip}C_{pjn})\mathbf{E}_n \tag{1.42}$$

Therefore:

$$T_{ijkn} = \frac{2}{3}(C_{ijl}C_{lkn} + C_{jkm}C_{min} + C_{kip}C_{pjn}) \tag{1.43}$$

where

$$T_{ijkn} = 0, \quad \text{if} \quad \{i,j,k\} = \begin{cases} \{1,2,3\}, & \{2,4,6\}, \quad \{3,5,6\}. \\ \{1,4,5\}, & \{2,5,7\}, \\ \{1,6,7\}, & \{3,4,7\}, \end{cases} \tag{1.44}$$

and

$T_{1247} = 2$	$T_{2345} = 2$
$T_{1256} = -2$	$T_{2367} = -2$
$T_{1346} = -2$	$T_{4567} = -2$
$T_{1357} = -2$	

Another operation should be presented (it will be useful for future analysis) is the anti-associator $\{\bullet, \bullet, \bullet\}$:

$$\{a, b, c\} = (ab)c + a(bc) \tag{1.45}$$

If $a, b, c \in \mathbb{O}$, the result of (1.45) can be null. It is a peculiarity of octonions because it result does not happen for any other division algebra.

For any three elements of the octonion's basis we have:

$$\begin{aligned} \{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k\} &= (\mathbf{E}_i\mathbf{E}_j)\mathbf{E}_k + \mathbf{E}_i(\mathbf{E}_j\mathbf{E}_k) \\ &= (-\delta_{ij}\mathbf{E}_0 + C_{ijl}\mathbf{E}_l)\mathbf{E}_k + \mathbf{E}_i(-\delta_{jk}\mathbf{E}_0 + C_{jkm}\mathbf{E}_m) \\ &= -\delta_{ij}\mathbf{E}_k - \delta_{jk}\mathbf{E}_i + C_{ijl}(-\delta_{lk} + C_{lkn}\mathbf{E}_n) + C_{jkm}(-\delta_{im} + C_{imn}\mathbf{E}_n) \\ &= -\delta_{ij}\mathbf{E}_k - \delta_{jk}\mathbf{E}_i - 2C_{ijk}\mathbf{E}_0 + (C_{ijl}C_{lkn} + C_{jkm}C_{imn})\mathbf{E}_n \end{aligned} \tag{1.46}$$

According to (1.46) one can verify that:

$$\{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k\} = S_{ijk} + A_{ijk} \tag{1.47}$$

where A_{ijk} is the part totally antisymmetric of (1.47) and S_{ijk} is the part partialy symmetric. By choosing i, j, k different indexes:

$$\begin{aligned}
\{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k\} &= -2C_{ijk}\mathbf{E}_0 + (C_{ijl}C_{lkn} + C_{jkm}C_{imn})\mathbf{E}_n = A_{ijk} \\
\{\mathbf{E}_j, \mathbf{E}_k, \mathbf{E}_i\} &= -2C_{jki}\mathbf{E}_0 + (C_{jkm}C_{min} + C_{kit}C_{jtn})\mathbf{E}_n = A_{jki} \\
\{\mathbf{E}_k, \mathbf{E}_i, \mathbf{E}_j\} &= -2C_{kij}\mathbf{E}_0 + (C_{kit}C_{tjn} + C_{ijl}C_{kln})\mathbf{E}_n = A_{kij}
\end{aligned} \tag{1.48}$$

Then, by adding the three equations of (1.48), we get:

$$A_{ijk} = -2C_{ijk}\mathbf{E}_0 \tag{1.49}$$

In order to determine an expression for S_{ijk} one can do:

$$\begin{aligned}
\{\mathbf{E}_i, \mathbf{E}_i, \mathbf{E}_k\} &= S_{iik} = -2\mathbf{E}_k, \\
\{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_j\} &= S_{ijj} = -2\mathbf{E}_i, \\
\{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_i\} &= S_{iji} = 2\mathbf{E}_j
\end{aligned} \tag{1.50}$$

Therefore:

$$S_{ijk} = -2\delta_{ij}\mathbf{E}_k - 2\delta_{jk}\mathbf{E}_i + 2\delta_{ik}\mathbf{E}_j \tag{1.51}$$

and

$$\begin{aligned}
\{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k\} &= -2\delta_{ij}\mathbf{E}_k - 2\delta_{jk}\mathbf{E}_i + 2\delta_{ik}\mathbf{E}_j - 2C_{ijk}\mathbf{E}_0 \\
&= -2(\delta_{ij}\delta_k{}^n + \delta_{jk}\delta_i{}^n - \delta_{ik}\delta_{jn})\mathbf{E}_n - 2C_{ijk}\mathbf{E}_0 \\
\frac{1}{2}\{\mathbf{E}_i, \mathbf{E}_j, \mathbf{E}_k\} &= H_{ijkn}\mathbf{E}_n - C_{ijk}\mathbf{E}_0
\end{aligned} \tag{1.52}$$

where

$$H_{ijkn} = -\delta_{ij}\delta_{kn} - \delta_{jk}\delta_{in} + \delta_{ik}\delta_{jn} \tag{1.53}$$

According to second line in (1.52) and (1.46),

$$\begin{aligned}
-\delta_{ij}\mathbf{E}_k - \delta_{jk}\mathbf{E}_i - 2C_{ijk}\mathbf{E}_0 + (C_{ijl}C_{lkn} - C_{jkm}C_{min})\mathbf{E}_n &= \\
-2(\delta_{ij}\delta_{kn} + \delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn})\mathbf{E}_n - 2C_{ijk}\mathbf{E}_0 & \\
(-\delta_{ij}\delta_{kn} - \delta_{jk}\delta_{in} + C_{ijl}C_{lkn} - C_{jkm}C_{min})\mathbf{E}_n &= \\
-2(\delta_{ij}\delta_{kn} + \delta_{jk}\delta_{in} - \delta_{ik}\delta_{jn})\mathbf{E}_n &
\end{aligned}$$

Therefore:

$$C_{ijl}C_{lkn} - C_{jkm}C_{min} = -\delta_{ij}\delta_{kn} - \delta_{jk}\delta_{in} + \delta_{ik}\delta_{jn} \tag{1.54}$$

1.6 Cayley-Dickson sedenions

Cayley-Dickson Sedenions are not a division algebra. It is a non-alternative algebra wich can be construct, by doubling octonions, as the same process done in the last algebras and using Cayley-dickson parameter $\epsilon = -1$. If we double octonions using $\epsilon = 1$, then we will get conic sedenions [12], [22], [23]. In this work let's focus only on Cayley-Dickson sedenions, since it is enough to understand how non-alternative arises in Cayley-Dickson doubling construction.

Therefore, elements of Cayley-Dickson sedenionic basis can be represented as:

$$\begin{gathered}
\mathbf{S}_0 = (\mathbf{E}_0, 1), \quad \mathbf{S}_1 = (\mathbf{E}_1, 0), \quad \mathbf{S}_2 = (\mathbf{E}_2, 0), \quad \mathbf{S}_3 = (\mathbf{E}_3, 0), \mathbf{S}_4 = (\mathbf{E}_4, 0) \\
\mathbf{S}_5 = (\mathbf{E}_5, 0), \quad \mathbf{S}_6 = (\mathbf{E}_6, 0), \quad \mathbf{S}_7 = (\mathbf{E}_7, 0) \\
\mathbf{S}_8 = (0, \mathbf{E}_0), \quad \mathbf{S}_9 = (0, \mathbf{E}_1), \quad \mathbf{S}_{10} = (0, \mathbf{E}_2), \quad \mathbf{S}_{11} = (0, \mathbf{E}_3), \quad \mathbf{S}_{12} = (0, \mathbf{E}_4) \\
\mathbf{S}_{13} = (0, \mathbf{E}_5), \quad \mathbf{S}_{14} = (0, \mathbf{E}_6), \quad \mathbf{S}_{15} = (0, \mathbf{E}_7)
\end{gathered} \tag{1.55}$$

Then, the multiplication between two elements of the sedenion base are:

$$\mathbf{S}_i \mathbf{S}_j = -\delta_{ij} \mathbf{S}_0 + K_{ijk} \mathbf{S}_k \tag{1.56}$$

where K_{ijk} sedenion structure constant, and it is a anti-symmetric symbol. The values of K_{ijk} are given in the following table: There is an important property about the basis of sede-

$K_{123} = 1$	$K_{2,8,10} = 1$	$K_{347} = 1$
$K_{145} = 1$	$K_{246} = 1$	$K_{356} = -1$
$K_{167} = -1$	$K_{257} = 1$	$K_{3,8,11} = 1$
$K_{189} = 1$	$K_{2,9,11} = 1$	$K_{3,9,10} = -1$
$K_{1,10,11} = 1$	$K_{2,12,14} = -1$	$K_{3,12,15} = -1$
$K_{1,12,13} = -1$	$K_{2,13,15} = -1$	$K_{3,14,13} = -1$
$K_{1,14,15} = 1$	$K_{5,8,13} = 1$	$K_{6,8,14} = 1$
$K_{4,8,12} = 1$	$K_{5,9,12} = -1$	$K_{6,9,15} = -1$
$K_{4,9,13} = 1$	$K_{5,10,15} = 1$	$K_{6,10,12} = -1$
$K_{4,10,14} = 1$	$K_{5,11,14} = -1$	$K_{6,11,13} = 1$
$K_{4,11,15} = 1$	$K_{7,8,15} = 1$	$K_{7,9,14} = 1$
$K_{7,10,13} = -1$	$K_{7,11,12} = -1$	

Table 1.2: Structure constant of Cayley-Dickson sedenions

nions and non-alternativity. Choosing any three elements of (1.55) we get:

$$[\mathbf{S}_i, \mathbf{S}_j, \mathbf{S}_k] = -\delta_{ij} \mathbf{S}_k + \delta_{jk} \mathbf{S}_i + (K_{ijl} K_{lkn} + K_{jkm} K_{min}) \mathbf{S}_n \tag{1.57}$$

Let's suppose now $i = j$, or $i = k$, or $j = k$. These choices mean that we are analyzing the alternativity of the sedenions basis elements. But even knowing that the sedenions are non-alternative, it is possible to notice that this result will be null. So, unlike the algebras presented so far, in which the characteristic degree of freedom of each one was already observed in multiplication between the elements of its basis, multiplication between elements of sedenions basis does not carry the main characteristic of this algebra.

Elements of the sedenionic basis are ordered pairs of the type $(0,*)$ or $(*,0)$. We can obtain non-alternativity by doing a multiplication between ordered pairs with both non-zero components $(*,*)$. Then, let's take the following three sedenions $x=(\mathbf{E}_i,\mathbf{E}_j)$, $y=(\mathbf{E}_k,\mathbf{E}_l)$ and $z=(\mathbf{E}_m,\mathbf{E}_n)$:

$$[x,y,z]=(xy)z-x(yz) \tag{1.58}$$

where

$$\begin{aligned}
(xy)z &= ((\mathbf{E}_i,\mathbf{E}_j)(\mathbf{E}_k,\mathbf{E}_l))(\mathbf{E}_m,\mathbf{E}_n)\\
&= (\mathbf{E}_i\mathbf{E}_k+\mathbf{E}_l\mathbf{E}_j,\mathbf{E}_l\mathbf{E}_i-\mathbf{E}_j\mathbf{E}_k)(\mathbf{E}_m,\mathbf{E}_n)\\
&= ((\mathbf{E}_i\mathbf{E}_k)\mathbf{E}_m+(\mathbf{E}_l\mathbf{E}_j)\mathbf{E}_m+\mathbf{E}_n(\mathbf{E}_l\mathbf{E}_i)-\mathbf{E}_n(\mathbf{E}_j\mathbf{E}_k),\\
&\quad\ \mathbf{E}_n(\mathbf{E}_i\mathbf{E}_k)-\mathbf{E}_n(\mathbf{E}_l\mathbf{E}_j)-(\mathbf{E}_l\mathbf{E}_i)\mathbf{E}_m+(\mathbf{E}_j\mathbf{E}_k)\mathbf{E}_m)
\end{aligned} \tag{1.59}$$

and

$$\begin{aligned}
x(yz) &= (\mathbf{E}_i,\mathbf{E}_j)((\mathbf{E}_k,\mathbf{E}_l)(\mathbf{E}_m,\mathbf{E}_n))\\
&= (\mathbf{E}_i,\mathbf{E}_j)((\mathbf{E}_k\mathbf{E}_m+\mathbf{E}_n\mathbf{E}_l,\mathbf{E}_n\mathbf{E}_k-\mathbf{E}_l\mathbf{E}_m))\\
&= (\mathbf{E}_i(\mathbf{E}_k\mathbf{E}_m)+\mathbf{E}_i(\mathbf{E}_n\mathbf{E}_l)-(\mathbf{E}_k\mathbf{E}_n)\mathbf{E}_j+(\mathbf{E}_m\mathbf{E}_l)\mathbf{E}_j,\\
&\quad\ (\mathbf{E}_n\mathbf{E}_k)\mathbf{E}_i-(\mathbf{E}_l\mathbf{E}_m)\mathbf{E}_i+\mathbf{E}_j(\mathbf{E}_m\mathbf{E}_k)+\mathbf{E}_j(\mathbf{E}_l\mathbf{E}_n))
\end{aligned} \tag{1.60}$$

We can now substitute (1.59) and (1.60) into (1.58), and we get the following result:

$$\begin{aligned}
[x,y,z] &= ([\mathbf{E}_i,\mathbf{E}_k,\mathbf{E}_m]-\{\mathbf{E}_m,\mathbf{E}_l,\mathbf{E}_j\}+\{\mathbf{E}_n,\mathbf{E}_l,\mathbf{E}_i\}-[\mathbf{E}_n,\mathbf{E}_k,\mathbf{E}_j],\\
&\quad\ [\mathbf{E}_i,\mathbf{E}_l,\mathbf{E}_m]-\{\mathbf{E}_n,\mathbf{E}_k,\mathbf{E}_i\}-\{\mathbf{E}_n,\mathbf{E}_l,\mathbf{E}_j\}+\{\mathbf{E}_j,\mathbf{E}_k,\mathbf{E}_m\})
\end{aligned} \tag{1.61}$$

Let's now set $i=k$ and $j=l$:

$$\begin{aligned}
[x,x,z] &= ([\mathbf{E}_i,\mathbf{E}_i,\mathbf{E}_m]-\{\mathbf{E}_m,\mathbf{E}_j,\mathbf{E}_j\}-\{\mathbf{E}_n,\mathbf{E}_j,\mathbf{E}_i\}-[\mathbf{E}_n,\mathbf{E}_i,\mathbf{E}_j],\\
&\quad\ [\mathbf{E}_i,\mathbf{E}_j,\mathbf{E}_m]-\{\mathbf{E}_n,\mathbf{E}_i,\mathbf{E}_i\}-\{\mathbf{E}_n,\mathbf{E}_j,\mathbf{E}_j\}+\{\mathbf{E}_j,\mathbf{E}_i,\mathbf{E}_m\})\\
&= (-2\mathbf{E}_m-\{\mathbf{E}_n,\mathbf{E}_j,\mathbf{E}_i\}-[\mathbf{E}_n,\mathbf{E}_i,\mathbf{E}_j],\\
&\quad\ [\mathbf{E}_i,\mathbf{E}_j,\mathbf{E}_m]+\{\mathbf{E}_j,\mathbf{E}_i,\mathbf{E}_m\}-4\mathbf{E}_n)
\end{aligned} \tag{1.62}$$

Thus, we were able to verify that the non-alternative nature of sedenions originates from the non-associative of elements of the octonion basis, which are used to form sedenion numbers that do not contain any null term in the ordered pair.

We can also prove how sedenions are a non-division algebra by using elements $x = (\mathbf{E}_i, \mathbf{E}_j)$ and $y = (\mathbf{E}_k, \mathbf{E}_l)$. By choosing $i = 1$, $j = 6$, $k = 2$ and $l = 5$, then:

$$\begin{aligned} x \cdot y &= (\mathbf{E}_1, \mathbf{E}_6)(\mathbf{E}_2, \mathbf{E}_5) \\ &= (\mathbf{E}_3 - \mathbf{E}_3, \mathbf{E}_4 - \mathbf{E}_4) = 0 \end{aligned}$$

1.7 Split-complex numbers

In the presentation of the Cayley-Dickson construction, it was reported that choosing the parameter $\epsilon = 1$ will result in the construction of a split-division algebra. For example, starting from the reals, we can obtain the complex numbers ($\mathbb{C}$) by choosing $\epsilon = -1$, or the split-complex numbers ($\widetilde{\mathbb{C}}$) for $\epsilon = 1$. The different choices will result in different properties for the multiplication and "norm".

For split-complex numbers, one can define the following elements of basis:

$$1 \equiv (1, 0), \quad j \equiv (0, 1) \tag{1.63}$$

Then, a split-complex number $\tilde{z} \in \widetilde{\mathbb{C}}$ can be defined as:

$$\tilde{z} = a + bj, \quad a, b \in \mathbb{R}, \tag{1.64}$$

where

$$j^2 = 1 \tag{1.65}$$

and

$$\mathcal{N}(\tilde{z}) = \tilde{z}^* \tilde{z} = a^2 - b^2 \tag{1.66}$$

It is important to note that, for a complex number z, its norm $\mathcal{N}(z) \in \mathbb{R}^+$; while for a split-complex number $\tilde{z}$, its "norm" $\mathcal{N}(\tilde{z}) \in \mathbb{R}$.

The main property of a split-division algebra is the possibility of division of zero. Let $a, b \in \mathbb{A}_+$, a split-division algebra. Even if both a and b are nonzero, the result of multiplying them can be zero.

Let's suppose $a, b \in \widetilde{\mathbb{C}}$, for $a = (x, y)$ and $b = (z, w)$, where $x, y, z, w \in \mathbb{R}^*$. Therefore,

$$a \cdot b = (x, y) \cdot (z, w) = (xz + wy, wx + yz) = 0 \tag{1.67}$$

From (1.67), we get the following system of equations:

$$\begin{cases} xz = -wy \\ wx = -yz \end{cases} \tag{1.68}$$

The solution for (1.68) goes through

$$z = \pm w \quad \text{or} \quad x = \pm y \tag{1.69}$$

The results (1.69) will be revisited later, in approach of matrix representation of Cayley-Dickson doubling construction.

1.8 Split-quaternions

It is possible to construct a $2N$-dimensional split-division algebra in two different ways: by doubling a N-dimension split-division algebra or doubling a N-dimension division algebra with $\epsilon = 1$. In this work, the second way will be more useful for future results. So, let's construct split-quaternions ($\widetilde{\mathbb{H}}$) by doubling complex numbers ($\mathbb{C}$) and choosing $\epsilon = 1$.

Analogous to the construction process of quaternions, the split-quaternion basis is formed by the following elements:

$$\mathbf{e}_0 = (1,0), \quad \widetilde{\mathbf{e}}_1 = (i,0), \quad \widetilde{\mathbf{e}}_2 = (0,1), \quad \widetilde{\mathbf{e}}_3 = (0,i) \tag{1.70}$$

Quaternion and split-quaternion basis elements are represented by the same ordered pairs, but its operations are not the same. For example, multiplication between two elements of (1.70) has the following result:

$$\widetilde{\mathbf{e}}_i \widetilde{\mathbf{e}}_j = -\eta_{ij}\mathbf{e}_0 + \widetilde{\varepsilon}_{ij}{}^{k}\widetilde{\mathbf{e}}_k \tag{1.71}$$

where η_{ij} is the ij-component of split-quaternionic metric, that is:

$$\eta_{ij} \rightarrow \boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.72}$$

$\widetilde{\varepsilon}_{ij}{}^{k}$ is the "split-Levi-Civita" pseudotensor, and it is not totally anti-symmetric, but if we lower the last index it becomes totally anti-symmetric. To do so, we must multiply $\widetilde{\varepsilon}_{ij}{}^{k}$ by split-quaternionic metric:

$$\widetilde{\varepsilon}_{ijk} = \widetilde{\varepsilon}_{ij}{}^{l}\eta_{lk} \tag{1.73}$$

where:

$$\begin{aligned} \widetilde{\varepsilon}_{12}{}^{3} &= 1, \quad \widetilde{\varepsilon}_{21}{}^{3} = -1 \\ \widetilde{\varepsilon}_{31}{}^{2} &= 1, \quad \widetilde{\varepsilon}_{13}{}^{2} = -1 \\ \widetilde{\varepsilon}_{23}{}^{1} &= -1, \quad \widetilde{\varepsilon}_{32}{}^{1} = 1 \end{aligned} \tag{1.74}$$

and

$$\begin{aligned}\tilde{\varepsilon}_{123} &= -1, \quad \tilde{\varepsilon}_{213}=1\\ \tilde{\varepsilon}_{312} &= -1, \quad \tilde{\varepsilon}_{132}=1\\ \tilde{\varepsilon}_{231} &= -1, \quad \tilde{\varepsilon}_{321}=1\end{aligned} \tag{1.75}$$

It is important to say that split-quaternions are a non-commutative algebra, as quaternions. Except for the norm, the mains properties of a division algebra also exists in the split form of this algebra.

1.9 Split-octonions

The split-octonions' elements basis are:

$$\begin{aligned}&\mathbf{E}_0=(\mathbf{e}_0,0), \quad \tilde{\mathbf{E}}_1=(\mathbf{e}_1,0), \quad \tilde{\mathbf{E}}_2=(\mathbf{e}_2,0), \quad \tilde{\mathbf{E}}_3=(\mathbf{e}_3,0)\\ &\tilde{\mathbf{E}}_4=(0,\mathbf{e}_0), \quad \tilde{\mathbf{E}}_5=(0,\mathbf{e}_1), \quad \tilde{\mathbf{E}}_6=(0,\mathbf{e}_2), \quad \tilde{\mathbf{E}}_7=(0,\mathbf{e}_3)\end{aligned} \tag{1.76}$$

The multiplication between two elements of (1.76) is:

$$\tilde{\mathbf{E}}_i\tilde{\mathbf{E}}_j=-\mu_{ij}\mathbf{E}_0+\tilde{C}_{ij}{}^k\tilde{\mathbf{E}}_k \tag{1.77}$$

where μ_{ij} is the ij-component of split-octonionic metric,

$$\tilde{\mu}_{ij}\to\boldsymbol{\mu}=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&-1\end{pmatrix} \tag{1.78}$$

and $\tilde{C}_{ijk}$ the structure constant of the split-octonions. The table of structure constant of split-octonions is:

It is easy to see that $\tilde{C}_{ij}{}^k$ is not totally anti-symmetric. But, by doing:

$$\tilde{C}_{ijk}=\tilde{C}_{ij}{}^l\tilde{\mu}_{lk} \tag{1.79}$$

we get the representation $\tilde{C}_{ijk}$ that is totally anti-symmetric.

$\widetilde{C}_{12^3} = 1$	$\widetilde{C}_{21}{}^{3} = -1$	$\widetilde{C}_{31}{}^{2} = 1$
$\widetilde{C}_{13}{}^{2} = -1$	$\widetilde{C}_{23}{}^{1} = 1$	$\widetilde{C}_{32}{}^{1} = -1$
$\widetilde{C}_{14}{}^{5} = 1$	$\widetilde{C}_{24}{}^{6} = 1$	$\widetilde{C}_{34}{}^{7} = 1$
$\widetilde{C}_{15}{}^{4} = -1$	$\widetilde{C}_{25}{}^{7} = 1$	$\widetilde{C}_{35}{}^{6} = -1$
$\widetilde{C}_{16}{}^{7} = -1$	$\widetilde{C}_{26}{}^{4} = -1$	$\widetilde{C}_{36}{}^{5} = 1$
$\widetilde{C}_{17}{}^{6} = 1$	$\widetilde{C}_{27}{}^{5} = -1$	$\widetilde{C}_{37}{}^{4} = -1$

$\widetilde{C}_{41}{}^{5} = -1$	$\widetilde{C}_{51}{}^{4} = 1$	$\widetilde{C}_{61}{}^{7} = 1$	$\widetilde{C}_{71}{}^{6} = -1$
$\widetilde{C}_{42}{}^{6} = -1$	$\widetilde{C}_{52}{}^{7} = -1$	$\widetilde{C}_{62}{}^{4} = 1$	$\widetilde{C}_{72}{}^{5} = 1$
$\widetilde{C}_{43}{}^{7} = -1$	$\widetilde{C}_{53}{}^{6} = 1$	$\widetilde{C}_{63}{}^{5} = -1$	$\widetilde{C}_{73}{}^{4} = 1$
$\widetilde{C}_{45}{}^{1} = -1$	$\widetilde{C}_{54}{}^{1} = 1$	$\widetilde{C}_{64}{}^{2} = 1$	$\widetilde{C}_{74}{}^{3} = 1$
$\widetilde{C}_{46}{}^{2} = -1$	$\widetilde{C}_{56}{}^{3} = 1$	$\widetilde{C}_{65}{}^{3} = -1$	$\widetilde{C}_{75}{}^{2} = 1$
$\widetilde{C}_{47}{}^{3} = -1$	$\widetilde{C}_{57}{}^{2} = -1$	$\widetilde{C}_{67}{}^{1} = 1$	$\widetilde{C}_{76}{}^{1} = -1$

Table 1.3: Structure constant of split-octoionios

$\widetilde{C}_{123} = 1$	$\widetilde{C}_{213} = -1$	$\widetilde{C}_{312} = 1$
$\widetilde{C}_{132} = -1$	$\widetilde{C}_{231} = 1$	$\widetilde{C}_{321} = -1$
$\widetilde{C}_{145} = -1$	$\widetilde{C}_{246} = -1$	$\widetilde{C}_{347} = -1$
$\widetilde{C}_{154} = 1$	$\widetilde{C}_{257} = -1$	$\widetilde{C}_{356} = 1$
$\widetilde{C}_{167} = 1$	$\widetilde{C}_{264} = 1$	$\widetilde{C}_{365} = -1$
$\widetilde{C}_{176} = -1$	$\widetilde{C}_{275} = 1$	$\widetilde{C}_{374} = 1$

$\widetilde{C}_{415} = 1$	$\widetilde{C}_{514} = -1$	$\widetilde{C}_{617} = -1$	$\widetilde{C}_{716} = 1$
$\widetilde{C}_{426} = 1$	$\widetilde{C}_{527} = 1$	$\widetilde{C}_{624} = -1$	$\widetilde{C}_{725} = -1$
$\widetilde{C}_{437} = 1$	$\widetilde{C}_{536} = -1$	$\widetilde{C}_{635} = 1$	$\widetilde{C}_{734} = -1$
$\widetilde{C}_{451} = -1$	$\widetilde{C}_{541} = 1$	$\widetilde{C}_{642} = 1$	$\widetilde{C}_{743} = 1$
$\widetilde{C}_{462} = -1$	$\widetilde{C}_{563} = 1$	$\widetilde{C}_{653} = -1$	$\widetilde{C}_{752} = 1$
$\widetilde{C}_{473} = -1$	$\widetilde{C}_{572} = -1$	$\widetilde{C}_{671} = 1$	$\widetilde{C}_{761} = -1$

Table 1.4: Totally anti-symmetric split-octonion's structure constants

Chapter 2

Matrix representation for (split-)division algebras

In addition to the vector representation, (split-)divisional algebras can also be compatible with the matrix representation. Initially, we will focus on the matrix representation of (split-)complex numbers and (split-)quaternions. Although (split-)octonions are non-associative, it is possible to describe them as matrices too, which obey a particular multiplication rule. However, this topic will be covered at the end of this section in detail. From this, we will finally be able to elaborate a matrix representation for the construction of the Cayley-Dickson algebras, having the doubling technique as background.

2.1 Matrix representation for (split-)complex numbers

As said before, a complex number z can be presented as the following vector:

$$z = x(1,0) + y(0,1), \quad x, y \in \mathbb{R} \tag{2.1}$$

where $(1,0)$ and $(0,1)$ are vector representations of real unit 1 and imaginary unit i, respectively. Now, let's construct two matrices that match the algebric properties of $(1,0)$ and $(0,1)$.

From start, we can associate vector (1,0) to unitary matrix $\mathbf{1}_{2\times2}$:

$$(1,0) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2}$$

For $(0,1)$, we have:

$$(0,1) \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.3}$$

where $A, B, C, D \in \mathbb{R}$. As we know:

$$(0,1) \cdot (0,1) = -(1,0) \tag{2.4}$$

Therefore:

$$\begin{pmatrix} A^2+CB & AC+CD \\ BA+DB & BC+D^2 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.5}$$

Then, we get the following system of equations:

$$\begin{aligned} A^2+CB &= -1 \\ C(A+D) &= 0 \\ B(A+D) &= 0 \\ BC+D^2 &= -1 \end{aligned} \tag{2.6}$$

To solve the set of equations (2.6) can be solved by imposing $C = B = 0$. But this choice will result in $A, D \not\in \mathbb{R}$. That is, the only possible solution is to assume that $A = -D$, reducing the system to an equation:

$$A^2+CB = -1 \tag{2.7}$$

Equation (2.7) explains freedom for the definition of the matrices associated with the vector $(0,1)$. A convenient choice is $A = 0$, $B = -1$ and $C = 1$. Thus,

$$(0,1) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.8}$$

Therefore, a complex number $z = (x, y)$, $x, y \in \mathbb{R}$ can be represented by the following matrix:

$$z = (x, y) \equiv \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \tag{2.9}$$

For split-complex numbers, the analogous procedure can be carried out. Matrices of $1 = (1,0)$ and $j = (0,1)$ are:

$$(1,0) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j = (0,1) \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.10}$$

So that $\tilde{z} \in \tilde{\mathbb{C}}$ will be represented by the following matrix:

$$\tilde{z} = (x, y) \equiv \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \quad x, y \in \mathbb{R} \tag{2.11}$$

2.2 Matrix representation for (split-)quaternions

Now let's try to define the quaternions in terms of a matrix 2×2 of complex entries. Then, for a quaternion number z:

$$z = (x, y) = a\mathbf{e}_0 + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3 \equiv aM_0 + bM_1 + cM_2 + dM_3 \tag{2.12}$$

where:

$$\begin{aligned} x &= a+ib, \\ y &= c+id \end{aligned} \tag{2.13}$$

As we know, coeficients a, b, c and d are real numbers. Element $\mathbf{e}_0 = (1,0)$ is the real unit and therefore:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.14}$$

For the other matrices, we have the following properties:

$$\begin{aligned} M_1^2 &= -\mathbf{1}_{2\times 2} & M_1M_2 &= M_3 \\ M_2^2 &= -\mathbf{1}_{2\times 2} & M_3M_1 &= M_2 \\ M_3^2 &= -\mathbf{1}_{2\times 2} & M_2M_3 &= M_1 \end{aligned} \tag{2.15}$$

Relations (2.15) are just a consequence of multiplication between basis elements of quaternions, such that:

$$M_iM_j = -\delta_{ij}M_0 + \varepsilon_{ijk}M_k \tag{2.16}$$

Informations just above are sufficient to determine matrices M, sendo:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \mathbb{C} \tag{2.17}$$

If $M^2 = -1$, then

$$M^2 = \begin{pmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.18}$$

It is possible to verify that $A = \pm D$. We then have two possibilities for choosing A, so that:

1) By choosing $A = D$:

$$M^2 = \begin{pmatrix} A^2 + BC & 2AB \\ 2AC & CB + A^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.19}$$

$$\begin{cases} A^2 + BC = -1 \\ AB = 0 \\ AC = 0 \end{cases}$$

1.1) We can impose $B = C = 0$. Then:

$$A = \pm i \Longrightarrow M = \pm i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.20}$$

So this option is useless because $M = \pm i\mathbf{1}_{2\times 2}$ and $\mathbf{1}_{2\times 2}$ is already related to the vector $(1, 0)$.

1.2) If $A = D = 0$, we have:

$$C = -\frac{1}{B} \tag{2.21}$$

Therefore, when choosing $A = D$, we will have the following matrix M:

$$M = \begin{pmatrix} 0 & B \\ -1/B & 0 \end{pmatrix} \tag{2.22}$$

2) By choosing $A = -D$,

$$M^2 = \begin{pmatrix} A^2 + BC & 0 \\ 0 & BC + A^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.23}$$

We get the following result for A:

$$A = i(1 + BC)^{1/2} \tag{2.24}$$

Then,

$$M = \begin{pmatrix} i(1 + BC)^{1/2} & B \\ C & -i(1 + BC)^{1/2} \end{pmatrix} \tag{2.25}$$

Finally, we define the two possible matrices:

$$M = \begin{pmatrix} 0 & X \\ -1/X & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} i(1 + BC)^{1/2} & B \\ C & -i(1 + BC)^{1/2} \end{pmatrix} \tag{2.26}$$

The produts between M and M' are:

$$MM' = \begin{pmatrix} XC & -iX(1 + BC)^{1/2} \\ -\frac{i}{X}(1 + BC)^{1/2} & -\frac{B}{X} \end{pmatrix} \tag{2.27}$$

$$M'M = \begin{pmatrix} -\frac{B}{X} & iX(1 + BC)^{1/2} \\ \frac{i}{X}(1 + BC)^{1/2} & CX \end{pmatrix} \tag{2.28}$$

$$MM' = -M'M \Longrightarrow C = \frac{B}{X^2} \tag{2.29}$$

such that:

$$M = \begin{pmatrix} 0 & X \\ -1/X & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} i\left(1 + \frac{B^2}{X^2}\right)^{1/2} & B \\ \frac{B}{X^2} & -i\left(1 + \frac{B^2}{X^2}\right)^{1/2} \end{pmatrix} \tag{2.30}$$

Now it will be necessary to determine the third matrix M'', which is defined by the product MM'. According to (2.30) and the value of $C = B/X^2$, M'' will be:

$$M'' = \begin{pmatrix} \frac{B}{X} & iX\left(1+\frac{B^2}{X^2}\right) \\ -\frac{i}{X}\left(1+\frac{B^2}{X^2}\right) & -\frac{B}{X} \end{pmatrix} \tag{2.31}$$

A plausible choice for defining the three matrices is to impose $X = 1$ and $B = i$. Thus, we get the following matrices:

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad M'' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2.32}$$

So let's consider the following choices:

$$M'' \to M_1 \equiv \mathbf{e}_1 = (i, 0)$$
$$M \to M_2 \equiv \mathbf{e}_2 = (0, 1)$$
$$M' \to M_3 \equiv \mathbf{e}_3 = (0, i)$$

We see that a quaternion z in (2.12) can be represented by the following matrix:

$$z \equiv a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$z \equiv \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \tag{2.33}$$

Generally, a quaternion $z = (x, y)$ can be represented by the following matrix:

$$q = (x, y) \equiv \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \quad x, y \in \mathbb{C} \tag{2.34}$$

For split-quaternions the same procedure can be done and the matrices of basis elements are:

$$\mathbf{e}_0 = (1, 0) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\mathbf{e}}_1 = (i, 0) \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{\mathbf{e}}_2 = (0, 1) \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\tilde{\mathbf{e}}_3 = (0, i) \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{2.35}$$

According to (2.35) a split-quaternion number $\tilde{z} = (x, y)$ has the following matrix representation:

$$\tilde{z} = (x, y) \equiv \begin{pmatrix} x & y \\ y^* & x \end{pmatrix}, \quad x, y \in \mathbb{C} \tag{2.36}$$

Next, we will see that (split-)quaternions can also be represented by matrices 4×4 of real inputs. This process can be performed using an algorithm based on a formula that generalizes the matrix representation of the Cayley-Dickson algebras. This formula can be realised by look backing for results (2.9), (2.11), (2.34) and (2.36). In principle, this general formula can not be used to represent octonions, since they are a non-associative algebra. But lately we will see how it would be possible to use this generalization for represent any Cayley-Dickson algebra, preserving all its degree of freedom.

2.3 General formula for matrix representation

Previously it has been shown that:

- For any given $z \in \mathbb{C}$,

$$z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \quad x, y \in \mathbb{R} \tag{2.37}$$

- For any given $\tilde{z} \in \tilde{\mathbb{C}}$,

$$\tilde{z} = \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \quad x, y \in \mathbb{R} \tag{2.38}$$

- For any given $w \in \mathbb{H}$,

$$w = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \quad x, y \in \mathbb{C} \tag{2.39}$$

- For any given $\tilde{w} \in \tilde{\mathbb{H}}$,

$$\tilde{w} = \begin{pmatrix} x & y \\ y^* & x^* \end{pmatrix}, \quad x, y \in \mathbb{C} \tag{2.40}$$

By induction, you can make the following hypothesis: Two elements of a divisional algebra $x, y in \mathbb{A}$, the vector resulting from doubling $(x, y) \in \mathbb{A}_\epsilon^2$ can be presented by the following matrix:

$$(x, y) \equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix} \tag{2.41}$$

where $\epsilon = -1$ for a division algebra and $\epsilon = +1$ for a split-division algebra.

However, this representation is incompatible with Cayley-Dickson multiplication for algebras whose inputs of ordered pairs are non-commutative elements.

Let's make it clear. We must remember that Cayley-Dickson multiplication for two vectors (x, y) and (z, w) of any algebra is

$$(x, y) \cdot (z, w) = (xz + \epsilon w^* y, wx + yz^*) \tag{2.42}$$

So, if (x, y) and (z, w) can be represented as a 2×2 matrices, it would must be possible a matrix representation for the result $(xz+\epsilon w^* y, wx+yz^*)$. Then, according to (2.41):

$$\begin{aligned}(x,y)(z,w) &\equiv \begin{pmatrix} xz+\epsilon w^* y & wx+yz^* \\ \epsilon(wx+yz^*)^* & (xz+\epsilon w^* y)^* \end{pmatrix} \\ &\equiv \begin{pmatrix} xz+\epsilon w^* y & wx+yz^* \\ \epsilon x^* w^* + \epsilon z y^* & z^* x^* + \epsilon y^* w \end{pmatrix}\end{aligned} \tag{2.43}$$

But the usual multiplication between matrices of (x, y) and (z, w) has the following result:

$$\begin{aligned}(x,y)(z,w) &\equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix}\begin{pmatrix} z & w \\ \epsilon w^* & z^* \end{pmatrix} \\ &\equiv \begin{pmatrix} xz+\epsilon w^* y & wx+yz^* \\ \epsilon x^* w^* + \epsilon y^* z & x^* z^* + \epsilon y^* w \end{pmatrix}\end{aligned} \tag{2.44}$$

The results (2.43) and (2.44) will only be equivalent for algebras in which x, y, z, w are numbers of a commutative algebra. If x, y, z, w are, for example, quaternions, so product for matrix representation needs to be defined as follows:

$$(x,y)(z,w) \equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix}\begin{pmatrix} z & w \\ \epsilon w^* & z^* \end{pmatrix} + \begin{pmatrix} \epsilon[w^*, y] & [w, x] \\ \epsilon[z, y^*] & [z^*, x^*] \end{pmatrix} \tag{2.45}$$

Definition (2.45) makes octonions and split-octonions compatible with the matrix representation and preserve its non-associative. Zorn has already demonstrated a matrix representation for octonions and split-octonions [11]. But the purpose of this work goes beyond demonstrating an alternative representation of Zorn's work. From (2.41) and (2.45), we will also demonstrate how an N-dimensional Cayley-Dickson algebra can be represented by an $N \times N$ matrix of real inputs and from this, we will explore its possible applications.

2.4 2 × 2 Matrix representation for (split-)octonions

According to (1.24) and (2.41), octonionic basis can be represented by following 2×2 with quaternions inputs:

$$\mathbf{E}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{E}_1 \equiv \begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & -\mathbf{e}_1 \end{pmatrix}, \quad \mathbf{E}_2 \equiv \begin{pmatrix} \mathbf{e}_2 & 0 \\ 0 & -\mathbf{e}_2 \end{pmatrix}, \quad \mathbf{E}_3 \equiv \begin{pmatrix} \mathbf{e}_3 & 0 \\ 0 & -\mathbf{e}_3 \end{pmatrix},$$

$$\mathbf{E}_4 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{E}_5 \equiv \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{e}_1 & 0 \end{pmatrix}, \quad \mathbf{E}_6 \equiv \begin{pmatrix} 0 & \mathbf{e}_2 \\ \mathbf{e}_2 & 0 \end{pmatrix}, \quad \mathbf{E}_7 \equiv \begin{pmatrix} 0 & \mathbf{e}_3 \\ \mathbf{e}_3 & 0 \end{pmatrix}. \tag{2.46}$$

So, any octonion $\mathbf{z} = a\mathbf{E}_0 + b\mathbf{E}_1 + c\mathbf{E}_2 + d\mathbf{E}_3 + f\mathbf{E}_4 + g\mathbf{E}_5 + h\mathbf{E}_6 + m\mathbf{E}_7$, where $a, b, c, d, f, g, h, m \in \mathbb{R}$, is equivalent to the matrix bellow:

$$\mathbf{u} \equiv \begin{pmatrix} a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3 & f + g\mathbf{e}_1 + h\mathbf{e}_2 + m\mathbf{e}_3 \\ -f + g\mathbf{e}_1 + h\mathbf{e}_2 + m\mathbf{e}_3 & a - b\mathbf{e}_1 - c\mathbf{e}_2 - d\mathbf{e}_3 \end{pmatrix}, \tag{2.47}$$

We can use some matrices in (2.46) to proof of how multiplication (2.45) are valid for octonions. Is just needed to remember the rules of multiplication between quaternions $\mathbf{e}_i$. A few examples:

$$\begin{aligned}
\mathbf{E}_1\mathbf{E}_3 &\equiv \begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & -\mathbf{e}_1 \end{pmatrix}\begin{pmatrix} \mathbf{e}_3 & 0 \\ 0 & -\mathbf{e}_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & [\mathbf{e}_3, \mathbf{e}_1] \end{pmatrix} \\
&\equiv -\begin{pmatrix} \mathbf{e}_2 & 0 \\ 0 & -\mathbf{e}_2 \end{pmatrix} = -\mathbf{E}_2
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
\mathbf{E}_1\mathbf{E}_4 &\equiv \begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & -\mathbf{e}_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & [1, \mathbf{e}_1] \\ 0 & 0 \end{pmatrix} \\
&\equiv -\begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{e}_1 & 0 \end{pmatrix} = \mathbf{E}_5
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\mathbf{E}_3\mathbf{E}_5 &\equiv \begin{pmatrix} \mathbf{e}_3 & 0 \\ 0 & -\mathbf{e}_3 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{e}_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & [\mathbf{e}_1, \mathbf{e}_3] \\ 0 & 0 \end{pmatrix} \\
&\equiv -\begin{pmatrix} 0 & \mathbf{e}_2 \\ \mathbf{e}_2 & 0 \end{pmatrix} = -\mathbf{E}_6
\end{aligned} \tag{2.50}$$

Matrices in (2.46) are just one of the three possibles representations for octonions in relation of matrix dimension. Let's remember that:

$$\begin{aligned}
\mathbb{C} &\equiv \mathbb{R}^2 \\
\mathbb{H} &\equiv \mathbb{C}^2 \equiv \mathbb{R}^4 \\
\mathbb{O} &\equiv \mathbb{H}^2 \equiv \mathbb{C}^4 \equiv \mathbb{R}^8
\end{aligned} \tag{2.51}$$

According to (2.51), we can see that complex numbers $\mathbb{C}$ has only one dimensional matrix representation: 2×2 matrices with real numbers inputs. For quaternions there are two possibilities: 2×2 with complex numbers inputs and 4×4 matrices with real numbers inputs. Finally, for octonions exist three possibilities: 2×2 matrices with quaternion inputs, 4×4 with complex numbers inputs and 8×8 with real inputs. In the following sections, it will be shown how to creat all these representations.

2.5 4×4 Matrix representation for (split-)quaternions

As already seen, an element $(x, y) \in \mathbb{A}_\epsilon^2$ can be associated with the following matrix:

$$(x, y)_{2\times 2} \equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix}; \quad x, y \in \mathbb{A} \tag{2.52}$$

where ϵ is the Cayley-Dickson parameter.

But the matrix entries can also be a vector element of a doubled algebra, that is, $x = (a,b)$ and $y = (c,d)$, so the doubling (x,y) becomes a matrix 4×4:

$$(x,y)_{4\times 4} \equiv \left(\begin{array}{c|c} (a,b)_{2\times 2} & (c,d)_{2\times 2} \\ \hline \epsilon(c,d)^*_{2\times 2} & (a,b)^*_{2\times 2} \end{array}\right) = \left(\begin{array}{c|c} (a,b)_{2\times 2} & (c,d)_{2\times 2} \\ \hline \epsilon(c^*,-d)_{2\times 2} & (a^*,-b)_{2\times 2} \end{array}\right) \tag{2.53}$$

Therefore, matrix (2.53) is:

$$(x,y)_{4\times 4} \equiv \left(\begin{array}{c|c} \begin{pmatrix} a & b \\ \bar{\epsilon}b^* & a^* \end{pmatrix} & \begin{pmatrix} c & d \\ \bar{\epsilon}d^* & c^* \end{pmatrix} \\ \hline \epsilon\begin{pmatrix} c^* & -d \\ -\bar{\epsilon}d^* & c \end{pmatrix} & \begin{pmatrix} a^* & -b \\ -\bar{\epsilon}b^* & a \end{pmatrix} \end{array}\right) \tag{2.54}$$

where $\bar{\epsilon}$ is the Cayley-Dickson parameter of algebra $\mathbb{A}$.

One can use a informal representation to define matrix (2.54). If $x = (a,b)$ and $y = (c,d)$, then:

$$(x,y)_{4\times 4} = ((a,b),(c,d)) \equiv (a,b,c,d) \equiv \left(\begin{array}{c|c} \begin{pmatrix} a & b \\ \bar{\epsilon}b^* & a^* \end{pmatrix} & \begin{pmatrix} c & d \\ \bar{\epsilon}d^* & c^* \end{pmatrix} \\ \hline \epsilon\begin{pmatrix} c^* & -d \\ -\bar{\epsilon}d^* & c \end{pmatrix} & \begin{pmatrix} a^* & -b \\ -\bar{\epsilon}b^* & a \end{pmatrix} \end{array}\right) \tag{2.55}$$

If we use complex numbers to construct quaternions, then $\epsilon = \bar{\epsilon} = -1$. So, let's rewrite quaternions $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ as a vector (a,b,c,d):

$$\begin{aligned} \mathbf{e}_0 &= (1,0) \equiv (1,0,0,0), \quad & \mathbf{e}_1 &= (i,0) \equiv (0,1,0,0) \\ \mathbf{e}_2 &= (0,1) \equiv (0,0,1,0), \quad & \mathbf{e}_3 &= (0,i) \equiv (0,0,0,1) \end{aligned} \tag{2.56}$$

According to (2.55) and (2.56), one can construct following matrices:

$$\begin{aligned} \mathbf{e}_0 &\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \mathbf{e}_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.57}$$

It is also possible to construct split-quaternions matrices in 4×4 form by doubling complex numbers. In this case for matrix (2.55) $\epsilon = 1$ and $\bar{\epsilon} = -1$. Split-quaternion basis can be represented in the form (a,b,c,d) as follows:

$$\begin{aligned} \mathbf{e}_0 &= (1,0) \equiv (1,0,0,0), \quad \tilde{\mathbf{e}}_1 = (i,0) \equiv (0,1,0,0) \\ \tilde{\mathbf{e}}_2 &= (0,1) \equiv (0,0,1,0) \quad \tilde{\mathbf{e}}_3 = (0,i) \equiv (0,0,0,1) \end{aligned} \tag{2.58}$$

And matrices of elements (2.58) are:

$$\begin{aligned} \mathbf{e}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{\mathbf{e}}_1 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \tilde{\mathbf{e}}_2 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_3 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.59}$$

2.6 4×4 and 8×8 Matrix representation for (split-)octonions

We have already seen the 2×2 matrices of octonion basis. Now, we can use the algoritm presented in last section to construct matrices 4×4 and 8×8 for $\mathbf{E}_i$.

First, for octonion basis matrices in 4×4 form, it is just needed to look back (2.55). We also can represent elements $\mathbf{E}_i$ as a vector (a,b,c,d). Then,

$$\begin{aligned} &\mathbf{E}_0 = (\mathbf{e}_0, 0) \equiv (1,0,0,0), \quad \mathbf{E}_1 = (\mathbf{e}_1, 0) \equiv (i,0,0,0), \quad \mathbf{E}_2 = (\mathbf{e}_2, 0) \equiv (0,1,0,0), \\ &\mathbf{E}_3 = (\mathbf{e}_3, 0) \equiv (0,i,0,0), \quad \mathbf{E}_4 = (0, \mathbf{e}_0) \equiv (0,0,1,0), \quad \mathbf{E}_5 = (0, \mathbf{e}_1) \equiv (0,0,i,0), \\ &\mathbf{E}_6 = (0, \mathbf{e}_2) \equiv (0,0,0,1), \quad \mathbf{E}_7 = (0, \mathbf{e}_3) \equiv (0,0,0,i) \end{aligned} \tag{2.60}$$

and the matrices of (2.60) are:

$$\begin{aligned} &\mathbf{E}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad \mathbf{E}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &\mathbf{E}_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \mathbf{E}_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \mathbf{E}_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \\ &\mathbf{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{E}_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.61}$$

To multiply matrices in (2.61), we must use the rule of multiplication presented in (2.45). For multiplication between any matrices of a Cayley-Dickson algebra let us use the symbol $\times$, that is what we will call from now on by *matrix Cayley-Dickson multiplication.*

So, for two 4×4 matrices of a given algebra $\mathbb{A}^2$, one can have:

$$\begin{aligned}\left(\begin{array}{c|c}(a,b)_{2\times2} & (c,d)_{2\times2}\\ \hline \epsilon(c,d)^*_{2\times2} & (a,b)^*_{2\times2}\end{array}\right)\times\left(\begin{array}{c|c}(e,f)_{2\times2} & (g,h)_{2\times2}\\ \hline \epsilon(g,h)^*_{2\times2} & (e,f)^*_{2\times2}\end{array}\right) &= \left(\begin{array}{c|c}(a,b)_{2\times2} & (c,d)_{2\times2}\\ \hline \epsilon(c,d)^*_{2\times2} & (a,b)^*_{2\times2}\end{array}\right)\left(\begin{array}{c|c}(e,f)_{2\times2} & (g,h)_{2\times2}\\ \hline \epsilon(g,h)^*_{2\times2} & (e,f)^*_{2\times2}\end{array}\right)\\ &+\left(\begin{array}{c|c}\epsilon[(g,h)^*_{2\times2},(c,d)_{2\times2}] & [(g,h)_{2\times2},(a,b)_{2\times2}]\\ \hline \epsilon[(e,f)_{2\times2},(c,d)^*_{2\times2}] & [(e,f)^*_{2\times2},(a,b)^*_{2\times2}]\end{array}\right)\end{aligned} \tag{2.62}$$

We can use two matrices in (2.61) as an example of (2.62):

$$\begin{aligned}\mathbf{E}_1\mathbf{E}_7 &= \begin{pmatrix} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & i\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix}+\left(\begin{array}{c|c}\mathbf{O} & \left[\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\right]\\ \hline \mathbf{O} & \mathbf{O}\end{array}\right)\\ &= \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}+\left(\begin{array}{c|c}\mathbf{O} & \begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\\ \hline \mathbf{O} & \mathbf{O}\end{array}\right)\\ &= \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}=\mathbf{E}_6\end{aligned} \tag{2.63}$$

As said before, we can also construct 8×8 matrices for octonion basis. In matrix (2.55), it is possible to see that a,b,c and d can be rewrite also as ordered pair and therefore as 2×2 matrix. For praticality, we will consider that (x,y), (a,b) and (c,d) are elements of a division algebra. So, if:

$$\begin{gathered} a=(\alpha,\beta),\quad b=(\gamma,\lambda),\quad c=(\xi,\sigma),\quad d=(\kappa,\eta),\\ a^*=(\alpha,-\beta),\quad b^*=(\gamma,-\lambda),\quad c^*=(\xi,-\sigma),\quad d^*=(\kappa,-\eta),\end{gathered} \tag{2.64}$$

and

$$\begin{aligned} a&=\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}, a^*=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix} & b&=\begin{pmatrix}\gamma & \lambda\\ -\lambda & \gamma\end{pmatrix}, b^*=\begin{pmatrix}\gamma & -\lambda\\ \lambda & \gamma\end{pmatrix}\\ c&=\begin{pmatrix}\xi & \sigma\\ -\sigma & \xi\end{pmatrix}, c^*=\begin{pmatrix}\xi & -\sigma\\ \sigma & \xi\end{pmatrix} & d&=\begin{pmatrix}\kappa & \eta\\ -\eta & \kappa\end{pmatrix}, d^*=\begin{pmatrix}\kappa & -\eta\\ \eta & \kappa\end{pmatrix}\end{aligned} \tag{2.65}$$

Then,

$$(a,b,c,d)=((\alpha,\beta),(\gamma,\lambda),(\xi,\sigma),(\kappa,\eta))\equiv(\alpha,\beta,\gamma,\lambda,\xi,\sigma,\kappa,\eta) \tag{2.66}$$

and, according to (2.55), the 8×8 matrix is:

$$(x, y))_{8\times 8} \equiv (\alpha, \beta, \gamma, \lambda, \xi, \sigma, \kappa, \eta) = \left(\begin{array}{cccc|cccc} \alpha & \beta & \gamma & \lambda & \xi & \sigma & \kappa & \eta \\ -\beta & \alpha & -\lambda & \gamma & -\sigma & \xi & -\eta & \kappa \\ -\gamma & \lambda & \alpha & -\beta & -\kappa & \eta & \xi & -\sigma \\ -\lambda & -\gamma & \beta & \alpha & -\eta & -\kappa & \sigma & \xi \\ \hline -\xi & \sigma & \kappa & \eta & \alpha & -\beta & -\gamma & -\lambda \\ -\sigma & -\xi & -\eta & \kappa & \beta & \alpha & \lambda & -\gamma \\ -\kappa & \eta & -\xi & -\sigma & \gamma & -\lambda & \alpha & \beta \\ -\eta & -\kappa & \sigma & -\xi & \lambda & \gamma & -\beta & \alpha \end{array}\right) \tag{2.67}$$

Matrix (2.67) can be divided into 4 blocks. The top left block is the 4×4 matrix representation for (a, b); the top right block is the 4×4 matrix representation for (c, d); the bottom left block is 4×4 matrix representation for $-(c, d)^*$ and the bottom right block is 4×4 matrix representation for $(a, b)^*$. This point of view will be important soon, when multiplication between octonion 8×8 matrices basis will be presented.

Therefore, let us also rewrite elements (2.60) as the follows:

$$\begin{gathered} \mathbf{E}_0 = (1,0,0,0,0,0,0,0), \quad \mathbf{E}_1 = (0,1,0,0,0,0,0,0), \quad \mathbf{E}_2 = (0,0,1,0,0,0,0,0), \\ \mathbf{E}_3 = (0,0,0,1,0,0,0,0), \quad \mathbf{E}_4 = (0,0,0,0,1,0,0,0), \quad \mathbf{E}_5 = (0,0,0,0,0,1,0,0), \\ \mathbf{E}_6 = (0,0,0,0,0,0,1,0), \quad \mathbf{E}_7 = (0,0,0,0,0,0,0,1). \end{gathered} \tag{2.68}$$

and according to (2.67) and (2.68), 8×8 matrices for octonion basis are:

$$
\begin{aligned}
&\mathbf{E}_0 \equiv \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad
\mathbf{E}_1 \equiv \begin{pmatrix} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0 \end{pmatrix},\\
&\mathbf{E}_2 \equiv \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad
\mathbf{E}_3 \equiv \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix},\\
&\mathbf{E}_4 = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{pmatrix}, \quad
\mathbf{E}_5 \equiv \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix},\\
&\mathbf{E}_6 \equiv \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{pmatrix}, \quad
\mathbf{E}_7 \equiv \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{pmatrix}
\end{aligned}
\tag{2.69}
$$

Multiplication between matrices (2.69) can be realized analogously to (2.62). But now, we have:

$$
\begin{aligned}
\left(\begin{array}{c|c} (a,b)_{4\times4} & (c,d)_{4\times4} \\ \hline \epsilon(c,d)^*_{4\times4} & (a,b)^*_{4\times4} \end{array}\right) \times \left(\begin{array}{c|c} (e,f)_{4\times4} & (g,h)_{4\times4} \\ \hline \epsilon(g,h)^*_{4\times4} & (e,f)^*_{4\times4} \end{array}\right) &= \left(\begin{array}{c|c} (a,b)_{4\times4} & (c,d)_{4\times4} \\ \hline \epsilon(c,d)^*_{4\times4} & (a,b)^*_{4\times4} \end{array}\right)\left(\begin{array}{c|c} (e,f)_{4\times4} & (g,h)_{4\times4} \\ \hline \epsilon(g,h)^*_{4\times4} & (e,f)^*_{4\times4} \end{array}\right) \\
&+ \left(\begin{array}{c|c} \epsilon[(g,h)^*_{4\times4},(c,d)_{4\times4}] & [(g,h)_{4\times4},(a,b)_{4\times4}] \\ \hline \epsilon[(e,f)_{4\times4},(c,d)^*_{4\times4}] & [(e,f)^*_{4\times4},(a,b)^*_{4\times4}] \end{array}\right)
\end{aligned} \tag{2.70}
$$

For split-octonions the same procedure can be done considering $\epsilon = 1$ and $\bar{\epsilon} = -1$ in formula (2.54). Then, matrices for split-octonions basis are:

$$
\mathbf{E}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \widetilde{\mathbf{E}}_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \qquad \widetilde{\mathbf{E}}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\widetilde{\mathbf{E}}_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \widetilde{\mathbf{E}}_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \widetilde{\mathbf{E}}_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}
$$

$$
\widetilde{\mathbf{E}}_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \widetilde{\mathbf{E}}_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{2.71}
$$

and for 8×8 representation, we have:

$$
\begin{aligned}
&\mathbf{E}_0 \equiv \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad
\widetilde{\mathbf{E}}_1 \equiv \begin{pmatrix} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0 \end{pmatrix},\\
&\widetilde{\mathbf{E}}_2 \equiv \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad
\widetilde{\mathbf{E}}_3 \equiv \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix},\\
&\widetilde{\mathbf{E}}_4 = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}, \quad
\widetilde{\mathbf{E}}_5 \equiv \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix},\\
&\widetilde{\mathbf{E}}_6 \equiv \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \quad
\widetilde{\mathbf{E}}_7 \equiv \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}
\end{aligned} \tag{2.72}
$$

For example:

$$
\begin{aligned}
\mathbf{E}_6\mathbf{E}_3 &= \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}+\\
&+\left(\begin{array}{c|c} \mathbf{O} & \mathbf{O} \\ \hline \left[\begin{pmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&1&0&0\\ -1&0&0&0 \end{pmatrix},\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ -1&0&0&0\\ 0&-1&0&0 \end{pmatrix}\right] & \mathbf{O} \end{array}\right)\\
&= \begin{pmatrix} 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix}+\left(\begin{array}{c|c} \mathbf{O} & \mathbf{O} \\ \hline \begin{pmatrix} 0&-2&0&0\\ 2&0&0&0\\ 0&0&0&2\\ 0&0&-2&0 \end{pmatrix} & \mathbf{O} \end{array}\right)\\
&= -\begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} = -\mathbf{E}_5
\end{aligned}
\tag{2.73}
$$

Matrix representation for octonions can be useful in Malcev's algebras and superalgebras [24],[25] because these matrices have graded structures. An other utility maybe to construct a non-associative gauge theory [26] by using octonions.

It is also possible to introduce this representation to rewrite matrices of Clifford Algebras $Cl_O(10,1)$, used in Octonionic M-Theory [13]:

$$\Gamma_i \equiv \begin{pmatrix} 0 & 0 & 0 & \mathbf{E}_i \\ 0 & 0 & -\mathbf{E}_i & 0 \\ 0 & \mathbf{E}_i & 0 & 0 \\ -\mathbf{E}_i & 0 & 0 & 0 \end{pmatrix}, \quad i = 1,...,7$$

$$\Gamma_8 \equiv \begin{pmatrix} 0 & 0 & 0 & \mathbf{E}_0 \\ 0 & 0 & \mathbf{E}_0 & 0 \\ 0 & \mathbf{E}_0 & 0 & 0 \\ \mathbf{E}_0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_9 \equiv \begin{pmatrix} 0 & 0 & \mathbf{E}_0 & 0 \\ 0 & 0 & 0 & -\mathbf{E}_0 \\ \mathbf{E}_0 & 0 & 0 & 0 \\ 0 & -\mathbf{E}_0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_{10} \equiv \begin{pmatrix} 0 & 0 & \mathbf{E}_0 & 0 \\ 0 & 0 & 0 & \mathbf{E}_0 \\ -\mathbf{E}_0 & 0 & 0 & 0 \\ 0 & -\mathbf{E}_0 & 0 & 0 \end{pmatrix}, \quad \Gamma_{11} \equiv \begin{pmatrix} \mathbf{E}_0 & 0 & 0 & 0 \\ 0 & \mathbf{E}_0 & 0 & 0 \\ 0 & 0 & -\mathbf{E}_0 & 0 \\ 0 & 0 & 0 & -\mathbf{E}_0 \end{pmatrix}. \tag{2.74}$$

Matrices (2.74) can be rewrite in 32×32 with real inputs and algebra $Cl_O(10,1)$ will be kept by using Cayley-Dickson matrix multiplication.

Lately, it will be shown how octonionic matrix representation can mimic Dirac's matrix algebra.

2.7 Matrix representation for Cayley-Dickson doubling construction

Now, we can write the main operations of Cayley-Dickson Doubling Construction suitable for matrix representation. As mentioned in section 1.1, the are five main operations that characterizes that construction: multiplication, conjugation, "norm", multiplication by a real number and conjugation of multiplication. As the last two operations are just a consequence of the first three, for simplicity, we will focus only on these three.

In order to understand formalism of matrix representation we can see (2.41), (2.45), (2.53), (2.54) and (2.62). From this formulas, one can make following conclusion: Let $x, y \in \mathbb{A}$, a Cayley-Dickson algebra, represented by:

$$x \equiv \mathbf{X}_{d/2}, \quad y \equiv \mathbf{Y}_{d/2} \tag{2.75}$$

where $\mathbf{X}_{d/2}$ and $\mathbf{Y}_{d/2}$ are $\frac{d}{2} \times \frac{d}{2}$ matrices, with $d = 2,4,8,16,...,2^n,...$. Thus, doubling $(x, y) \in \mathbb{A}^2$ is represented by the $d \times d$ matrix as follows:

$$(x, y) \equiv \left(\begin{array}{c|c} \mathbf{X}_{d/2} & \mathbf{Y}_{d/2} \\ \hline \epsilon \mathbf{Y}^*_{d/2} & \mathbf{X}^*_{d/2} \end{array}\right)_d \tag{2.76}$$

and it must obey following operations:

- **i) Multiplication:**

$$(x,y)(z,w) \equiv \left(\begin{array}{c|c} \mathbf{X}_{d/2} & \mathbf{Y}_{d/2} \\ \hline \epsilon\mathbf{Y}^*_{d/2} & \mathbf{X}^*_{d/2} \end{array}\right)\left(\begin{array}{c|c} \mathbf{Z}_{d/2} & \mathbf{W}_{d/2} \\ \hline \epsilon\mathbf{W}^*_{d/2} & \mathbf{Z}^*_{d/2} \end{array}\right)_d + \left(\begin{array}{c|c} \epsilon[\mathbf{W}^*,\mathbf{Y}]_{d/2} & [\mathbf{W},\mathbf{X}]_{d/2} \\ \hline \epsilon[\mathbf{Z},\mathbf{Y}^*]_{d/2} & [\mathbf{Z}^*,\mathbf{X}^*]_{d/2} \end{array}\right)_d \tag{2.77}$$

- **ii) Conjugation**:

$$(x,y)^* \equiv \left(\begin{array}{c|c} \mathbf{X}^*_{d/2} & -\mathbf{Y}_{d/2} \\ \hline -\epsilon\mathbf{Y}^*_{d/2} & \mathbf{X}_{d/2} \end{array}\right)_d \tag{2.78}$$

- **iii) Norm:**

$$\mathcal{N}(x,y) \equiv \frac{1}{d}\mathrm{Tr}\left[\left(\begin{array}{c|c} \mathbf{X}^*_{d/2} & -\mathbf{Y}_{d/2} \\ \hline -\epsilon\mathbf{Y}^*_{d/2} & \mathbf{X}_{d/2} \end{array}\right)_d \left(\begin{array}{c|c} \mathbf{X}_{d/2} & \mathbf{Y}_{d/2} \\ \hline \epsilon\mathbf{Y}^*_{d/2} & \mathbf{X}^*_{d/2} \end{array}\right)_d + \left(\begin{array}{c|c} \mathbf{O} & [\mathbf{Y},\mathbf{X}^*]_{d/2} \\ \hline -\epsilon[\mathbf{X},\mathbf{Y}^*]_{d/2} & \mathbf{O} \end{array}\right)_d\right] \tag{2.79}$$

For norm, just needed to remember that:

$$\mathcal{N}(x,y) = (x,y)^*(x,y) \tag{2.80}$$

Chapter 3

Quaternions, octonions and $SU(2)$ group

3.1 Quaternions and $SU(2)$ Pauli matrices

A simple connection between quaternions and physics lies in its relation with matrices of $SU(2)$. The $SU(2)$ group is the group of complex unitary matrices and of determinant 1. Its generators can be the Pauli matrices:

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

where

$$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = -2i\varepsilon_{jkm}\boldsymbol{\sigma}_m \tag{3.2}$$

According to (2.33) ,

$$\mathbf{e}_j = i\boldsymbol{\sigma}_j \tag{3.3}$$

and one can have:

$$[\mathbf{e}_j, \mathbf{e}_k] = 2\varepsilon_{jkm}\mathbf{e}_m. \tag{3.4}$$

It's easy to see that (3.2) and (3.4) are equivalents algebras. Therefore, it's possible to use quaternions to represent $SU(2)$ and the physics grounded in it.

Quaternions can be represented by 4×4 matrices and this can spontaneously induce a four-dimensional representation of SU(2). According to (2.57) and (3.3), Pauli matrices in 4×4 representation are:

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Matrices in (3.5) are not a minimal representation of Pauli matrices since all inputs are complex numbers. There is a minimal representation with 4×4 real matrices [27] and it can be obtained by writing matrices of 1 and i, defined in (2.2) and (2.8), inside matrices (3.1).

It is known that isospin operators I_+,I_- and I act on quark spins up $|u\rangle$ and down $|d\rangle$ as follows [28]:

$$\mathbf{I}_+|u\rangle = 0, \quad \mathbf{I}_-|u\rangle = \frac{1}{2}|d\rangle, \quad \mathbf{I}|u\rangle = \frac{1}{2}|u\rangle,$$
$$\mathbf{I}_+|d\rangle = \frac{1}{2}|u\rangle, \quad \mathbf{I}_-|d\rangle = 0, \quad \mathbf{I}|d\rangle = -\frac{1}{2}|d\rangle. \tag{3.6}$$

where:

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.7}$$

By using (3.1), (3.3) and (3.7), one can see that following definitions of $\mathbf{I}_+$, $\mathbf{I}_-$ and $\mathbf{I}$ satisfy the equations in (3.6):

$$\mathbf{I}_+ = \frac{\mathbf{e}_2 - i\mathbf{e}_3}{4}, \quad \mathbf{I}_- = -\frac{\mathbf{e}_2 + i\mathbf{e}_3}{4}, \quad \mathbf{I} = -i\frac{\mathbf{e}_1}{2} \tag{3.8}$$

But if we use the 4×4 representation of the quaternion matrices, we can build a four-dimensional representation of spins up $|u\rangle$ and down $|d\rangle$, as follows:

$$|u\rangle = \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ i \\ 1 \end{pmatrix}, \quad |d\rangle = \frac{1}{2}\begin{pmatrix} i \\ 1 \\ i \\ -1 \end{pmatrix}. \tag{3.9}$$

3.2 Quaternionic Dirac Lagrangian

Kugo and Townsend shows how to represent spinors in terms of elements of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, to relation supersymmetry and algebras [29]. But here, we will use their idea of representation spinors to construct Dirac's lagragian from quaternionic matrices.

Lagrangian density for two spins $\frac{1}{2}$ a and b, with no interaction, is:

$$\mathscr{L} = i[\bar{\psi}_a\gamma^\mu\partial_\mu\psi_a + \bar{\psi}_b\gamma^\mu\partial_\mu\psi_b] - m[\bar{\psi}_a\psi_a + \bar{\psi}_b\psi_b] \tag{3.10}$$

where γ^μ are the Dirac matrices, $\bar{\psi}_{a,b}$ are the adjoint representation of $\psi_{a,b}$ and m is the mass of particle. Adjoint representation of spins are defined as follows:

$$\bar{\psi}_{a,b} = \psi^\dagger_{a,b}\gamma^0 \tag{3.11}$$

where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.12}$$

and spinor ψ are often written as column matrix, such that:

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \Longrightarrow \quad \bar{\psi} = \begin{pmatrix} \bar{\psi}_a & \bar{\psi}_b \end{pmatrix}. \tag{3.13}$$

To $SU(2)$ we can construct a quaternion spinor [29], such that

$$\psi = \psi_0 \mathbf{e}_0 + \sum_j \psi_j \mathbf{e}_j, \quad j = 1,2,3 \tag{3.14}$$

and

$$\psi^\dagger = \psi_0 \mathbf{e}_0 - \sum_j \psi_j \mathbf{e}_j. \tag{3.15}$$

Components ψ_j are spinors in Real ($\mathbb{R}$), so that

$$\begin{aligned} \psi &= \psi_0(1,0) + \psi_1(i,0) + \psi_2(0,1) + \psi_3(0,i) \\ &= (\psi_0 + i\psi_1, \psi_2 + i\psi_3) \end{aligned} \tag{3.16}$$

According to matrix representation of division algebras, it is possible to establish a representation for ψ different from (3.13) and based on (3.15). So, according to Cayley-Dickson representation of quaternion as ordered pairs we can define consistently

$$\psi_a = \psi_0 + i\psi_1, \psi_b = \psi_2 + i\psi_3. \tag{3.17}$$

Therefore, the result (3.16) can be written as the following matrix:

$$\psi = \begin{pmatrix} \psi_0 + i\psi_1 & \psi_2 + i\psi_3 \\ -\psi_2 + i\psi_3 & \psi_0 - i\psi_1 \end{pmatrix} = \begin{pmatrix} \psi_a & \psi_b \\ -\psi_b^* & \psi_a^* \end{pmatrix}. \tag{3.18}$$

For $\bar{\psi}$, we have

$$\bar{\psi} \quad = \quad \psi^\dagger \gamma^0 = (\psi_0 \mathbf{e}_0 - \psi_1 \mathbf{e}_1 - \psi_2 \mathbf{e}_2 - \psi_3 \mathbf{e}_3)\gamma_0. \tag{3.19}$$

For $\bar{\psi}$, we have

$$\bar{\psi} \quad = \quad \psi^\dagger \gamma^0 = (\psi_0 \mathbf{e}_0 - \psi_1 \mathbf{e}_1 - \psi_2 \mathbf{e}_2 - \psi_3 \mathbf{e}_3)\gamma_0. \tag{3.20}$$

By matrix representation of $\mathbf{e}_j$, it is easy to see that $[\mathbf{e}_j, \gamma^0] = 0$. Therefore,

$$\begin{aligned} \bar{\psi} &= \bar{\psi}_0 \mathbf{e}_0 - \bar{\psi}_1 \mathbf{e}_1 - \bar{\psi}_2 \mathbf{e}_2 - \bar{\psi}_3 \\ &= (\bar{\psi}_0 - i\bar{\psi}_1, -\bar{\psi}_2 - \bar{\psi}_3). \end{aligned} \tag{3.21}$$

Using (3.11) and (3.17), it's easy to see that:

$$\bar{\psi}_a = \bar{\psi}_0 - i\bar{\psi}_1, \quad \bar{\psi}_b^* = \bar{\psi}_2 + i\bar{\psi}_3 \tag{3.22}$$

then,

$$\bar{\psi} = (\bar{\psi}_a, -\bar{\psi}_b^*) = \begin{pmatrix} \bar{\psi}_a & -\bar{\psi}_b^* \\ \bar{\psi}_b & \bar{\psi}_a^* \end{pmatrix}. \tag{3.23}$$

By matrix representation of product $\bar{\psi}\psi$, we get:

$$\bar{\psi}\psi = \begin{pmatrix} \bar{\psi}_a\psi_a + \bar{\psi}_b^*\psi_b^* & \bar{\psi}_a\psi_b - \bar{\psi}_b^*\psi_a^* \\ \bar{\psi}_b\psi_a - \bar{\psi}_a^*\psi_b^* & \bar{\psi}_b\psi_b + \bar{\psi}_a^*\psi_a^* \end{pmatrix}. \tag{3.24}$$

It's easy to prove that $\bar{\psi}_a^*\psi_a^* = \bar{\psi}_a\psi_a$ and $\bar{\psi}_b^*\psi_b^* = \bar{\psi}_b\psi_b$; so that,

$$\frac{1}{2}\mathrm{Tr}(\bar{\psi}\psi) = \bar{\psi}_a\psi_a + \bar{\psi}_b\psi_b. \tag{3.25}$$

By the same way, one can prove that

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}(\bar{\psi}\gamma^\mu\partial_\mu\psi) &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} \bar{\psi}_a & -\bar{\psi}_b^* \\ \bar{\psi}_b & \bar{\psi}_a^* \end{pmatrix}\gamma^\mu\begin{pmatrix} \partial_\mu\psi_a & \partial_\mu\psi_b \\ -\partial_\mu\psi_b^* & \partial_\mu\psi_a^* \end{pmatrix}\right] \\ &= \bar{\psi}_a\gamma^\mu\partial_\mu\psi_a + \bar{\psi}_b\gamma^\mu\partial_\mu\psi_b. \end{aligned} \tag{3.26}$$

Then, the quaternionic formulation for Lagrangian (3.10) can be written as

$$\mathscr{L}_{\mathbb{H}} = \frac{1}{2}\mathrm{Tr}[\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi]. \tag{3.27}$$

3.3 Quaternionic $SU(2)$ symmetries

For a global symmetry, let's suppose that a quaternion spinor transforms by the following way

$$\psi' = \mathbf{M}\psi. \tag{3.28}$$

If $\psi \in SU(2)$, $\mathbf{M}$ necessarily is a unitary tranformation, independent of space-time, if (3.28) is a symmetric transformation that preserves invariance of the Dirac Lagrangian. So that, M transformation can be defined as:

$$\mathbf{M} = \exp\{i\hat{\theta}\} \tag{3.29}$$

where $\hat{\theta} \in SU(2)$ matrix, such that

$$\hat{\boldsymbol{\theta}} = \theta_0\mathbf{1} + \theta_j\boldsymbol{\sigma}_j \tag{3.30}$$

According to (3.3), we can see that $\hat{\theta}$ can be described as a quaternion

$$\hat{\boldsymbol{\theta}} \longmapsto \hat{\boldsymbol{\theta}} = \theta_0\mathbf{e}_0 + i\theta_j\mathbf{e}_j. \tag{3.31}$$

By (3.31) is easy to see that $\hat{\theta}^\dagger = \hat{\theta}$. Replacing (3.31) in (3.29) we get

$$\mathbf{M} = \exp\{i\theta_0\mathbf{e}_0\}\exp\{-i\theta_j\mathbf{e}_j\}. \tag{3.32}$$

The term $\exp\{i\theta_0\mathbf{e}_0\}$ indicates a gauge transformation $U(1)$, so, we can just make

$$\psi' = \exp\{-\theta_j\mathbf{e}_j\}\psi. \tag{3.33}$$

So, (3.33) indicates a non-Abelian $SU(2)$ transformation. To adjoint spinor $\bar{\psi}$, we get

$$\begin{aligned}\bar{\psi}' &= \psi'^{\dagger}\gamma^0 \\ &= \psi^{\dagger}\exp\{\theta_j\mathbf{e}_j\}\gamma^0.\end{aligned} \tag{3.34}$$

By expanding $\exp\{\theta_j\mathbf{e}_j\}$ in MacLaurin series and knowing following results

$$\mathbf{e}_j^{2n} = (-1)^n, \quad [\mathbf{e}_i,\gamma^0] = 0 \tag{3.35}$$

we get

$$\bar{\psi}' = \bar{\psi}\exp\{\theta\mathbf{e}_j\}. \tag{3.36}$$

Applying this transformation to Lagrangian (3.10), we get

$$\bar{\psi}'\psi' = \bar{\psi}\psi \tag{3.37}$$

and, to preserve lagrangian invariance,

$$\exp\{\theta_j\mathbf{e}_j\}\gamma^{\mu}\exp\{-\theta_j\mathbf{e}_j\} = \gamma^{\mu}. \tag{3.38}$$

Other consequence of (3.38) is the non-invariance of quadri-current j^{μ} defined as

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi. \tag{3.39}$$

One can easily prove that

$$j\text{'}'^{\mu} = \bar{\psi}'\gamma^{\mu}\psi' = j^{\mu}. \tag{3.40}$$

For a gauge transformation [30]:

$$\delta\mathscr{L} = 0. \tag{3.41}$$

As we know, $SU(2)$ Lagrangian is a function $\mathscr{L}(\psi,\bar{\psi},\partial_{\mu}\psi)$, then

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\psi}\delta\psi + \frac{\partial\mathscr{L}}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\psi)}\delta(\partial_{\mu}\psi) = 0 \tag{3.42}$$

If we use Euler-Lagrange equation, we get

$$\frac{\partial\mathscr{L}}{\partial\psi} = \partial_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\psi)}\right), \quad \frac{\partial\mathscr{L}}{\partial\bar{\psi}} = 0. \tag{3.43}$$

Therefore,

$$\begin{aligned}\delta\mathscr{L} &= \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\right)\delta\psi+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\partial(\delta\psi)\\ &= \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\delta\psi\right)=0.\end{aligned} \tag{3.44}$$

In quanternionic representation,

$$\delta\mathscr{L}_{\mathbb{H}}=\frac{1}{2}\mathrm{Tr}[\delta\mathscr{L}] \tag{3.45}$$

where

$$\mathscr{L}=\bar{\psi}\gamma^\mu\partial_\mu\psi-m\bar{\psi}\psi. \tag{3.46}$$

Therefore,

$$\delta\mathscr{L}_{\mathbb{H}}=\partial_\mu\left[\frac{1}{2}\mathrm{Tr}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\delta\psi\right)\right]=0. \tag{3.47}$$

Considering infinitesimal transformation

$$\begin{aligned}\psi' &= \psi+\delta\psi\\ \exp\{-\theta_j\mathbf{e}_j\}\psi &= \psi+\delta\psi\\ &\therefore\\ \delta\psi &\approx -\theta_j\mathbf{e}_j\psi\end{aligned} \tag{3.48}$$

then,

$$\partial_\mu\left(\frac{1}{2}\mathrm{Tr}[\bar{\psi}\gamma^\mu\mathbf{e}_j\psi]\right)=0. \tag{3.49}$$

According to (3.49) we can define the quaternionic gauge current, that is conserved by a global gauge transformation

$$j_j^\mu=\frac{1}{2}\mathrm{Tr}[\bar{\psi}\gamma^\mu\mathbf{e}_j\psi]. \tag{3.50}$$

For a local symmetry $SU(2)$, we must consider the generalization of phase rotation, where the parameter is spacetime dependent. First, we start with a Dirac doublet as

$$\Psi=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \tag{3.51}$$

which transforms as

$$\Psi=V(x)\Psi=e^{i\alpha^i(x)\mathbf{e}_i}\Psi \tag{3.52}$$

where $\alpha^i(x)$ are the parameters on the field transformations that depends of spacetime coordinates. In order to define the covariant derivative, we have to define a comparator [31] which hold the folowing relation

$$U(y,x) \to V(y)U(y,x)V^{\dagger}(x) \tag{3.53}$$

where we set $U(y,y) = 1$. Now, we are able to define a infitesimal expression for the comparator above, leaning on the fact that near $U = 1$, we can expand in terms of the Hermitian quaternionic generators.

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^{\mu} A^{i}_{\mu}\mathbf{e}_i + \mathcal{O}(\epsilon^2) \tag{3.54}$$

where ϵ is the infinetesimal increment, g is a convenient constant and n^{μ} is a vector used to define a covariant derivative. The definition is

$$n^{\mu}D_{\mu}\Psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\Psi(x+\epsilon n) - U(x+\epsilon n)\Psi\right]. \tag{3.55}$$

Using the expression for the comparator given in equation (3.54), we have

$$D_{\mu} = \partial_{\mu} - igA^{i}_{\mu}\mathbf{e}_i. \tag{3.56}$$

We notice that this covariant derivative requires one vector field to each generator of the group, that is, the quaternionic generators.

Now, we focus on the gauge transformation law of the connection A^{i}_{μ} by substituting the expansion of the comparator (3.54) into the transformation law given in (3.53)

$$1 + ig\epsilon n^{\mu}A^{i}_{\mu}\mathbf{e}_i \to V(x+\epsilon n)(1 + ig\epsilon n^{\mu}A^{i}_{\mu}\mathbf{e}_i)V^{\dagger}(x). \tag{3.57}$$

Using the following identity

$$V(x+\epsilon n)V^{\dagger}(x) = 1 + \epsilon n^{\mu}V(x)\left(-\frac{\partial}{\partial x^{\mu}}V^{\dagger}(x)\right) + \mathcal{O}(\epsilon^2) \tag{3.58}$$

we find by comparison with (3.57),

$$A^{i}_{\mu}\mathbf{e}_i \to V(x)\left(A^{i}_{\mu}(x)\mathbf{e}_i + \frac{i}{g}\partial_{\mu}\right)V^{\dagger}(x). \tag{3.59}$$

For infinitesimal transformations, we can expand $V(x)$ to first order in α,

$$A^{i}_{\mu}\mathbf{e}_i \to A^{i}_{\mu}\mathbf{e}_i + \frac{1}{g}(\partial_{\mu}\alpha^{i})\mathbf{e}_i + i\left[\alpha^{i}\mathbf{e}_i, \alpha^{j}\mathbf{e}_j\right] + \ldots \tag{3.60}$$

where the last term arises from the noncommutativity of the local transformations. Considering the transformation of the fermion and the last transformation, we have

$$D_\mu \Psi \to (1 + i\alpha^i \mathbf{e}_i) D_\mu \Psi. \tag{3.61}$$

To write a complete lagrangian, we must consider gauge-invariant terms that depends only on A^i_μ. We write the transformation of the commutator considering the transformation of the covariant derivatives as

$$[D_\mu, D_\nu]\Psi(x) \to V(x)[D_\mu, D_\nu]\Psi \tag{3.62}$$

and we find

$$[D_\mu, D_\nu] = -igF^i_{\mu\nu}\mathbf{e}_i \tag{3.63}$$

where

$$F^i_{\mu\nu}\mathbf{e}_i = \partial_\mu A^i_\nu \mathbf{e}_i - \partial_\nu A^i_\mu \mathbf{e}_i - ig[A^i_\mu \mathbf{e}_i, A^j_\nu \mathbf{e}_j]. \tag{3.64}$$

We can use the quaternionic commutations relations given by

$$[\mathbf{e}_i, \mathbf{e}_j] = 2\varepsilon^{ijk}\mathbf{e}_k \tag{3.65}$$

and then

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - 2ig\varepsilon^{jki} A^j_\mu A^k_\nu. \tag{3.66}$$

The transformation of the $F^i_{\mu\nu}$ follows from above definitions and transformations laws and we write as

$$F^i_{\mu\nu}\mathbf{e}_i \to V(x) F^j_{\mu\nu}\mathbf{e}_j V^\dagger(x) \tag{3.67}$$

where the infinitesimal transformation is

$$F^i_{\mu\nu}\mathbf{e}_i \to F^i_{\mu\nu}\mathbf{e}_i + [i\alpha^i \mathbf{e}_i, F^j_{\mu\nu}\mathbf{e}_j]. \tag{3.68}$$

We can see explicitly that the tensor above is no longer gauge-invariant. However, we can construct gauge-invariant theories using combinations of this tensor field. Now, we construct a theory of interacting fermions.

$$\mathscr{L}_{\mathbb{H}} = \frac{1}{2}\mathrm{Tr}\left[\bar{\Psi}(i\not{D})\Psi - \frac{1}{4}(F^i_{\mu\nu})^2 - m\bar{\Psi}\Psi\right] \tag{3.69}$$

where this is the quaternionic Yang-Mills lagrangian density. Using the variation of this lagrangian we reach to

$$\partial^\mu F^i_{\mu\nu} + 2g\varepsilon^{ijk} A^{j\mu} F^k_{\mu\nu} = -g\frac{1}{2}\mathrm{Tr}\left[\Psi\gamma_\nu \mathbf{e}_i \Psi\right] \tag{3.70}$$

where this result is the classical equations of motion of the gauge theory for the vector field.

3.4 Octonions and Dirac matrices' algebra

Before we start the investigation of the relation between octonions and $SU(2)$, let's keep in mind these two important relations that lies in $SU(2)$

$$[\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j] = -2\varepsilon_{ijk}\boldsymbol{i}\boldsymbol{\sigma}_k \tag{3.71}$$

and for quantum electrodynamic's algebra

$$\begin{aligned} [\gamma_i, \gamma_j] &= 2\varepsilon_{ijk} i \begin{pmatrix} \boldsymbol{\sigma}_k & 0 \\ 0 & \boldsymbol{\sigma}_k \end{pmatrix} = 2\varepsilon_{ijk}\boldsymbol{i}\boldsymbol{\sigma}_k \otimes \mathbf{1}, \\ \{\gamma_i, \gamma_j\} &= \delta_{ij}. \end{aligned} \tag{3.72}$$

The Dirac γ_i matrices are defined as

$$\gamma_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix}. \tag{3.73}$$

Octonion basis can be separated in two parts: $\mathbf{E}_i$ is the associative part if $i = 1,2,3$ and the non-associative part if $i = 4,5,6,7$. Resorting to (3.3), to the associative part, we can do

$$\mathbf{E}_i = \begin{pmatrix} \boldsymbol{i}\boldsymbol{\sigma}_i & 0 \\ 0 & -\boldsymbol{i}\boldsymbol{\sigma}_i \end{pmatrix} = \boldsymbol{i}\boldsymbol{\Sigma}_i, \quad i = 1,2,3. \tag{3.74}$$

To the non-associative sector, we have

$$\mathbf{E}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{E}_{i+4} = \begin{pmatrix} 0 & \boldsymbol{i}\boldsymbol{\sigma}_i \\ \boldsymbol{i}\boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad i = 1,2,3. \tag{3.75}$$

If we define $\boldsymbol{\Gamma}_i$ as the following matrix:

$$\boldsymbol{\Gamma}_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix} \tag{3.76}$$

then,

$$\mathbf{E}_{i+4} = \boldsymbol{i}\boldsymbol{\Gamma}_i, \quad i = 1,2,3. \tag{3.77}$$

If we use the Cayley-Dickson multiplication for $[\mathbf{\Sigma}_i, \mathbf{\Sigma}_j]$ we have

$$[\mathbf{\Sigma}_i, \mathbf{\Sigma}_j] = \mathbf{\Sigma}_i \times \mathbf{\Sigma}_j - \mathbf{\Sigma}_j \times \mathbf{\Sigma}_i = -2\varepsilon_{ijk}\boldsymbol{i}\mathbf{\Sigma}_k. \tag{3.78}$$

Result (3.78) is pretty consistent with the commutator of the associative elements of the basis $[\mathbf{E}_i, \mathbf{E}_j]$

$$[\mathbf{E}_i, \mathbf{E}_j] = \mathbf{E}_i \times \mathbf{E}_j - \mathbf{E}_j \times \mathbf{E}_i = 2C_{ijk}E_k \tag{3.79}$$

In table 1.1, we can see that $C_{ijk} = \varepsilon_{ijk}$ if $i, j, k = 1, 2, 3$. By comparing (3.71) and (3.78), we can suppose that the associative sector of octonions basis behaves as a "mimicry" of $SU(2)$ algebras to Pauli matrices σ_i.

As we know,

$$[\mathbf{E}_{i+4}, \mathbf{E}_{j+4}] = 2C_{i+4,j+4,k}\mathbf{E}_k, \quad i, j, k = 1, 2, 3. \tag{3.80}$$

Carefully analyzing Table 1.1, one can see that:

$$C_{i+4,j+4,k} = -\varepsilon_{ijk}. \tag{3.81}$$

Therefore,

$$[\mathbf{\Gamma}_i, \mathbf{\Gamma}_j] = 2\varepsilon_{ijk}\boldsymbol{i}\mathbf{\Sigma}_k \tag{3.82}$$

and also,

$$\{\mathbf{\Gamma}_i, \mathbf{\Gamma}_j\} = 2\delta_{ij}. \tag{3.83}$$

In brief:

$SU(2)$ and QED	Octonions
$\boldsymbol{\sigma}_j$	$\mathbf{\Sigma}_j$
$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = -2\varepsilon_{jkm}\boldsymbol{i}\boldsymbol{\sigma}_m$	$[\mathbf{\Sigma}_j, \mathbf{\Sigma}_k] = -2\varepsilon_{jkm}\boldsymbol{i}\mathbf{\Sigma}_m$
γ_j	$\mathbf{\Gamma}_j$
$[\gamma_j, \gamma_k] = 2\varepsilon_{jkm}\boldsymbol{i}\boldsymbol{\sigma}_m \otimes \mathbf{1}$	$[\mathbf{\Gamma}_j, \mathbf{\Gamma}_k] = 2\varepsilon_{jkm}\boldsymbol{i}\mathbf{\Sigma}_m$
$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$	$\{\mathbf{\Gamma}_i, \mathbf{\Gamma}_j\} = 2\delta_{ij}$

Table 3.1: Comparison between the algebras of elements of $SU(2)$ and Quantum Electrodynamics (on the left) and the algebra of elements of the octonionic basis (on the right).

Some works are intended to use octonions related to supersymmetry [32], [33] and Quantum Chromodynamics [34], [14], but our intention here is to use octonions Σ_i and Γ_i to construct algebraic basis for a non-associative quantum electrodynamics theory for future works. We can consider construction dyonic QED a possible application, since octonions already been used to construct electromagnetism of dyons [8]. Therefore, by considering a octonionic spinor ψ, given by:

$$\psi = \psi_0 \mathbf{E}_0 + \sum_{j=1}^{7} \psi_j \mathbf{E}_j \tag{3.84}$$

By redoing same procedure of section **3.2** it is expected that octonionic version of Lagrangian (3.10) is:

$$\mathscr{L}_{\mathbb{O}} = \frac{1}{2}\mathrm{Tr}[(\bar{\psi}\boldsymbol{\Gamma}^{\mu})\partial_{\mu}\psi - m\bar{\psi}\psi] \tag{3.85}$$

where:

$$\boldsymbol{\Gamma}^0 = \begin{pmatrix} 0 & \mathbf{e}_0 \\ -\mathbf{e}_0 & 0 \end{pmatrix}, \quad \boldsymbol{\Gamma}^i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix} \tag{3.86}$$

Chapter 4

Graded Lie Algebras

A graded Lie algebra is a vector space that can be decomposed into subspaces indicated by gradings. A Lie algebra is defined by the Lie bracket $(\bullet,\bullet)$, so for a graded Lie algebra, this operation will be defined according to the gradings of the elements that participate in the operation [5], [35]. Specifically, for A_α and A_β in a graded Lie algebra, the Lie bracket is given by

$$(A_\alpha, A_\beta) = A_\alpha A_\beta - (-1)^{<\alpha,\beta>} A_\beta A_\alpha, \tag{4.1}$$

where α and β are the gradings of A_α and A_β, respectively. One can write $\alpha = \deg(A_\alpha)$ and $\beta = \deg(A_\beta)$. $<\alpha,\beta>$ Is the mapping onto the gradings of the algebra in question. For physical theories, the mapping $<\alpha,\beta>$ will indicates how particles in the systems will interact.

For a Lie algebra, it is required that $<\alpha,\beta>= 0$ for any A and B, so that the Lie bracket reduces to the commutation operation on A and B; that is, $(A_\alpha, A_\beta) = [A_\alpha, A_\beta]$

More precisely, a vector space $\mathscr{G}$ over $\mathbb{C}$ is a graded Lie algebra if it can be decomposed into homogeneous subspaces. That is:

$$\mathscr{G} = \bigoplus_{\alpha\geq 0} \mathscr{G}_\alpha \tag{4.2}$$

where $\mathscr{G}_\alpha$ are the subspaces that make up $\mathscr{G}$, and

$$[\mathscr{G}\alpha, \mathscr{G}_\beta] \subseteq \mathscr{G}_{\alpha+\beta} \tag{4.3}$$

A graded Lie superalgebra corresponds to nothing more than a generalization of operation (4.1), in the sense that it can play the role of either a commutator or an anticommutator depending on the values of the mapping $<\alpha,\beta>$. The general Jacobi identity is:

$$(-1)^{<\gamma,\alpha>}(A_\alpha,(A_\beta,A_\gamma)) + (-1)^{<\alpha,\beta>}(A_\beta,(A_\gamma,A_\alpha)) + (-1)^{<\beta,\gamma>}(A_\gamma,(A_\alpha,A_\beta)) = 0 \tag{4.4}$$

where $\alpha = \deg(A_\alpha)$, $\beta = \deg(A_\beta)$ and $\gamma = \deg(A_\gamma)$. The definitions of mappings cannot be arbitrary. There are constraints that must be followed, which are known as the Leibniz rules:

$$< \alpha, \beta > + < \beta, \alpha > = 2r \tag{4.5}$$

$$< \alpha, \beta + \gamma > = < \alpha, \beta > + < \alpha, \gamma > + 2s \tag{4.6}$$

$$< \alpha + \gamma, \beta > = < \alpha, \beta > + < \gamma, \beta > + 2s \tag{4.7}$$

Values of r and s are arbitrary integers.

It is important to mention that, for a standard graded Lie superalgebra, (A, B) cannot assume both roles, of $[A, B]$ or $\{A, B\}$, for the same pair of elements A and B. However, in the last section of this chapter, it is demonstrate how $(\bullet, \bullet)$ can be simultaneously $[\bullet, \bullet]$ and $\{\bullet, \bullet\}$ by introducing a perturbation on the mapping $< \alpha, \beta >$.

Graded Lie algebras play a very important role in the context of mathematical physics. They are important tools for describing symmetries in physical systems, as well as for generalizing symmetries that involve fermionic degrees of freedom [36]. In string theory, the superconformal algebra is a graded super Lie algebra that describes the conformal and supersymmetric symmetries in supersymmetric string theories [37]. Graded Lie algebras are also used to describe symmetries in quantum field theories. For example, the superalgebra of supersymmetry is a graded super Lie algebra that describes supersymmetric symmetry in supersymmetric field theories [38], [39].

In this work, we will not focus on a fundamental mathematical approach to these (super)algebras. The main purpose of this section is to demonstrate and explain the main aspects of the approaches developed by our study group in the context of the $\mathbb{Z}_2^n$ algebras, with a focus on their relations to Cayley-Dickson algebras and their application to theories involving fermions, parafermions, bosons, and parabosons. The last two sections of this chapter consist of presenting a different way to create matrix representations for $\mathbb{Z}_2^n$ and how we can obtain a Volichenko [15] algebra from a graded Lie algebra.

4.1 $\mathbb{Z}_2$ (Super)Algebras

A (super)algebra $\mathbb{Z}_2$ is a graded Lie algebra whose grading values are $\alpha = 0, 1$. Thus, a (super)algebra $\mathbb{Z}_2$ is a vector space $\mathscr{G}$ that can be divided as follows:

$$\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \tag{4.8}$$

The grading 0 is defined as even sector and 1 as odd sector [4], [5]. Commonly, for physical particles theories, grading 0 is the bosonic one and grading 1 is the fermionic [4], [7], [40], [41]. Rittenberg and Wyler [5] and [42], according to Leibniz rules, shows that there are two possible definitions for the mapping of a $\mathbb{Z}_2$ (super)algebra:

$$<\alpha,\beta> \ = \ 0 \tag{4.9}$$

$$<\alpha,\beta> \ = \ \alpha\beta \tag{4.10}$$

The definition (4.9) gives a $\mathbb{Z}_2$ algebra and (4.10) gives a $\mathbb{Z}_2$ superalgebra.

It is important to mention that a (super)algebra $\mathbb{Z}_2^n$ has a total of 2^n generators. Therefore, there are two generators for (super)algebras $\mathbb{Z}_2$, A_α and A_β, which commute or anticommute according to the following rules:

$$\mathbb{Z}_2 \text{ Algebra}: \quad (A_\alpha, A_\beta) = A_\alpha A_\beta - (-1)^0 A_\beta A_\alpha = [A_\alpha, A_\beta] \tag{4.11}$$

and

$$\mathbb{Z}_2 \text{ Superalgebra}: \quad (A_\alpha, A_\beta) = A_\alpha A_\beta - (-1)^{\alpha\beta} A_\beta A_\alpha \tag{4.12}$$

The gradings $\alpha = \deg(A_\alpha)$ and $\beta = \deg(A_\beta)$ can take on values of 0 or 1. Therefore, there is only one superalgebra $\mathbb{Z}_2$ and the commutation and anticommutation rule works as follows:

	A_0	A_1
A_0	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_1	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$

Table 4.1: Superalgebra $<\alpha,\beta>=\alpha\beta$.

4.2 $\mathbb{Z}_2^2$ (Super)Algebras

A graded Lie (super)algebra $\mathbb{Z}_2^2$ is a vector space $\mathcal{G}$ whose gradings α can be classified as following pairs $\alpha = 00, 01, 10, 11$. Vector space $\mathcal{G}$ is a divided into four subspaces:

$$\mathcal{G} = \mathcal{G}_{00} \oplus \mathcal{G}_{01} \oplus \mathcal{G}_{10} \oplus \mathcal{G}_{11} \tag{4.13}$$

The grading 00 is even and it is commonly agreed to be the bosonic sector of particle physical theories. The other sectors are classified as even or odd according to the choosing (super)algebra.

The generators for $\mathbb{Z}_2^3$ will be represented by: A_{00}, A_{01}, $\mathbb{A}_{10}$ and A_{11}. For (super)algebras $\mathbb{Z}_2^2$ vectors grandings α and β are two dimensional:

$$\alpha = (\alpha_1, \alpha_2), \quad \beta = (\beta_1, \beta_2) \tag{4.14}$$

where α_i and β_i are 0 or 1.

According to Leibniz rules, there are one possible algebra $\mathbb{Z}_2^2$ and three superalgebras $\mathbb{Z}_2^2$:

$$< \alpha, \beta > \; = \; 0 \tag{4.15}$$

$$< \alpha, \beta > \; = \; \alpha_1 \beta_1 \tag{4.16}$$

$$< \alpha, \beta > \; = \; \alpha_1 \beta_1 + \alpha_2 \beta_2 \tag{4.17}$$

$$< \alpha, \beta > \; = \; \alpha_1 \beta_2 - \alpha_2 \beta_1 \tag{4.18}$$

Below we have tables that indicate the commutation and anticommutation rules for $\mathbb{Z}_2^2$ algebras and superalgebras:

	A_{00}	A_{01}	A_{10}	A_{11}
A_{00}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{01}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{10}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{11}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$

Table 4.2: Algebra $< \alpha, \beta >= 0$

	A_{00}	A_{01}	A_{10}	A_{11}
A_{00}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{01}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{10}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{11}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$

Table 4.3: Superalgebra $< \alpha, \beta >= \alpha_1 \beta_1$

	A_{00}	A_{01}	A_{10}	A_{11}
A_{00}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{01}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$
A_{10}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{11}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$

Table 4.4: Superalgebra $< \alpha, \beta >= \alpha_1 \beta_1 + \alpha_2 \beta_2$

	A_{00}	A_{01}	A_{10}	A_{11}
A_{00}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{01}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{10}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$
A_{11}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$

Table 4.5: Superalgebra $< \alpha, \beta >= \alpha_1 \beta_2 - \alpha_2 \beta_1$

The product of operators belonging to distinct grading sectors, neither of which is in the bosonic grading, should result in an operator belonging to a grading different from the three previously mentioned gradings. For instead, if an operator from grading sector 10 is multiplied by one from 01, the result will be an operator from 11. According to definition presented in (4.1), in the fourth case, the grading sectors 10, 01, and 11 are equivalent, and their grading assignments can be rearranged without changing the anticommutators under the S_3 permutation group. In the second and third cases, the grading sectors 01, 11 and 10, 01 respectively, can be rearranged under the S_2 permutation group.

4.3 $\mathbb{Z}_2^3$ (Super)Algebras

A $\mathbb{Z}_2^3$ (super)algebra is a vector space $\mathscr{G}$ that can be divided in 8 subspaces as follows:

$$\mathscr{G} = \mathscr{G}_{000} \oplus \mathscr{G}_{001} \oplus \mathscr{G}_{010} \oplus \mathscr{G}_{011} \oplus \mathscr{G}_{100} \oplus \mathscr{G}_{101} \oplus \mathscr{G}_{110} \oplus \mathscr{G}_{111} \tag{4.19}$$

The generators for $\mathbb{Z}_2^3$ will be represented by: A_{000}, A_{001}, A_{010}, A_{011}, A_{100}, A_{101}, A_{110}, and A_{111}. The gradings vectors of a $\mathbb{Z}_2^3$ (super)algebra are three-dimensional, such that $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and α_i is 0 or 1.

Mapping $<\alpha,\beta>$ has five possible definitions and generate a algebra and four superalgebra. According to [42], possible five mappings of $\mathbb{Z}_2^3$ are:

$$<\alpha,\beta> \;=\; 0 \tag{4.20}$$
$$<\alpha,\beta> \;=\; \alpha_1\beta_1 \tag{4.21}$$
$$<\alpha,\beta> \;=\; \alpha_1\beta_1+\alpha_2\beta_2 \tag{4.22}$$
$$<\alpha,\beta> \;=\; \alpha_1\beta_2-\alpha_2\beta_1 \tag{4.23}$$
$$<\alpha,\beta> \;=\; \alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 \tag{4.24}$$

where (4.20) is a $\mathbb{Z}_2^3$ algebra and all the others are mapping of superalgebras. The mapping (4.21) makes it explicit that a $\mathbb{Z}_2^3$ superalgebra has a $\mathbb{Z}_2$ superalgebra as a subalgebra. The same applies to (4.22) and (4.23), which show how the $\mathbb{Z}_2^2$ superalgebras are also subalgebras of $\mathbb{Z}_2^3$.

The tables below contain the commutation and anticommutation rules for the generators of the $\mathbb{Z}_2^3$ (super)algebras.

	A_{000}	A_{001}	A_{010}	A_{011}	A_{100}	A_{101}	A_{110}	A_{111}
A_{000}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{001}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{010}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{011}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{100}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{101}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{110}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{111}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$

Table 4.6: Algebra $<\alpha,\beta>=0$

	A_{000}	A_{001}	A_{010}	A_{011}	A_{100}	A_{101}	A_{110}	A_{111}
A_{000}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{001}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{010}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{011}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{100}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{101}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{110}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{111}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$

Table 4.7: Superalgebra $<\alpha,\beta>=\alpha_1\beta_1$

	A_{000}	A_{001}	A_{010}	A_{011}	A_{100}	A_{101}	A_{110}	A_{111}
A_{000}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{001}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{010}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{011}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{100}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{101}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{110}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{111}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$

Table 4.8: Superalgebra $<\alpha,\beta>=\alpha_1\beta_1+\alpha_2\beta_2$

	A_{000}	A_{001}	A_{010}	A_{011}	A_{100}	A_{101}	A_{110}	A_{111}
A_{000}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{001}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{010}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{011}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{100}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{101}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{110}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{111}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$

Table 4.9: Superalgebra $<\alpha,\beta>=\alpha_1\beta_2-\alpha_2\beta_1$

	A_{000}	A_{001}	A_{010}	A_{011}	A_{100}	A_{101}	A_{110}	A_{111}
A_{000}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{001}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$
A_{010}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{011}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$
A_{100}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$
A_{101}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$
A_{110}	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$
A_{111}	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$	$[\bullet,\bullet]$	$[\bullet,\bullet]$	$\{\bullet,\bullet\}$

Table 4.10: Superalgebra $<\alpha,\beta>=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3$

As the dimension n of $\mathbb{Z}_2^n$ increases, it becomes increasingly difficult to analyze. To simplify the process of distinguishing between non-equivalent tables of Lie brackets, we can define the following variables: C for the total number of lines that have only commutators $[\bullet,\bullet]$; D for the total number of anticommutators $\{\bullet,\bullet\}$ on diagonal and, finally, A for total number of $\{\bullet,\bullet\}$ on table. Therefore, it is possible to see that these variables are related as follows:

$$A=\left(\frac{D}{2}-C\right)D \tag{4.25}$$

For a graded (super)algebra $\mathbb{Z}_2^n$, the value of D is restricted to either zero or 2^{n-1} due to the requirement that it contains smaller graded algebras as subalgebras. The fact that $\mathbb{Z}_2^n$

contains $\mathbb{Z}_2^{n-1}$ as a subalgebra is a crucial property for understanding the reasoning in the next chapter. This property enables us to obtain graded (super)algebras of dimension n by doubling algebras with dimensions smaller than n.

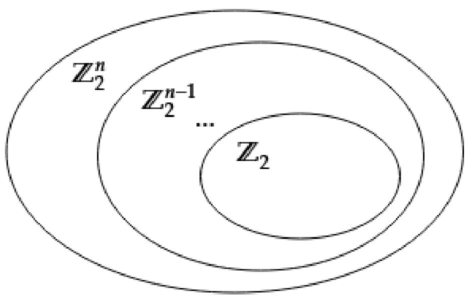
4.4 Doubling construction for $\mathbb{Z}_2^n$ (super)algebras

The (super)algebras $\mathbb{Z}_2^n$ have very characteristic matrix representations that relate to the classifications of the gradings of their elements as belonging to even or odd sectors [43], [44]. Therefore, in this chapter, we will adopt the following block representations for $\mathbb{Z}_2^n$ matrices, based on the classification of gradings as even or odd:

$$\text{Even sector:} \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right), \tag{4.26}$$

$$\text{Odd sector:} \quad \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right). \tag{4.27}$$

Let's now try to create a specific type of $\mathbb{Z}_2^n$ algebras whose 2^n generators can be obtained by doing $n-1$ successive doublings, such that $\mathbb{Z}_2^{n-1}$, $\mathbb{Z}_2^{n-2}$, ..., $\mathbb{Z}_2$ are subalgebras of $\mathbb{Z}_2^n$.



...	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
	0	0	0
	0	0	1
	0	1	0
	0	1	1
	1	0	0
	1	0	1
	1	1	0
	1	1	1

Figure 4.1: Illustrative diagram of the doubling of graded Lie (super)algebras. One can consider a $\mathbb{Z}_2^n$ having $\mathbb{Z}_2^{n-1}$, ..., $\mathbb{Z}_2$ as subalgebras. In the table on the right, the gradings also follow the logic of doubling.

A doubling in graded algebras means regrouping all generators of $\mathbb{Z}_2^{n-1}$ in the even and odd sectors of the algebra $\mathbb{Z}_2^n$. That is, if we consider $n=2$, we can obtain an algebra $\mathbb{Z}_2^2$ by means of an algebra $\mathbb{Z}_2$, which has the generators f_0 and f_1 , in the following way:

$$\text{Even sector:} \quad g_0 = \left(\begin{array}{c|c} f_0 & 0 \\ \hline 0 & f_0 \end{array}\right), \quad g_1 = \left(\begin{array}{c|c} f_1 & 0 \\ \hline 0 & f_1 \end{array}\right) \tag{4.28}$$

$$\text{Odd sector:} \quad g_2 = \left(\begin{array}{c|c} 0 & f_0 \\ \hline -f_0 & 0 \end{array}\right), \quad g_3 = \left(\begin{array}{c|c} 0 & f_1 \\ \hline -f_1 & 0 \end{array}\right) \tag{4.29}$$

where g_0, g_1, g_2 e g_3 generators of algebra $\mathbb{Z}_2^2$. For this we can think in the following representation:

$$\begin{aligned} f_0 &\to 0 \qquad g_0 \to 00 \\ f_1 &\to 1 \qquad g_1 \to 01 \\ & \qquad\quad\; g_2 \to 10 \\ & \qquad\quad\; g_3 \to 11 \end{aligned} \tag{4.30}$$

Therefore, from the generators of $\mathbb{Z}_2^2$ above one can construct an algebra $\mathbb{Z}_2^3$ as follows:

$$\text{Even sector}: \quad G_0 = \left(\begin{array}{c|c} g_0 & 0 \\ \hline 0 & g_0 \end{array}\right), \quad G_1 = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_1 \end{array}\right), \quad G_2 = \left(\begin{array}{c|c} g_2 & 0 \\ \hline 0 & g_2 \end{array}\right), \quad G_3 = \left(\begin{array}{c|c} g_3 & 0 \\ \hline 0 & g_3 \end{array}\right) \tag{4.31}$$

$$\text{Odd sector}: G_4 = \left(\begin{array}{c|c} 0 & g_0 \\ \hline -g_0 & 0 \end{array}\right), G_5 = \left(\begin{array}{c|c} 0 & g_1 \\ \hline -g_1 & 0 \end{array}\right), G_6 = \left(\begin{array}{c|c} 0 & g_2 \\ \hline -g_2 & 0 \end{array}\right), G_7 = \left(\begin{array}{c|c} 0 & g_3 \\ \hline -g_3 & 0 \end{array}\right), \tag{4.32}$$

which G_0, ...,G_7 are generators of a $\mathbb{Z}_2^n$.

For instance, let us take supersymmetry $S_2^{\mathbb{Z}_2}$ generators [45]:

$$H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix} \tag{4.33}$$

where H is the hamiltonian, Q is the supercharge operator and ∂_t is the time derivative. One can have:

$$[H,Q] = 0, \quad \{Q,Q\} = 2H \tag{4.34}$$

and

$$\begin{aligned} HQ = QH &= \partial_t Q, \\ Q^2 &= H, \\ H^2 &= \partial_t H \end{aligned} \tag{4.35}$$

Wherever one can define $f_0 = H$ and $f_1 = Q$, such that:

$$\begin{aligned} f_0 f_0 &= \frac{c_{00}}{2} f_0 \\ f_0 f_1 &= \frac{c_{01}}{2} f_1 \\ f_1 f_0 &= \frac{c_{10}}{2} f_1 \\ f_1 f_1 &= \frac{c_{11}}{2} f_0 \end{aligned} \tag{4.36}$$

The coefficients c_{ij} are:

$$c_{00} = c_{01} = c_{10} = 2\partial_t, \quad c_{11} = 2. \tag{4.37}$$

From $S_2^{\mathbb{Z}_2}$, we can construct a $\mathbb{Z}_2^2$ algebra by doubling as follows:

$$g_0 = \begin{pmatrix} f_0 & 0 \\ 0 & f_0 \end{pmatrix}, \quad g_1 = \begin{pmatrix} f_1 & 0 \\ 0 & f_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & f_0 \\ -f_0 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & f_1 \\ -f_1 & 0 \end{pmatrix} \tag{4.38}$$

where

$$\begin{aligned} g_0 &\to 00 \\ g_1 &\to 01 \\ g_2 &\to 10 \\ g_3 &\to 11 \end{aligned} \tag{4.39}$$

The multiplication between generators $g_0, \ldots, g_3$ has the following results:

$$g_0 g_i = \frac{b_{0i}}{2} g_i, \quad g_i g_0 = \frac{b_{i0}}{2} g_i \tag{4.40}$$

and

$$g_i g_j = \frac{a_{ij}}{2} \delta_{ij} g_0 + \frac{b_{ij}}{2} \varepsilon_{ijk} g_k \tag{4.41}$$

where coefficients a_{ij}, a_{00}, b_{ij} and b_{00} are:

$a_{00} = 2\partial_t$ $a_{11} = 2$ $a_{22} = -2\partial_t$ $a_{33} = -2$	$b_{01} = 2\partial_t$ $b_{02} = 2\partial_t$ $b_{03} = 2\partial_t$	$b_{10} = 2\lambda$ $b_{12} = 2\partial_t$ $b_{13} = -2$	$b_{20} = 2\partial_t$ $b_{21} = -2\partial_t$ $b_{23} = -2\partial_t$	$b_{30} = 2\partial_t$ $b_{31} = -2$ $b_{32} = 2\partial_t$

Table 4.11: List of coefficients of $\mathbb{Z}_2^2$ constructed by doubling $S_2^{\mathbb{Z}_2}$.

The matrices of this $\mathbb{Z}_2^2$ algebra are, more explicitly:

$$g_0 = \begin{pmatrix} \partial_t & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & \partial_t \end{pmatrix}, \quad g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \partial_t & 0 \end{pmatrix}$$
$$g_2 = \begin{pmatrix} 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & \partial_t \\ -\partial_t & 0 & 0 & 0 \\ 0 & -\partial_t & 0 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \partial_t & 0 \\ 0 & -1 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 \end{pmatrix} \tag{4.42}$$

Therefore, to construct an algebra $\mathbb{Z}_2^3$, we will do:

$$G_0 = \begin{pmatrix} g_0 & 0 \\ 0 & g_0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} g_2 & 0 \\ 0 & g_2 \end{pmatrix}, \quad G_3 = \begin{pmatrix} g_3 & 0 \\ 0 & g_3 \end{pmatrix}$$
$$G_4 = \begin{pmatrix} 0 & g_0 \\ -g_0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 0 & g_1 \\ -g_1 & 0 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 0 & g_2 \\ -g_2 & 0 \end{pmatrix}, \quad G_7 = \begin{pmatrix} 0 & g_3 \\ -g_3 & 0 \end{pmatrix} \tag{4.43}$$

From G_0 to G_3, we have the even sector of $\mathbb{Z}_2^3$, while from G_4 to G_7 we have the odd sector. The multiplication formula of generators given in (4.43) is:

$$G_i G_j = \frac{A_{ij}}{2}\delta_{ij} G_0 + \frac{B_{ij}}{2} C_{ijk} G_k \tag{4.44}$$

where A_{ij} and B_{ij} are coeficients of multiplication.

It is important to mention that C_{ijk} in (4.44) is the structure constant of octonions. Despite that, algebra $\mathbb{Z}_2^n$ cannot be a non-associative algebra. So, equation (4.44) explicit the property of *quasi-non-associativity* for a kind of $\mathbb{Z}_2^3$ algebras. Wherever, by imposing:

$$[G_i, G_j, G_k] = 0, \tag{4.45}$$

the condition for *quasi-non-associative* is given by the following relations:

$$\begin{aligned} B_{ij} A_{lk} C_{ijl} \delta_{lk} &= B_{jk} A_{im} C_{jkm} \delta_{im} \\ \frac{B_{ij} B_{lk}}{4} C_{ijl} C_{lkm} - \frac{A_{ij}}{2} \delta_{ij} \delta_{kn} &= \frac{B_{jk} B_{im}}{4} C_{jkm} C_{imn} + \frac{A_{jk}}{2} \delta_{jk} \delta_{in} \end{aligned} \tag{4.46}$$

The matrices in (4.43) can also be write, more explicitly, in 8×8 form using the 4×4 matrices in (4.42).

$$G_0 = \begin{pmatrix} \partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_t \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 0 & 0 & \partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_t & 0 & 0 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_t \\ 0 & 0 & 0 & 0 & -\partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial_t & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_t & 0 & 0 & 0 \end{pmatrix}$$

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & \partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_t \\ -\partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_t & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_t & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_t \\ 0 & 0 & 0 & 0 & -\partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial_t & 0 & 0 \\ 0 & 0 & -\partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_t & 0 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_t & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, G_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_t & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.47}$$

Let's take now the definition of two matrices:

$$\mathbb{1}_{2n\times 2n} = \left(\begin{array}{c|c} \mathbb{1}_{n\times n} & 0 \\ \hline 0 & \mathbb{1}_{n\times n} \end{array}\right), \quad I_{2n\times 2n} = \left(\begin{array}{c|c} 0 & \mathbb{1}_{n\times n} \\ \hline -\mathbb{1}_{n\times n} & 0 \end{array}\right) \tag{4.48}$$

Matrices $\mathbb{1}_{2n\times 2n}$ and $I_{2n\times 2n}$ are useful to understand the method of doubling by a point of view of tensor products. Taking a $\mathbb{Z}_2^{n-1}$ algebra, one can construct the even sector of a $\mathbb{Z}_2^n$ algebra by doing the tensor product of all generators of $\mathbb{Z}_2^{n-1}$ with $\mathbb{1}_{(2n-1)\times(2n-1)}$. For the odd sector of $\mathbb{Z}_2^n$, one will use the matrices $I_{(2n-1)\times(2n-1)}$ for tensor product.

Let's use g_0, g_1, g_2 and g_3 of (4.42) as an example. We can reach to this matrices by doing:

$$\text{Even sector:} \quad g_0 = f_0 \otimes \mathbb{1}_{2\times 2}, \quad g_1 = f_1 \otimes \mathbb{1}_{2\times 2} \tag{4.49}$$

$$\text{Odd sector:} \quad g_2 = f_0 \otimes I_{2\times 2}, \quad g_3 = f_1 \otimes I_{2\times 2} \tag{4.50}$$

We can do the same for Z_2^3 matrices exposed in (4.16). Nonetheless, by using (4.21), it is not more necessary starting from matrices g_0, g_1, g_2 and g_3. We can start from f_0 and f_1 and multiplying then by $\mathbb{1}_{4\times 4}$ or $\mathbb{1}_{2\times 2}$ and $I_{4\times 4}$ or $I_{2\times 2}$ to construct even and odd sectors, respectively. That is:

$$\text{Even sector:} \quad G_0 = f_0 \otimes \mathbb{1}_{4\times 4}, \quad G_1 = f_1 \otimes \mathbb{1}_{4\times 4}, \quad G_2 = f_0 \otimes I_{2\times 2} \otimes \mathbb{1}_{2\times 2}, \quad G_3 = f_1 \otimes I_{2\times 2} \otimes \mathbb{1}_{2\times 2} \tag{4.51}$$

Odd sector: $G_4 = f_0 \otimes I_{4\times4}, \quad G_5 = f_1 \otimes \mathbb{1}_{2\times2} \otimes I_{2\times2}, \quad G_6 = f_0 \otimes I_{2\times2} \otimes I_{2\times2}, \quad G_7 = f_1 \otimes I_{2\times2} \otimes I_{2\times2}$ (4.52)

The algebra $S_2^{\mathbb{Z}_2}$ has the structure of complex numbers ($\mathbb{C}$). This characteristic is evident in the commutation and anti-commutation relations exposed in (4.34). When we double this algebra, we obtain a $\mathbb{Z}_2^2$ algebra with quaternionic structure, a fact that is exposed in equation (4.42). Doubling once again, we obtain a representation for $\mathbb{Z}_2^3$ that has the structure constants of octonions. These mentioned relations are not mere coincidences when we compare the logic of doubling for $\mathbb{Z}_2^n$ (super)algebras and the Cayley-Dickson matrix representation. The procedures are very similar and both can be performed using the matrices (2.76).

The complex number matrices are:

$$\mathbb{1}_{2\times2} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.53}$$

Therefore, the quaternions matrices can be constructed as follows:

$$\text{Commutative sector:} \quad \mathbf{e}_0 = \mathbb{1}_{2\times2} \otimes \mathbb{1}_{2\times2}, \quad \mathbf{e}_1 = i \otimes \mathbb{1}_{2\times2} \tag{4.54}$$

$$\text{Non-commutative sector:} \quad \mathbf{e}_2 = I_{2\times2} \otimes \mathbb{1}_{2\times2}, \quad \mathbf{e}_3 = i \otimes I_{2\times2} \tag{4.55}$$

And the octonions matrices are constructed as:

$$\begin{aligned} \text{Associative sector:} \quad & \mathbf{E}_0 = \mathbb{1}_{2\times2} \otimes \mathbb{1}_{4\times4}, \quad \mathbf{E}_1 = i \otimes \mathbb{1}_{4\times4}, \quad \mathbf{E}_2 = \mathbb{1}_{2\times2} \otimes I_{2\times2} \otimes \mathbb{1}_{2\times2}, \\ & \mathbf{E}_3 = i \otimes I_{2\times2} \otimes \mathbb{1}_{2\times2} \end{aligned} \tag{4.56}$$

$$\begin{aligned} \text{Non-associative sector:} \quad & \mathbf{E}_4 = I_{4\times4} \otimes \mathbb{1}_{2\times2}, \quad \mathbf{E}_5 = i \otimes \mathbb{1}_{2\times2} \otimes I_{2\times2}, \quad \mathbf{E}_6 = \mathbb{1}_{2\times2} \otimes I_{2\times2} \otimes I_{2\times2}, \\ & \mathbf{E}_7 = i \otimes I_{2\times2} \otimes I_{2\times2} \end{aligned} \tag{4.57}$$

But for Cayley-Dickson algebras, the multiplication for matrices should follow the rule (2.77).

4.5 Perturbation in graded $\mathbb{Z}_2^n$ algebra systems

Before starting the main calculus of this section, is important to present an operator T, here called by *permutation operator*, that act in a product generators of a graded algebra $\mathbb{Z}_2^n$ represented by A, B and C as follows:

$$T(\gamma) \longrightarrow T(\gamma)(AB) = e^{i\pi\gamma} BA \tag{4.58}$$

For simplicity, we can define $T(\gamma) \equiv T_\gamma$. And let's impose some important properties concerning operator T_γ:

- $T_\gamma 1 = e^{i\pi\gamma} 1$
- $T_\gamma A = e^{i\pi\gamma} A$
- $T_\gamma(AB) = e^{i\pi\gamma} BA$
- $T_\gamma(ABC) = e^{i\pi\gamma} CBA$
- $T_{\gamma'} T_\gamma(AB) = e^{i\pi(\gamma'+\gamma)} AB$

Now, let's consider a physical system with (super)algebra $\mathbb{Z}_2^n$. Generators of $\mathbb{Z}_2^n$ will be represented by A and B. (Anti-)commutating rule is given by:

$$(A,B) = AB - (-1)^{<\alpha,\beta>} BA \tag{4.59}$$

where $<\alpha,\beta>$ is the mapping. Considering $n-mod$:

$$<\alpha,\beta>=\gamma \Longrightarrow \begin{cases} 0 \\ 1 \end{cases} \tag{4.60}$$

But now, let's introduce a perturbation in the system acting on the mapping, since it is this operation that indicates how the particles in the system will interact:

$$<\alpha,\beta>=\gamma + \epsilon\frac{\theta_\gamma}{\pi} \tag{4.61}$$

For now, only minor perturbation - that is $\epsilon << 1$ - are considering. Term θ_γ is a operator that will act on A and B. Therefore:

$$\begin{aligned} (A,B) &= AB - (e^{i\pi})^{\left(\gamma+\epsilon\frac{\theta_\gamma}{\pi}\right)} BA \\ &= AB - e^{i\pi\gamma} e^{i\epsilon\theta_\gamma} BA \\ &= AB - e^{i\pi\gamma}(1 + i\epsilon\theta_\gamma) BA \\ &= AB - e^{i\pi\gamma} BA - i\epsilon e^{i\pi\gamma}\theta_\gamma(BA) \end{aligned} \tag{4.62}$$

Operator θ_γ in terms of T_γ is defined as follows:

$$\theta_\gamma = i(1 + T_\gamma) \tag{4.63}$$

Then, equation (4.62) becomes:

$$(A,B) = AB - e^{i\pi\gamma} BA + \epsilon e^{i\pi\gamma}(BA + T_\gamma(BA)) \tag{4.64}$$

Choosing $e^{i\pi\gamma} = 1$,

$$\begin{aligned} (A,B) &= AB - BA + \epsilon(T(1)(BA) + BA) \\ &= [A,B] + \epsilon(T(1)(BA) + BA) \\ &= [A,B] + \epsilon\{A,B\} \end{aligned} \tag{4.65}$$

But if we choose $e^{i\pi\gamma} = -1$:

$$\begin{aligned}(A,B) &= AB + BA - \epsilon(T(-1)(BA) + BA) \\ &= \{A,B\} - \epsilon(T(-1)(BA) + BA) \\ &= \{A,B\} + \epsilon[A,B] \end{aligned} \tag{4.66}$$

In this section, an expansion of the term $e^{i\epsilon\theta_\gamma}$ up to first order was considered. Now, we will consider perturbations beyond this order, in order to obtain an analytical result for any degree of perturbation.

Expansion up to second order

Considering now a expansion up to second order:

$$e^{i\epsilon\theta_\gamma} \approx 1 + i\epsilon\theta_\gamma - \frac{\epsilon^2}{2}\theta_\gamma^2 \tag{4.67}$$

and

$$\theta_\gamma^2 = -(1+T_\gamma)^2 = -(1+2T_\gamma + T_\gamma^2) \tag{4.68}$$

According to (4.62), (4.67) and (4.68),

$$\begin{aligned}(A,B) &= AB - e^{i\pi\gamma}\left(BA + -\epsilon(1+T_\gamma)BA + \frac{\epsilon^2}{2}(1+2T_\gamma+T_\gamma^2)BA\right) \\ &= AB - e^{i\pi\gamma}\left(BA - \epsilon(BA + e^{i\pi\gamma AB}) + \frac{\epsilon^2}{2}(BA + 2e^{i\pi\gamma}AB + e^{2i\pi\gamma}BA)\right)\end{aligned} \tag{4.69}$$

For $e^{i\pi\gamma} = 1$,

$$\begin{aligned}(A,B) &= AB - BA + \epsilon(BA + AB) - \frac{\epsilon^2}{2}(BA + 2AB + BA) \\ &= [A,B] + (\epsilon - \epsilon^2)\{A,B\}\end{aligned} \tag{4.70}$$

And for $e^{i\pi\gamma} = -1$,

$$\begin{aligned}(A,B) &= AB + (BA - \epsilon(BA - AB)) + \frac{\epsilon^2}{2}(BA - 2AB + BA) \\ &= \{A,B\} + (\epsilon - \epsilon^2)[A,B]\end{aligned} \tag{4.71}$$

Expansion up to third order

For this procedure we must do:

$$e^{i\epsilon\theta_\gamma} \approx 1 + i\epsilon\theta_\gamma - \frac{\epsilon^2}{2}\theta_\gamma^2 - \frac{\epsilon^3}{3!}\theta_\gamma^3 \tag{4.72}$$

where

$$\theta_\gamma^3 = -i(1 + 3T_\gamma + 3T_\gamma^2 + T_\gamma^3) \tag{4.73}$$

Then,

$$(A,B) = \mathcal{O}(\epsilon^2) + e^{i\pi\gamma}\frac{\epsilon}{3!}(BA + 3e^{i\pi\gamma}AB + 3e^{2i\pi\gamma}BA + e^{3i\pi\gamma}AB) \tag{4.74}$$

By using $e^{i\pi\gamma} = 1$, one get:

$$\begin{aligned}(A,B) &= [A,B] + \left(\epsilon - \frac{2}{2!}\epsilon^2\right)\{A,B\} + \frac{\epsilon^3}{3!}(BA + 3AB + 3BA + AB)\\ &= [A,B] + \left(\epsilon - \frac{2}{2!}\epsilon^2 + \frac{4}{3!}\epsilon^3\right)\{A,B\}\end{aligned} \tag{4.75}$$

And $e^{i\pi\gamma} = -1$:

$$\begin{aligned}(A,B) &= \{A,B\} + \left(\epsilon - \frac{2}{2!}\epsilon^2\right)[A,B] + \frac{\epsilon^3}{3!}(BA - 3AB + 3BA - AB)\\ &= \{A,B\} + \left(\epsilon - \frac{2}{2!}\epsilon^2 + \frac{4}{3!}\epsilon^3\right)[A,B]\end{aligned} \tag{4.76}$$

Expansion up to fourth order

By expanding $e^{i\epsilon\theta_\gamma}$ up to ϵ^4, we will get:

$$e^{i\epsilon\theta_\gamma} \approx 1 + i\epsilon\theta_\gamma - \frac{\epsilon^2}{2}\theta_\gamma^2 - \frac{i\epsilon^3}{3!}\theta_\gamma^3 + \frac{\epsilon^4}{4!}\theta_\gamma^4 \tag{4.77}$$

and

$$\theta_\gamma^4 = 1 + 4T_\gamma + 6T_\gamma^2 + 4T_\gamma^3 + T_\gamma^4 \tag{4.78}$$

Therefore,

$$\begin{aligned}(A,B) &= \mathcal{O}(\epsilon^3) - e^{i\pi\gamma}\frac{\epsilon^4}{4!}(1 + 4T_\gamma + 6T_\gamma^2 + 4T_\gamma^3 + T_\gamma^4)BA\\ &= \mathcal{O}(\epsilon^3) - e^{i\pi\gamma}\frac{\epsilon^4}{4!}(BA + 4e^{i\pi\gamma}AB + 6e^{2i\pi\gamma}BA + 4e^{3i\pi\gamma}AB + e^{i\pi\gamma}BA)\end{aligned} \tag{4.79}$$

For $e^{i\pi\gamma} = 1$,

$$\begin{aligned}(A,B) &= [A,B] + \left(\epsilon - \frac{2}{2!}\epsilon^2 + \frac{4}{3!}\epsilon^3\right)\{A,B\} - \frac{\epsilon^4}{4!}(BA + 4AB + 6BA + 4AB + BA)\\ &= [A,B] + \left(\epsilon - \frac{2}{2!}\epsilon^2 + \frac{4}{3!}\epsilon^3 - \frac{8}{4!}\epsilon^4\right)\{A,B\}\end{aligned} \tag{4.80}$$

For $e^{i\pi\gamma} = -1$,

$$\begin{aligned}(A,B) &= \{A,B\} + \left(\epsilon - \frac{2}{2!}\epsilon^2 + \frac{4}{3!}\epsilon^3\right)[A,B] + \frac{\epsilon^4}{4!}(BA - 4AB + 6BA - 4AB + BA)\\ &= \{A,B\} + \left(\epsilon - \frac{2}{2!}\epsilon^2 + \frac{4}{3!}\epsilon^3 - \frac{8}{4!}\epsilon^4\right)[A,B]\end{aligned} \tag{4.81}$$

General result

According to results obtained until now, we can induce a general result for any order of perturbation. For any algebra characterized by:

$$(A,B) = AB - (-1)^{<\alpha,\beta>} BA \tag{4.82}$$

By introducing a perturbation

$$<\alpha,\beta> = \gamma + \frac{\epsilon}{\pi}\theta_\gamma, \quad (1-\mathrm{mod})$$

where

$$\theta_\gamma = i(1+T_\gamma) \tag{4.83}$$

and $T_\gamma(AB) = e^{i\pi\gamma}BA$.

Wherefore,

- For $e^{i\pi\gamma} = 1$,
$$(A,B) = [A,B] + \left(\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{n!}\epsilon^n\right)\{A,B\} \tag{4.84}$$
- For $e^{i\pi\gamma} = -1$,
$$(A,B) = \{A,B\} + \left(\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{n!}\epsilon^n\right)[A,B] \tag{4.85}$$

Results (4.84) e (4.85) can be rewritten as, respectively:

$$(A,B) = [A,B] + \left(\frac{e^{-2\epsilon}}{2} - \frac{1}{2}\right)\{A,B\} \tag{4.86}$$

and

$$(A,B) = \{A,B\} + \left(\frac{e^{-2\epsilon}}{2} - \frac{1}{2}\right)[A,B] \tag{4.87}$$

This approach is expected to be particularly useful in systems of graded Lie (super)algebras that exhibit broken symmetries or dynamics involving perturbations. The technique allows for a more precise and flexible description of these systems, as it takes into account the interaction between the generators of the algebra, which may have different behaviors before and after the perturbation. In addition, the approach allows for the construction of new algebras from already known algebras. A very similar formalism was demonstrated by Mohapatra and Greenberg [46], [47].

Results (4.86) and (4.87) are very similar to a Volichenko Algebra [15]. It is expected that this result can also be applied to systems of particles under a perturbation - maybe a external field - for example, in the phenomena of electron fractionalization [48].

Chapter 5

Conclusions

In this master's thesis, new tools and formalisms were presented in Cayley-Dickson and Graded Lie (Super)Algebras, as well as their potential applications in topics such as particle physics, classical electromagnetism and dyons, and M-theory. Regarding Cayley-Dickson Algebras, the intention was to briefly present the constructions of complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, octonions $\mathbb{O}$, and Cayley-Dickson sedenions $\mathbb{S}$, focusing on exploring the emergence of new properties through constructions and operations such as commutator, associator, and anti-associator.

Using the Cayley-Dickson construction, we double the real numbers to obtain (split-)complex numbers, complex numbers to obtain (split-)quaternions, quaternions to obtain (split-)octonions, and octonions to obtain Cayley-Dickson sedenions. Thus, we see that the non-commutativity of the quaternion algebra is the direct cause of the non-associativity of the octonion algebra, just as quaternions do not commute because the conjugation operation of a complex number generates a different number. Similarly, the non-associativity of octonions is responsible for the non-alternativity of sedenions. These findings were only possible to verify by applying commutation and association operations to quaternions and octonions. However, if we use the associator on two elements of the sedenionic basis, we find that the result is zero for any two elements. This is an intriguing result, as the zero value of this operation is characteristic of alternative algebras. However, if we use sedenionic numbers that are a linear combination of the elements $(\mathbf{E}_i, 0)$ and $(0, \mathbf{E}_j)$ of the basis, then the non-alternativity will be verified.

After this, the matrix representation of complex and quaternion numbers was presented. The matrices of the bases of $\mathbb{C}$ and $\mathbb{H}$ were formed through simple calculations, using only the algebraic properties characteristic of each one. Thus, a pattern of 2×2 matrix representation for quaternionic complex numbers was verified, whose entries were the elements of the ordered pairs that form these numbers (see 2.41). From this pattern, we induced a matrix representation for any ordered pair in the Cayley-Dickson algebras, by expanding the definition of matrix multiplication to make it compatible with non-associative algebras, as shows equation (2.45).

Another possibility that the matrix representation 2.41 offers is that its entries can be rewritten as blocks, since these are also Cayley Dickson algebras and can be rewritten as other matrices. Thus, we see that the Cayley-Dickson algebras have a number of possible matrix representations that is equal to the number of doublings required to create these algebras from the reals. Therefore, we create quaternionic matrices in 2×2 and 4×4 forms, and octonionic matrices in 2×2, 4×4, and 8×8 forms. Using the Cayley-Dickson matrix multiplication, we have seen that the octonionic matrices obey the octonions algebra presented in the first chapter, such that the multiplication between matrices corresponding to the basis elements is compatible with the octonions structure constant. These octonionic matrices can be used to rewrite the matrices of the octonionic M-theory algebra (the Clifford algebra $Cl_0(10,1)$) presented in (2.74), so that they are represented only with real entries. Finally, the matrix representation of the Cayley-Dickson Doubling Construction was presented.

After the presentation of the formalism and mathematical techniques in Cayley- Dickson algebras, some possible applications were demonstrated. Initially, the already known relationship between the elements of the $SU(2)$ algebra, the Pauli matrices, and quaternions was explored. Using the quaternionic matrix representation of size 4×4 and the definition of isospins in terms of the quaternionic basis (see (3.8)), it was possible to demonstrate a representation of spin-up and spin-down in the form of column matrices with four complex numbers entries. After this, following the ideas of Kugo and Townsend [29], the quaternionic spin was constructed for the construction of the quaternionic Dirac Lagrangian and presented global and local $SU(2)$ symmetries. These approaches were only a motivation to verify how the complete basis of octonions can simulate an $SU(2)$ algebra and Dirac matrices, as summarized in 3.1. According these results we construct the octonionic Lagrangian of dyons, following ideas of Chanyal *et al* [8].

For the topic of graded Lie algebras and superalgebras, the thesis provides a brief introduction to the subject, explaining the general rules and possible mappings of $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, and $\mathbb{Z}_2^3$. However, the main focus of this chapter was not on the detailed presentation of these algebras, but rather on techniques for creating possible algebras and the formalism of systems of graded (super)algebras subject to perturbations.

The procedure named by Doubling Construction for $\mathbb{Z}_2^n$ (Super)algebras was an idea very similar to Cayley-Dickson doubling construction to create a $\mathbb{Z}^n$ from a $\mathbb{Z}_2^{n-1}$. This amounts to projecting the generators of $\mathbb{Z}_2^{n-1}$ onto the even and odd sectors of a $\mathbb{Z}_2^n$. However, when using the matrices defined in 4.48, we see that it is possible to obtain a $\mathbb{Z}_2^n$ from a $\mathbb{Z}_2$ without worrying about intermediate doublings. From this procedure we could use a $\mathbb{Z}_2$ supersymmetry superalgebra ($S_2^{\mathbb{Z}_2^2}$) to construct a $\mathbb{Z}_2^3$ superalgebra for supersymmetry $\mathcal{N} =$

2.

In the end, the construction of a formalism was presented in which it is assumed that graded (super)algebra systems $\mathbb{Z}_2^n$ are subject to a perturbation. In this case, the nature of the perturbation was not thoroughly discussed, as the intention is for this topic to have a more general application. However, initially, we can think of this perturbation as, for example, an external field being applied to systems of bosons, fermions, para-bosons, and para-fermions that will change the interaction between these particles. This perturbation are indicates by operator θ_γ, defined in 4.63 in terms of permutation operator T_γ. The first result found was considering the minor perturbation regime, as shown by formulas 4.65 and 4.66. Just like in the first case, for higher-order perturbations, the results always indicated that the Lie bracket would always be a result that contained one part as being from the commutator and another as being from the anticommutator. In summary, the results were as follows: if the mapping determines that the Lie bracket should be a commutator, the perturbation will add an anticommutator to the result and vice versa. The results followed a certain pattern so that they could be written as a Taylor series of the perturbation parameter, which corresponds to an expansion of an exponential, as shown in results 4.86 and 4.87. It is important to note that the obtained formulas resemble a Volichenko algebra. This is because this formalism was initially conceived as an attempt to obtain a Volichenko algebra through a Lie (super)algebra.

Bibliography

[1] R. D. Schafer, "On the Algebras Formed by the Cayley-Dickson Process," *American Journal of Mathematics*, vol. 76, pp. 435–446, 1954.

[2] K. McCrimmon, *A Taste of Jordan Algebras*, 1st ed. San Francisco, CA: Pearson, 2003.

[3] G. M. Dixon, *Division Algebras:: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics.* Springer Science & Business Media, 2013, vol. 290.

[4] M. Scheunert, "Generalized Lie algebras," *Journal of Mathematical Physics*, vol. 20, no. 4, pp. 712–720, 1979.

[5] V. Rittenberg and D. Wyler, "Generalized superalgebras," *Nuclear Physics B*, vol. 139, no. 3, pp. 189–202, 1978.

[6] R. Ree, "Generalized lie elements," *Canadian Journal of Mathematics*, vol. 12, pp. 493–502, 1960.

[7] F. Toppan, "Inequivalent quantizations from gradings and $\mathbb{Z}_2 \times \mathbb{Z}_2$-parabosons," *Journal of Physics A: Mathematical and Theoretical*, vol. 54, no. 35, p. 355202, 2021.

[8] B. Chanyal, "Sedenion unified theory of gravi-electromagnetism." *Indian Journal of Physics*, vol. 88, pp. 1197–1205, 2014.

[9] C. Flaut and V. Shpakivskyi, "Real matrix representations for the complex quaternions," 2012. [Online]. Available: https://arxiv.org/abs/1209.0585

[10] W. D. Smith, "Quaternions, octonions, and now, 16-ons and 2 n -ons; new kinds of numbers," 2004.

[11] M. Zorn, "Alternativkörper und quadratische systeme," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 9, pp. 395–402, 1933.

[12] R. D. Schafer, *An introduction to nonassociative algebras.* Courier Dover Publications, 2017.

[13] J. Lukierski and F. Toppan, "Generalized space–time supersymmetries, division algebras and octonionic m-theory," *Physics Letters B*, vol. 539, no. 3-4, pp. 266–276, 2002.

[14] Pushpa, P. S. Bisht, T. Li, and O. P. S. Negi, "Quaternion octonion reformulation of quantum chromodynamics," *International Journal of Theoretical Physics*, vol. 50, no. 2, pp. 594–606, nov 2010. [Online]. Available: https://doi.org/10.1007%2Fs10773-010-0579-8

[15] D. Leites and V. Serganova, " Symmetries Wider than Supersymmetry." *Noncommutative Structures in Mathematics and Physics. NATO Science Series*, vol. 22, pp. 13–30, 2001. [Online]. Available: https://link.springer.com/chapter/10.1007/978-94-010-0836-5_2

[16] V. Christianto and F. Smarandache, "A derivation of maxwell equations in quaternion space," *Progress in Physics*, vol. 6, 04 2010.

[17] I. K. Hong and C. S. Kim, "Quaternion electromagnetism and the relation with two-spinor formalism," *Universe*, vol. 5, no. 6, p. 135, jun 2019. [Online]. Available: https://doi.org/10.3390%2Funiverse5060135

[18] M. Arribas, A. Elipe, and M. Latasa, "Quaternions and the rotation of a rigid body," *Celestial Mechanics and Dynamical Astronomy*, vol. 96, pp. 239–251, 11 2006.

[19] S. Giardino, "‘‘four-dimensional conformal field theory using quaternions’’," *Advances in Applied Clifford Algebras*, vol. 27, p. 2457–2471, 2017. [Online]. Available: https://doi.org/10.1007/s00006-017-0781-3

[20] C. Furey, "Standard model physics from an algebra?" 2016. [Online]. Available: https://arxiv.org/abs/1611.09182

[21] Z. Kuznetsova and F. Toppan, "Superalgebras of (split-)division algebras and the split octonionic m-theory in (6,5)-signature," 2006. [Online]. Available: https://arxiv.org/abs/hep-th/0610122

[22] K. Imaeda and M. Imaeda, "Sedenions: algebra and analysis," *Applied Mathematics and Computation*, vol. 115, no. 2, pp. 77–88, 2000. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S009630039900140X

[23] D. K. Biss, J. D. Christensen, D. Dugger, and D. C. Isaksen, "Large annihilators in cayley-dickson algebras ii," 2007. [Online]. Available: https://arxiv.org/abs/math/0702075

[24] H. Albuquerque, "Malcev superalgebras," *Mathematics and its Aplications (MAIA)*, vol. 303, pp. 1–7.

[25] A. A. Sagle, "Malcev superalgebras," *https://www.ams.org/journals/tran/1961-101-03/S0002-9947-1961-0143791-X/S0002-9947-1961-0143791-X.pdf.*

[26] T. Ootsuka, E. Tanaka, and E. Loginov, "Non-associative gauge theory," 2005. [Online]. Available: https://arxiv.org/abs/hep-th/0512349

[27] J. Harris, "Forming groups with 4 × 4 matrices," *The Mathematical Gazette*, vol. 94, pp. 426–429, 11 2010.

[28] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. San Francisco: Pearson, 2011.

[29] T. Kugo and P. Townsend, "Supersymmetry and the division algebras; rev. version," *Nucl. Phys. B*, vol. 221, pp. 357–380. 32 p, Nov 1982. [Online]. Available: https://cds.cern.ch/record/140183

[30] P. Zizzi and E. Pessa, "From su(2) gauge theory to spin 1/2 quantum mechanics," 2011. [Online]. Available: https://arxiv.org/abs/1104.0114

[31] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995.

[32] T. Dray and C. A. Manogue, "Octonionic cayley spinors and e6," 2009. [Online]. Available: https://arxiv.org/abs/0911.2255

[33] C. R. Preitschopf, "Octonions and supersymmetry," 1995. [Online]. Available: https://arxiv.org/abs/hep-th/9512134

[34] B. C. Chanyal, P. S. Bisht, T. Li, and O. P. S. Negi, "Octonion quantum chromodynamics," *International Journal of Theoretical Physics*, vol. 51, no. 11, pp. 3410–3422, jun 2012. [Online]. Available: https://doi.org/10.1007%2Fs10773-012-1222-7

[35] M. I. Dillon, "Constructing graded lie algebras," 2004. [Online]. Available: https://arxiv.org/abs/math/0412179

[36] L. Corwin, Y. Ne'eman, and S. Sternberg, "Graded lie algebras in mathematics and physics (bose-fermi symmetry)," *Rev. Mod. Phys.*, vol. 47, pp. 573–603, Jul 1975. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.47.573

[37] T. Lada and J. Stasheff, "Introduction to SH lie algebras for physicists," *International Journal of Theoretical Physics*, vol. 32, no. 7, pp. 1087–1103, jul 1993. [Online]. Available: https://doi.org/10.1007%2Fbf00671791

[38] R. Haag, J. T. Łopuszański, and M. Sohnius, "All possible generators of supersymmetries of the s-matrix," *Nuclear Physics B*, vol. 88, no. 2, pp. 257–274, 1975. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0550321375902795

[39] M. Moshinsky, C. Quesne, and Y. F. Smirnov, "Supersymmetry and superalgebra for the two-body system with a dirac oscillator interaction," *Journal of Physics A: Mathematical and General*, vol. 28, no. 22, pp. 6447–6457, nov 1995. [Online]. Available: https://doi.org/10.1088%2F0305-4470%2F28%2F22%2F020

[40] Aizawa, N., Kuznetsova, Z., and Toppan, F., "$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: the classical theory," *Eur. Phys. J. C*, vol. 80, no. 7, p. 668, 2020. [Online]. Available: https://doi.org/10.1140/epjc/s10052-020-8242-x

[41] N. Aizawa, P. Isaac, and J. Segar, "$\mathbb{Z}_2 \times \mathbb{Z}_2$-generalizations of infinite-dimensional lie superalgebra of conformal type with complete classification of central extensions," *Reports on Mathematical Physics*, vol. 85, no. 3, pp. 351–373, 2020.

[42] M. M. Balbino, "$\mathbb{Z}_2^n$ graded structures: Lie (super)algebras and super division algebras," Master's thesis, Centro Brasileiro de Pesquisas Físicas, 2023.

[43] A. Pashnev and F. Toppan, "On the classification of n-extended supersymmetric quantum mechanical systems," *Journal of Mathematical Physics*, vol. 42, no. 11, pp. 5257–5271, 2001.

[44] Z. Kuznetsova, M. Rojas, and F. Toppan, "Classification of irreps and invariants of the n-extended supersymmetric quantum mechanics," *Journal of High Energy Physics*, vol. 2006, no. 03, p. 098, 2006.

[45] Z. Kuznetsova and F. Toppan, "Classification of minimal z2×z2-graded lie (super)algebras and some applications," *Journal of Mathematical Physics*, vol. 62, no. 6, p. 063512, jun 2021. [Online]. Available: https://doi.org/10.1063%2F5.0050200

[46] R. N. Mohapatra, "Infinite statistics and a possible small violation of the pauli principle," *Physics Letters B*, vol. 242, no. 3, pp. 407–411, 1990. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0370269390917838

[47] O. W. Greenberg, "Particles with small violations of fermi or bose statistics," *Phys. Rev. D*, vol. 43, pp. 4111–4120, Jun 1991. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevD.43.4111

[48] S. A. Kivelson, "Electron fractionalization." *Synthetic Metal*, vol. 125, pp. 99–106, Nov 2001. [Online]. Available: www.sciencedirect.com/science/article/abs/pii/S0379677901005161