$\mathbb{Z}_2^n$ graded structures: Lie (super)algebras and super division algebras

Matheus de Miranda Balbino

Master thesis

Supervisor:
Prof. Francesco Toppan

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Thesis submitted to the post-graduate program at the Centro Brasileiro de Pesquisas Físicas in partial fulfillment of the requirements for the degree of the Master in Physics.

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Resumo

Esta tese apresenta uma visão geral em Álgebras de Clifford, Álgebras de Cayley-Dickson e a relação com as álgebras de Lie $\mathbb{Z}_2^n$ graduadas. As propriedades mais importantes são discutidas e uma classificação completa das álgebras super divisionais até a graduação $\mathbb{Z}_2^3$ é apresentada.

A motivação para este trabalho é de estabelecer uma base de referência sólida para os tópicos fundamentais que são necessários para trabalhar com álgebras $\mathbb{Z}_2^n$ graduadas. Portanto, uma base matemática simples e clara está sendo desenvolvida.

Para trabalhos futuros, estamos estudando a não associatividade com modelos físicos e desenvolvendo o que é chamado de (super)álgebra graduada de Malcev. Artigos com modelos físicos usando uma (super)algebra $\mathbb{Z}_2^3$ estão sendo escritos e a classificação das super álgebras divisionais logo sera submetida para publicação.

Palavras chave: Álgebras de Clifford; Álgebras de Cayley-Dickson; Álgebras graduadas; Super álgebras divisionais; Álgebras graduadas de Malcev

Abstract

This thesis presents an overview on Clifford algebras, Cayley-Dickson algebras and the relation of them with the $\mathbb{Z}_2^n$ graded Lie algebras. The main properties are thoroughly discussed and a complete classification of the super division algebras up to the $\mathbb{Z}_2^3$ grading is presented.

The motivation for this work is to establish a solid reference base for the fundamental topics that are necessary to work with $\mathbb{Z}_2^n$ algebras. Therefore, a simple and clear mathematical framework is being developed.

For future work we are studying non-associativity, with physics models and developing what we call a graded Malcev (super)algebra. Papers with physics models using a $\mathbb{Z}_2^3$ (super)algebra are already being written and the classification of the super division algebras will be soon submitted for publication.

Keywords: Clifford algebras; Cayley-Dickson algebras; $\mathbb{Z}_2^n$ Graded algebras; Super division algebras; Graded Malcev algebras

I dedicate this work to my family and my fiancé.

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List of Figures

List of Tables

Contents

Chapter 1

Introduction

This thesis presents an introduction on the important topics necessary to study graded algebras and super division algebras. This thesis and I. P. de Freitas master's degree thesis [1] are complementary and present the work done by the group, which also includes R. G. Rana.

The Cayley-Dickson algebras, as a doubling iteration of a field, were introduced by Arthur Cayley and Leonard Dickson [[2],[3]]. Of these algebras, the one that is getting more attention in recent years is the octonions, see [[4],[2]]. As a non-associative algebra, its application in physics are still obscure. However, it is known that the octonions are responsible for the generation of the exceptional Jordan algebras and exceptional Lie algebras, both of which are believed to play a important role in physics, see [[4], [3], [5]].

A number of authors have attempted to use the octonions to generalize the standard model of particle physics, see [[6], [7], [8]] and references within. Others have attempted to use octonions in a Clifford algebra framework, see [[9],[10]] and references therein. References on Clifford algebras include [[11], [12],[13],[14],[15]]. This thesis presents a matrix realization for the octonions, also discussed in [[16],[17]], which can potentially be applied in this area.

The $\mathbb{Z}_2^n$ graded (super)algebras were introduced in 1978 by Rittenberg-Wyler in two papers [[18],[19]], see also [20]. They were a generalization of [[21],[22]]. Since then, they have been studied mostly by mathematicians, [[23],[24],[25],[26],[27],[28]]. Some early attempts to find physical applications were made by [[29],[30],[31]], but in the recent years this area grew rapidly.

In [[32],[33]], it is shown that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded (super)algebras describe symmetries of Lévi-Leblond equations. After that, it was discussed in the literature, invariant world-line models [34], 2-dimensional sigma models [[35],[36],[37]], quantum mechanics [[38],[39]], superconformal quantum mechanics [40] and superspace [[41],[36],[42]]. The extension and bosonization of double graded supersymmetric quantum mechanics are analyzed in [[43],[44], [45]]. The $\mathbb{Z}_2^3$ graded (super)algebras started to be analyzed in physics very recently. In [46], the authors analyze a $\mathbb{Z}_2^3$ graded superconformal quantum mechanics constructed using a relation with Clifford algebras.

The theories discussed above are a generalization of ordinary supersymmetric and superconformal quantum mechanics. For that and root multiplets, that appears in this type of models, see [[47],[48], [49],[50]] and [[51],[52],[53]].

$\mathbb{Z}_2^n$ graded (super)algebras have been shown to have a relationship with parastatistics. This relationship has been explored in references [[54],[55],[56]] and references within. While most papers focus only on superalgebras, $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebras with parabosons have been studied and classified in [36]. Moreover, the theoretical detectability of parafermions was analyzed in [54] and the extension to parabosons in [57].

Another important area of study is the grading of division algebras. A super division algebra represents a $\mathbb{Z}_2^n$ graded (super)algebra in which all homogeneous elements admits inverse. The $\mathbb{Z}_2$ super division algebras in addition to the three associative division algebras form what is known as the Tenfold way, see references [[58],[59]]. The Tenfold way is appearing to be essential to the understanding of the "periodic table of topological insulators and superconductors". In this topic, see [60] for the physical significance of the periodic table, [[61],[62]] for the relation of Cartan's classification of symmetric spaces with random-matrix theory and [63] for the classification of generic Hamiltonians. For a more mathematical study analyzing implementations of the Tenfold way see [64].

While the Tenfold way is formed by the $\mathbb{Z}_2$ graded super division algebras, an extension to it exists when considering a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded structure, as analyzed in [65]. In this last reference, the authors use what is called the alphabetic (re)presentation of Clifford algebras, first introduced in [66], to classify the $\mathbb{Z}_2^2$ graded super division algebras. It consists in assigning a letter to the 2x2 invertible real matrices that generate all matrix representations of Clifford algebras. Therefore, the generators will be represented by words with equal length in a four letter alphabet and the tensor product symbol is skipped.

Studying real super division algebras highlights the importance of considering other types of graded (super)algebras beyond the Lie ones. One important example are the graded Malcev (super)algebras which are already being defined by mathematicians [[67], [68], [69], [70]]. This thesis presents an idea for a mapping for the $\mathbb{Z}_2^3$ graded Malcev (super)algebras that works only for the octonions and split-octonions.

The scheme of the thesis is the following. Chapter 2 present the definition of Clifford algebras and their classification following Okubo [[13],[14]]. Chapter 3 introduces the definition of the Cayley-Dickson algebras and their relation to Clifford algebras. Chapter 4 fully discusses the alphabetic (re)presentation using the algebras presented in chapters 2 and 3. The last section of chapter 4 presents the Octonionic M-algebra, see references [[71],[72],[73]]. The purpose is to show the application of the alphabetic (re)presentation and the matrix representations for octonions presented in sections before.

Chapter 5 is the main core of the thesis, it presents the $\mathbb{Z}_2^n$ graded (super)algebras and the super division algebras. The last sections of the chapter presents a possible guide to study non-associative graded (super)algebras, focusing on the graded Malcev (super)algebras. The conclusion summarizes the main results of the thesis and discusses ongoing and future research. Appendix A presents the structure constants for the octonions and split-octonions. Appendix B shows the new matrix representation for the octonions found by I. P. de Freitas and M. M. Balbino. Appendix C shows how to use the left and right action to find the matrix representation of the Clifford algebras, which are related to the (split)octonions. Appendix D gives the Octonionic M-algebra matrices in the usual 4x4 octonionic representation. For last the Appendix E gives all possible table of brackets for the study of the $\mathbb{Z}_2^3$ graded Malcev (super)algebras.

Chapter 2

Clifford algebras

2.1 Definition

There are more then one way to define Clifford algebras, in [12] the authors define them as a quotient algebra from a tensor algebra with a two sided ideal, see also [11]. The other way is to define them as the enveloping algebra of a quadratic vector space, see [11] and the classification by Okubo [13].

Definition: A universal Clifford algebra $(\mathscr{A}, \gamma)$ is generated by a quadratic vector space $(\mathscr{V}, g)$ with $\gamma(v) \in \mathscr{A}$ for $v \in \mathscr{V}$, the generators obey the following relation:

$$\gamma(v)\gamma(u) + \gamma(u)\gamma(v) = 2\eta(u, v)\mathbf{1} \tag{2.1}$$

Where **1** is the identity. The dimension of a universal Clifford algebra is:

$$dim(\mathscr{A}) = 2^n, \quad \text{where} \quad n = dim(\mathscr{V}) \tag{2.2}$$

Every universal Clifford algebra is an associative algebra with the multiplication being the geometric product or Clifford product. In a matrix representation the product becomes the standard matrix multiplication. This means that the matrix multiplication is the analog of the Clifford multiplication acting on the matrix representation of the algebra.

A quadratic vector space $\mathscr{V}$ of a real Clifford algebra will be the reals, of a complex Clifford algebra it will the complex with a p, q signature, where p is the number of generators which are space-like and q which are time-like. In this thesis, the Clifford algebras will be denoted by $Cl_{(p,q)}$; some authors use the notation $Cl(p, q)$, [13] and [15], but the algebra is the same, therefore, it won't generate any confusion.

For every Clifford algebra, there is a minimal dimension matrix representation. Minimal and non-minimal representations have different properties, this is why Okubo's paper only use the minimal representation for the classification.

2.2 Fundamental Clifford algebra and doubling process

The fundamental Clifford algebra that generates every other one is $Cl_{(1,0)}$. With a particular Clifford algebra, there are two ways to create a new one. These two algorithms are called the doubling process, the algorithm assures that the generating matrices anti-commute, see [15]. Doubling $Cl_{(1,0)}$ generates $Cl_{(2,1)}$, the minimal matrix representation for the generators of $Cl_{(2,1)}$ plus the scalar identity is given by:

$$Cl_{(2,1)} \quad \gamma(v): \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.3}$$

Therefore, the general algorithm is given by:

1^{st}: $Cl_{(p,q)} \to Cl_{(p+1,q+1)}$:

$$\Gamma_i = \begin{pmatrix} 0_d & \gamma_i \\ \gamma_i & 0_d \end{pmatrix}, \quad \Gamma_{n+1} = \begin{pmatrix} 0_d & 1_d \\ -1_d & 0_d \end{pmatrix}, \quad \Gamma_{n+2} = \begin{pmatrix} 1_d & 0_d \\ 0_d & -1_d \end{pmatrix}, \quad d = dim(\gamma_i) \tag{2.4}$$

2^{nd}: $Cl_{(p,q)} \to Cl_{(q+2,p)}$:

$$\Gamma_i = \begin{pmatrix} 0_d & \gamma_i \\ -\gamma_i & 0_d \end{pmatrix}, \quad \Gamma_{n+1} = \begin{pmatrix} 0_d & 1_d \\ 1_d & 0_d \end{pmatrix}, \quad \Gamma_{n+2} = \begin{pmatrix} 1_d & 0_d \\ 0_d & -1_d \end{pmatrix}, \quad d = dim(\gamma_i) \tag{2.5}$$

where γ_i are the matrices that generates $Cl_{(p,q)}$ and $n = p + q$

Both algorithm comes from 2.3, from the doubling of $Cl_{(1,0)}$ to $Cl_{(2,1)}$, see figure 2.1. There are two ways of doubling it given by both algorithms.

The first relativistic quantum mechanics theory was created by P. Dirac, see [74] for a review. The famous Dirac matrices were produced by him using the algorithms above. Dirac used the Pauli matrices, which are generators of $Cl_{(3,0)}$, to create the Clifford algebra $Cl_{(1,3)}$, by excluding one generator of $Cl_{(2,3)}$.

The maximal Clifford algebras are presented in the figure 2.1 below. A maximal Clifford algebra is odd dimensional, hence $n = p + q$ is odd. The non-maximal are obtained by excluding generators from the maximal ones. However, when a matrix representation for a Clifford algebra is found by excluding generators, it can be non-minimal in dimension. For example, in figure 2.1, one can see that $Cl_{(4,3)}$ is represented by 8x8 real matrices. By excluding the four space-like elements, it then becomes a 8x8 matrix representation for $Cl_{(0,3)}$. But, as one can see, there is also a 4x4 real representation for $Cl_{(0,3)}$; this is the minimal representation.

Later, it will be shown the matrix representation of the Clifford algebras and therefore it will be clear why these are the maximal ones.

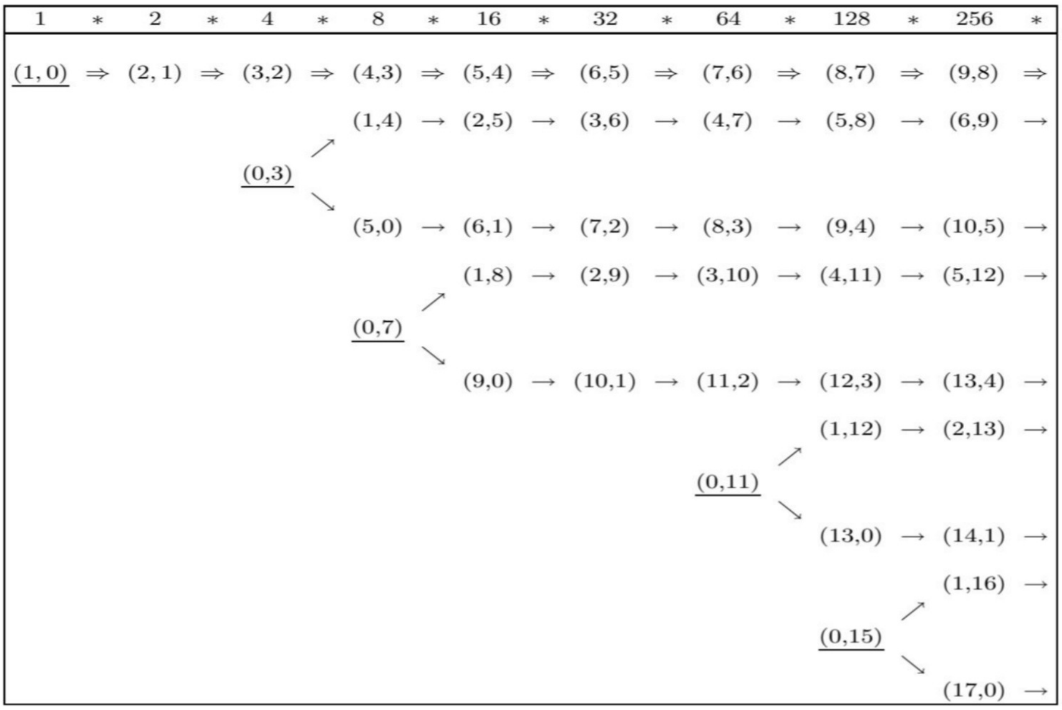

Figure 2.1: Maximal Clifford algebras

2.3 Classification

A Clifford algebra can be normal, almost complex or quaternionic regarding their minimal matrix representation, according to Okubo in [13] and [14]. Assume that there is a real matrix B that commutes with all the generators of the Clifford algebras:

$$[B, \gamma_\mu] = 0, \quad \text{where} \quad \mu = p + q \tag{2.6}$$

- In a normal Clifford algebra, the most general matrix B will be given by:

$$B = a\mathbf{1} \tag{2.7}$$

where a is a real constant and $\mathbf{1}$ is the matrix of the identity element.

- In a almost complex Clifford algebra, the most general matrix B will be given by:

$$B = a\mathbf{1} + bJ \tag{2.8}$$

where a and b are real constants and J satisfies the following properties:

$$J^2 = -\mathbf{1} \quad \text{and} \quad [J, \gamma_\mu] = 0 \tag{2.9}$$

- In a quaternionic Clifford algebra, the most general matrix B will be given by:

$$B = a_0\mathbf{1} + \sum_{j=1}^{3} a_j E_j \tag{2.10}$$

where a_0 and a_j are real constants and E_j satisfies the following properties:

$$[E_j, \gamma_\mu] = 0 \tag{2.11}$$

$$E_j E_k = -\delta_{jk}\mathbf{1} + \sum_{r=1}^{3} \varepsilon_{jkr} E_r \tag{2.12}$$

where ε is the Levi-Civita symbol.

- Normal Clifford algebra: $p - q(\text{mod}8) = 0, 1, 2$
- Almost complex Clifford algebra: $p - q(\text{mod}8) = 3, 7$
- Quaternionic Clifford algebra: $p - q(\text{mod}8) = 4, 5, 6$

where mod8 means that it is taken modulo 8.

Hence, on figure 2.1, we have the following. $Cl_{(1,0)}, Cl_{(0,7)}, Cl_{(0,15)}$ and so on along with every Clifford algebra generated by them are normal Clifford algebras. $Cl_{(0,3)}, Cl_{(0,11)}$ and so on along with every Clifford algebra generated by them are quaternionic Clifford algebras. This is only true for the minimal representation, the irreducible one.

Instead of talking about the minimal matrix representation of the Clifford algebras, first let's introduce the Cayley-Dickson algebras because they are directly related to them. Therefore, they are important to the understanding and presentation of the minimal representations.

Chapter 3

Cayley-Dickson algebras

A Cayley-Dickson construction is a doubling of an algebra with itself. The fundamental algebra is the reals $\mathbb{R}$ and all Cayley-Dickson algebras constructed through this process will double the dimension of the algebra, see references [[2],[3]] for introduction on the subject.

Let $\mathbb{A}$ be a Cayley-Dickson algebra over the reals, then $\mathbb{A}^2$ will be defined by the following relations:

1. Multiplication: $(x, y)\cdot(z, w) = (xz + \epsilon w^* y, wx + yz^*)$.
2. Conjugation: $(x, y)^* = (x^*, -y)$.
3. "Norm": $\mathcal{N}(x, y) = \mathcal{N}(x) - \epsilon\mathcal{N}(y)$.

Definitions 1-3 generates the following properties:

Multplication by a real number: $a(x, y) = (ax, ay), \quad a \in \mathbb{R}$.

Conjugation of multiplication: $[(x, y)\cdot(z, w)]^* = (z, w)^* \cdot (x, y)^*$

where x, y, z and $w \in \mathbb{A}$. If $\mathbb{A}$ is a division algebra, then $\epsilon = -1$ creates a division algebra and $\epsilon = 1$ creates a split-division algebra.

A division algebra is an algebra that obeys the following relation:

$$\text{If} \quad ab = 0 \quad \text{then} \quad a \ \text{ or } \ b = 0 \tag{3.1}$$

A Cayley-Dickson algebra can be commutative, associative and alternative. Consider x, y and $z \in \mathbb{A}$, then it is:

$$\text{Commutative} \quad \text{if} \quad xy - yx = 0 \tag{3.2}$$

$$\text{Associative} \quad \text{if} \quad (xy)z - x(yz) = 0 \tag{3.3}$$

$$\text{Alternative} \quad \text{if} \quad x(xy) = x^2 y \quad \text{and} \quad (yx)x = yx^2 \tag{3.4}$$

Here is a list of the important division algebras for further use:

- $\mathbb{R}$: Commutative, associative and alternative.
- $\mathbb{C}$: Commutative, associative and alternative.
- $\mathbb{H}$: Non-commutative, associative and alternative.
- $\mathbb{O}$: Non-commutative, non-associative and alternative.

The split-division algebras will be denoted by a tilde above the division algebra symbols, for example, $\tilde{\mathbb{C}}$ is the split-complex numbers.

- Another important property of Cayley-Dickson algebras is that every one of them except the fundamental, which is the reals, contain as a sub-algebra another Cayley-Dickson division algebra. For example, the octonions and split-octonions have as a sub-algebra the quaternions. The quaternions together with the split-quaternions have the complex numbers as a sub-algebra. This is important for the study of the topic of graded algebras

The important Cayley-Dickson algebras above are the only division algebras. Therefore, beyond the octonions, the term split doesn't make sense anymore because both doubling values for ϵ will generate non-division algebras. However, it will continue to be used. The split algebras will denote the Cayley-Dickson algebras which has space-like vectors. The first Cayley-Dickson algebra which is not a division algebra is the sedenions. They are generated by the doubling of the octonions.

- $\mathbb{S}$: Non-commutative, non-associative and non-alternative.

3.1 Complex and Split-complex numbers

A complex number can be written in form of a Cayley-Dickson doubling in the following way:

$$z = xe_0 + ye_1 = x(1,0) + y(0,1) \qquad x, y \in \mathbb{R} \tag{3.5}$$

Where e_0 is the identity and e_1 is the imaginary, $e_1^2 = -e_0$. By forcing a 2x2 matrix representation, the obvious matrix for e_0 is the identity matrix. Using the square equals minus the identity relation, one can define a condition for the imaginary matrix. One particular choice for it is the anti-symmetric matrix "A" in 2.3. Hence a general complex number is given by:

$$z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \quad x, y \in \mathbb{R} \tag{3.6}$$

The split-complex numbers are the same but with the relation $e_1^2 = e_0$, doing the same procedure will give for $\tilde{e}_1$ matrix "X" in 2.3. Therefore, a split-complex number is given by:

$$\tilde{z} = \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \quad x, y \in \mathbb{R} \tag{3.7}$$

3.2 Quaternions and Split-quaternions

A general quaternion is given by:

$$z = ae_0 + be_1 + ce_2 + de_3 = a(1,0) + b(i,0) + c(0,1) + d(0,i)$$

$$a, b, c, d \in \mathbb{R}, \quad 1, i \in \mathbb{C} \tag{3.8}$$

where e_0 is always the identity and the vectors obey the following properties:

$$\begin{aligned} e_1^2 = e_2^2 = e_3^2 &= -e_0 \\ e_1 e_2 &= e_3 \\ e_3 e_1 &= e_2 \\ e_2 e_3 &= e_1 \\ e_i e_j &= -e_j e_i \end{aligned} \tag{3.9}$$

Using these relations one can find a matrix representation for the quaternions:

$$z = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$z = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \quad x, y \in \mathbb{C} \tag{3.10}$$

The quaternionic multiplication can be written in the following form:

$$e_i e_j = -\delta_{ij} 1 + \varepsilon_{ijk} \delta^{kl} e_l \qquad \text{where} \quad \varepsilon_{123} = 1 \tag{3.11}$$

epsilon is the Levi-Civita totally anti-symmetric tensor and the sum convention is adopted here unless told otherwise.

For split-quaternions there are different relations:

$$\begin{aligned} e_0^2 &= e_0 & \tilde{e}_1 \tilde{e}_2 &= \tilde{e}_3 \\ \tilde{e}_1^2 &= -e_0 & \tilde{e}_3 \tilde{e}_1 &= \tilde{e}_2 \\ \tilde{e}_2^2 &= e_0 & \tilde{e}_2 \tilde{e}_3 &= -\tilde{e}_1 \\ \tilde{e}_3^2 &= e_0 & \tilde{e}_2 \tilde{e}_1 &= -\tilde{e}_3 \\ \tilde{e}_3 \tilde{e}_2 &= \tilde{e}_1 & \tilde{e}_1 \tilde{e}_3 &= -\tilde{e}_2 \\ \tilde{e}_i \tilde{e}_j &= -\tilde{e}_j \tilde{e}_i \end{aligned} \tag{3.12}$$

Using these relations a split-quaternion is given by:

$$z = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$z = \begin{pmatrix} a+ib & c+id \\ c-id & a-ib \end{pmatrix} = \begin{pmatrix} x & y \\ y^* & x^* \end{pmatrix}, \quad x, y \in \mathbb{C} \tag{3.13}$$

The split-quaternionic multiplication can be written in the following form, defining the metric:

$$\tilde{\eta}_{ij} = \tilde{\eta}^{ij} \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.14}$$

$$\tilde{e}_i \tilde{e}_j = \tilde{\eta}_{ij} 1 + \varepsilon_{ijk} \tilde{\eta}^{kl} \tilde{e}_l \tag{3.15}$$

3.13 is the complex representation of the split-quaternions. They have a real 2x2 matrix representation given by 2.3. There is also a 4x4 real matrix representation for both quaternions and split-quaternions, more on them later.

There is a way to unify the quaternions and split-quaternions results using the ϵ from the Cayley-Dickson doubling, defining:

$$\eta_{ij} = \eta^{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \tag{3.16}$$

Where $\epsilon = -1$ for a division algebra and $\epsilon = 1$ for a split-division algebra, the multiplication becomes:

$$e_i e_j = \eta_{ij} 1 + \epsilon \varepsilon_{ijk} \eta^{kl} e_l \tag{3.17}$$

The general matrix representation is given by:

$$z = (x, y) \equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix} \tag{3.18}$$

The 4x4 real matrix representation can be created in two ways. One is to use 3.18 and instead of x and y, substitute for the complex numbers 2x2 matrices, "I" and "A". The other way is described in [[6],[75],[76]]. It is the relation that the left multiplication for the octonions produces the matrices generators of $Cl_{(0,7)}$. F. Toppan's, Z. Kuznetsova and N. Aizawa called that quasi-nonassociativity, here it will be the same. A left multiplication on a general quaternion or split-quaternion can be written in matrix form and $Cl_{(0,3)}$ and $Cl_{(2,1)}$ will appear. However, they are isomorphic to the quaternions and split-quaternions so they are a 4x4 real matrix representation.

Left multiplication is the action of a vector of the algebra on a general element of the algebra with the vector being on the left. Right multiplication is the dual of the left and is also possible, but the common convention in the literature is the left multiplication.

Here is the result via the second way using the generalized structure constants, it is derived using 3.17:

$$e_i\vec{x} = e_i x_0 + (\eta_{ij}1 + \epsilon\varepsilon_{ijk}\eta^{kl}e_l)x_j = \eta_{ij}x_j + (x_0\delta_i^l + \epsilon\varepsilon_{ijk}\eta^{kl}x_j)e_l$$

It can be put in matrix form:

$$\vec{x}_i' = \gamma_i^L\vec{x} \qquad \text{with} \qquad (\gamma_i^L)_{MN} = \left(\begin{array}{c|c} 0 & \eta_{ij} \\ \hline \delta_i^l & \epsilon\varepsilon_{ijk}\eta^{kl} \end{array}\right)_{MN} \tag{3.19}$$

Where $M = 0, l$ and $N = 0, j$ with $l, m = 1,2,3$ and γ_i^L represents the matrices found by left multiplication. The matrices produced by right multiplication will only change the sign of the vector multiplication:

$$(\gamma_i^R)_{MN} = \left(\begin{array}{c|c} 0 & \eta_{ij} \\ \hline \delta_i^l & -\epsilon\varepsilon_{ijk}\eta^{kl} \end{array}\right)_{MN} \tag{3.20}$$

It can be shown that the right multiplication matrices are related to the ones produced by left multiplication via a similarity transformation. Confirming that right multiplication does not add any information. Matrices 3.19 for the quaternions are given by:

$$e_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.21}$$

And for the split-quaternions by:

$$\tilde{e}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \tilde{e}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.22}$$

3.3 Octonions and Split-octonions

A general octonion will be the same than the quaternions:

$$e_0 = (1,0),\ e_1 = (i,0),\ e_2 = (j,0),\ e_3 = (k,0),\ e_4 = (0,1),\ e_5 = (0,i),\ e_6 = (0,j),\ e_7 = (0,k) \tag{3.23}$$

where $i, j, k \in \mathbb{H}$

The only relation that will change is the multiplication which is now non-associative:

$$e_i e_j = -\delta_{ij} + C_{ijk}\delta^{kr} e_r \tag{3.24}$$

Where C_{ijk} is the structure constant of the octonions, it is a completely anti-symmetric tensor. There are 480 possible conventions for this structure constant. The one that will be used in this thesis is the following:

$$C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = 1 \tag{3.25}$$

Another important convention is the inverse of the above:

$$C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = -1 \tag{3.26}$$

For the split-octonion,s the metric becomes $\tilde{\eta}_{ij} = diag(-1,-1,-1,1,1,1,1)$ and the multiplication is given by:

$$\tilde{e}_i \tilde{e}_j = \tilde{\eta}_{ij} 1 + \tilde{C}_{ij}{}^k \tilde{e}_r = \tilde{\eta}_{ij} 1 + \tilde{C}_{ijk}\tilde{\eta}^{kr}\tilde{e}_r \tag{3.27}$$

the structure constant of the split-octonions $\tilde{C}_{ijk}$ is completely anti-symmetric, more on them in the appendix A. The structure constant is:

$$\tilde{C}_{132} = \tilde{C}_{145} = \tilde{C}_{176} = \tilde{C}_{246} = \tilde{C}_{257} = \tilde{C}_{347} = \tilde{C}_{365} = 1 \tag{3.28}$$

Everything that was done for the quaternions and split-quaternions can be replicated for octonions and split-octonions, the only difference is the matrix representation for the vector elements. Here is only the results. Defining the metric:

$$\eta_{ij} = \eta^{ij} \to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{3.29}$$

The multiplication is given by:

$$e_i e_j = \eta_{ij} + C_{ijk}\eta^{kl} e_l \tag{3.30}$$

it is implied that, in the split-octonionic case, there is a tilde on every element of the formula.

The matrices that generates $Cl_{(0,7)}$ and $Cl_{(4,3)}$ are given by:

$$(\gamma_i^L)_{MN} = \left(\begin{array}{c|c} 0 & \eta_{ij} \\ \hline \delta_i^l & C_{ijk}\eta^{kl} \end{array}\right)_{MN} \qquad (\gamma_i^R)_{MN} = \left(\begin{array}{c|c} 0 & \eta_{ij} \\ \hline \delta_i^l & -C_{ijk}\eta^{kl} \end{array}\right)_{MN} \tag{3.31}$$

Now a matrix representation, with the standard matrix multiplication, is not possible for the octonions because the standard product is always associative. Max Zorn [77] created a 2x2 vector-matrix representation mixing real numbers and vectors in $\mathbb{R}^3$. However, in [[16],[17]], it is presented 2x2 matrices for the vector basis of Cayley-Dickson algebras using the doubling construction, defining a new matrix multiplication. Together with I.P. de Freitas it was found the same result, but we studied also representations of bigger dimension than 2. The Cayley-Dickson doubling already gives us the correct matrix representation for every Cayley-Dickson algebra. If we look at 3.18, we can try to rewrite the Cayley-Dickson multiplication in matrix form:

$$(x,y)(z,w) = \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix}\begin{pmatrix} z & w \\ \epsilon w^* & z^* \end{pmatrix} = \begin{pmatrix} xz+\epsilon y w^* & xw+yz^* \\ \epsilon x^* w^* + \epsilon y^* z & x^* z^* + \epsilon y^* w \end{pmatrix} \tag{3.32}$$

But from the defined multiplication:

$$(x,y)(z,w) = (xz+\epsilon w^* y, wx+yz^*) \tag{3.33}$$

If you put it on matrix form 3.18:

$$(x,y)(z,w) = \begin{pmatrix} xz+\epsilon w^* y & wx+yz^* \\ \epsilon(wx+yz^*)^* & (xz+\epsilon w^* y)^* \end{pmatrix} = \begin{pmatrix} xz+\epsilon w^* y & wx+yz^* \\ \epsilon x^* w^* + \epsilon z y^* & z^* x^* + \epsilon y^* w \end{pmatrix} \tag{3.34}$$

Comparing 3.32 and 3.34 show us that it is not the same, there is a subtle difference that changes everything. This means that the definition for the multiplication is not just the usual one. We are going to follow [17] with the sign for the generalized multiplication.

Multiplication of the matrix representation of the Cayley-Dickson doubling:

$$(x,y)\circ(z,w) \equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix}\begin{pmatrix} z & w \\ \epsilon w^* & z^* \end{pmatrix} + \begin{pmatrix} \epsilon[w^*,y] & [w,x] \\ \epsilon[z,y^*] & [z^*,x^*] \end{pmatrix} \tag{3.35}$$

Being [,] the commutator, 3.35 consists of the usual matrix multiplication plus a correction term. This term is zero up to the quaternions and split-quaternions. From the octonions and split-octonions this term won't be always zero because the quaternions and split-quaternions are anti-commutative, therefore the commutators can be non-zero. Using this multiplication and defining norm and conjugation we can create a matrix realization for every Cayley-Dickson algebra.

In resume, for the octonions and split-octonions there are matrix realizations, 2x2 quaternionic, 4x4 complex or even 8x8 real representation, more on them later.

3.4 Important algebras

This section is dedicated to the relation between Clifford algebras and division algebras. But first it is important to talk about the matrix representation for universal Clifford algebras because it relates with division algebras, for this section see references [[11],[12]].

Clifford algebras matrix representation

Let's come back to the Clifford algebras and their representation. The figures below are 4.1 and 4.2 of [11], in pages 101 and 102. Considering that $\mathcal{M}(N,\mathbb{K})$ is the complete group of all NxN real matrices with entries that belong to the $\mathbb{K}$ algebra. Also $n = p + q$:

$p-q$ mod 8	0	1	2	3
$\mathcal{C}\ell_{p,q}$	$\mathcal{M}(2^{[n/2]},\mathbb{R})$	$\mathcal{M}(2^{[n/2]},\mathbb{R}) \oplus \mathcal{M}(2^{[n/2]},\mathbb{R})$	$\mathcal{M}(2^{[n/2]},\mathbb{R})$	$\mathcal{M}(2^{[n/2]},\mathbb{C})$
$p-q$ mod 8	4	5	6	7
$\mathcal{C}\ell_{p,q}$	$\mathcal{M}(2^{[n/2]-1},\mathbb{H})$	$\mathcal{M}(2^{[n/2]-1},\mathbb{H}) \oplus \mathcal{M}(2^{[n/2]-1},\mathbb{H})$	$\mathcal{M}(2^{[n/2]-1},\mathbb{H})$	$\mathcal{M}(2^{[n/2]},\mathbb{C})$

Figure 3.1: Real Clifford algebras

And now the complex Clifford algebras or the tensor product of the complex with the real Clifford algebras:

n even	$\mathcal{C}\ell_{\mathbb{C}}(2k) = \mathcal{M}(2^k,\mathbb{C})$
n odd	$\mathcal{C}\ell_{\mathbb{C}}(2k+1) = \mathcal{M}(2^k,\mathbb{C}) \oplus \mathcal{M}(2^k,\mathbb{C})$

Figure 3.2: Complex Clifford algebras

By looking at figure 3.1, it is clear why the maximal Clifford algebras are the ones with odd dimension n, see 2.1. They are the algebras in which $p - q(mod 8) = 1,3,5,7$. Their matrix representation are the most general ones.

For the sake of clarity and for further use here is a list of important Clifford algebras in both conventions:

$$Cl_{(0,0)} \simeq \mathbb{R} \qquad Cl_{(1,1)} \simeq Cl_{(2,0)} \simeq \tilde{\mathbb{H}} \simeq \mathcal{M}(2,\mathbb{R})$$

$$Cl_{(0,1)} \simeq \mathbb{C} \qquad Cl_{(0,3)} \simeq \mathbb{H} \oplus \mathbb{H}$$

$$Cl_{(1,0)} \simeq \tilde{\mathbb{C}} \qquad Cl_{(2,1)} \simeq \mathcal{M}(2,\mathbb{R}\oplus\mathbb{R})$$
$$Cl_{(0,2)} \simeq \mathbb{H} \qquad Cl_{(0,7)} \simeq Cl_{(4,3)} \simeq \mathcal{M}(8,\mathbb{R}\oplus\mathbb{R}) \tag{3.36}$$

The relation between division algebras and Clifford algebras is given. Regarding the classification of Clifford algebras, the complex numbers are a normal Clifford algebra and the quaternions are a quaternionic Clifford algebra. The octonions are not isomorphic to the Clifford algebra $Cl_{(0,7)}$, however, they have a relation through the left and right actions. Curiously, once it is lost the direct relation between Cayley-Dickson algebras and Clifford algebras, in octonions and split-octonions, one can see that the Clifford algebras with the same signature than the Cayley-Dickson algebras becomes isomorphic, they are all normal Clifford algebras. There is the following:

$$\begin{array}{cc} Cl_{(0,1)} \neq Cl_{(1,0)} & Cl_{(0,3)} \neq Cl_{(2,1)} \\ Cl_{(0,7)} \simeq Cl_{(4,3)} & Cl_{(0,15)} \simeq Cl_{(8,7)} \\ Cl_{(0,31)} \simeq Cl_{(16,15)} & Cl_{(0,63)} \simeq Cl_{(32,31)} \end{array} \tag{3.37}$$

$Cl_{(0,1)}$ is not isomorphic to $Cl_{(1,0)}$ because there is only one generator, each with different signature. $Cl_{(0,3)}$ is not isomorphic to $Cl_{(2,1)}$ because one is a quaternionic Clifford algebra and the other is a normal Clifford algebra. The rest of the relations of isomorphism are true because all are normal Clifford algebras, figure 2.1 helps to visualize this.

3.5 Non-associative algebras

From the Octonions and split-octonions all Cayley-Dickson algebras are non associative [7]. Therefore, a direct isomorphism is not possible, but there is a relation. To see that, one needs to understand multiplication, however, the order of a multiplication is now important. The usual convention in literature is to use the left multiplication, which means the following.

Let $w \in \mathbb{A}$ be a general vector of a division or split-division algebra, then the left multiplication of a vector $e_i \in \mathbb{A}$ on w is given by:

$$\text{left multiplication:} \quad e_i w \tag{3.38}$$

This was already discussed in section 3.2. Another important definition is the left multiplication chain:

$$\overleftarrow{e_i e_j} w = e_i(e_j w) \tag{3.39}$$

C.Furey showed in [7] the properties of this definition, section 6.3. She showed that a left multiplication in the bi-octonions can be expressed as a sum of right multiplication chains, concluding that no information is lost when one uses only one type of multiplication.

The relation between the Octonions and split-octonions with $Cl_{(0,7)}$ and $Cl_{(4,3)}$ is given by[[6],[75],[76]]:

- A left or right multiplication of a Octonionic or split-octonionic vector on a general element of the respective algebra will generate the matrices that represents Clifford algebras $Cl_{(0,7)}$ and $Cl_{(4,3)}$ respectively, 3.31.

To see that, one needs to separate the equation in a multiplication of two matrices. One will be a matrix with the octonionic vectors and the other will be a 8x8 matrix with the signs of the multiplication.

Another very important property is that the sum of all multiplication chains forms an associative algebra, which can be showed to be isomorphic to a certain Clifford algebra. C. furey in [7] talks about the bi-quaternions and bi-octonions, but to every Cayley-Dickson algebra up to the octonions and split-octonions this is true. C. Furey denotes this sum of chains with the left arrow above the Cayley-Dickson symbol.

Before showing some examples of algebras generated by sum of multiplication chains, there is a relation called the Hodge duality that is important here. What it states is that the product of $n-r = p+q-r$ orthogonal elements of a Clifford algebra is equivalent to the product of elements of rank r. For example, the case of $Cl_{(0,7)}$, the multiplication of 4 elements will be equivalent than of 3 elements and so on because $7-3=4$.

Associative Cayley-Dickson algebras will always generate itself with multiplication chains, so it is redundant to put here. C. Furey used the bi-quaternions in [7], their left multiplication chain gives $\mathbb{C}\otimes\overleftarrow{\mathbb{H}} \simeq Cl_{(0,3)} \simeq \mathbb{C}l_2$, but with octonions and split-octonions there is the following:

$$\begin{gathered} \overleftarrow{\mathbb{O}} \simeq Cl_{0,6} \simeq Cl_{3,3} \\ \mathbb{C}\otimes\overleftarrow{\mathbb{O}} \simeq \mathbb{C}l_6 \\ \tilde{\mathbb{C}}\otimes\overleftarrow{\mathbb{O}} \simeq Cl_{0,7} \simeq Cl_{4,3} \end{gathered} \qquad \begin{gathered} \overleftarrow{\tilde{\mathbb{O}}} \simeq Cl_{3,3} \simeq Cl_{0,6} \\ \mathbb{C}\otimes\overleftarrow{\tilde{\mathbb{O}}} \simeq \mathbb{C}l_6 \\ \tilde{\mathbb{C}}\otimes\overleftarrow{\tilde{\mathbb{O}}} \simeq Cl_{4,3} \simeq Cl_{0,7} \end{gathered} \tag{3.40}$$

C. Furey also showed why the first relation is not isomorphic to $Cl_{(0,7)}$ instead of $Cl_{(0,6)}$. There is an additional relation that connects every multiplication chain. One can say that one multiplication chain is linear dependent of all the other ones. Additionally, because of the Hodge duality it can only create an algebra with 64 elements, $1+7+21+35=64$.

Chapter 4

Alphabetic (Re)presentation

The alphabetic (Re)presentation comes from the properties of the tensor product between matrices. Named by F. Toppan and P. Vierbeek [66], the idea is to assign words in a letter alphabet to represent the matrices and then analyze their properties faster and with less difficulties. It is defined as follows:

- First, define a set of fundamental matrices that will generate every other matrix.
- Second, assign letters of one's preference to each of the fundamental matrices, there can be no repetition of letters.
- All matrices one can create with the tensor product between the fundamental ones have a unique alphabetic (Re)presentation.
- To create words in the alphabet one only need to hide the tensor product symbol between letters; it is understood that between letters that form a word there is a tensor product.

A set of fundamental matrices of great interest is the vector space of real 2x2 matrices with null trace plus the identity, denoted here by $\mathscr{M}(2,\mathbb{R})$:

$$\mathscr{M}(2,\mathbb{R}): \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.1}$$

Below there is a list of the properties of the alphabetic (re)presentation that makes it so useful:

1^{st}: If a matrix possesses an even number of letters A, then it is space-like or symmetric, if it possesses an odd number it is time-like or anti-symmetric.

2^{nd}: The number of letters in a word gives the dimension of the matrix, if a matrix have "n" letters than it is 2^nx2^n in size.

3^{rd}: The first letter of the word, that represents a matrix, gives if the matrix is block-diagonal or block-anti-diagonal, being I and Z diagonal, X and A anti-diagonal.

4^{th}: Allow us to see if two matrices commute or anti-commute without having to multiply them. One just need to see if letters in the same position from both words commute or anti-commute. If there is an even number of anti-commutations then the matrices commute, if there is an odd number of anti-commutations then they anti-commute.

5^{th}: Matrix multiplication can be done by multiplying letters in the same position, using the multiplication table 4.1.

6^{th}: The algorithm of creation of new bases can be put in an alphabetic form.

On 5^{th} the table is the multiplication of the matrices in 4.1:

—	I	Z	X	A
I	I	Z	X	A
Z	Z	I	A	X
X	X	$-A$	I	$-Z$
A	A	$-X$	Z	$-I$

Table 4.1: Multiplication of the 2x2 real matrices

4.1 Quaternions and Split-quaternions

Using as an example the matrices 3.21 and 3.22, their alphabetic (re)presentation is given by:

$$e_1 = -I \otimes A = -IA, \quad e_2 = -A \otimes Z = -AZ, \quad e_3 = -A \otimes X = -AX \tag{4.2}$$

$$\tilde{e}_1 = -I \otimes A = -IA, \quad \tilde{e}_2 = X \otimes Z = XZ, \quad \tilde{e}_3 = X \otimes X = XX \tag{4.3}$$

By looking at their alphabetic (re)presentation it is explicit properties 1, 2 and 3 of the last section. If one wants to check for anti-commutation, just follow property 4. Example, AZ and AX anti-commute because A commutes with itself and Z anti-commutes with X leaving an odd number of anti-commutations. However, AZ commutes with XX because A and Z anti-commute with X, hence there are two anti-commutations.

Property 5 is the same, $e_1 e_2 = e_3$, so $-IA * -AZ$, the minus signs cancel each other, $I * A = A$ and $A * Z = -X$, the result is $-IA * -AZ = -AX$ which gives exactly e_3. It will always work because this is a property of the tensor product between matrices.

4.2 Covariant and Contravariant

The algebra that they create have a metric, in the case of split algebras this metric is non-trivial. It makes sense to define then covariant and contravariant elements, Matrices

3.21 and 3.22 represents covariant vectors, with the metric given by 3.16, one can construct the contravariant vectors:

$$e^i = \eta^{ij} e_j \qquad \text{and} \qquad \gamma^i = \eta^{ij} \gamma_i \tag{4.4}$$

$$e^1 = IA, \qquad e^2 = AZ, \qquad e^3 = AX \tag{4.5}$$

$$\tilde{e}^1 = IA, \qquad \tilde{e}^2 = XZ, \qquad \tilde{e}^3 = XX \tag{4.6}$$

4.3 Clifford doubling

The algorithm for the Clifford doubling discussed in section 2.2, 2.4 and 2.5, can be put in alphabetic form. It is pretty simple, the matrices in 2.4 can be written in the following form, $\Gamma_i = X \otimes \gamma_i$, $\Gamma_{n+1} = A \otimes 1_d$ and $\Gamma_{n+2} = Z \otimes 1_d$. In the same way, the matrices in 2.5 can be written in the following form, $\Gamma_i = A \otimes \gamma_i$, $\Gamma_{n+1} = X \otimes 1_d$ and $\Gamma_{n+2} = Z \otimes 1_d$. Therefore, the two algorithms become:

1^{st}: $Cl_{(p,q)} \to Cl_{(p+1,q+1)}$:

$$\Gamma_i = X\gamma_i, \quad \Gamma_{n+1} = AI^m, \quad \Gamma_{n+2} = ZI^m, \quad n = p+q, \quad m = \log_2[dim(\gamma_i)] \tag{4.7}$$

2^{nd}: $Cl_{(p,q)} \to Cl_{(q+2,p)}$:

$$\Gamma_i = A\gamma_i, \quad \Gamma_{n+1} = XI^m, \quad \Gamma_{n+2} = ZI^m, \quad n = p+q, \quad m = \log_2[dim(\gamma_i)] \tag{4.8}$$

It becomes easy to create a new Clifford algebra basis with this algorithm. For example, the Clifford algebra $Cl_{(1,3)}$ was used in Dirac's theory, to create it, one can just use the first algorithm with the quaternions 4.5; using $Cl_{(0,3)}$ to create $Cl_{(1,4)}$:

$$Cl_{(1,4)}: \qquad e^1 = XIA, \qquad e^2 = XAZ, \qquad e^3 = XAX, \qquad e^4 = AII, \qquad e^5 = ZII \tag{4.9}$$

Excluding e^4 generates the real 8x8 representation of $Cl_{(1,3)}$. If one wants the exact Dirac matrices, the 4x4 complex representation, one can take the imaginary number of the second Pauli matrix and an alphabetic (re)presentation is possible, nothing changes:

$$\text{Pauli Matrices}: \qquad \sigma_x = X, \qquad \sigma_y = -iA, \qquad \sigma_z = Z \tag{4.10}$$

where between "i" and "A" there is the usual multiplication of a number with a matrix. Treating "i" like a minus sign leave all six properties of the alphabetic (re)presentation unchanged. To reach $Cl_{(1,3)}$ from $Cl_{(3,0)}$ the second algorithm needs to be used:

$$Cl_{(2,3)}: \qquad e^1 = AX, \qquad e^2 = -iAA, \qquad e^3 = AZ, \qquad e^4 = XI, \qquad e^5 = ZI \tag{4.11}$$

These 5 matrices are the Dirac matrices, e^4 is the fermion operator. There is a possibility to use e^5 as a fermion operator instead; then the Clifford algebra $Cl_{(1,3)}$ becomes a Weyl Clifford algebra, meaning that every generator is represented by a block anti-diagonal matrix.

4.4 Cayley-Dickson doubling

The algorithm for the Cayley-Dickson doubling discussed in chapter 3 can be put in alphabetic form. Here is the general matrix representation saw in 3.18, it is the same definition of the references [[16],[17]]:

$$(x, y) \equiv \begin{pmatrix} x & y \\ \epsilon y^* & x^* \end{pmatrix} \tag{4.12}$$

The identity always remain the same. When $x = 1$ then $x^* = x$, hence the identity matrix won't change signs. Forgetting about the epsilon for a moment, when x is an imaginary vector, then $x^* = -x$, one can see that the structure of the matrix Z appears here just as it did in the Clifford doubling. When $y = 1$ then $y^* = y$, that structure is from the matrix X. When y is an imaginary then $y^* = -y$, then the structure of the matrix A appear. One can see that it is easy to put the doubling in an alphabetic form:

Consider a Cayley-Dickson algebra $\mathbb{A}$, with dimension $d/2$ and an "i" number of vector elements "e_i". Using the doubling process to create a new algebra $\mathbb{A}^2$, with dimension d. The matrix realization of the elements of the new algebra, "E_μ", can be found via the following algorithms:

Division algebra:

$$E_0 = I \otimes e_0, \quad E_i = Z \otimes e_i, \quad E_{d/2} = A \otimes e_0 \quad \text{and} \quad E_{i+d/2} = X \otimes e_i \quad \text{with} \quad 1 \leq i < d/2 \tag{4.13}$$

Split-division algebra:

$$E_0 = I \otimes e_0, \quad E_i = Z \otimes e_i, \quad E_{d/2} = X \otimes e_0 \quad \text{and} \quad E_{i+d/2} = A \otimes e_i \quad \text{with} \quad 1 \leq i < d/2 \tag{4.14}$$

Considering that x and y are isomorphic to the matrices $\mathbf{X}$ and $\mathbf{Y}$:

$$(x, y) \equiv \left(\begin{array}{c|c} \mathbf{X} & \mathbf{Y} \\ \hline \epsilon\mathbf{Y}^* & \mathbf{X}^* \end{array}\right) \tag{4.15}$$

Matrices 4.13 and 4.14 obey the relations:

1^{st} **Multiplication:**

$$(x, y)(z, w) \equiv \left(\begin{array}{c|c} \mathbf{X} & \mathbf{Y} \\ \hline \epsilon\mathbf{Y}^* & \mathbf{X}^* \end{array}\right)\left(\begin{array}{c|c} \mathbf{Z} & \mathbf{W} \\ \hline \epsilon\mathbf{W}^* & \mathbf{Z}^* \end{array}\right) + \left(\begin{array}{c|c} \epsilon[\mathbf{W}^*, \mathbf{Y}] & [\mathbf{W}, \mathbf{X}] \\ \hline \epsilon[\mathbf{Z}, \mathbf{Y}^*] & [\mathbf{Z}^*, \mathbf{X}^*] \end{array}\right) \tag{4.16}$$

2^{nd} **Conjugation:**

$$(x, y)^* \equiv \left(\begin{array}{c|c} \mathbf{X}^* & -\mathbf{Y} \\ \hline -\epsilon\mathbf{Y}^* & \mathbf{X} \end{array}\right) \tag{4.17}$$

3^{rd} **Norm:**

$$\mathcal{N}(x,y) \equiv \frac{1}{d}\mathrm{Tr}\left[\left(\begin{array}{c|c} \mathbf{X}^* & -\mathbf{Y} \\ \hline -\epsilon\mathbf{Y}^* & \mathbf{X} \end{array}\right)\left(\begin{array}{c|c} \mathbf{X} & \mathbf{Y} \\ \hline \epsilon\mathbf{Y}^* & \mathbf{X}^* \end{array}\right) + \left(\begin{array}{c|c} -\epsilon[\mathbf{Y}^*,\mathbf{Y}] & [\mathbf{Y},\mathbf{X}^*] \\ \hline -\epsilon[\mathbf{X},\mathbf{Y}^*] & [\mathbf{X}^*,\mathbf{X}^*] \end{array}\right)\right] \tag{4.18}$$

therefore, the construction of the division and split-division algebras can be done easily:

Complex and split-complex numbers:

The complex numbers are generated via the reals, the element is only $e_0 = 1$; there are no vectors. Using 4.13 and 4.14 the matrices 3.6 and 3.7 appear:

Complex number: $E_0 = I$, $E_1 = A$ **Split-Complex number**: $E_0 = I$, $E_1 = X$

Quaternions and split-quaternions:

Quaternions and split-quaternions are created from the complex numbers. Doing the same procedure, one can create the 2x2 complex representation by setting $e_0 = 1$ and $e_1 = i$ or the 4x4 real representation by setting $e_0 = I$ and $e_1 = A$:

Quaternions: $E_0 = II$, $E_1 = ZA$, $E_2 = AI$ e $E_3 = XA$

Split-Quaternions: $\tilde{E}_0 = II$, $\tilde{E}_1 = ZA$, $\tilde{E}_2 = XI$ e $\tilde{E}_3 = AA$

these matrices form a different representation for the quaternions and split-quaternions.

4.5 Octonions and Split-octonions

References for this section are [[2],[4]]. The octonions and split-octonions are created from the quaternions, therefore there are three vectors. The one dimensional representation of the quaternions generates, using 4.13 and 4.14, a 2x2 quaternionic representation isomorphic to the (split)octonionic algebra. The 2x2 complex representation in 3.10 for the quaternions generates the 4x4 complex representation. However, the most interesting is to use the 4x4 real representation, of the last section, to generate the 8x8 real representation isomorphic to the (split)octonions:

Octonions: $E_0 = III$, $E_1 = ZZA$, $E_2 = ZAI$, $E_3 = ZXA$, $E_4 = AII$, $E_5 = XZA$, $E_6 = XAI$, $E_7 = XXA$

Split-octonions: $\tilde{E}_0 = III$, $\tilde{E}_1 = ZZA$, $\tilde{E}_2 = ZAI$, $\tilde{E}_3 = ZXA$, $\tilde{E}_4 = XII$, $\tilde{E}_5 = AZA$, $\tilde{E}_6 = AAI$, $\tilde{E}_7 = AXA$

Check the appendix C for more information on these matrices. It is important to show that these matrices are not the matrices that generates $Cl_{(0,7)}$, below is the two groups for comparison:

$$\begin{array}{cc} Cl_{(0,7)}: & \text{Octonions:} \\ ZZA & ZZA \\ ZAI & ZAI \\ ZXA & ZXA \\ AII & AII \\ XIA & XZA \\ XAZ & XAI \\ XAX & XXA \end{array} \tag{4.19}$$

Important notes:

- One can check, using the fourth process explained in chapter 6 that, every matrix in $Cl_{(0,7)}$ anti-commutes, but it is not true for the octonionic ones, for example, ZZA and XAI commute.
- The octonionic matrices must anti-commute because the octonionic vectors anti - commute. What is happening is that the octonionic matrices obey the multiplication rule 4.16 with the correction term. Even both groups being very similar, the multiplication is different for each.
- The fact that the octonions anti-commute, but regarding to a non-associative multiplication separates them to the properties of Clifford algebras. In the literature it is called the octonionic Clifford algebra.
- Another important property is that the multiplication of the $Cl_{(0,7)}$ generators is not closed, but with the octonionic ones it must be closed. Also, on $Cl_{(0,7)}$, there is no complex conjugation, but in the octonionic matrices, even though they are real matrices, relation 4.17 allow us to define complex conjugation and transpose conjugation.
- The commutator of the $Cl_{(0,7)}$ matrices generates the Lorentz group. For the octonionic ones their commutator does not generate a Lie algebra, instead the coset $SO(p,q)/G_2$, where G_2 is the 14 dimensional exceptional Lie group of automorphisms of the octonions and p is the number of space-like generators, q is the number of temporal-like generators.

To prove that the multiplication works here is an example, $E_1 * E_7 = E_6$ according to our convention, to continue notice that:

$$ZZA = Z \otimes ZA = \left(\begin{array}{c|c} ZA & \mathbf{0} \\ \hline \mathbf{0} & -ZA \end{array}\right) \tag{4.20}$$

where zero is the 4x4 null matrix, so using the multiplication rule 4.16:

$$ZZA \times XXA = -AAI + \left(\begin{array}{c|c} \mathbf{0} & [XA, ZA] \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \qquad \text{with} \qquad XA * ZA = AI \tag{4.21}$$

Therefore:

$$ZZA \times XXA = \left(\begin{array}{c|c} \mathbf{O} & -AI \\ \hline AI & \mathbf{O} \end{array}\right) + \left(\begin{array}{c|c} \mathbf{O} & 2AI \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{O} & AI \\ \hline AI & \mathbf{O} \end{array}\right) = XAI \tag{4.22}$$

Exactly the right answer, now the opposite multiplication:

$$XXA \times ZZA = -AAI + \left(\begin{array}{c|c} \mathbf{O} & \mathbf{O} \\ \hline -[ZA, -XA] & \mathbf{O} \end{array}\right) \tag{4.23}$$

where the conjugation of XA was done using 4.17, remember that XA represents the quaternionic vector e_3.

Therefore:

$$XXA \times ZZA = \left(\begin{array}{c|c} \mathbf{O} & -AI \\ \hline AI & \mathbf{O} \end{array}\right) + \left(\begin{array}{c|c} \mathbf{O} & \mathbf{O} \\ \hline -2AI & \mathbf{O} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{O} & -AI \\ \hline -AI & \mathbf{O} \end{array}\right) = -XAI \tag{4.24}$$

Exactly the right answer, showing that in regard to the multiplication rule 4.16, the matrices ZZA and XXA anti-commute.

Sedenions:

The same process works for the sedenions and beyond, but it is important to know that one must be careful when using the multiplication rule 4.16. On the sedenions case, the correction matrix is going to feature commutation of the octonions instead of a normal commutator, it is the commutator according to the octonionic multiplication. Instead of using 4.16 one can just look at the structure constants of the octonions to help calculate.

4.6 Alphabetic (re)presentation of octonions

Why does the Alphabetic (re)presentation appears in all those algebras? The answer is that it doesn't have to appear. The matrices that can be written in alphabetic form are special because they can be generated via tensor product of the fundamental matrices. However, this doesn't happen all the time, actually, only if one works with a particular set of structure constant conventions the alphabetic (re)presentation will appear for the Clifford algebras $Cl_{(0,7)}$ and $Cl_{(4,3)}$ via right action 3.31.

The division algebras have an alphabetic (re)presentation because on the demonstration of the general matrix representation for the Cayley-Dickson algebras, it was used the fundamental matrices that form the alphabet as a representation for the complex numbers, split-complex numbers, quaternions and split-quaternions.

It is a choice to use the alphabetic (re)presentation because it helps a lot. Studying properties of the octonionic M-algebra without a program is only possible in the alphabetic form. Therefore, it is a recommendation to adopt the structure constant convention that produces the alphabetic represented matrices for Clifford algebras like we showed. However, there 480 structure constant conventions for the octonions, which ones produces alphabetic (re)presented matrices?

It is clear that the following convention that is being used generates alphabetic (re)-presented matrices with right action:

$$C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = 1 \tag{4.25}$$

The alphabetic (re)presented matrices of $Cl_{(0,7)}$ generated by it are given by:

Rep. 0:

$$\begin{array}{c} ZZA \\ ZAI \\ ZXA \\ AII \\ XIA \\ XAZ \\ XAX \end{array} \tag{4.26}$$

To answer that question one needs to see how the structure constant can be changed. By changing the sign of the generators of the basis, different structure constants will appear from 4.25. There is just one change of sign, two changes and so on.

Possible transformations:

1^{st}: Just one: $\begin{pmatrix} 7 \\ 1 \end{pmatrix} = 7$ 2^{nd}: Two transf. : $\begin{pmatrix} 7 \\ 2 \end{pmatrix} = 21$

3^{rd}: Three transf. : $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35$ 4^{th}: Four transf. : $\begin{pmatrix} 7 \\ 4 \end{pmatrix} = 35$

1^{st} Case

The seven possibilities break the alphabetic (re)presentation because the first column and line won't change signs. Here is an example, let's transform e_4, $e'_4 = -e_4$ then 145 becomes 154, 246 becomes 264 and 347 becomes 374 in 4.25:

$$\Gamma_1^R = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \Gamma_1^R = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

One can see that the alphabetic (re)presentation is lost because the constant structure 123 and 176 does not change, but 145 changes to 154. This will happen in all 7 possibilities.

2^{nd} Case

Of the 21 possibilities, 7 are the possible transformations, each can be obtained by three different ways. Surprisingly, all these 7 possibilities generates a representation that can be alphabetically (re)presented. The parenthesis with two elements below represents the elements that is going to be transformed via sign inversion.

1^{st} Rep.: $(e_2, e_3) = (e_4, e_5) = (e_3, e_6)$

2^{nd} Rep.: $(e_1, e_3) = (e_4, e_6) = (e_5, e_7)$

3^{rd} Rep.: $(e_1, e_2) = (e_4, e_7) = (e_5, e_6)$

4^{th} Rep.: $(e_1, e_5) = (e_2, e_6) = (e_3, e_7)$

5^{th} Rep.: $(e_1, e_4) = (e_2, e_7) = (e_3, e_6)$

6^{th} Rep.: $(e_1, e_7) = (e_2, e_4) = (e_3, e_5)$

7^{th} Rep.: $(e_1, e_6) = (e_2, e_5) = (e_3, e_4)$

$$\text{Rep. } 0: C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = 1 \tag{4.27}$$

$$\text{Rep. } 1: C_{123} = C_{145} = C_{176} = C_{264} = C_{275} = C_{374} = C_{356} = 1 \tag{4.28}$$

$$\text{Rep. } 2: C_{123} = C_{154} = C_{167} = C_{246} = C_{257} = C_{374} = C_{356} = 1 \tag{4.29}$$

$$\text{Rep. } 3: C_{123} = C_{154} = C_{167} = C_{264} = C_{275} = C_{347} = C_{365} = 1 \tag{4.30}$$

$$\text{Rep. } 4: C_{132} = C_{145} = C_{167} = C_{246} = C_{275} = C_{347} = C_{356} = 1 \tag{4.31}$$

$$\text{Rep. } 5: C_{132} = C_{145} = C_{167} = C_{264} = C_{257} = C_{374} = C_{365} = 1 \tag{4.32}$$

$$\text{Rep. } 6: C_{132} = C_{154} = C_{176} = C_{246} = C_{275} = C_{374} = C_{365} = 1 \tag{4.33}$$

$$\text{Rep. } 7: C_{132} = C_{154} = C_{176} = C_{264} = C_{257} = C_{347} = C_{356} = 1 \tag{4.34}$$

Rep. 0	Rep. 1	Rep. 2	Rep. 3	Rep. 4	Rep. 5	Rep. 6	Rep. 7
ZZA	ZZA	IZA	IZA	ZIA	ZIA	IIA	IIA
ZAI	IAI	ZAI	IAI	ZAZ	IAZ	ZAZ	IAZ
ZXA	IXA	IXA	ZXA	ZAX	IAX	IAX	ZAX
AII	AZI	AIZ	AZZ	AII	AZI	AIZ	AZZ
XIA	XZA	AIX	AZX	XZA	XIA	AZX	AIX
XAZ	AXZ	XAI	AXI	XAI	AXI	XAZ	AXZ
XAX	AXX	AAA	XXA	XXA	AAA	AXX	XAX

A property that holds here is that every convention differs from the others in 4 constants from the structure constant.

3^{rd} Case

There are 7 transformations that totally invert the convention 4.27, which are given by the structure constants:

Inverse convention:

$$(e_1, e_3, e_2) = (e_1, e_5, e_4) = (e_1, e_6, e_7) = (e_2, e_6, e_4) = (e_2, e_7, e_5) = (e_3, e_7, e_4) = (e_3, e_5, e_6)$$

$$C_{132} = C_{154} = C_{167} = C_{264} = C_{275} = C_{374} = C_{356} = 1 \tag{4.35}$$

This convention has an alphabetic (re)presentation, but with left action, not right action.

All the other 28 possibilities will break the alphabetic (re)presentation because it will be like the first case. For example, let's invert 247:

By inverting 2 and 4 one gets Rep. 6. However, if one inverts 2 and 7 one gets Rep. 5. In both cases, when the last transformation is made, it will break the alphabetic (re)presentation in the same way that was showed in case one.

4^{th} Case

If one gets the 7 seven transformations that inverts convention 4.27 and apply one more transformation, there are $7 * 4 = 28$ ways to do it; all will end up in one of the seven other conventions that are alphabetically written.

The other 7 possibilities are transformations that do not alter the convention, each possibility is given by the structure constant with 4 indices:

Convention 0: $6C_{ijkl} = \varepsilon_{ijklmnp} C_{mnp}$ where $\varepsilon_{1234567} = 1$

therefore: $C_{4576} = C_{2376} = C_{2345} = C_{1375} = C_{1364} = C_{1265} = C_{1247} = 1$

therefore (e_4, e_5, e_7, e_6), (e_2, e_3, e_7, e_6), (e_2, e_3, e_4, e_5), (e_1, e_3, e_6, e_4), (e_1, e_2, e_6, e_5), (e_1, e_2, e_4, e_7) bring Rep. 0 to itself.

Beyond four transformations is redundant to do because it comes back to the other cases. The conclusion is that, there are 8, among 480 different structure constant conventions for the octonions, that generates a representation of Clifford algebra $Cl_{(0,7)}$ via right action that can be alphabetically written. Curiously the same number of the dimension of the octonions and its matrix representation.

4.7 Octonionic M-algebra

E. Witten proposed in 1995 a framework to unify all consistent versions of superstring theory and supergravity in 11 dimensions, it is called today the M-theory. The dimension of the space-time is 11 and the spinors are real majorana spinors, that is why the Clifford algebra is the normal $Cl_{(10,1)}$ Clifford algebra.

The octonionic M-algebra is generated from two Clifford doublings of the octonions, see appendix D for their representation and also references [[71],[72],[73],[15]]. This means that there are two multiplications to account for, the usual matrix multiplication from the Clifford doubling and the octonionic multiplication from the vectors inside the matrices. This is possible by the fact that the octonions form the octonionic Clifford algebra in terms of the definition with relation 2.1, the notation will be $Cl_0(0,7)$. To create the Clifford algebra used in M-theory, $Cl_0(10,1)$, one needs to use both algorithms of section 4.3, first the second algorithm 4.8 to create $Cl_0(9,0)$:

$$Cl_0(9,0):$$

$$\begin{array}{ccc} AZZA, & AZAI, & AZXA, \\ AAII, & AXZA, & AXAI, \\ AXXA, & XIII, & ZIII \end{array} \tag{4.36}$$

The first letter for each word multiplies according to the usual matrix multiplication using the table in chapter 4, the last three letters represents the octonions and must multiply using 4.16:

$$AZZA * AXXA = -IXAI \tag{4.37}$$

$Cl_0(10,1)$ is generated by using the first algorithm 4.7 on $Cl_0(9,0)$:

$$\begin{array}{cccccc} Cl_0(10,1): & XAZZA, & XAZAI, & XAZXA, & XAAII, & XAXZA, \\ XAXAI, & XAXXA, & XXIII, & XZIII, & AIIII, & ZIIII \end{array} \tag{4.38}$$

Now the two first letters of each word multiply with the usual matrix multiplication, the last three represents the octonions hence multiplies with rule 4.16. To certify that this

32x32 real representation of the octonionic M-algebra is indeed correct, a definition of the Â, B and C matrices according to Kugo-Townsend [78] must be possible, another good reference is by F. Toppan and M.A. de Andrade [79]. The definition of a conjugation 4.17 allows us to properly define these matrices, let's compare the octonionic M-algebra with $Cl_{(10,1)}$:

$$\begin{array}{cc} Cl_{(10,1)}: & Cl_0(10,1): \\ XAZZA & XAZZA \\ XAZAI & XAZAI \\ XAZXA & XAZXA \\ XAAII & XAAII \\ XAXIA & XAXZA \\ XAXAZ & XAXAI \\ XAXAX & XAXXA \\ XXIII & XXIII \\ XZIII & XZIII \\ AIIII & AIIII \\ ZIIII & ZIIII \end{array} \tag{4.39}$$

They are very similar but extremely different in its properties. The matrices from the octonionic M-algebra only anti-commutes with the multiplication being the mix of the matrix and the octonionic one. On the Clifford algebra the multiplication is:

$$XAZZA * XAXAZ = IIAXX$$

however the octonionic multiplication is:

$$XAZZA * XAXAI = IIXXA$$

Let's calculate the Kugo-Townsend matrices Â, B and C, here the matrix Â has a circumflex to differentiate it from the alphabetic represented anti-symmetric matrix A. The definitions of the Kugo-Townsend matrices are the following:

$$\hat{A}_t = \Gamma_{p+1}\Gamma_{p+2}...\Gamma_q, \quad \hat{A}_{sp} = \Gamma_1, \Gamma_2...\Gamma_p, \quad C_s = \prod_i \Gamma_i^s, \quad C_a = \prod_j \Gamma_j^a, \quad B^T = C\hat{A}^{-1} \tag{4.40}$$

where "t" is temporal, "sp" is spacial, "s" is symmetric and "a" is anti-symmetric. Of all possible combinations to create B, there will be just two different $B's$, where the following relation holds:

$$B^T = \epsilon B \qquad \text{and} \qquad B^* B = 1\epsilon \tag{4.41}$$

For basis of even dimension the definitions will generate different matrices, on the other hand, for basis of odd dimension the matrices will merge leaving just one of each, the matrices Â, B and C.

Matrix Â is easy to calculate because the base is odd dimension, meaning that one can choose the multiplication of time-like elements, which in this case is just one:

$$\hat{A} = AIe_0 = AIIII \tag{4.42}$$

The matrix C can be the multiplication of the symmetric elements. The order of the multiplication will be changed to give a positive sign because it is irrelevant:

$$XZe_0 * XXe_0 = IAe_0 \quad \rightarrow \quad IAe_0 * ZIe_0 = ZAe_0$$

Therefore:

$$C = ZAe_0 = ZAIII \tag{4.43}$$

Using the definition of B gives us:

$$B = ZAe_0 * Aie_0 = XAe_0 = XAIII \tag{4.44}$$

Let's apply these matrices in a linearity transformation and see the results:

$$AIe_0 * XAe_i * (-AIe_0) = -XAe_i$$

$$AIe_0 * XXe_0 * (-AIe_0) = -XXe_0$$

$$AIe_0 * AIe_0 * (-AIe_0) = AIe_0$$

$$ZAe_0 * XAe_i * (-ZAe_0) = -XAe_i$$

$$ZAe_0 * XXe_0 * (-ZAe_0) = XXe_0$$

$$ZAe_0 * AIe_0 * (-ZAe_0) = -AIe_0$$

$$XAe_0 * XAe_i * (-XAe_0) = XAe_i$$

$$XAe_0 * XXe_0 * (-XAe_0) = -XXe_0$$

$$XAe_0 * AIe_0 * (-XAe_0) = -AIe_0$$

Synthesizing the results:

$$\hat{A}\Gamma_\mu \hat{A}^{-1} = -\Gamma_\mu^\dagger, \qquad C\Gamma_\mu C^{-1} = \Gamma_\mu^T, \qquad B\Gamma_\mu B^{-1} = -\Gamma_\mu^* \tag{4.45}$$

Because of the definition of a conjugation, matrix Â related to transpose conjugate, C related with transpose and B related with complex conjugate can be properly defined for every representation of the octonionic M-algebra. This is incredible because, without the definitions 4.16-4.18, it wouldn't be possible to define the Kugo-Townsend matrices, this shows that there are 32x32 real spinors related with this representation. With these matrices one can define the following spinors for $Cl_0(10,1)$:

Conjugate spinor: Defined as:

$$\bar{\Psi} = \Psi^{\dagger} \hat{A} \tag{4.46}$$

Charge conjugate spinor: Defined as:

$$\Psi^{c} = B^{\dagger} \Psi^{*} \tag{4.47}$$

Chapter 5

Graded structures

5.1 Graded Lie (super)algebras

Generalized Lie (super)algebras were introduced in the late 70's, here it will be used the definitions according to Rittenberg-Wyler and Scheunert [18], [19] and [20]. A grading is a separation of a vector space in two, four and so on vector spaces with defined properties. If one grades a particular algebra, but one of the following properties does not hold, it is not a graded Lie algebra.

Let's consider a vector space $\mathscr{G}$ over $\mathbb{R}$ or $\mathbb{C}$ and $\vec{\alpha}$ an n-dimensional vector, called grading vector, that is also over the $\mathbb{R}$ or $\mathbb{C}$, if $\mathscr{G}$ is given by:

$$\mathscr{G} = \bigoplus_{\vec{\alpha}} \mathscr{G}_{\vec{\alpha}} \tag{5.1}$$

where $\mathscr{G}_{\vec{\alpha}}$ are the graded subspaces, then one can define the generalized product between operators of this vector space:

$$(A_{\vec{\alpha}}, B_{\vec{\beta}}) = AB - (-1)^{(\vec{\alpha},\vec{\beta})} BA \tag{5.2}$$

where $A, B \in \mathscr{G}$, $\vec{\alpha} = deg(A)$ and $\vec{\beta} = deg(B)$ are the grading of the operators and $deg((A_{\vec{\alpha}}, B_{\vec{\beta}})) = \vec{\alpha} + \vec{\beta}$. $(\vec{\alpha}, \vec{\beta})$ is the bilinear mapping between the grading vectors. Introducing another operator C, with grading vector $\vec{\gamma}$, the generalization of the Jacobi identity is given by:

$$(-1)^{(\gamma,\alpha)}(A,(B,C)) + (-1)^{(\alpha,\beta)}(B,(C,A)) + (-1)^{(\beta,\gamma)}(C,(A,B)) = 0 \tag{5.3}$$

The parenthesis represents a possible commutation or anti-commutation, depending on the bilinear mapping between the grading vectors. This means that, conditions must be defined on the bilinear mapping, they are called the Leibniz rules and come from 5.2 and the fact that the graded vector space is associative, see reference [80].

Leibniz Rules:

$$(\alpha,\beta)+(\beta,\alpha)=2r \tag{5.4}$$

$$(\alpha,\beta+\gamma)=(\alpha,\beta)+(\alpha,\gamma)+2s \tag{5.5}$$

$$(\alpha+\gamma,\beta)=(\alpha,\beta)+(\gamma,\beta)+2s \tag{5.6}$$

where r and s are arbitrary integers. If the product of the grading vectors, also called the mapping, is symmetric $(\alpha,\beta)=(\beta,\alpha)$, then the space is called quadratic; if the mapping is anti-symmetric $(\alpha,\beta)=-(\beta,\alpha)$, then the space is called symplectic. The grading vectors values will always be 0 or 1. The sum of the elements of grading vectors will be taken modulo 2, this gives graded algebras a relation with binary language and logic portals, more on this later. Let's see then the most general $\mathbb{Z}_2^n$ graded structures, they don't need to obey the graded Jacobi identity 5.3, when it does, it is called a graded Lie structure.

$\mathbb{Z}_2$ graded (super)algebras

A $\mathbb{Z}_2$ graded (super)algebra is a graded space that is a sum of two subspaces, one is the bosonic and the other fermionic grading subspaces:

$$\mathscr{G}=\mathscr{G}_0\oplus\mathscr{G}_1 \tag{5.7}$$

The even grading "0" is the bosonic one and the odd "1" is the fermionic one. The Leibniz rules defines the mappings of the grading vectors, in this case there are only two mappings that obeys them:

$$1: \qquad (\alpha,\beta)=0 \tag{5.8}$$

$$2: \qquad (\alpha,\beta)=\alpha\beta \tag{5.9}$$

The first is the null mapping and it gives the Lie algebras, the second gives the $\mathbb{Z}_2$ graded superalgebra. The table of brackets (A,B) is very important in graded algebras, here is for the two cases above:

A/B	0	1
0	[,]	[,]
1	[,]	[,]

Table 5.1: Lie algebra

A/B	0	1
0	[,]	[,]
1	[,]	{,}

Table 5.2: $\mathbb{Z}_2$ graded superalgebra

When a anti-commutation appears in the diagonal of the brackets table, then the algebra is called a graded superalgebra, if there is not a anti-commutator in the diagonal, then it is called a graded algebra. It will be very useful to put this table of brackets in one's and zero's instead of anti-commutators and commutators, this will help classify all graded algebras in next sections. Below are the same tables above:

A/B	0	1
0	0	0
1	0	0

Table 5.3: Lie algebra

A/B	0	1
0	0	0
1	0	1

Table 5.4: $\mathbb{Z}_2$ graded superalgebra

The operators of the grading vectors must multiply with the following rules:

- The multiplication of operators from the bosonic sector 0 must give an operator from the same grading 0.

- The multiplication of operators from the fermionic grading 1 must give an operator of the bosonic sector 0.

If a $\mathbb{Z}_2$ graded superalgebra does not multiply obeying the above rules, then it is not a graded Lie superalgebra. On the next section it will be discussed about the classification of the $\mathbb{Z}_2^n$ graded algebras.

$\mathbb{Z}_2^2$ graded (super)algebras

A $\mathbb{Z}_2^2$ graded (super)algebra is a graded space that is a sum of four subspaces:

$$\mathscr{G} = \mathscr{G}_{00} \oplus \mathscr{G}_{10} \oplus \mathscr{G}_{01} \oplus \mathscr{G}_{11} \tag{5.10}$$

The even grading 00 is the bosonic one, the others depends on the graded algebra. Doing the same from the $\mathbb{Z}_2$ case, the Leibniz rules, applied to the table of brackets, gives us four mappings. The grading vectors are now two dimensional:

$$1: \quad (\alpha, \beta) = 0 \tag{5.11}$$

$$2: \quad (\alpha, \beta) = \alpha_2\beta_2 \tag{5.12}$$

$$3: \quad (\alpha, \beta) = \alpha_1\beta_1 + \alpha_2\beta_2 \tag{5.13}$$

$$4: \quad (\alpha, \beta) = \alpha_1\beta_2 - \alpha_2\beta_1 \tag{5.14}$$

The first is the null mapping and it gives the Lie algebras. The second could have also been $\alpha_1\beta_1$ and gives the $\mathbb{Z}_2^2$ graded superalgebra. The third gives the $\mathbb{Z}_2^2$ graded color superalgebra and the fourth gives the $\mathbb{Z}_2^2$ graded color algebra. The table of brackets are:

A/B	00	10	01	11
00	0	0	0	0
10	0	0	0	0
01	0	0	0	0
11	0	0	0	0

Table 5.5: Mapping 5.11

A/B	00	10	01	11
00	0	0	0	0
10	0	0	0	0
01	0	0	1	1
11	0	0	1	1

Table 5.6: Mapping 5.12

A/B	00	10	01	11
00	0	0	0	0
10	0	1	0	1
01	0	0	1	1
11	0	1	1	0

Table 5.7: Mapping 5.13

A/B	00	10	01	11
00	0	0	0	0
10	0	0	1	1
01	0	1	0	1
11	0	1	1	0

Table 5.8: Mapping 5.14

There is another rule that is added here and remains to the rest of the $\mathbb{Z}_2^n$ graded algebras:

- The multiplication of operators from a grading sector with operators from a different grading, both not being in the bosonic grading, must give an operator from a grading which is not the three gradings mentioned before.

Hence, for example, the multiplication of a operator from grading 10 with one from 01 will give a operator from 11. Because of the definition 5.2, on the fourth case, the grading sectors 10, 01 and 11 are on equal footing, their grading assignment can be permuted under the S_3 group of permutations without changing the anti-commutators. On the second and third cases, the grading sectors 01, 11 and 10, 01, respectively, can be permuted under the S_2 group of permutations.

Another important property is that a $\mathbb{Z}_2^n$ graded (super)algebra must have as subalgebras a corresponding $\mathbb{Z}_2^{n-1}$ until a $\mathbb{Z}_2$.

$\mathbb{Z}_2^3$ graded (super)algebras

A $\mathbb{Z}_2^3$ graded (super)algebra is a graded space that is a sum of eight subspaces:

$$\mathscr{G} = \mathscr{G}_{000} \oplus \mathscr{G}_{001} \oplus \mathscr{G}_{011} \oplus \mathscr{G}_{010} \oplus \mathscr{G}_{100} \oplus \mathscr{G}_{101} \oplus \mathscr{G}_{111} \oplus \mathscr{G}_{110} \tag{5.15}$$

The demonstration that there are only 5 mappings, using the Leibniz rules on the brackets table, is going to be published in the future along with other ideas by F. Toppan, I. P. de Freitas, M. M. Balbino and R. G. Rana. The table of brackets are bigger now, hence the first case is trivial and will be omitted:

$$1: \quad (\alpha,\beta) = 0 \tag{5.16}$$

$$2: \quad (\alpha,\beta) = \alpha_1\beta_1 \tag{5.17}$$

$$3: \quad (\alpha,\beta) = \alpha_1\beta_1 + \alpha_2\beta_2 \tag{5.18}$$

$$4: \quad (\alpha,\beta) = \alpha_1\beta_2 - \alpha_2\beta_1 \tag{5.19}$$

$$5: \quad (\alpha,\beta) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \tag{5.20}$$

A/B	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	0	0
011	0	0	0	0	0	0	0	0
010	0	0	0	0	0	0	0	0
100	0	0	0	0	1	1	1	1
101	0	0	0	0	1	1	1	1
111	0	0	0	0	1	1	1	1
110	0	0	0	0	1	1	1	1

Table 5.9: Mapping 5.17

A/B	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	0	0
011	0	0	1	1	0	0	1	1
010	0	0	1	1	0	0	1	1
100	0	0	0	0	1	1	1	1
101	0	0	0	0	1	1	1	1
111	0	0	1	1	1	1	0	0
110	0	0	1	1	1	1	0	0

Table 5.10: Mapping 5.18

A/B	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	0	0
011	0	0	0	0	1	1	1	1
010	0	0	0	0	1	1	1	1
100	0	0	1	1	0	0	1	1
101	0	0	1	1	0	0	1	1
111	0	0	1	1	1	1	0	0
110	0	0	1	1	1	1	0	0

Table 5.11: Mapping 5.19

A/B	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	1	1	0	0	1	1	0
011	0	1	0	1	0	1	0	1
010	0	0	1	1	0	0	1	1
100	0	0	0	0	1	1	1	1
101	0	1	1	0	1	0	0	1
111	0	1	0	1	1	0	1	0
110	0	0	1	1	1	1	0	0

Table 5.12: Mapping 5.20

With the increase of n in $\mathbb{Z}_2^n$ it becomes harder to analyze, to help differentiate non-equivalent tables of brackets easily, let's define the following variables:

$$N = (\text{Number of lines with all elements zero}) \tag{5.21}$$

$$D1 = (\text{Number of "1's" on the diagonal}) \tag{5.22}$$

$$T1 = (\text{Total number of "1's"}) \tag{5.23}$$

where $D1$ can be only equal to zero or 2^{n-1} because of the property that it must contain as subalgebras smaller graded algebras and:

$$T1 = (2^n - N) * 2^{n-1} \tag{5.24}$$

Here are the cases up to $\mathbb{Z}_2^4$:

$\mathbb{Z}_2$- Graded

Mappings:

$$1 : (\alpha, \beta) = 0 \qquad \text{e} \qquad 2 : (\alpha, \beta) = \alpha\beta \tag{5.25}$$

Properties:

1: N=2, D1=0 e T1=0

2: N=1, D1=1 e T1=1

$\mathbb{Z}_2^2$- Graded

Mappings:

$$1 : (\alpha, \beta) = 0 \qquad 2 : (\alpha, \beta) = \alpha\beta \qquad 3 : (\alpha, \beta) = \alpha_1\beta_1 + \alpha_2\beta_2 \qquad 4 : (\alpha, \beta) = \alpha_1\beta_2 - \alpha_2\beta_1 \tag{5.26}$$

Properties:

1: N=4, D1=0 e T1=0

2: N=2, D1=2 e T1=4

3: N=1, D1=2 e T1=6

4: N=1, D1=0 e T1=6

$\mathbb{Z}_2^3$- Graded

Mappings:

$$1:(\alpha,\beta)=0 \qquad 2:(\alpha,\beta)=\alpha\beta \qquad 3:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2 \qquad 4:(\alpha,\beta)=\alpha_1\beta_2-\alpha_2\beta_1$$
$$5:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 \tag{5.27}$$

Properties:

1: N=8, D1=0 e T1=0

2: N=4, D1=4 e T1=16

3: N=2, D1=4 e T1=24

4: N=2, D1=0 e T1=24

5: N=1, D1=4 e T1=28

$\mathbb{Z}_2^4$- Graded

Mappings:

$$1:(\alpha,\beta)=0 \qquad 2:(\alpha,\beta)=\alpha\beta \qquad 3:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2 \qquad 4:(\alpha,\beta)=\alpha_1\beta_2-\alpha_2\beta_1$$
$$5:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 \qquad 6:(\alpha,\beta)=\alpha_1\beta_2-\alpha_2\beta_1+\alpha_3\beta_4-\alpha_4\beta_3 \tag{5.28}$$
$$7:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\alpha_4\beta_4$$

Properties:

1: N=16, D1=0 e T1=0

2: N=8, D1=8 e T1=64

3: N=4, D1=8 e T1=96

4: N=4, D1=0 e T1=96

5: N=2, D1=8 e T1=112

6: N=1, D1=0 e T1=120

7: N=1, D1=8 e T1=120

XOR/AND logic portals

$\mathbb{Z}_2^n$ graded algebras has a direct relation with logic portals, this was indicated and studied by R. G. Rana. For example, the second table of brackets in the $\mathbb{Z}_2$ case is related to the logic portal AND:

α	β	(α,β)
0	0	0
0	1	0
1	0	0
1	1	1

Table 5.13: Mapping 5.9

Therefore, α and β are the entries on the portal AND and the output is (α,β). In the $\mathbb{Z}_2^2$ case, there are two logic portals AND from which the outputs goes into a logic portal XOR, the output of that is the mapping. In the color superalgebra case the entries of the portals AND are from the same number, α_1 and β_1 for example, in the color algebra case the entries are with different number, therefore α_1 and β_2.

This direct relation really helps to check if a table of brackets produces a valid mapping in terms of the Leibniz rules; there are sites in the internet that gives the Karnaugh map, like for example, "http://www.32x8.com/index.html". Hence, one can put the table of brackets and it will generate the combination of logic portals for that multiplication. With a program, like mathematica, the result can be translated in terms of the mapping language and then the particular mapping, that generates that table of brackets, is found. this is very useful to help study graded algebras, more on this later.

5.2 Matrix representation

On the appendix of [36], F. Toppan and Z. Kuznetsova describe how to create the matrix representation of the $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$ graded (super)algebras; on the appendix of [65] they use the alphabetic (re)presentation to create the matrix representation for $\mathbb{Z}_2^2$. Therefore, here it will be put only the main results.

The creation of a matrix representation for graded (super)algebras comes from the realization that a matrix can be seen as it is made of sectors, see [81]. In a 2x2 matrix it is simple, one can check that the multiplication between diagonal and anti-diagonal matrices are exactly the multiplication rules of the $\mathbb{Z}_2$ graded (super)algebras.

For $\mathbb{Z}_2^n$ graded (super)algebras there are not just one possible way to grade a matrix. Each possibility will work, but the (super)algebra changes depending on the choice, see [[81],[57]]. A complete study on the possible gradings for $\mathbb{Z}_2^n$ and their effect on the (super)algebras is still not done. The general matrix representation for $\mathbb{Z}_2$ graded (super)algebras is given by.

$$M_0 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \tag{5.29}$$

where d_i and a_i belongs to the $\mathbb{R}$ or $\mathbb{C}$. On the next section, it will be discussed about how to find the values of d_i and a_i depending on the superalgebra.

The convention that will be used in this thesis of the grading of 4x4 matrices is the following.

$$M_{00} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}, \qquad M_{11} = \begin{pmatrix} 0 & m_5 & 0 & 0 \\ m_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_7 \\ 0 & 0 & m_8 & 0 \end{pmatrix},$$
$$M_{10} = \begin{pmatrix} 0 & 0 & m_9 & 0 \\ 0 & 0 & 0 & m_{10} \\ m_{11} & 0 & 0 & 0 \\ 0 & m_{12} & 0 & 0 \end{pmatrix}, \qquad M_{01} = \begin{pmatrix} 0 & 0 & 0 & m_{13} \\ 0 & 0 & m_{14} & 0 \\ 0 & m_{15} & 0 & 0 \\ m_{16} & 0 & 0 & 0 \end{pmatrix} \tag{5.30}$$

Here the matrix representation is created thinking of the graded (super)algebras 5.13 and 5.14 because for the color superalgebra case, the sector 11 represents the exotic bosons. Therefore, the matrix representation is chosen to be block diagonal. See reference [57] for other possible gradings. For the algebra case it doesn't matter, but if we are studying the superalgebra case 5.12, then the sector 11 is now fermionic, this means that the matrix representation for it will be the same from the above with the change between sectors 11 and 10.

To create a matrix representation for the $\mathbb{Z}_2^3$ case one can just look at the octonionic matrices in the appendix B.3 and change the 1's with m_i. In this case there are 64 variables to solve, but as it will be mentioned later, there are more equations than variables in most of the cases.

Grading of Clifford algebras

There are nuances that one must not forget when studying graded (super)algebras. The matrix representation not only can change, but there are also different conventions on what is a graded (super)algebra, see [[64],[82]]. In [64], the authors use that the grading of a Clifford algebra is made by, the even sector being the field $\mathbb{R}$ and the odd sector are given by all the generators of the Clifford algebra.

Let's use the example given in [64]. The Clifford algebras $Cl_{(1,1)} \simeq Cl_{(2,0)}$:

$$Cl_{(1,1)}: \quad e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.31}$$

$$Cl_{(2,0)}: \quad e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.32}$$

According to the convention of reference [64], the Clifford algebra $Cl_{(1,1)}$ has a $\mathbb{Z}_2$ grading, but $Cl_{(2,0)}$ doesn't because of the diagonal matrix in the generators. The grading is represented by $Cl_0 : e_0, e_{12}$ and $Cl_1 : e_1, e_2$. Therefore, e_1 in the Clifford algebra $Cl_{(2,0)}$ is represented by the matrix Z, which is diagonal and cannot be put in Cl_1 in this convention.

In [82] it is analyzed all possible choices of the grading of a Clifford algebra. They also showed that, if one use the grading of a Clifford algebra that is defined by the mxm matrices as generators, as showed in chapter 2, like in 5.29 and 5.30, then it does not preserve the multivector structure of the Grassmann algebra. This is not a problem but one needs to know that this is the case.

Therefore, according to [82], the Clifford algebra $Cl_{(2,0)}$ in the minimal matrix representation above can be graded. Using the arbitrary grading, $Cl_0 : e_0, e_1$ and $Cl_1 : e_2 e_{12}$. This is the convention that will be used further in the super division algebras. The Clifford algebras $Cl_{(1,1)}$ and $Cl_{(2,0)}$ are isomorphic to the split-quaternions. A $\mathbb{Z}_2$ grading of the split-quaternions will be $\mathcal{G}_0 : e_0, e_1$ and $\mathcal{G}_1 : e_2, e_3$. Although a $\mathbb{Z}_2$ grading is possible for the matrix representation of the Clifford algebras above; a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading is not possible for both because the minimal representation does not have four grading sectors. Hence, the 4x4 representation must be used.

5.3 Classification

To classify all possible graded (super)algebras, the multiplication between the operators must be used; the algebra must obey the fundamental properties discussed in the introduction. Let's use the convention that, a operator from the bosonic grading is represented by the letter H, operators from the fermionic gradings will be represented by Q_i and from the bosonic ones by Z_j, following reference [36]. Hence for the $\mathbb{Z}_2$ graded superalgebra, the most general algebra is given by:

$$[H, Q] = rQ, \quad \{Q, Q\} = 2sH \tag{5.33}$$

where r and s are integers that are constrained by the Jacobi identity, using it gives us the condition:

$$rs = 0 \tag{5.34}$$

This condition generates 3 superalgebras:

$$S_1^{\mathbb{Z}_2} : \quad [H, Q] = \{Q, Q\} = 0 \tag{5.35}$$

$$S_2^{\mathbb{Z}_2} : \quad [H, Q] = 0, \quad \{Q, Q\} = 2H \tag{5.36}$$

$$S_3^{\mathbb{Z}_2} : \quad [H, Q] = Q, \quad \{Q, Q\} = 0 \tag{5.37}$$

The first is called the $\mathbb{Z}_2$ graded "abelian" superalgebra and enters the simplest example of superspace, see [83]. The second is called the $\mathcal{N} = 1$ one-dimensional supersymmetry algebra, it is the simplest superalgebra related to the one dimensional supersymmetry quantum mechanics, see [84]. The last one is called the $\mathbb{Z}_2$ graded Lie superalgebra with a Grassmann generator and relates to the topological quantum mechanics, see [85]. Their matrix representation is given by:

$$S_1^{\mathbf{Z}_2}: \qquad H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} \tag{5.38}$$

$$S_2^{\mathbf{Z}_2}: \qquad H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & p \\ \frac{\lambda}{p} & 0 \end{pmatrix} \tag{5.39}$$

$$S_3^{\mathbf{Z}_2}: \qquad H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda - 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} \tag{5.40}$$

The $\mathbb{Z}_2^2$ case is completely done for the color graded (super)algebras on paper [36]. Hence, it will be presented the beginning of the classification of the $\mathbb{Z}_2^3$ case for the mapping 5.20. The table of brackets for this mapping is:

A/B	H	Q_1	Z_1	Q_2	Q_3	Z_2	Q_4	Z_3
H	0	0	0	0	0	0	0	0
Q_1	0	1	1	0	0	1	1	0
Z_1	0	1	0	1	0	1	0	1
Q_2	0	0	1	1	0	0	1	1
Q_3	0	0	0	0	1	1	1	1
Z_2	0	1	1	0	1	0	0	1
Q_4	0	1	0	1	1	0	1	0
Z_3	0	0	1	1	1	1	0	0

Table 5.14: $\mathbb{Z}_2^3$ graded superalgebra 5.20

The most general superalgebra that obeys the multiplication relations for this mapping is given by:

$$[H, Q_i] = a_i Q_i, \quad [H, Z_i] = b_i Z_i, \quad \{Q_i, Q_i\} = \alpha_i H$$

$$[Q_1, Q_2] = c Z_1, \quad [Q_1, Q_3] = d Z_2, \quad [Q_3, Q_2] = e Z_3,$$

$$\{Q_i, Q_j\} = \beta_i |\varepsilon_{jk}| Z_k, \quad jk = \begin{vmatrix} 43 & \text{se} & i=1 \\ 42 & \text{se} & i=2 \\ 41 & \text{se} & i=3 \end{vmatrix}$$

$$\{Z_i, Q_j\} = \gamma_i |\varepsilon_{jk}| Q_k, \quad jk = \begin{vmatrix} 12 & \text{se} & i=1 \\ 13 & \text{se} & i=2 \\ 23 & \text{se} & i=3 \end{vmatrix}$$

$$\{Z_i, Q_j\} = \bar{\gamma}_i |\varepsilon_{jk}| Q_k, \quad jk = \begin{vmatrix} 21 & \text{se} & i=1 \\ 31 & \text{se} & i=2 \\ 32 & \text{se} & i=3 \end{vmatrix} \tag{5.41}$$

$$[Z_i, Q_j] = \theta_i |\varepsilon_{jk}| Q_k, \quad jk = \begin{vmatrix} 24 & \text{se} & i=2 \\ 14 & \text{se} & i=3 \end{vmatrix}$$

$$[Z_i, Q_j] = \bar{\theta}_i |\varepsilon_{jk}| Q_k, \quad jk = \begin{vmatrix} 42 & \text{se} & i=2 \\ 41 & \text{se} & i=3 \end{vmatrix}$$

$$[Q_3, Z_1] = \theta_1 Q_4, \quad [Q_4, Z_1] = \bar{\theta}_1 Q_3$$

$$\{Z_i, Z_j\} = \sigma_i |\varepsilon_{jk}| Z_k, \quad jk = \begin{vmatrix} 23 & \text{se} & i=1 \\ 31 & \text{se} & i=2 \\ 12 & \text{se} & i=3 \end{vmatrix}$$

the sum over k is understood and ε_{jk} is normalized to 1.

There are approximately one hundred constraints given by the Jacobi graded identity. Therefore, to do a complete classification, one needs to use all these constraints, plus the fact that the four fermionic sectors are interchangeable and the three bosonic ones are also interchangeable. It is a lot of calculation to do, to find every matrix representation of all those superalgebras is even worse. However, there is a way to find a superalgebra that obeys all the constraints and solve it. Here it begins the discussion of the relation between graded algebras and Cayley-Dickson and Clifford algebras.

In the $\mathbb{Z}_2^3$ case, the graded (super)algebras has at least 8 operators, one from each sector, for now that's the only case that will be considered. The Cayley-Dickson algebras that has 8 vectors are the bi-quaternions, the tri-complex($\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$), the octonions, etc. The octonions and split-octonions are nonassociative, hence, they won't be of use here, but the bi-quaternions and the tri-complex will be.

The $\mathcal{N} = 1$ one-dimensional supersymmetry algebra 5.36 has the complex and split-complex numbers as subcases. If one set $\lambda = -1$ and $p = 1$ and make the transformation $H' = -H$ it will give the complex numbers matrix representation. This show us that the Cayley-Dickson algebras can also be graded, more on that on the next section. The tri-complex is given by $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$, if one makes the tensor product of that superalgebra with itself, $S_2^{\mathbb{Z}_2} \otimes S_2^{\mathbb{Z}_2} \otimes S_2^{\mathbb{Z}_2}$, the resulting superalgebra will englobe the tri-complex:

$$H = \begin{pmatrix} \mu\lambda\chi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu\lambda\chi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu\lambda\chi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\lambda\chi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu\lambda\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu\lambda\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu\lambda\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\lambda\chi \end{pmatrix}$$

$$Z_1 = \begin{pmatrix} 0 & q\lambda\chi & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\lambda\chi}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q\lambda\chi & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu\lambda\chi}{q} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q\lambda\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mu\lambda\chi}{q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q\lambda\chi \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu\lambda\chi}{q} & 0 \end{pmatrix}$$

$$Z_2 = \begin{pmatrix} 0 & 0 & p\mu\chi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p\mu\chi & 0 & 0 & 0 & 0 \\ \frac{\mu\lambda\chi}{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu\lambda\chi}{p} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p\mu\chi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p\mu\chi \\ 0 & 0 & 0 & 0 & \frac{\mu\lambda\chi}{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\mu\lambda\chi}{p} & 0 & 0 \end{pmatrix}$$

$$Z_3 = \begin{pmatrix} 0 & 0 & 0 & pq\chi & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{p\mu\chi}{q} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q\lambda\chi}{p} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\lambda\chi}{pq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pq\chi \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{p\mu\chi}{q} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q\lambda\chi}{p} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mu\lambda\chi}{pq} & 0 & 0 & 0 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & r\mu\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r\mu\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r\mu\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r\mu\lambda \\ \frac{\mu\lambda\chi}{r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu\lambda\chi}{r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu\lambda\chi}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu\lambda\chi}{r} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & qr\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{r\mu\lambda}{q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & qr\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{r\mu\lambda}{q} & 0 \\ 0 & \frac{q\lambda\chi}{r} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\lambda\chi}{qr} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q\lambda\chi}{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu\lambda\chi}{qr} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & pr\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pr\mu \\ 0 & 0 & 0 & 0 & \frac{r\mu\lambda}{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{r\mu\lambda}{p} & 0 & 0 \\ 0 & 0 & \frac{p\mu\chi}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p\mu\chi}{r} & 0 & 0 & 0 & 0 \\ \frac{\mu\lambda\chi}{pr} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu\lambda\chi}{pr} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & pqr \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{pr\mu}{q} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{qr\lambda}{p} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{r\mu\lambda}{pq} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{pq\chi}{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{p\mu\chi}{qr} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q\lambda\chi}{pr} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\lambda\chi}{pqr} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.42}$$

where each matrix representation for the tensor product has a different letter for the matrices values, to make the most general representation, this superalgebra is given by:

$$\begin{gathered} [H,Q_i]=0, \quad [H,Z_i]=0, \quad \{Q_1,Q_1\}=2\mu\lambda H, \quad \{Q_2,Q_2\}=2\lambda H \\ \{Q_3,Q_3\}=2\mu H, \quad \{Q_4,Q_4\}=2H, \quad [Q_1,Q_2]=0, \quad [Q_1,Q_3]=0 \\ [Q_3,Q_2]=0, \quad \{Q_1,Q_4\}=2\mu\lambda Z_3, \quad \{Q_2,Q_4\}=2\lambda Z_2, \quad \{Q_3,Q_4\}=2\mu Z_1 \\ \{Z_1,Q_1\}=2\mu\lambda\chi Q_2, \quad \{Z_2,Q_1\}=2\mu\lambda\chi Q_3, \quad \{Z_3,Q_2\}=2\lambda\chi Q_3 \\ \{Z_1,Q_2\}=2\lambda\chi Q_1, \quad \{Z_2,Q_3\}=2\mu\chi Q_1, \quad \{Z_3,Q_3\}=2\mu\chi Q_2 \\ [Q_3,Z_1]=0, \quad [Z_2,Q_2]=0, \quad [Z_3,Q_1]=0 \\ [Q_4,Z_1]=0, \quad [Z_2,Q_4]=0, \quad [Z_3,Q_4]=0 \\ \{Z_1,Z_2\}=2\mu\lambda\chi Z_3, \quad \{Z_3,Z_1\}=2\lambda\chi Z_2, \quad \{Z_2,Z_3\}=2\mu\chi Z_1 \end{gathered} \tag{5.43}$$

Using the graded Jacobi identity confirms that this is indeed a Lie superalgebra. This superalgebra paves the way to an even more general superalgebra representation; by looking at 5.43, one can see the correct signs that obeys the graded Jacobi identities. Hence in this case:

$$\begin{aligned}&\alpha_1=1,\quad \alpha_2=1,\quad \alpha_3=1,\quad \alpha_4=1,\quad \beta_1=1,\quad \beta_2=1,\quad \beta_3=1,\quad \gamma_1=1\\&\gamma_2=1,\quad \gamma_3=1,\quad \bar{\gamma}_1=1,\quad \bar{\gamma}_2=1,\quad \bar{\gamma}_3=1,\quad \sigma_1=1,\quad \sigma_2=1,\quad \sigma_3=1\end{aligned}\tag{5.44}$$

or other combination depending on the values of the constants. With these constants one can start with the superalgebra, use the general matrix representation with the m_i as values and find the matrix representation of it, just like in the paper of F. Toppan and Z. Kuznetsova. In the bi-quaternions case, use 5.39 and the matrix representation of the color superalgebra named $S10_\varepsilon$ in [36], that englobes the quaternions and split-quaternions.

However, there is an even more simple way. The matrix representation of the tri-complex, bi-quaternions, etc, can be used to discover the superalgebras related with them, only using matrices with one's and zero's. In [36], it is mentioned that the quaternions and split-quaternions obey the color graded algebra $A7$ and color graded superalgebra $S10_\varepsilon$. They are also a especial case of one of the superalgebras from the second mapping 5.12, that was not done in the paper [36]. This will happen for $\mathbb{Z}_2^3$ too. Hence, the bi-quaternions and tri-complex, for example, will produce correct (super)algebras.

With a particular superalgebra one can study physics with it. See the references that were mentioned in the introduction [[34]-[40],[43]-[54],[57]] for how to use them.

5.4 Super division algebras

On the last section, it was showed that a Cayley-Dickson algebra can be graded, they are subcases of particular graded algebras. In this section, it will be introduced the super division algebras, which has the same properties of all graded algebras, however, the operators that compose each sector must form a division algebra. In other way, the sum between operators from the same grading sector must be invertible. The split algebras are not a division algebra because when one sum the elements $(e_i,0)$ with the elements $(0,e_i)$, the resulting matrix is singular and does not have an inverse. When we grade the Cayley-Dickson algebras and separate these elements it becomes a super division algebra.

The $\mathbb{Z}_2$ graded super division algebras form what is called the tenfold way, a good reference is by J. Baez [59]. The $\mathbb{Z}_2^2$ graded super division algebras form the thirteen fold way, in [65], F. Toppan and Z. Kuznetsova use the alphabetic (re)presentation to find the thirteen super division algebras. They create a classification of the super division algebras that will be used here. However, instead of using the alphabetic (re)presentation, here it will be used the tensor product between Cayley-Dickson algebras because it is easier to analyze and classify.

5.5 Classification

On [65], it was introduced important concepts to classify all super division algebras, but first let's start with the conventions. To designate a particular $\mathbb{Z}_2^n$ graded (super)algebra it will be used the notation $\mathbb{D}_{\mathbb{K},i}^{[p]}$, where $\mathbb{K}$ is the field of the (super)algebra, i is the number of the (super)algebra and p is the $\mathbb{Z}_2^p$ grading.

Regarding the field $\mathbb{K}$, if a super division algebra is real, then it must have only one operator for each sector, if it is complex, then it must have two operators for each sector and so on. A quaternionic super division algebra cannot be made of a tensor product finishing in the complex field, for example, $\mathbb{H}\otimes\mathbb{C}$ because the bosonic grading won't be made of the quaternions. Another thing, a quaternionic super division algebra can be $\mathbb{C}\otimes\mathbb{O}$ because, as it was said in the Cayley-Dickson algebras chapter, they have as a sub-algebra a smaller division algebra, in this case the quaternions.

In both references cited here, it is only discussed the associative super division algebras, here it will be presented a complete classification with the nonassociative ones. Also the term super division algebras will be abbreviated to SDA.

Projection and inequivalent super division algebras

In the same way that the signature of a Clifford algebra represents that particular algebra, in the Cayley-Dickson algebras and super division algebras it will be the same. The classification of the $\mathbb{Z}_2$ graded algebras will be given by the different types of signatures, the signatures give the inequivalent super division algebras.

In the $\mathbb{Z}_2^n$ case, with $n > 1$, the inequivalent SDA's will be given by the projection defined in paper [65], let's define it with an example, the octonions:

$$\mathbb{O} = III,\ ZZA,\ ZAI,\ ZXA,\ AII,\ XZA,\ XAI,\ XXA \tag{5.45}$$

In a $\mathbb{Z}_2^3$ grading each element will be in one sector, hence the $\mathbb{Z}_2$ graded subalgebras are III with ZZA, III with ZAI and so on. Therefore, all the subalgebras are the $\mathbb{Z}_2$ grading of the complex numbers, this gives a unique projection for this super division algebra. Other $\mathbb{Z}_2^3$ graded super division algebras will have as a projection a mix between the $\mathbb{Z}_2$ grading of the complex and split-complex numbers, this show us that two SDA's are non-equivalent.

In the last example, what about the $\mathbb{Z}_2^2$ subalgebra? It has only one, III, ZZA, ZAI, ZXA, the quaternions. III, XZA, XAI, XXA is not a subalgebra because it does not obey the multiplication rule. The multiplication of XZA and XAI gives ZXA which is not in the subalgebra, hence the multiplication is not closed.

The only projection that matters is the $\mathbb{Z}_2$ grading because, as it will be clear, by looking at it one can see the $\mathbb{Z}_2^2$ projection and the $\mathbb{Z}_2^3$ and so on, it is the more fundamental one.

The other projections are important and need to be discovered too, as it will be explained later, but the visualization will come from the $\mathbb{Z}_2$ grading projection.

The convention for the projections will be to use a parenthesis with the number of the respective subalgebra. Using the example above, it will be shown that there are two real $\mathbb{Z}_2$ graded super division algebras, the grading of the complex and the split-complex numbers. The representation of them will be given by $\mathbb{D}^{[1]}_{\mathbb{R},1}$ and $\mathbb{D}^{[1]}_{\mathbb{R},2}$, respectively. Therefore, the projection of the real $\mathbb{Z}_2^3$ graded super division algebra, given by the grading of the octonions, will be, as it was mentioned before, the grading of the complex numbers seven times. Hence, the projection will be:

$$(\mathbb{D}^{[1]}_{\mathbb{R},1},\mathbb{D}^{[1]}_{\mathbb{R},1},\mathbb{D}^{[1]}_{\mathbb{R},1};\mathbb{D}^{[1]}_{\mathbb{R},1},\mathbb{D}^{[1]}_{\mathbb{R},1},\mathbb{D}^{[1]}_{\mathbb{R},1},\mathbb{D}^{[1]}_{\mathbb{R},1}) \tag{5.46}$$

Instead of using this notation, it will be easier to analyze if one uses just the number of the SDA. The number is given by "i" in $\mathbb{D}^{[1]}_{\mathbb{K},i}$. Therefore the projection above can be simplified to:

$$(1,1,1;1,1,1,1) \tag{5.47}$$

This means that all subalgebras are the $\mathbb{Z}_2$ grading of the complex numbers. The dot-comma is a aesthetic tool to help us visualize and analyze the projections. It marks, in the example above, the $\mathbb{Z}_2^2$ graded subalgebra, meaning that the first three $\mathbb{Z}_2$ subalgebras $(1,1,1)$ gives the $\mathbb{Z}_2^2$ graded subalgebra, which in this case is the $\mathbb{Z}_2^2$ grading of the quaternions.

Important notes:

An important concept that needs to be discussed here is the rearrangements of the terms. For example, instead of AII from the octonions being in the grading 100, one could change it with XZA in grading 101. But the problem is that the multiplication rules will be broken, in the sense that the multiplication of the grading is already fixed by the mapping. Hence, to rearrange terms, one needs to respect the multiplication rule, the correct rearrangements will be shown later.

The second note is that there are two ways of doing a tensor product, here the bi-quaternions will be used as an example:

$$\mathbb{C}: I \text{ e } A, \qquad \mathbb{H}: II,\ IA,\ AZ \text{ e } AX$$

one can make the matrix representation of the bi-quaternions in two ways:

1: III, IIA, IAZ, IAX, AII, AIA, AAZ, AAX

2: III, AII, IAZ, AAZ, IIA, AIA, IAX, AAX

The second way is equivalent to $\mathbb{H}\otimes\mathbb{C}$, but it won't create a new SDA, as it will be discussed later, hence it will be used the rearrangement. To maintain a convention, here it will be used only the first way to do a tensor product. When the second way derive another inequivalent SDA it will be marked with a comment after the classification.

5.6 $\mathbb{Z}_2$ Super division algebras

Real $\mathbb{Z}_2$ SDA

1: $\mathbb{D}^{[1]}_{\mathbb{R},1}$ is a $\mathbb{Z}_2$ grading of $\mathbb{C}$

2: $\mathbb{D}^{[1]}_{\mathbb{R},2}$ is a $\mathbb{Z}_2$ grading of $\tilde{\mathbb{C}}$

Complex $\mathbb{Z}_2$ SDA

1: $\mathbb{D}^{[1]}_{\mathbb{C};1}$ is a $\mathbb{Z}_2$ grading of the quaternions $\mathbb{H}$.

2: $\mathbb{D}^{[1]}_{\mathbb{C};2}$ is a $\mathbb{Z}_2$ grading of the split-quaternions $\tilde{\mathbb{H}}$.

3: $\mathbb{D}^{[1]}_{\mathbb{C};3}$ is a $\mathbb{Z}_2$ grading of $\mathbb{C}\otimes\mathbb{C}$.

where the inequivalent SDA's are given by the signature of the odd sector:

$$\mathbb{D}^{[1]}_{\mathbb{C};1}: \quad - \quad - \tag{5.48}$$

$$\mathbb{D}^{[1]}_{\mathbb{C};2}: \quad + \quad + \tag{5.49}$$

$$\mathbb{D}^{[1]}_{\mathbb{C};3}: \quad - \quad + \tag{5.50}$$

there is a equivalence of signature between:

$$\tilde{\mathbb{C}}\otimes\mathbb{C} \simeq \mathbb{C}\otimes\mathbb{C} \tag{5.51}$$

The $\simeq$ symbol will be used throughout this classification to represent when two SDA's are equivalent in terms of the rearrangements, it does not represent an isomorphism. The signature of $\tilde{\mathbb{C}}\otimes\mathbb{C}$ is given by $+ \quad -$, this is equivalent to the signature of $\mathbb{D}^{[1]}_{\mathbb{C};3}$.

Quaternionic $\mathbb{Z}_2$ SDA

1: $\mathbb{D}^{[1]}_{\mathbb{H};1}$ is a $\mathbb{Z}_2$ grading of $\mathbb{C}\otimes\mathbb{H}$.

2: $\mathbb{D}^{[1]}_{\mathbb{H};2}$ is a $\mathbb{Z}_2$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}$.

3: $\mathbb{D}^{[1]}_{\mathbb{H};3}$ is a $\mathbb{Z}_2$ grading of the Octonions $\mathbb{O}$.

4: $\mathbb{D}^{[1]}_{\mathbb{H};4}$ is a $\mathbb{Z}_2$ grading of the Split-octonions $\tilde{\mathbb{O}}$.

with signatures:

$$\mathbb{D}^{[1]}_{\mathbb{H};1}: \quad - \quad + \quad + \quad + \tag{5.52}$$

$$\mathbb{D}^{[1]}_{\mathbb{H};2}: \quad + \quad - \quad - \quad - \tag{5.53}$$

$$\mathbb{D}^{[1]}_{\mathbb{H};3}: \quad - \quad - \quad - \quad - \tag{5.54}$$

$$\mathbb{D}^{[1]}_{\mathbb{H};4}: \quad + \quad + \quad + \quad + \tag{5.55}$$

Octonionic $\mathbb{Z}_2$ SDA

1: $\mathbb{D}^{[1]}_{\mathbb{O};1}$ is a $\mathbb{Z}_2$ grading of $\mathbb{C}\otimes\mathbb{O}$.

2: $\mathbb{D}^{[1]}_{\mathbb{O};2}$ is a $\mathbb{Z}_2$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{O}$.

3: $\mathbb{D}^{[1]}_{\mathbb{O};3}$ is a $\mathbb{Z}_2$ grading of $\mathbb{S}$.

4: $\mathbb{D}^{[1]}_{\mathbb{O};4}$ is a $\mathbb{Z}_2$ grading of $\tilde{\mathbb{S}}$.

with signatures:

$$\mathbb{D}^{[1]}_{\mathbb{O};1}: \quad - \ + \ + \ + \ + \ + \ + \ + \tag{5.56}$$
$$\mathbb{D}^{[1]}_{\mathbb{O};2}: \quad + \ - \ - \ - \ - \ - \ - \ - \tag{5.57}$$
$$\mathbb{D}^{[1]}_{\mathbb{O};3}: \quad - \ - \ - \ - \ - \ - \ - \ - \tag{5.58}$$
$$\mathbb{D}^{[1]}_{\mathbb{O};4}: \quad + \ + \ + \ + \ + \ + \ + \ + \tag{5.59}$$

where $\mathbb{S}$ and $\tilde{\mathbb{S}}$ are the sedenions and split-sedenions, the name split-sedenions doesn't make sense anymore because the sedenions are not a division algebra. The split will continue to be used to represent the sedenions that has space-like vectors.

5.7 $\mathbb{Z}_2^2$ Super division algebras

Real $\mathbb{Z}_2^2$ SDA

Now the inequivalent SDA's will be given by the projection. Hence, the number inside the parenthesis is the number of the subalgebra, as explained before. In the real case there are only two subalgebras, therefore it will be number 1 and 2. In the complex case it will be three subalgebras and so on.

$\mathbb{Z}_2$: 1: 0+ , 2: 1+

where 1: is the grading of the complex numbers and 2: of the split-complex numbers. 1+ means that there is one space like element in the odd sector and 0+ means that there is not a space like element.

1: $\mathbb{D}^{[2]}_{\mathbb{R};1}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{H}$.

2: $\mathbb{D}^{[2]}_{\mathbb{R};2}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{H}}$.

3: $\mathbb{D}^{[2]}_{\mathbb{R};3}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C}\otimes\mathbb{C}$.

4: $\mathbb{D}^{[2]}_{\mathbb{R};4}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}$.

1: (1,1,1) 2: (1,2,2) 3: (1,1,2) 4: (2,2,2)

the projections show that they are not equivalent. One can check that there is also an extension of equivalence 5.51:

$$\tilde{\mathbb{C}} \otimes \mathbb{C} \simeq \mathbb{C} \otimes \tilde{\mathbb{C}} \simeq \mathbb{C} \otimes \mathbb{C} \tag{5.60}$$

Complex $\mathbb{Z}_2^2$ SDA

$\mathbb{Z}_2$: 1: 0+ , 2: 2+ , 3: 1+

1: $\mathbb{D}^{[2]}_{\mathbb{C};1}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \mathbb{H}$

2: $\mathbb{D}^{[2]}_{\mathbb{C};2}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{H}$

3: $\mathbb{D}^{[2]}_{\mathbb{C};3}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{H}}$

4: $\mathbb{D}^{[2]}_{\mathbb{C};4}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{H}$

5: $\mathbb{D}^{[2]}_{\mathbb{C};5}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$

6: $\mathbb{D}^{[2]}_{\mathbb{C};6}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{O}$.

7: $\mathbb{D}^{[2]}_{\mathbb{C};7}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{O}}$.

1: (1,3,1) 2: (1,3,2) 3: (2,3,2) 4: (3,3,3) 5: (3,3,3) 6: (1,1,1) 7: (1,2,2)

where the non-equivalence between 4 and 5 is given by the fact that, all matrices from the odd sector commute with each other in 5 and anti-commute with each other in 4. Also 4 is the rearrangement of 2. $\mathbb{H} \otimes \mathbb{C}$ won't work because the matrices in the grading sectors will commute, therefore, it is equivalent to 5.

There are also the following equivalences:

$$\mathbb{C} \otimes \tilde{\mathbb{H}} \simeq \mathbb{C} \otimes \mathbb{H} \tag{5.61}$$

$$\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \mathbb{C} \simeq \tilde{\mathbb{C}} \otimes \mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \tag{5.62}$$

Quaternionic $\mathbb{Z}_2^2$ SDA

$\mathbb{Z}_2$: 1: 3+ , 2: 1+ , 3: 0+ , 4: 4+

1: $\mathbb{D}^{[2]}_{\mathbb{H};1}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{H} \otimes \mathbb{H}$.

2: $\mathbb{D}^{[2]}_{\mathbb{H};2}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{H}$.

3: $\mathbb{D}^{[2]}_{\mathbb{H};3}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{H}} \otimes \mathbb{H}$.

4: $\mathbb{D}^{[2]}_{\mathbb{H};4}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \mathbb{H}$.

5: $\mathbb{D}^{[2]}_{\mathbb{H};5}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{O}$.

6: $\mathbb{D}^{[2]}_{\mathbb{H};6}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \mathbb{O}$.

7: $\mathbb{D}^{[2]}_{\mathbb{H};7}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{O}}$.

8: $\mathbb{D}^{[2]}_{\mathbb{H};8}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{S}$.

9: $\mathbb{D}^{[2]}_{\mathbb{H};9}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{S}}$.

1: (1,1,1) 2: (1,1,2) 3: (1,2,2) 4: (2,2,2) 5: (3,1,4) 6: (3,2,3) 7: (4,2,4) 8: (3,3,3) 9: (3,4,4)

Equivalence 5.60 remains here and there is also:

$$\mathbb{C} \otimes \tilde{\mathbb{O}} \simeq \mathbb{C} \otimes \mathbb{O} \tag{5.63}$$

Octonionic $\mathbb{Z}_2^2$ SDA

$\mathbb{Z}_2$: 1: 7+ , 2: 1+ , 3: 0+ , 4: 8+

1: $\mathbb{D}^{[2]}_{\mathbb{O};1}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{H} \otimes \mathbb{O}$.

2: $\mathbb{D}^{[2]}_{\mathbb{O};2}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{O}$.

3: $\mathbb{D}^{[2]}_{\mathbb{O};3}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{H}} \otimes \mathbb{O}$.

4: $\mathbb{D}^{[2]}_{\mathbb{O};4}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \mathbb{O}$.

5: $\mathbb{D}^{[2]}_{\mathbb{O};5}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{C} \otimes \mathbb{S}$.

6: $\mathbb{D}^{[2]}_{\mathbb{O};6}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \mathbb{S}$.

7: $\mathbb{D}^{[2]}_{\mathbb{O};7}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{S}}$.

8: $\mathbb{D}^{[2]}_{\mathbb{O};8}$ is a $\mathbb{Z}_2^2$ grading of $\mathbb{TR}$.

9: $\mathbb{D}^{[2]}_{\mathbb{O};9}$ is a $\mathbb{Z}_2^2$ grading of $\tilde{\mathbb{TR}}$.

1: (1,1,1) 2: (1,1,2) 3: (1,2,2) 4: (2,2,2) 5: (3,1,4) 6: (3,2,3) 7: (4,2,4) 8: (3,3,3) 9: (3,4,4)

where $\mathbb{TR}$ and $\tilde{\mathbb{TR}}$ are the trigentaduonions and the split-trigentaduonions.

There are also 5.60 and the following equivalence:

$$\mathbb{C} \otimes \tilde{\mathbb{S}} \simeq \mathbb{C} \otimes \mathbb{S} \tag{5.64}$$

5.8 $\mathbb{Z}_2^3$ Super division algebras

Real $\mathbb{Z}_2^3$ SDA

Here one has to consider that $\mathbb{Z}_2^3$ SDA's has a $\mathbb{Z}_2^2$ graded subalgebra while having also $\mathbb{Z}_2$ graded subalgebras. Therefore, now it is necessary to see every possible $\mathbb{Z}_2^2$ graded subalgebra inside the projection, based on the rearrangements. It can be done by looking at the $\mathbb{Z}_2$ graded projection and knowing the possible rearrangements that will be discussed below.

The parenthesis will give the $\mathbb{Z}_2$ graded subalgebras. However, the dot-comma will mark the $\mathbb{Z}_2^2$ graded subalgebras, reminding that the bosonic grading is not in the parenthesis.

$\mathbb{Z}_2$: 1: 0+ , 2: 1+

1: $\mathbb{D}^{[3]}_{\mathbb{R};1}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{H}$.

2: $\mathbb{D}^{[3]}_{\mathbb{R};2}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}$.

3: $\mathbb{D}^{[3]}_{\mathbb{R};3}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}$.

4: $\mathbb{D}^{[3]}_{\mathbb{R};4}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$.

5: $\mathbb{D}^{[3]}_{\mathbb{R};5}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}$.

6: $\mathbb{D}^{[3]}_{\mathbb{R};6}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{O}$.

7: $\mathbb{D}^{[3]}_{\mathbb{R};7}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{O}}$.

1: (1,1,1;1,2,2,2) 2: (1,1,1;2,1,1,1) 3: (1,2,2;2,1,2,2) 4: (1,1,2;1,2,2,1)

5: (2,2,2;2,2,2,2) 6: (1,1,1;1,1,1,1) 7: (1,1,1;2,2,2,2)

equivalences 5.61 and 5.62 continue here. In 5.62 there is the addition of all possible combinations of tensor product of the complex and split-complex numbers. There is not the rearrangement of 1 because it won't change anything.

Rearrangements

Now it is the time to talk about the possible rearrangements that can be done without changing the multiplication rule. Here it will be only about the $\mathbb{Z}_2^3$ case, but this can be extended to further cases. Let's see then why 1 and 4 are not equivalent.

There are three rearrangements that does not break the multiplication rule. The first is of two adjacent elements. In the example below, the dot-commas marks the group of two that can be exchanged. The first number cannot be changed in this type of rearrangement because of the bosonic grading.

$$(1;\overline{1,1};\overline{1,2};\overline{2,2})$$

therefore, in this example, the rearrangements of this type that changes the subalgebra are given by.

1: (1;1,2;1,1;2,2) or (1;2,2;1,2;1,1)

The second possibility is of one element, with the following conditions:

$$(1;1,1;1,2;2,2) \qquad (1;1,1;1,2;2,2)$$

One can change the last element from the group of two elements inside the dot-commas. However, when one trades two elements, the other two must be trade either. Therefore, in this example, the only rearrangement of this type that changes the subalgebra is given by.

1: (2;1,2;1,1;2,1)

This is why a rearrangement of the elements that are the first in the group of two, delimited by the dot-commas, cannot be changed; otherwise one would have to change the bosonic sector that contains the identity.

The last type of rearrangement consists in a trade of the elements inside the dot-commas. The only condition is that, one can only change two groups of two, not all three. There are three possibilities, below are two of them:

$(1;\overset{\frown}{1,1};\overset{\frown}{1,2};2,2)$ $(1;\overset{\frown}{1,1};1,2;\overset{\frown}{2,2})$

This show us that it is necessary to check all the possible combinations of rearrangements and see the complete set of subalgebras. For example, in 1, one rearrangement gives the same $\mathbb{Z}_2^2$ graded subalgebra then 4, but what matters is the complete set of rearrangements:

1: (1,1,1;1,2,2,2)

1: (1,1,2;1,1,2,2)

1: (1,2,2;1,2,1,1)

4: (1,1,2;1,2,2,1)

4: (2,2,2;1,1,1,1)

In the second parenthesis of 4 there is a mix of two rearrangements. The commutation and anti-commutation relations between operators also matters for the subalgebra to be correct. The $\mathbb{Z}_2^2$ graded subalgebras of them are given by:

1: $\mathbb{Z}_2^2$: $\mathbb{H}$ or $\tilde{\mathbb{H}}$ or $\mathbb{C}\otimes\mathbb{C}$

4: $\mathbb{Z}_2^2$: $\mathbb{C}\otimes\mathbb{C}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}$

Complex $\mathbb{Z}_2^3$ SDA

$\mathbb{Z}_2$: 1: 0+ , 2: 2+ , 3: 1+

1: $\mathbb{D}_{\mathbb{C};1}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H}\otimes\mathbb{H}$.

2: $\mathbb{D}_{\mathbb{C};2}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{H}$.

3: $\mathbb{D}_{\mathbb{C};3}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$.

4: $\mathbb{D}_{\mathbb{C};4}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}$.

5: $\mathbb{D}_{\mathbb{C};5}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}$.

6: $\mathbb{D}_{\mathbb{C};6}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$.

7: $\mathbb{D}_{\mathbb{C};7}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{O}$.

8: $\mathbb{D}_{\mathbb{C};8}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{O}$.

9: $\mathbb{D}_{\mathbb{C};9}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{O}$.

10: $\mathbb{D}_{\mathbb{C};10}^{[3]}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{O}}$.

11: $\mathbb{D}^{[3]}_{\mathbb{C};11}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{S}$.

12: $\mathbb{D}^{[3]}_{\mathbb{C};12}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{S}}$.

1: (1,3,2;3,2,3,2) 2: (1,3,2;3,1,3,1) 3: (1,3,2;3,2,3,1) 4: (1,3,1;3,1,3,1) 5: (2,3,2;3,2,3,2)

6: (3,3,3;3,3,3,3) 7: (1,1,1;3,2,2,2) 8: (3,3,3;3,3,3,3) 9: (1,1,1;3,1,1,1)

10: (1,2,2;3,1,2,2) 11: (1,1,1;1,1,1,1) 12: (1,1,1;2,2,2,2)

The non-equivalence between 9 and 11 is because one is associative and the other is non-associative, also the elements from the sectors commute in 9 and anti-commute in 11.

Equivalence 5.60 continue and there are also:

$$\mathbb{H}\otimes\tilde{\mathbb{H}} \simeq \tilde{\mathbb{H}}\otimes\mathbb{H} \tag{5.65}$$

$$\tilde{\mathbb{H}}\otimes\tilde{\mathbb{H}} \simeq \mathbb{H}\otimes\mathbb{H} \tag{5.66}$$

$$\mathbb{C}\otimes\mathbb{C}\otimes\tilde{\mathbb{H}} \simeq \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H} \tag{5.67}$$

$$\tilde{\mathbb{C}}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C} \simeq \ldots \simeq \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C} \tag{5.68}$$

where in the last one above it is all possible combinations of tensor products.

The equivalence 5.63 continues here, to see it, just change the position of the last four bi-octonionic elements with the elements 4-8.

Quaternionic $\mathbb{Z}_2^3$ SDA

$\mathbb{Z}_2$: 1: 3+ , 2: 1+ , 3: 0+ , 4: 4+

1: $\mathbb{D}^{[3]}_{\mathbb{H};1}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{H}$.

2: $\mathbb{D}^{[3]}_{\mathbb{H};2}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{H}$.

3: $\mathbb{D}^{[3]}_{\mathbb{H};3}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{H}$.

4: $\mathbb{D}^{[3]}_{\mathbb{H};4}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$.

5: $\mathbb{D}^{[3]}_{\mathbb{H};5}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}$.

6: $\mathbb{D}^{[3]}_{\mathbb{H};6}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H}\otimes\mathbb{O}$.

7: $\mathbb{D}^{[3]}_{\mathbb{H};7}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H}\otimes\mathbb{O}$.

8: $\mathbb{D}^{[3]}_{\mathbb{H};8}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{O}$.

9: $\mathbb{D}^{[3]}_{\mathbb{H};9}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H}\otimes\tilde{\mathbb{O}}$.

10: $\mathbb{D}^{[3]}_{\mathbb{H};10}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H}\otimes\tilde{\mathbb{O}}$.

11: $\mathbb{D}^{[3]}_{\mathbb{H};11}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{H}}\otimes\tilde{\mathbb{O}}$.

12: $\mathbb{D}^{[3]}_{\mathbb{H};12}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{O}$.

13: $\mathbb{D}^{[3]}_{\mathbb{H};13}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{O}$.

14: $\mathbb{D}^{[3]}_{\mathbb{H};14}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \tilde{\mathbb{O}}$.

15: $\mathbb{D}^{[3]}_{\mathbb{H};15}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C} \otimes \mathbb{S}$.

16: $\mathbb{D}^{[3]}_{\mathbb{H};16}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \mathbb{S}$.

17: $\mathbb{D}^{[3]}_{\mathbb{H};17}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{S}}$.

18: $\mathbb{D}^{[3]}_{\mathbb{H};18}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{TR}$.

19: $\mathbb{D}^{[3]}_{\mathbb{H};19}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{T}}\mathbb{R}$.

1: (1,1,1;1,2,2,2) 2: (1,1,1;2,1,1,1) 3: (1,2,2;2,1,2,2) 4: (1,1,2;1,2,2,1) 5: (2,2,2;2,2,2,2)

6: (3,1,4;1,4,1,4) 7: (1,1,1;1,1,1,1) 8: (3,1,4;2,3,2,3) 9: (4,1,3;1,3,1,3) 10: (1,1,1;2,2,2,2)

11: (4,1,3;2,4,2,4) 12: (3,1,4;1,4,2,3) 13: (3,2,3;2,3,2,3) 14: (4,2,4;2,4,2,4) 15: (3,3,3;1,4,4,4)

16: (3,3,3;2,3,3,3) 17: (3,4,4;2,3,4,4) 18: (3,3,3;3,3,3,3) 19: (3,3,3;4,4,4,4)

Now it appears the rearrangement of $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{H} \otimes \tilde{\mathbb{O}}$, it works in the same way then the bi-quaternions and others. In this case it can be $\mathbb{O} \otimes \mathbb{H}$ and $\tilde{\mathbb{O}} \otimes \mathbb{H}$.

Octonionic $\mathbb{Z}_2^3$ SDA

$\mathbb{Z}_2$: 1: 7+ , 2: 1+ , 3: 0+ , 4: 8+

1: $\mathbb{D}^{[3]}_{\mathbb{O};1}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$.

2: $\mathbb{D}^{[3]}_{\mathbb{O};2}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \mathbb{H} \otimes \mathbb{O}$.

3: $\mathbb{D}^{[3]}_{\mathbb{O};3}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{H}} \otimes \mathbb{O}$.

4: $\mathbb{D}^{[3]}_{\mathbb{O};4}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{O}$.

5: $\mathbb{D}^{[3]}_{\mathbb{O};5}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \mathbb{O}$.

6: $\mathbb{D}^{[3]}_{\mathbb{O};6}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{O} \otimes \mathbb{O}$.

7: $\mathbb{D}^{[3]}_{\mathbb{O};7}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{O}} \otimes \mathbb{O}$.

8: $\mathbb{D}^{[3]}_{\mathbb{O};8}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H} \otimes \mathbb{S}$.

9: $\mathbb{D}^{[3]}_{\mathbb{O};9}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{H}} \otimes \mathbb{S}$.

10: $\mathbb{D}^{[3]}_{\mathbb{O};10}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{H} \otimes \tilde{\mathbb{S}}$.

11: $\mathbb{D}^{[3]}_{\mathbb{O};11}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{H}} \otimes \tilde{\mathbb{S}}$.

12: $\mathbb{D}^{[3]}_{\mathbb{O};12}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{S}$.

13: $\mathbb{D}^{[3]}_{\mathbb{O};13}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \mathbb{S}$.

14: $\mathbb{D}^{[3]}_{\mathbb{O};14}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \tilde{\mathbb{S}}$.

15: $\mathbb{D}^{[3]}_{\mathbb{O};15}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{C} \otimes \mathbb{TR}$.

16: $\mathbb{D}^{[3]}_{\mathbb{O};16}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \mathbb{TR}$.

17: $\mathbb{D}^{[3]}_{\mathbb{O};17}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{TR}}$.

18: $\mathbb{D}^{[3]}_{\mathbb{O};18}$ is a $\mathbb{Z}_2^3$ grading of $\mathbb{SE}$.

19: $\mathbb{D}^{[3]}_{\mathbb{O};19}$ is a $\mathbb{Z}_2^3$ grading of $\tilde{\mathbb{SE}}$.

1: (1,1,1;1,2,2,2) 2: (1,1,1;2,1,1,1) 3: (1,2,2;2,1,2,2) 4: (1,1,2;1,2,2,1) 5: (2,2,2;2,2,2,2)

6: (1,1,1;1,1,1,1) 7: (1,1,1;2,2,2,2) 8: (3,1,4;1,4,1,4) 9: (3,1,4;2,3,2,3) 10: (4,1,3;1,3,1,3)

11: (4,1,3;2,4,2,4) 12: (3,1,4;1,4,2,3) 13: (3,2,3;2,3,2,3) 14: (4,2,4;2,4,2,4) 15: (3,3,3;1,4,4,4)

16: (3,3,3;2,3,3,3) 17: (3,4,4;2,3,4,4) 18: (3,3,3;3,3,3,3) 19: (3,3,3;4,4,4,4)

Where $\mathbb{SE}$ is the Cayley-Dickson algebra that has a dimension of 64. $\tilde{\mathbb{SE}}$ is the Cayley-Dickson algebra of dimension 64 that has space-like elements. There is also the additional equivalence:

$$\mathbb{C} \otimes \tilde{\mathbb{TR}} \simeq \mathbb{C} \otimes \mathbb{TR} \tag{5.69}$$

5.9 $\mathbb{Z}_2^4$ Super division algebras

Real $\mathbb{Z}_2^4$ SDA

A $\mathbb{Z}_2^4$ graded SDA has a $\mathbb{Z}_2^3$ graded subalgebra, together with a $\mathbb{Z}_2^2$ and $\mathbb{Z}_2$ graded subalgebras. Therefore, the projection now has a combination of three subalgebras.

$\mathbb{Z}_2$: 1: 0+ , 2: 1+

1: $\mathbb{D}^{[4]}_{\mathbb{R};1}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H} \otimes \mathbb{H}$.

2: $\mathbb{D}^{[4]}_{\mathbb{R};2}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}} \otimes \mathbb{H}$.

3: $\mathbb{D}^{[4]}_{\mathbb{R};3}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{H}$.

4: $\mathbb{D}^{[4]}_{\mathbb{R};4}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \mathbb{H}$.

5: $\mathbb{D}^{[4]}_{\mathbb{R};5}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \tilde{\mathbb{H}}$.

6: $\mathbb{D}^{[4]}_{\mathbb{R};6}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$.

7: $\mathbb{D}^{[4]}_{\mathbb{R};7}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}} \otimes \tilde{\mathbb{C}}$.

8: $\mathbb{D}^{[4]}_{\mathbb{R};8}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C} \otimes \mathbb{O}$.

9: $\mathbb{D}^{[4]}_{\mathbb{R};9}$ is a $\mathbb{Z}_2^4$grading of $\tilde{\mathbb{C}} \otimes \mathbb{O}$.

10: $\mathbb{D}^{[4]}_{\mathbb{R};10}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}} \otimes \tilde{\mathbb{O}}$.

11: $\mathbb{D}^{[4]}_{\mathbb{R};11}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{S}$.

12: $\mathbb{D}^{[4]}_{\mathbb{R};12}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{S}}$.

1: (1,1,1;1,2,2,2;1,2,2,2;1,2,2,2) 2: (1,1,1;1,2,2,2;2,1,1,1;2,1,1,1) 3: (1,1,1;1,2,2,2;1,2,2,2;2,1,1,1)

4: (1,1,1;2,1,1,1;2,1,1,1;2,1,1,1) 5: (1,2,2;2,1,2,2;2,1,2,2;2,1,2,2) 6: (1,1,2;1,2,2,1;1,2,2,1;2,1,1,2)

7: (2,2,2;2,2,2,2;2,2,2,2;2,2,2,2) 8: (1,1,1;1,1,1,1;1,2,2,2;2,2,2,2) 9: (1,1,1;1,1,1,1;2,1,1,1;1,1,1,1)

10: (1,1,1;2,2,2,2;2,1,1,1;2,2,2,2) 11: (1,1,1;1,1,1,1;1,1,1,1;1,1,1,1) 12: (1,1,1;1,1,1,1;2,2,2,2;2,2,2,2)

Equivalences 5.63 and 5.65-5.68 continues here. One can check, see the next section, that by making the rearrangements on 3 and 6, the $\mathbb{Z}_2^3$ subalgebras are given by:

3: $\mathbb{Z}_2^3$: $\mathbb{C}\otimes\mathbb{H}$ or $\tilde{\mathbb{C}}\otimes\mathbb{H}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}$

6: $\mathbb{Z}_2^3$: $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}$

Looking at the $\mathbb{Z}_2^3$ graded subalgebras is enough to see the inequivalence of the SDA's. The way to check for the equivalence is to see if, by using the possible rearrangements, the complete projection becomes exactly like the other one that is being compared to.

Rearrangements

The rearrangements in the $\mathbb{Z}_2^3$ case will transfer to $\mathbb{Z}_2^4$, case but with more possibilities. The rearrangement of the group of two in the $\mathbb{Z}_2^3$ case becomes here a rearrangement of the group of four, marked by the dot-commas. Hence, in this case, the first three numbers cannot be changed.

The rearrangements of one number, with a condition in the $\mathbb{Z}_2^3$ case, becomes here a rearrangement of a group of two, with the same condition. Therefore, the last two numbers of the group of four, delimited by the dot-commas, can be changed. By changing two, the other two needs to be changed either. In this case the first two numbers of each group cannot be changed because of the bosonic sector.

$$(1,\overline{1,1};1,2,\overline{2,2};1,2,\overline{2,2};1,2,\overline{2,2}) \qquad (1,\overline{1,1};1,2,\overline{2,2};1,2,\overline{2,2};1,2,\overline{2,2})$$

The rearrangement of the group of two in the $\mathbb{Z}_2^3$ case, becomes here a rearrangement of two numbers inside the group of four. The condition remains, one must change two of the three groups of four delimited by the dot-commas. there are three possibilities, below is one of them.

$$(1,1,1;\overline{1,2},\overline{2,2};\overline{1,2},\overline{2,2};1,2,2,2)$$

The new rearrangements that can be done are of one number. By dividing the group of four into groups of two, this type of rearrangement obeys the following conditions:

$$(1;1,1;1,2;2,2;1,2;2,2;1,2;2,2) \qquad (1;1,1;1,2;2,2;1,2;2,2;1,2;2,2)$$

$$(1;1,1;1,2;2,2;1,2;2,2;1,2;2,2)$$

The condition is the same than the second type of rearrangement. By changing two numbers, one must change the other two to respect the multiplication rules.

And the last type of rearrangement is the trade of the elements of the group of two. There are now two conditions. The first three numbers cannot be changed and one must change four groups; there are 6 in total. Below is an example:

$$(1,1,1;\overset{\frown}{1,2};\overset{\frown}{2,2};\overset{\frown}{1,2};\overset{\frown}{2,2};1,2;2,2)$$

All these possibilities doesn't brake the multiplication rules. One can use a mix of the five types of rearrangements and it won't brake the multiplication rules either.

Complex $\mathbb{Z}_2^4$ SDA

$\mathbb{Z}_2$: 1: 0+ , 2: 2+ , 3: 1+

1: $\mathbb{D}_{\mathbb{C};1}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{H}$.

2: $\mathbb{D}_{\mathbb{C};2}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{H}$.

3: $\mathbb{D}_{\mathbb{C};3}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{H}$.

4: $\mathbb{D}_{\mathbb{C};4}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$.

5 $\mathbb{D}_{\mathbb{C};5}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}$.

6: $\mathbb{D}_{\mathbb{C};6}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}$.

7: $\mathbb{D}_{\mathbb{C};7}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$.

8: $\mathbb{D}_{\mathbb{C};8}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\mathbb{O}$.

9: $\mathbb{D}_{\mathbb{C};9}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{O}$.

10: $\mathbb{D}_{\mathbb{C};10}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\tilde{\mathbb{O}}$.

11: $\mathbb{D}_{\mathbb{C};11}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\tilde{\mathbb{O}}$.

12: $\mathbb{D}_{\mathbb{C};12}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{O}$.

13: $\mathbb{D}_{\mathbb{C};13}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{O}$.

14: $\mathbb{D}_{\mathbb{C};14}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{O}}$.

15: $\mathbb{D}_{\mathbb{C};15}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{S}$.

16: $\mathbb{D}_{\mathbb{C};16}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{S}$.

17: $\mathbb{D}_{\mathbb{C};17}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{S}$.

18: $\mathbb{D}_{\mathbb{C};18}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{S}}$.

19: $\mathbb{D}_{\mathbb{C};19}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{T}\mathbb{R}$.

20: $\mathbb{D}_{\mathbb{C};20}^{[4]}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{T}}\mathbb{R}$.

1: (1,3,2;3,2,3,2;3,2,3,1;3,1,3,1) 2: (1,3,2;3,2,3,2;3,1,3,2;3,2,3,2) 3: (1,3,2;3,1,3,1;3,1,3,2;3,1,3,1)

4: (1,3,2;3,2,3,1;3,2,3,1;3,1,3,2) 5: (1,3,1;3,1,3,1;3,1,3,1;3,1,3,1) 6: (2,3,2;3,2,3,2;3,2,3,2;3,2,3,2)

7: (3,3,3;3,3,3,3;3,3,3,3;3,3,3,3) 8: (1,1,1;3,2,2,2;3,2,2,2;3,2,2,2) 9: (1,1,1;3,2,2,2;3,1,1,1;3,1,1,1)

10: (1,2,2;3,2,1,1;3,2,1,1;3,2,1,1) 11: (1,2,2;3,2,1,1;3,1,2,2;3,1,2,2) 12: (1,1,1;3,2,2,2;3,2,2,2;3,1,1,1)

13: (1,1,1;3,1,1,1;3,1,1,1;3,1,1,1) 14: (1,2,2;3,1,2,2;3,1,2,2;3,1,2,2) 15: (1,1,1;1,1,1,1;3,2,2,2;2,2,2,2)

16: (3,3,3;3,3,3,3;3,3,3,3;3,3,3,3) 17: (1,1,1;1,1,1,1;3,1,1,1;1,1,1,1) 18: (1,1,1;2,2,2,2;3,1,1,1;2,2,2,2)

19: (1,1,1;1,1,1,1;1,1,1,1;1,1,1,1) 20: (1,1,1;1,1,1,1;2,2,2,2;2,2,2,2)

One can check that, by making the rearrangements on 1 and 4, the $\mathbb{Z}_2^3$ subalgebras are given by:

1: $\mathbb{Z}_2^3$: $\mathbb{H}\otimes\mathbb{H}$ or $\tilde{\mathbb{H}}\otimes\mathbb{H}$ or $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$

4: $\mathbb{Z}_2^3$: $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}$

Quaternionic $\mathbb{Z}_2^4$ SDA

$\mathbb{Z}_2$: 1: 3+ , 2: 1+ , 3: 0+ , 4: 4+

1: $\mathbb{D}^{[4]}_{\mathbb{H};1}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$.

2: $\mathbb{D}^{[4]}_{\mathbb{H};2}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{H}\otimes\mathbb{H}$.

3: $\mathbb{D}^{[4]}_{\mathbb{H};3}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{H}$.

4: $\mathbb{D}^{[4]}_{\mathbb{H};4}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{H}$.

5: $\mathbb{D}^{[4]}_{\mathbb{H};5}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{H}$.

6: $\mathbb{D}^{[4]}_{\mathbb{H};6}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$.

7: $\mathbb{D}^{[4]}_{\mathbb{H};7}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}$.

8: $\mathbb{D}^{[4]}_{\mathbb{H};8}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$.

9: $\mathbb{D}^{[4]}_{\mathbb{H};9}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{O}$.

10: $\mathbb{D}^{[4]}_{\mathbb{H};10}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{O}$.

11: $\mathbb{D}^{[4]}_{\mathbb{H};11}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\tilde{\mathbb{O}}$.

12: $\mathbb{D}^{[4]}_{\mathbb{H};12}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\tilde{\mathbb{O}}$.

13: $\mathbb{D}^{[4]}_{\mathbb{H};13}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{O}$.

14: $\mathbb{D}^{[4]}_{\mathbb{H};14}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{O}$.

15: $\mathbb{D}^{[4]}_{\mathbb{H};15}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{O}}$.

16: $\mathbb{D}^{[4]}_{\mathbb{H};16}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{O}\otimes\mathbb{O}$.

17: $\mathbb{D}^{[4]}_{\mathbb{H};17}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{O}}\otimes\mathbb{O}$.

18: $\mathbb{D}^{[4]}_{\mathbb{H};18}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{O}\otimes\tilde{\mathbb{O}}$.

19: $\mathbb{D}^{[4]}_{\mathbb{H};19}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{O}}\otimes\tilde{\mathbb{O}}$.

20: $\mathbb{D}^{[4]}_{\mathbb{H};20}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\mathbb{S}$.

21: $\mathbb{D}^{[4]}_{\mathbb{H};21}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\mathbb{S}$.

22: $\mathbb{D}^{[4]}_{\mathbb{H};22}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{S}$.

23: $\mathbb{D}^{[4]}_{\mathbb{H};23}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\tilde{\mathbb{S}}$.

24: $\mathbb{D}^{[4]}_{\mathbb{H};24}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\tilde{\mathbb{S}}$.

25: $\mathbb{D}^{[4]}_{\mathbb{H};25}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\tilde{\mathbb{S}}$.

26: $\mathbb{D}^{[4]}_{\mathbb{H};26}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{S}$.

27: $\mathbb{D}^{[4]}_{\mathbb{H};27}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{S}$.

28: $\mathbb{D}^{[4]}_{\mathbb{H};28}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{S}}$.

29: $\mathbb{D}^{[4]}_{\mathbb{H};29}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{T}\mathbb{R}$.

30: $\mathbb{D}^{[4]}_{\mathbb{H};30}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{T}\mathbb{R}$.

31: $\mathbb{D}^{[4]}_{\mathbb{H};31}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\widetilde{\mathbb{T}\mathbb{R}}$.

32: $\mathbb{D}^{[4]}_{\mathbb{H};32}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{S}\mathbb{E}$.

33: $\mathbb{D}^{[4]}_{\mathbb{H};33}$ is a $\mathbb{Z}_2^4$ grading of $\widetilde{\mathbb{S}\mathbb{E}}$.

1: (1,1,1;1,2,2,2;1,2,2,2;1,2,2,2) 2: (1,1,1;1,2,2,2;2,1,1,1;2,1,1,1) 3: (1,1,1;1,2,2,2;1,2,2,2;2,1,1,1)

4: (1,1,1;2,1,1,1;2,1,1,1;2,1,1,1) 5: (1,2,2;2,1,2,2;2,1,2,2;2,1,2,2) 6: (1,1,2;1,2,2,1;1,2,2,1;2,1,1,2)

7: (2,2,2;2,2,2,2;2,2,2,2;2,2,2,2) 8: (3,1,4;1,4,1,4;1,4,2,3;2,3,2,3) 9: (3,1,4;1,4,1,4;2,3,1,4;1,4,1,4)

10: (3,1,4;2,3,2,3;2,3,1,4;2,3,2,3) 11: (4,1,3;1,3,1,3;2,4,1,3;1,3,1,3) 12: (4,1,3;2,4,2,4;2,4,1,3;2,4,2,4)

13: (3,1,4;1,4,2,3;1,4,2,3;2,3,1,4) 14: (3,2,3;2,3,2,3;2,3,2,3;2,3,2,3) 15: (4,2,4;2,4,2,4;2,4,2,4;2,4,2,4)

16: (3,1,4;1,4,1,4;1,4,1,4;1,4,1,4) 17: (3,1,4;1,4,1,4;2,3,2,3;2,3,2,3) 18: (4,1,3;1,3,1,3;1,3,1,3;1,3,1,3)

19: (4,1,3;1,3,1,3;2,4,2,4;2,4,2,4) 20: (3,3,3;1,4,4,4;1,4,4,4;1,4,4,4) 21: (1,1,1;1,1,1,1;1,1,1,1;1,1,1,1)

22: (3,3,3;1,4,4,4;2,3,3,3;2,3,3,3) 23: (3,4,4;1,4,3,3;1,4,3,3;1,4,3,3) 24: (1,1,1;1,1,1,1;2,2,2,2;2,2,2,2)

25: (3,4,4;1,4,3,3;2,3,4,4;2,3,4,4) 26: (3,3,3;1,4,4,4;1,4,4,4;2,3,3,3) 27: (3,3,3;2,3,3,3;2,3,3,3;2,3,3,3)

28: (3,4,4;2,3,4,4;2,3,4,4;2,3,4,4) 29: (3,3,3;3,3,3,3;1,4,4,4;4,4,4,4) 30: (3,3,3;3,3,3,3;2,3,3,3;3,3,3,3)

31: (3,3,3;4,4,4,4;2,3,3,3;4,4,4,4) 32: (3,3,3;3,3,3,3;3,3,3,3;3,3,3,3) 33: (3,3,3;3,3,3,3;4,4,4,4;4,4,4,4)

It was showed in the real case that 3 and 6 have different subalgebras. Here will be the same.

Octonionic $\mathbb{Z}_2^4$ SDA

$\mathbb{Z}_2$: 1: 7+ , 2: 1+ , 3: 0+ , 4: 8+

1: $\mathbb{D}^{[4]}_{\mathbb{O};1}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{O}$.

2: $\mathbb{D}^{[4]}_{\mathbb{O};2}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{H}\otimes\mathbb{O}$.

3: $\mathbb{D}^{[4]}_{\mathbb{O};3}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$.

4: $\mathbb{D}^{[4]}_{\mathbb{O};4}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{O}$.

5: $\mathbb{D}^{[4]}_{\mathbb{O};5}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{O}$.

6: $\mathbb{D}^{[4]}_{\mathbb{O};6}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{O}$.

7: $\mathbb{D}^{[4]}_{\mathbb{O};7}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{O}$.

8: $\mathbb{D}^{[4]}_{\mathbb{O};8}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{O}\otimes\mathbb{O}$.

9: $\mathbb{D}^{[4]}_{\mathbb{O};9}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{O}\otimes\mathbb{O}$.

10: $\mathbb{D}^{[4]}_{\mathbb{O};10}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{O}}\otimes\mathbb{O}$.

11: $\mathbb{D}^{[4]}_{\mathbb{O};11}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{S}$.

12: $\mathbb{D}^{[4]}_{\mathbb{O};12}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{S}$.

13: $\mathbb{D}^{[4]}_{\mathbb{O};13}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{S}$.

14: $\mathbb{D}^{[4]}_{\mathbb{O};14}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\tilde{\mathbb{S}}$.

15: $\mathbb{D}^{[4]}_{\mathbb{O};15}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\tilde{\mathbb{S}}$.

16: $\mathbb{D}^{[4]}_{\mathbb{O};16}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{S}$.

17: $\mathbb{D}^{[4]}_{\mathbb{O};17}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{S}$.

18: $\mathbb{D}^{[4]}_{\mathbb{O};18}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{S}}$.

19: $\mathbb{D}^{[4]}_{\mathbb{O};19}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{O}\otimes\mathbb{S}$.

20: $\mathbb{D}^{[4]}_{\mathbb{O};20}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{O}\otimes\mathbb{S}$.

21: $\mathbb{D}^{[4]}_{\mathbb{O};21}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{O}}\otimes\mathbb{S}$.

22: $\mathbb{D}^{[4]}_{\mathbb{O};22}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{O}\otimes\tilde{\mathbb{S}}$.

23: $\mathbb{D}^{[4]}_{\mathbb{O};23}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{O}\otimes\tilde{\mathbb{S}}$.

24: $\mathbb{D}^{[4]}_{\mathbb{O};24}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{O}}\otimes\tilde{\mathbb{S}}$.

25: $\mathbb{D}^{[4]}_{\mathbb{O};25}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\mathbb{TR}$.

26: $\mathbb{D}^{[4]}_{\mathbb{O};26}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\mathbb{TR}$.

27: $\mathbb{D}^{[4]}_{\mathbb{O};27}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{H}\otimes\tilde{\mathbb{T}}\mathbb{R}$.

28: $\mathbb{D}^{[4]}_{\mathbb{O};28}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{H}}\otimes\tilde{\mathbb{T}}\mathbb{R}$.

29: $\mathbb{D}^{[4]}_{\mathbb{O};29}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{T}\mathbb{R}$.

30: $\mathbb{D}^{[4]}_{\mathbb{O};30}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{T}\mathbb{R}$.

31: $\mathbb{D}^{[4]}_{\mathbb{O};31}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\widetilde{\mathbb{T}\mathbb{R}}$.

32: $\mathbb{D}^{[4]}_{\mathbb{O};32}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\otimes\mathbb{S}\mathbb{E}$.

33: $\mathbb{D}^{[4]}_{\mathbb{O};33}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\mathbb{S}\mathbb{E}$.

34: $\mathbb{D}^{[4]}_{\mathbb{O};34}$ is a $\mathbb{Z}_2^4$ grading of $\tilde{\mathbb{C}}\otimes\widetilde{\mathbb{S}\mathbb{E}}$.

35: $\mathbb{D}^{[4]}_{\mathbb{O};35}$ is a $\mathbb{Z}_2^4$ grading of $\mathbb{C}\mathbb{E}$.

36: $\mathbb{D}^{[4]}_{\mathbb{O};36}$ is a $\mathbb{Z}_2^4$ grading of $\widetilde{\mathbb{C}\mathbb{E}}$.

1: (1,1,1;1,2,2,2;1,2,2,2;1,2,2,2) 2: (1,1,1;1,2,2,2;2,1,1,1;2,1,1,1) 3: (1,1,1;1,2,2,2;1,2,2,2;2,1,1,1)

4: (1,1,1;2,1,1,1;2,1,1,1;2,1,1,1) 5: (1,2,2;2,1,2,2;2,1,2,2;2,1,2,2) 6: (1,1,2;1,2,2,1;1,2,2,1;2,1,1,2)

7: (2,2,2;2,2,2,2;2,2,2,2;2,2,2,2) 8: (1,1,1;1,1,1,1;1,2,2,2;2,2,2,2) 9: (1,1,1;1,1,1,1;2,1,1,1;1,1,1,1)

10: (1,1,1;2,2,2,2;2,1,1,1;2,2,2,2) 11: (3,1,4;1,4,1,4;1,4,2,3;2,3,2,3) 12: (3,1,4;1,4,1,4;2,3,1,4;1,4,1,4)

13: (3,1,4;2,3,2,3;2,3,1,4;2,3,2,3) 14: (4,1,3;1,3,1,3;2,4,1,3;1,3,1,3) 15: (4,1,3;2,4,2,4;2,4,1,3;2,4,2,4)

16: (3,1,4;1,4,2,3;1,4,2,3;2,3,1,4) 17: (3,2,3;2,3,2,3;2,3,2,3;2,3,2,3) 18: (4,2,4;2,4,2,4;2,4,2,4;2,4,2,4)

19: (3,1,4;1,4,1,4;1,4,1,4;1,4,1,4) 20:(1,1,1;1,1,1,1;1,1,1,1;1,1,1,1) 21: (3,1,4;1,4,1,4;2,3,2,3;2,3,2,3)

22: (4,1,3;1,3,1,3;1,3,1,3;1,3,1,3) 23: (1,1,1;1,1,1,1;2,2,2,2;2,2,2,2) 24: (4,1,3;1,3,1,3;2,4,2,4;2,4,2,4)

25: (3,3,3;1,4,4,4;1,4,4,4;1,4,4,4) 26: (3,3,3;1,4,4,4;2,3,3,3;2,3,3,3) 27: (3,4,4;1,4,3,3;1,4,3,3;1,4,3,3)

28: (3,4,4;1,4,3,3;2,3,4,4;2,3,4,4) 29: (3,3,3;1,4,4,4;1,4,4,4;2,3,3,3) 30: (3,3,3;2,3,3,3;2,3,3,3;2,3,3,3)

31: (3,4,4;2,3,4,4;2,3,4,4;2,3,4,4) 32: (3,3,3;3,3,3,3;1,4,4,4;4,4,4,4) 33: (3,3,3;3,3,3,3;2,3,3,3;3,3,3,3)

34: (3,3,3;4,4,4,4;2,3,3,3;4,4,4,4) 35: (3,3,3;3,3,3,3;3,3,3,3;3,3,3,3) 36: (3,3,3;3,3,3,3;4,4,4,4;4,4,4,4)

Where $\mathbb{C}\mathbb{E}$ is the Cayley-Dickson algebra that has a dimension of 128. $\widetilde{\mathbb{C}\mathbb{E}}$ is the Cayley-Dickson algebra of dimension 128 that has space-like elements. There are the following new equivalences:

$$\mathbb{C}\otimes\mathbb{C}\otimes\widetilde{\mathbb{T}\mathbb{R}} \simeq \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{T}\mathbb{R} \tag{5.70}$$

$$\mathbb{C}\otimes\widetilde{\mathbb{S}\mathbb{E}} \simeq \mathbb{C}\otimes\mathbb{S}\mathbb{E} \tag{5.71}$$

It was already discussed that 3 and 6 are non-equivalent. However, here 3, 6 and 8 are a non-associative SDA. In the real case, 8 was non-associative while 3 and 6 were associative. Therefore let's do the subalgebras of the three.

3: $\mathbb{Z}_2^3$: $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ or $\tilde{\mathbb{C}}\otimes\mathbb{H}\otimes\mathbb{O}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}\otimes\mathbb{O}$

6: $\mathbb{Z}_2^3$: $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{O}$ or $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\mathbb{O}$

8: $\mathbb{Z}_2^3$: $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ or $\mathbb{O}\otimes\mathbb{O}$ or $\tilde{\mathbb{O}}\otimes\mathbb{O}$

One can check that 1 and 10 are non-equivalent, 11 and 17 either.

Table with the SDA's

To sum up the results let's present the tables with all the super division algebras:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{Z}_2$	2	3	4	4
$\mathbb{Z}_2^2$	4	7	9	9
$\mathbb{Z}_2^3$	7	12	19	19
$\mathbb{Z}_2^4$	12	20	33	36

Table 5.15: Total SDA's.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{Z}_2$	2	3	2	-
$\mathbb{Z}_2^2$	4	5	4	-
$\mathbb{Z}_2^3$	5	6	5	-
$\mathbb{Z}_2^4$	7	7	7	-

Table 5.16: Associative.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{Z}_2$	0	0	2	4
$\mathbb{Z}_2^2$	0	2	5	9
$\mathbb{Z}_2^3$	2	6	14	19
$\mathbb{Z}_2^4$	5	13	26	36

Table 5.17: N-associative.

A very curious property that one can see is that, the number of inequivalent real associative SDA's is exactly the number of mappings for the $\mathbb{Z}_2^n$ graded Lie (super)algebras.

Tenfold way and the thirteen fold way

The tenfold way and the thirteen fold way that are defined in the papers of [59] and [65] are in the classification. The tenfold way is given by the associative $\mathbb{Z}_2$ super division algebras, hence $2+3+2=7$, plus the associative division algebras $7+3=10$. The thirteen fold way is given by the associative $\mathbb{Z}_2^2$ super division algebras, so $4+5+4=13$.

5.10 Inequivalent SDA's and mappings

It was discussed in the last sections that the inequivalent SDA's are determined base on their signatures. The signature could also relate to the mappings of graded algebras. This idea was created by finding the commutation and anti-commutation relations of the $\mathbb{Z}_2^n$ real super division algebras. However, this is not possible formally speaking; there is not a defined multiplication between a division algebra vector with a split-division vector. Neither there is a definition of a multiplication of a vector, from the quaternions for example, with one from the bi-complex. If one tries to create a algebra that is the sum of the division and the correspondent split-division there will be problems in the metric and in the multiplication.

Therefore, what is presented here is a curiosity, a game that may be explained later with future works. The only solid base for this game up until now is the fact that one can commute the generators of the SDA's, one cannot commute elements from two different SDA's. Therefore, instead of using the super division algebra generators to commute, one can use its matrix representation to commute.

The idea is to use a Clifford algebra framework and extract from this Clifford algebra, the elements whose matrix representation is equal to the matrix representation of the SDA's; commuting them it will appear the table of brackets. Here in this section it will be used a parenthesis with the number of space-like generators from each SDA's odd sector. Which represent the signature of the matrices.

Let's use the Clifford algebras that have as a representation $\mathcal{M}(2^{[n/2]},\mathbb{R})\oplus\mathcal{M}(2^{[n/2]},\mathbb{R})$. In this way there are matrices to represent the SDA's elements inside the Clifford algebra.

Clifford algebra $Cl_{(2,1)}$:

$$Cl_{(2,1)}: \quad e_0 \simeq I, \quad e_1 \simeq Z, \quad e_2 \simeq X, \quad e_3 \simeq A, \quad e_{12} \simeq A, \quad e_{13} \simeq X, \quad e_{23} \simeq -Z, \quad e_{123} \simeq -I, \tag{5.72}$$

The properties are:

$$e_i e_j = -e_j e_i, \qquad e_0^2 = e_1^2 = e_2^2 = -e_3^2 = e_0 \tag{5.73}$$

Clifford algebra $Cl_{(3,2)}$:

$$\begin{gathered} Cl_{(3,2)}: \quad e_0 \simeq II, \quad e_1 \simeq XZ, \quad e_2 \simeq XX, \quad e_3 \simeq XA, \quad e_4 \simeq AI, \quad e_5 \simeq ZI, \quad e_{12} \simeq IA, \quad e_{13} \simeq IX, \\ e_{14} \simeq -ZZ, \quad e_{15} \simeq -AZ, \quad e_{23} \simeq -IZ, \quad e_{24} \simeq -ZX, \quad e_{25} \simeq -AX, \quad e_{34} \simeq -ZA, \quad e_{35} \simeq -AA, \\ e_{45} \simeq -XI, \quad e_{123} \simeq -XI, \quad e_{124} \simeq AA, \quad e_{125} \simeq ZA, \quad e_{134} \simeq AX, \quad e_{135} \simeq ZX, \quad e_{145} \simeq -IZ, \\ e_{234} \simeq -AZ, \quad e_{235} \simeq -ZZ, \quad e_{245} \simeq -IX, \quad e_{345} \simeq -IA, \quad e_{1234} \simeq ZI, \quad e_{1235} \simeq AI, \\ e_{1245} \simeq -XA, \quad e_{1345} \simeq -XX, \quad e_{2345} \simeq XZ, \quad e_{12345} \simeq II \end{gathered} \tag{5.74}$$

The properties are:

$$e_i e_j = -e_j e_i, \qquad e_0^2 = e_1^2 = e_2^2 = e_3^2 = -e_4^2 = -e_5^2 = e_0 \tag{5.75}$$

Clifford algebra $Cl_{(0,7)}$ will be used too, but it is too big. Therefore, to help, the 7 generators will be given by the matrices in 4.19.

Real $\mathbb{Z}_2$ graded algebras

There are only two SDA's, the complex and the split-complex numbers. Therefore, the matrices that one could use are I, A, X or Z. The commutation of the matrices I and A with themselves, or the commutation of I and X with themselves, represents the commutation of signatures with the same number of space-like elements, in this case zero or one.

The commutation of I and A with I and X is the commutation of a signature with zero space-like elements with one that has one space-like element. Every time that the signatures doesn't have the same number of space-like elements, it will give a superalgebra. Using then $Cl_{(2,1)}$:

A/B	e_{123}	e_{12}
e_0	[,]	[,]
e_3	[,]	[,]

Table 5.18: (0;0) and (1;1)

A/B	e_{123}	e_{13}
e_0	[,]	[,]
e_3	[,]	{,}

Table 5.19: (0;1)

the parenthesis only represents the element in the odd sector. Hence (0;0) represents that it is a commutation of elements which has a temporal element in the odd sector. The first minus sign comes from e_0 and e_3, the second from e_{123} and e_{12}. The second parenthesis (1;1) is the commutation of e_0 and e_2 with e_{123} and e_{13}, which generates the same table of brackets. If, instead of using e_{13} in this last example, one use e_1, it won't generate the bracket table of the lie algebra, but instead the superalgebra case. Therefore, the matrix chosen must be the one related with the SDA's matrix representation, otherwise it can lead to different table of brackets.

It would be stronger if one could define the sum between the division algebras and split-division algebras. However, the matrix representation show us that the different signatures might have a relation with the mappings of graded algebras.

Real $\mathbb{Z}_2^2$ graded algebras

Using $Cl_{(3,2)}$ one can find the commutations using the 4 SDA's as guiding:

	e_{12345}	e_{345}	e_{234}	e_{134}
e_0	0	0	0	0
e_{12}	0	0	1	1
e_{15}	0	1	0	1
e_{25}	0	1	1	0

Table 5.20: (0;0) and (2;2)

	e_{12345}	e_1	e_2	e_{345}
e_0	0	0	0	0
e_{15}	0	1	0	1
e_{25}	0	0	1	1
e_{12}	0	1	1	0

Table 5.21: (0;2),(1;3) and (0;3)

	e_{12345}	e_{345}	e_{1235}	e_{124}
e_0	0	0	0	0
e_{12}	0	0	0	0
e_4	0	0	0	0
e_{35}	0	0	0	0

Table 5.22: (1;1) and (3;3)

	e_{12345}	e_{345}	e_4	e_{35}
e_0	0	0	0	0
e_{12}	0	0	0	0
e_{15}	0	0	1	1
e_{25}	0	0	1	1

Table 5.23: (0;1), (2;1) and (2;3)

the tables above are the example of the first parenthesis that generates the table of brackets; the others will generate the same result, but with different elements. When the difference of space-like elements is zero, it generates a graded algebra, when it is different than zero it generates a graded superalgebra.

Important note: If one changes the matrix representation of the quaternions or other SDA, it will change the table of brackets, however, if one respect the commutations and anti-commutations, for example, the split-quaternions guided to the choice of e_{12345}, e_1, e_2 and e_{345}. The last three all anti-commute, the first one commute with everyone. If this relation of the SDA and the signature is respected, it will generate a table of brackets related to a mapping. If one changes the matrix representation of the split-quaternions it will generate a different table of brackets, but inside all possible mappings for the $\mathbb{Z}_2^2$ case. This will be true for the next case too.

Real $\mathbb{Z}_2^3$ graded algebras

Because the bi-quaternions and the tri-complex have 3 spacelike-elements, the tri-complex will have a $*$ on the parenthesis. Therefore, using the Clifford algebra $Cl_{(0,7)}$.

$\mathbb{C}\otimes\mathbb{H}$: 3, $\tilde{\mathbb{C}}\otimes\mathbb{H}$: 1, $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}}$: 5, $\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$: 3*, $\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}$: 7, $\mathbb{O}$: 0, $\tilde{\mathbb{O}}$: 4

(3;3), (1;1) and (5;5):

	$e_{1234567}$	e_{12345}	e_{12347}	e_{12346}	e_{123567}	e_{1235}	e_{1237}	e_{1236}
e_0	0	0	0	0	0	0	0	0
e_{67}	0	0	1	1	0	0	1	1
e_{56}	0	1	0	1	0	1	0	1
e_{57}	0	1	1	0	0	1	1	0
e_4	0	0	0	0	0	0	0	0
e_{467}	0	0	1	1	0	0	1	1
e_{456}	0	1	0	1	0	1	0	1
e_{457}	0	1	1	0	0	1	1	0

Table 5.24: Mapping 5.19

(3*;3*) and (7;7):

	$e_{1234567}$	e_{12345}	e_{24567}	e_{1367}	e_{123567}	e_{1235}	e_{2567}	e_{13467}
e_0	0	0	0	0	0	0	0	0
e_{67}	0	0	0	0	0	0	0	0
e_{13}	0	0	0	0	0	0	0	0
e_{245}	0	0	0	0	0	0	0	0
e_4	0	0	0	0	0	0	0	0
e_{467}	0	0	0	0	0	0	0	0
e_{134}	0	0	0	0	0	0	0	0
e_{25}	0	0	0	0	0	0	0	0

Table 5.25: Mapping 5.16

(3*;7), (3;5), (1;5), (3;1), (3;7), (1;3*):

	$e_{1234567}$	e_{247}	e_{345}	e_{146}	e_{567}	e_{137}	e_{125}	e_{236}
e_0	0	0	0	0	0	0	0	0
e_{67}	0	1	0	1	0	1	0	1
e_{13}	0	0	1	1	0	0	1	1
e_{245}	0	1	1	0	0	1	1	0
e_4	0	0	0	0	1	1	1	1
e_{467}	0	1	0	1	1	0	1	0
e_{134}	0	0	1	1	1	1	0	0
e_{25}	0	1	1	0	1	0	0	1

Table 5.26: Mapping 5.20

(3;3*) and (5;7):

	$e_{1234567}$	e_{12345}	e_{13}	e_{245}	e_{123567}	e_{1235}	e_{134}	e_{25}
e_0	0	0	0	0	0	0	0	0
e_{57}	0	1	0	1	0	1	0	1
e_{67}	0	0	0	0	0	0	0	0
e_{56}	0	1	0	1	0	1	0	1
e_4	0	0	0	0	0	0	0	0
e_{457}	0	1	0	1	0	1	0	1
e_{467}	0	0	0	0	0	0	0	0
e_{456}	0	1	0	1	0	1	0	1

Table 5.27: Mapping 5.17

(1;7) and (5;3*):

	$e_{1234567}$	e_{247}	e_{345}	e_{146}	e_{1234}	e_{137}	e_{125}	e_{236}
e_0	0	0	0	0	0	0	0	0
e_{67}	0	1	0	1	0	1	0	1
e_{56}	0	0	1	1	0	0	1	1
e_{57}	0	1	1	0	0	1	1	0
e_{567}	0	0	0	0	0	0	0	0
e_5	0	1	0	1	0	1	0	1
e_7	0	0	1	1	0	0	1	1
e_6	0	1	1	0	0	1	1	0

Table 5.28: Mapping 5.18

These tables of brackets are not exactly the same from the mappings. However, they have the same classification according to the number of one's showed in the final part of the first section. The remaining signatures will be different because they are related with the octonions and split-octonions. They are non-associative, hence it will be treated separately.

5.11 Graded Malcev algebras

Malcev algebras

The Malcev algebras, first defined by A. Malcev [86], are well described in reference [87], the fundamental relations of Malcev algebras are the following:

$$X^2 = 0 \quad \text{where} \quad X \cdot X = [X, X] \tag{5.76}$$

$$(XY)(XZ) = ((XY)Z)X + ((YZ)X)X + ((ZX)X)Y \quad \text{or} \tag{5.77}$$

$$J(X, Y, Z)X = J(X, Y, XZ) \quad \text{where} \quad J(X, Y, Z) = (XY)Z + (YZ)X + (ZX)Y$$

where the product is anti-commutative. The octonions are a Malcev algebra, therefore, to study the non-associative super division algebras, one will need to study the Malcev algebras. The sedenions and beyond does not obey the Malcev identity, therefore this section will be only about the real $\mathbb{Z}_2^3$ graded algebras.

Sagle Identity

A Malcev algebra over a field of characteristic different than 2 can be given by the Sagle identity:

$$(XY)(ZW) = ((XZ)Y)W + ((WX)Z)Y + ((YW)X)Z + ((ZY)W)X \tag{5.78}$$

Let's remember then the properties of the Lie (super)algebras.

Lie (super)algebra

A Lie Superalgebra is given by:

$$(A, B) = -(-1)^{(\alpha,\beta)}(B, A) \tag{5.79}$$

$$(-1)^{(\gamma,\alpha)}(A, (B, C)) + (-1)^{(\alpha,\beta)}(B, (C, A)) + (-1)^{(\beta,\gamma)}(C, (A, B)) = 0 \tag{5.80}$$

The graded vectors are known to obey the Leibniz Rules, see [80]:

$$(\alpha, \beta) + (\beta, \alpha) = 2r \tag{5.81}$$

$$(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma) + 2s \tag{5.82}$$

$$(\alpha + \gamma, \beta) = (\alpha, \beta) + (\gamma, \beta) + 2s \tag{5.83}$$

Malcev (super)algebras

To define a Malcev (super)algebra, it will be the same from the Lie (super)algebras defined in the introduction, good references on the subject are [[67], [68], [69], [70]]. A Malcev (super)algebra can be defined by the following relations:

$$(A,B) = AB - (-1)^{(\alpha,\beta)} BA \tag{5.84}$$

$$\bar{J}(A,B,AC) = \bar{J}(A,B,C)A \tag{5.85}$$

$$\bar{J}(A,B,C) = (-1)^{(\gamma,\alpha)}(A,(B,C)) + (-1)^{(\alpha,\beta)}(B,(C,A)) + (-1)^{(\beta,\gamma)}(C,(A,B)) \tag{5.86}$$

Malcev Superalgebra via Sagle identity

The Malcev (super)algebra is given by:

$$\begin{aligned}(-1)^{(\gamma,\beta)}(A,B)(C,D) &= (((A,C),B),D) + (-1)^{(\alpha,\beta+\gamma+\sigma)}(((C,B),D),A) + \\ &+ (-1)^{(\alpha+\gamma,\beta+\sigma)}(((B,D),A),C) + (-1)^{(\sigma,\alpha+\beta+\gamma)}(((D,A),C),B)\end{aligned} \tag{5.87}$$

Leibniz rules for Malcev (super)algebras

It was shown that all possible brackets are given by the Leibniz rules on the Lie (super)algebra case. In the last section, it was shown that, with the super division algebras, one can find the table of brackets using the Clifford algebras to mimic the SDA's and it's signature. Let's try then to use that to define the Leibniz rules.

There are two non-associative real $\mathbb{Z}_2^3$ super division algebras, the octonions and the split-octonions. Hence, there must be two brackets associated with them, $(0,0)$ or $(4,4)$ and the bracket from $(0,4)$. The first one gives a graded algebra and the second one a graded superalgebra. Using the $Cl_{(0,7)}$ Clifford algebra.

Non-associative $\mathbb{Z}_2^3$ graded algebra

(0;0) and (4;4):

	$e_{1234567}$	e_{123567}	e_{234567}	e_{123467}	e_{123457}	e_{134567}	e_{123456}	e_{124567}
e_0	0	0	0	0	0	0	0	0
e_4	0	0	1	1	1	1	1	1
e_1	0	1	0	1	1	1	1	1
e_5	0	1	1	0	1	1	1	1
e_6	0	1	1	1	0	1	1	1
e_2	0	1	1	1	1	0	1	1
e_7	0	1	1	1	1	1	0	1
e_3	0	1	1	1	1	1	1	0

Table 5.29: Table of brackets for the (split)octonionic $\mathbb{Z}_2^3$ graded algebra

Non-associative $\mathbb{Z}_2^3$ graded superalgebra

(0;4):

	$e_{1234567}$	e_{567}	e_{234567}	e_{467}	e_{457}	e_{134567}	e_{456}	e_{124567}
e_0	0	0	0	0	0	0	0	0
e_4	0	1	1	0	0	1	0	1
e_1	0	1	0	1	1	1	1	1
e_5	0	0	1	1	0	1	0	1
e_6	0	0	1	0	1	1	0	1
e_2	0	1	1	1	1	0	1	1
e_7	0	0	1	0	0	1	1	1
e_3	0	1	1	1	1	1	1	0

Table 5.30: Table of brackets for the (split)octonionic $\mathbb{Z}_2^3$ graded superalgebra

Here the elements were not chosen to mimic the matrices in the appendix B. Instead, they were chosen based on the anti-commutation relations, hence the generators of $Cl_{(0,7)}$ and $Cl_{(4,3)}$.

The first table is more easy to accept because it is the generalization of the quaternionic table of brackets, the graded color algebra. If one commutes the octonions with themselves the same table of brackets will appear. Regarding the second table of brackets, there is the need to verify its properties, to see if it is a viable option for the superalgebra case.

Finding the map

Both tables above must represent some mapping, as it is on the Lie (super)algebra case, the difference is that it is not known if they obey the Leibniz Rules and if they are generated by the Graded Malcev identity. The correct approach is to use the graded Malcev or Sagle identity to try and demonstrate the generalized Leibniz rules, or use non-associative monoids to derive it. However, it is very difficult to analyze the graded Sagle identity. Hence, together with R. G. Rana, this approach was taken. Due to the relation of the mappings to logic portals, as it was explained later, the idea is to put the table of brackets above in a program, like the one that was mentioned in the logic portal section and use the result to get to the mapping. This is basically a reverse process, it is not a demonstration, it is a very precise guess.

The first Leibniz rule 5.81 remains the same because it comes from 5.79. However, 5.82 and 5.83 must be generalized to something that relates to the associative case. Hence, one can use a site that converts the results of the brackets to a computer language:

http://www.32x8.com/var6.html

Putting both tables on the site will give the corresponding equations that generates them. Therefore, one can use a program like mathematica to translate it to a map language and the result is:

Non-associative $\mathbb{Z}_2^3$ Graded algebra map

The mapping that generates the first table of brackets is given by:

$$(\alpha,\beta)_{\mathbb{Z}_2^3} = (\alpha_1\beta_2+\alpha_2\beta_1)+(\alpha_1\beta_3+\alpha_3\beta_1)+(\alpha_2\beta_3+\alpha_3\beta_2)+\alpha_1\alpha_2\beta_3+\alpha_3\alpha_1\beta_2+ \\ +\alpha_2\alpha_3\beta_1+\alpha_1\beta_2\beta_3+\alpha_3\beta_1\beta_2+\alpha_2\beta_3\beta_1 \quad (5.88)$$

The name non-associative here is because of the relation of the table of brackets with the octonions and split-octonions. Also possibly because of the graded Malcev identity, not because the multiplication is non-associative.

Non-associative $\mathbb{Z}_2^3$ Graded Superalgebra map

The mapping that generates the second table of brackets is given by:

$$(\alpha,\beta)_{\mathbb{Z}_2^3} = (\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)+\alpha_1\alpha_2\beta_3+\alpha_3\alpha_1\beta_2+\alpha_2\alpha_3\beta_1+\alpha_1\beta_2\beta_3+\alpha_3\beta_1\beta_2+\alpha_2\beta_3\beta_1 \quad (5.89)$$

One can see that there is a structure in both cases. Let's define then the following parameter:

$$\mathbf{S}_3 = \alpha_1\alpha_2\beta_3+\alpha_3\alpha_1\beta_2+\alpha_2\alpha_3\beta_1+\alpha_1\beta_2\beta_3+\alpha_3\beta_1\beta_2+\alpha_2\beta_3\beta_1 \quad (5.90)$$

The mapping is given by the sum of the real associative $\mathbb{Z}_2^3$ graded mapping, with a term that is the cyclic permutation of the multiplication of three elements. Do not confuse with the S_3 permutation group, this parameter was named after it. In the algebra case it is actually the sum of the associative mapping of the color algebra three times, going through all possibilities.

It is easy to realize that the Leibniz rules are the same for both cases. One can also notice that the two common products appear here, the inter and outer product. Let's emphasize then the results. Using both mappings, it is easy to calculate $(\alpha,\beta+\gamma)_{\mathbb{Z}_2^3}$. The parameter $\mathbf{S}_3$ will change the Leibniz rules, the other terms already obey them.

5.12 Generalized Leibniz Rules

Therefore, what we found from "reverse deduction" is that the generalized Leibniz rules are given by the following relations.

Generalized Leibniz Rules:

$$(\alpha,\beta)_{\mathbb{Z}_2^3} + (\beta,\alpha)_{\mathbb{Z}_2^3} = 2r \tag{5.91}$$

$$(\alpha,\beta+\gamma)_{\mathbb{Z}_2^3} = (\alpha,\beta)_{\mathbb{Z}_2^3} + (\alpha,\gamma)_{\mathbb{Z}_2^3} + \sum_{perm.} \alpha_i\beta_j\gamma_k + 2s \quad \text{where} \quad i,j,k = 1,2,3 \tag{5.92}$$

$$(\alpha+\gamma,\beta)_{\mathbb{Z}_2^3} = (\alpha,\beta)_{\mathbb{Z}_2^3} + (\gamma,\beta)_{\mathbb{Z}_2^3} + \sum_{perm.} \alpha_i\beta_j\gamma_k + 2s \quad \text{where} \quad i,j,k = 1,2,3 \tag{5.93}$$

there is a sum of all possible permutations of 123 on the tri-linear term.

The assumption that these generalized Leibniz rules comes from non-associative monoids, or from the graded Malcev or Sagle identity needs to be proved. However, this idea gives a guide to where it may go when one tries to prove it by using the fundamental identities 5.84-5.87. One thing to look for, that is not done, is if the generalized Leibniz rules above are separate from the associative ones, or if one can find a way to vanish the extra term. Another thing that can also be discussed is in regard to the classification of the Malcev (super)algebras. This is a work in progress and the idea of this section is to increase the readers curiosity towards non-associative algebras and super division algebras.

5.13 Complete Classification

Let's classify all the mappings that obeys the generalized Leibniz rules. The first six mappings will be the associative ones, plus $\mathbf{S}_3$, that obeys the usual Leibniz rules. To be able to classify them the following variables must be defined:

$$N = (\text{Number of lines with all elements zero}) \tag{5.94}$$

$$D1 = (\text{Number of "1's" on the diagonal}) \tag{5.95}$$

$$T1 = (\text{Total number of "1's"}) \tag{5.96}$$

$$N_i = (\text{Number of lines with i "1's"}) \tag{5.97}$$

$$T1 = \sum i * N_i \tag{5.98}$$

Below, the mappings will be written using the 5 associative ones. For example, $4+\mathbf{S}_3$ means the color algebra mapping plus the parameter $\mathbf{S}_3$, doesn't matter if it is $\alpha_1\beta_2+\alpha_2\beta_1$ or $\alpha_1\beta_3+\alpha_3\beta_1$ or the other case.

$\mathbb{Z}_2^3$- **Graded Mappings**:

$$\begin{gathered} 1:(\alpha,\beta)=0 \qquad 2:(\alpha,\beta)=\alpha\beta \qquad 3:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2 \qquad 4:(\alpha,\beta)=\alpha_1\beta_2+\alpha_2\beta_1 \\ 5:(\alpha,\beta)=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 \qquad 6:\mathbf{S}_3 \qquad 7:4+4+4+\mathbf{S}_3 \qquad 8:5+\mathbf{S}_3 \\ 9:2+\mathbf{S}_3 \qquad 10:3+\mathbf{S}_3 \qquad 11:4+\mathbf{S}_3 \qquad 12:4+4+\mathbf{S}_3 \end{gathered} \tag{5.99}$$

Classification:

1: N=8, N_4=0, D1=0 e T1=0

2: N=4, N_4=4, D1=4 e T1=16

3: N=2, N_4=6, D1=4 e T1=24

4: N=2, N_4=6, D1=0 e T1=24

5: N=1, N_4=7, D1=4 e T1=28

6: N=1, N_2=6, N_7=1, D1=1 e T1=19

7: N=1, N_6=7, D1=0 e T1=42

8: N=1, N_4=4, N_6=3, D1=4 e T1=34

9: N=1, N_2=3, N_4=3, N_6=1, D1=3 e T1=24

10: N=1, N_2=1, N_3=1, N_4=4, N_6=1, D1=6 e T1=27

11: N=1, N_2=5, N_3=1, N_6=1, D1=2 e T1=21

12: N=1, N_2=3, N_3=2, N_5=1, N_6=1, D1=1 e T1=23

These are the nonequivalent mappings that obeys the generalized Leibniz rules, except for the first 6. The other possibilities, that were not mentioned before, are the commutation of the signatures of the associative SDA's with the non-associative ones. Using the idea mentioned in the last section it can be done. See the appendix E for the table of brackets:

(0;3): N=2, N_4=6, D1=3 e T1=24

(0;1): N=4, N_4=4, D1=1 e T1=16

(0;5): N=2, N_4=6, D1=5 e T1=24

(0;3*): N=2, N_4=6, D1=3 e T1=24

(0;7): N=1, N_4=7, D1=7 e T1=28

(4;3): N=4, N_4=4, D1=1 e T1=28

(4;1): N=2, N_4=6, D1=3 e T1=24

(4;5): N=2, N_4=6, D1=3 e T1=24

(4;3*): N=2, N_4=6, D1=3 e T1=24

(4;7): N=2, N_4=6, D1=3 e T1=24

It can be seen that they are not related to any of the mappings that obeys the Leibniz rules, which was expected. A complete analysis of the mappings are still in progress. The last thing to see is if the multiplication tables for the SDA's produce a bracket, that can be associated with one of the 12 brackets, which obey the Leibniz rules. The main interest, however, was to see if it helps visualize the difference from an associative multiplication to a non-associative one. It's two brackets for each multiplication table, one with all the positive elements being 1 and - being 0. The other one the inverse, +=0 and -=1. Here are only the results, the table of brackets are in appendix E:

$\mathbb{O}_1$: N=0, N_4=7, N_8=1, D1=1 e T1=36

$\mathbb{O}_2$: N=1, N_4=7, D1=7 e T1=28

$\tilde{\mathbb{O}}_1$: N=0, N_2=1, N_4=3, N_6=3, N_8=1, D1=5 e T1=40

$\tilde{\mathbb{O}}_2$: N=1, N_2=3, N_4=3, N_6=1, D1=3 e T1=24

$(\mathbb{C}\otimes\mathbb{H})_1$: N=0, N_4=7, N_8=1, D1=4 e T1=36

$(\mathbb{C}\otimes\mathbb{H})_2$: N=1, N_4=7, D1=4 e T1=28

$(\tilde{\mathbb{C}}\otimes\mathbb{H})_1$: N=0, N_4=6, N_8=2, D1=2 e T1=40

$(\tilde{\mathbb{C}}\otimes\mathbb{H})_2$: N=2, N_4=6, D1=6 e T1=24

$(\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}})_1$: N=0, N_4=4, N_8=4, D1=6 e T1=48

$(\tilde{\mathbb{C}}\otimes\tilde{\mathbb{H}})_2$: N=4, N_4=4, D1=2 e T1=16

$(\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C})_1$: N=0, N_4=7, N_8=1, D1=4 e T1=36

$(\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C})_2$: N=1, N_4=7, D1=4 e T1=28

$(\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}})_1$: N=0, N_8=8, D1=8 e T1=64

$(\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}}\otimes\tilde{\mathbb{C}})_2$: N=8, D1=0 e T1=0

One can see that the bi-quaternions are equal to the tri-complex; the difference is the position of the elements on the gradings, both second brackets gives the mapping 5, which is the associative superalgebra. Another association is that the second bracket of the split-octonions gives 9 and the second bracket of the octonions are related to (0;7).

Everything that was done here was with the intention to find if there is a relation of non-associativity and associativity with the mappings and the table of brackets. For now there is not an apparent relation and the work is still in progress. The main thing to pursue is which are the properties of Malcev (super)algebras, which are the mappings and how they are related with Lie (super)algebras.

Chapter 6

Conclusions

In this thesis we presented a classification of the super division algebras up to a $\mathbb{Z}_2^4$ grading and the table of brackets up to a $\mathbb{Z}_2^3$ grading. We also used the alphabetic (re)presentation [66] because of their power and practicality in analyzing and performing calculations with large algebras and matrices.

Physical applications of $\mathbb{Z}_2^3$ and $\mathbb{Z}_2^n$ graded Lie (super)algebras have just started to be analyzed [[43],[44],[46]]. On the other hand, $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded (super)algebras have been extensively studied in recent years[[29]-[42],[45],[54]-[57]]. The difference in terms of work and possibilities to analyze between $\mathbb{Z}_2^3$ graded (super)algebras and $\mathbb{Z}_2^2$ graded (super)algebras is enormous. For example, the complete classification of Lie (super)algebras for the table of brackets, given by the mapping 5.20, involves solving approximately one hundred equations given by the Jacobi graded identity. After that, the matrix representation of all the non-equivalent (super)algebras must be found. This is just for one table of brackets; there are 5 in total. Therefore, a clear and precise review on the fundamental topics was needed.

In the first three chapters, we briefly discussed the fundamental algebras that relate to graded (super)algebras, focusing on the applications for chapter 5. In chapter 5, we used an idea to relate the mappings of graded (super)algebras with logic portals. This was used to propose a possible generalization of the Leibniz rules. It is known that graded Lie (super)algebras are associative and therefore, the mapping obeys three conditions called the Leibniz rules, see [[18],[20]].

The generalization of the Leibniz rules comes from the fact that we can grade a non-associative Cayley-Dickson algebra. Therefore, the graded Jacobi identity and Leibniz rules must be changed. For the octonions and split-octonions, we know that they obey the Malcev identity [87]. Graded Malcev (super)algebras have already been defined in the literature, see [[67],[68],[69], [70]]. However, these references only discuss the $\mathbb{Z}_2$ grading of the Malcev identity. In contrast, we focused on a possible mapping and table of brackets for the $\mathbb{Z}_2^3$ grading.

Our approach involved the realization that the commutation of the real super division algebras generators gives all possible table of brackets for $\mathbb{Z}_2^n$ graded Lie algebras, this does not apply to Lie superalgebras. The reason is that one cannot commute a generator from a particular super division algebra with one from a different SDA. Therefore, to include the $\mathbb{Z}_2^n$ graded Lie superalgebras, we used Clifford algebras as a framework to extract the matrices that represent the super division algebras and commute them. In this way, we have a defined multiplication that allow us to check the commutation of matrices that represents generators from different super division algebras. However, our findings can be explicitly applied only to the (split)octonions, not to Malcev (super)algebras in general.

The super division algebras appear in the periodic table of topological insulators and superconductors [[63],[60]]. There are five different classes of topological insulators, within these classes the phases are characterized by a topological invariant. These topological invariant belong to one of the super division algebras that forms the tenfold way. It can be argued that more general $\mathbb{Z}_2^n$ graded super division algebras can be applicable to physics involving parastatistics, which is the statistics of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ para-particles that obey the trilinear relation by Green [88]. Therefore, a complete classification of them is necessary.

In the last section of chapter 5, several tables of brackets were presented and briefly commented on. They are helping on the classification of the $\mathbb{Z}_2^3$ graded Malcev (super)algebras, which is a work in progress.

Another important idea reviewed in the thesis, chapter 3 and 4, was the matrix realization of the Cayley-Dickson doubling, see appendix B. Many possible applications are envisaged. The main one is if this realization adds something to the understanding of the octonionic M-algebra and M-theory. Concerning a possible application to the SU(2) group, it will be discussed in the master thesis that will be presented by I. P. de Freitas [1].

Finally, there are two papers that are currently under finalization. The first one concerning the inequivalent $\mathbb{Z}_2^n$ graded Lie brackets and applications, the second concerning the classification of the $\mathbb{Z}_2^3$ graded super division algebras, both associative and non-associative.

Appendix A

Structure constants of octonions and split-octonions

To construct the octonions and split-octonions structure constants, I.P. de Freitas used the convention for the quaternion structure constant $\varepsilon_{ijk} = 1$ and used the Cayley-Dickson doubling process in chapter 3, see [2], to perform every calculation and form a table. Here is the results:

Table A.1: Octonions structure constants

$C_{123} = 1$	$C_{213} = -1$	$C_{312} = 1$
$C_{132} = -1$	$C_{231} = 1$	$C_{321} = -1$
$C_{145} = 1$	$C_{246} = 1$	$C_{347} = 1$
$C_{154} = -1$	$C_{257} = 1$	$C_{356} = -1$
$C_{167} = -1$	$C_{264} = -1$	$C_{365} = 1$
$C_{176} = 1$	$C_{275} = -1$	$C_{374} = -1$

$C_{415} = -1$	$C_{514} = 1$	$C_{617} = 1$	$C_{716} = -1$
$C_{426} = -1$	$C_{527} = -1$	$C_{624} = 1$	$C_{725} = 1$
$C_{437} = -1$	$C_{536} = 1$	$C_{635} = -1$	$C_{734} = 1$
$C_{451} = 1$	$C_{541} = -1$	$C_{642} = -1$	$C_{743} = -1$
$C_{462} = 1$	$C_{563} = -1$	$C_{653} = 1$	$C_{752} = -1$
$C_{473} = 1$	$C_{572} = 1$	$C_{671} = -1$	$C_{761} = 1$

Table A.2: Split-octonions structure constants

$\tilde{C}_{12}{}^{3} = 1$	$\tilde{C}_{21}{}^{3} = -1$	$\tilde{C}_{31}{}^{2} = 1$
$\tilde{C}_{13}{}^{2} = -1$	$\tilde{C}_{23}{}^{1} = 1$	$\tilde{C}_{32}{}^{1} = -1$
$\tilde{C}_{14}{}^{5} = 1$	$\tilde{C}_{24}{}^{6} = 1$	$\tilde{C}_{34}{}^{7} = 1$
$\tilde{C}_{15}{}^{4} = -1$	$\tilde{C}_{25}{}^{7} = 1$	$\tilde{C}_{35}{}^{6} = -1$
$\tilde{C}_{16}{}^{7} = -1$	$\tilde{C}_{26}{}^{4} = -1$	$\tilde{C}_{36}{}^{5} = 1$
$\tilde{C}_{17}{}^{6} = 1$	$\tilde{C}_{27}{}^{5} = -1$	$\tilde{C}_{37}{}^{4} = -1$

$\tilde{C}_{41}{}^{5} = -1$	$\tilde{C}_{51}{}^{4} = 1$	$\tilde{C}_{61}{}^{7} = 1$	$\tilde{C}_{71}{}^{6} = -1$
$\tilde{C}_{42}{}^{6} = -1$	$\tilde{C}_{52}{}^{7} = -1$	$\tilde{C}_{62}{}^{4} = 1$	$\tilde{C}_{72}{}^{5} = 1$
$\tilde{C}_{43}{}^{7} = -1$	$\tilde{C}_{53}{}^{6} = 1$	$\tilde{C}_{63}{}^{5} = -1$	$\tilde{C}_{73}{}^{4} = 1$
$\tilde{C}_{45}{}^{1} = -1$	$\tilde{C}_{54}{}^{1} = 1$	$\tilde{C}_{64}{}^{2} = 1$	$\tilde{C}_{74}{}^{3} = 1$
$\tilde{C}_{46}{}^{2} = -1$	$\tilde{C}_{56}{}^{3} = 1$	$\tilde{C}_{65}{}^{3} = -1$	$\tilde{C}_{75}{}^{2} = 1$
$\tilde{C}_{47}{}^{3} = -1$	$\tilde{C}_{57}{}^{2} = -1$	$\tilde{C}_{67}{}^{1} = 1$	$\tilde{C}_{76}{}^{1} = -1$

there is another way to put these results that is more common in the literature:

Table A.3: Octonions multiplication

$e_i e_j$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table A.4: Split-octonions multiplication

$\tilde{e}_i \tilde{e}_j$	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{e}_3$	$\tilde{e}_4$	$\tilde{e}_5$	$\tilde{e}_6$	$\tilde{e}_7$
$\tilde{e}_1$	-1	$\tilde{e}_3$	$-\tilde{e}_2$	$\tilde{e}_5$	$-\tilde{e}_4$	$-\tilde{e}_7$	$\tilde{e}_6$
$\tilde{e}_2$	$-\tilde{e}_3$	-1	$\tilde{e}_1$	$\tilde{e}_6$	$\tilde{e}_7$	$-\tilde{e}_4$	$-\tilde{e}_5$
$\tilde{e}_3$	$\tilde{e}_2$	$-\tilde{e}_1$	-1	$\tilde{e}_7$	$-\tilde{e}_6$	$\tilde{e}_5$	$-\tilde{e}_4$
$\tilde{e}_4$	$-\tilde{e}_5$	$-\tilde{e}_6$	$-\tilde{e}_7$	1	$-\tilde{e}_1$	$-\tilde{e}_2$	$-\tilde{e}_3$
$\tilde{e}_5$	$\tilde{e}_4$	$-\tilde{e}_7$	$\tilde{e}_6$	$\tilde{e}_1$	1	$\tilde{e}_3$	$-\tilde{e}_2$
$\tilde{e}_6$	$\tilde{e}_7$	$\tilde{e}_4$	$-\tilde{e}_5$	$\tilde{e}_2$	$-\tilde{e}_3$	1	$\tilde{e}_1$
$\tilde{e}_7$	$-\tilde{e}_6$	$\tilde{e}_5$	$\tilde{e}_4$	$\tilde{e}_3$	$\tilde{e}_2$	$-\tilde{e}_1$	1

to check if the structure constants are correct, the table of multiplication for both algebras must be anti-symmetric.

There is a property that maintains to every Cayley-Dickson algebra, the table of multiplication of the split algebra is always the division one, but with a complete change of signs of the multiplication between space-like vectors. Another interesting property, only for division algebras, is that the $Cl(0,7)$ matrices appear considering a change of sign on the first column, below $e_i e_j$.

$\tilde{C}_{ij}{}^{k}$ is not totally anti-symmetric like C_{ijk} is but $\tilde{C}_{ijk} = \tilde{C}_{ij}{}^{l}\tilde{\eta}_{kl}$ is totally anti-symmetric:

Table A.5: Split-octonions structure constants

$\tilde{C}_{123} = -1$	$\tilde{C}_{213} = 1$	$\tilde{C}_{312} = -1$
$\tilde{C}_{132} = 1$	$\tilde{C}_{231} = -1$	$\tilde{C}_{321} = 1$
$\tilde{C}_{145} = 1$	$\tilde{C}_{246} = 1$	$\tilde{C}_{347} = 1$
$\tilde{C}_{154} = -1$	$\tilde{C}_{257} = 1$	$\tilde{C}_{356} = -1$
$\tilde{C}_{167} = -1$	$\tilde{C}_{264} = -1$	$\tilde{C}_{365} = 1$
$\tilde{C}_{176} = 1$	$\tilde{C}_{275} = -1$	$\tilde{C}_{374} = -1$

$\tilde{C}_{415} = -1$	$\tilde{C}_{514} = 1$	$\tilde{C}_{617} = 1$	$\tilde{C}_{716} = -1$
$\tilde{C}_{426} = -1$	$\tilde{C}_{527} = -1$	$\tilde{C}_{624} = 1$	$\tilde{C}_{725} = 1$
$\tilde{C}_{437} = -1$	$\tilde{C}_{536} = 1$	$\tilde{C}_{635} = -1$	$\tilde{C}_{734} = 1$
$\tilde{C}_{451} = 1$	$\tilde{C}_{541} = -1$	$\tilde{C}_{642} = -1$	$\tilde{C}_{743} = -1$
$\tilde{C}_{462} = 1$	$\tilde{C}_{563} = -1$	$\tilde{C}_{653} = 1$	$\tilde{C}_{752} = -1$
$\tilde{C}_{473} = 1$	$\tilde{C}_{572} = 1$	$\tilde{C}_{671} = -1$	$\tilde{C}_{761} = 1$

The structure constant for the octonions is:

$$C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = 1 \tag{A.1}$$

and for the split-octonions is:

$$\tilde{C}_{132} = \tilde{C}_{145} = \tilde{C}_{176} = \tilde{C}_{246} = \tilde{C}_{257} = \tilde{C}_{347} = \tilde{C}_{365} = 1 \tag{A.2}$$

Appendix B

Matrix realization of the octonions

In sections 3.3 and 4.5, it was developed the matrix realization for the Cayley-Dickson algebras, but only showed the octonionic matrices in the alphabetic form. Therefore, here is the matrices that obeys rules 4.16-4.18. They all come from the general matrix of the Cayley-Dickson doubling 3.18:

$$E_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 \equiv \begin{pmatrix} e_1 & 0 \\ 0 & e_1^* \end{pmatrix}, \quad E_2 \equiv \begin{pmatrix} e_2 & 0 \\ 0 & e_2^* \end{pmatrix}, \quad E_3 \equiv \begin{pmatrix} e_3 & 0 \\ 0 & e_3^* \end{pmatrix},$$

$$E_4 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_5 \equiv \begin{pmatrix} 0 & e_1 \\ -e_1^* & 0 \end{pmatrix}, \quad E_6 \equiv \begin{pmatrix} 0 & e_2 \\ -e_2^* & 0 \end{pmatrix}, \quad E_7 \equiv \begin{pmatrix} 0 & e_3 \\ -e_3^* & 0 \end{pmatrix}. \tag{B.1}$$

where e_i are the three quaternionic vectors. One might check that the multiplication rule in 4.16 works for B.1. Now the 4x4 complex matrices:

$$E_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad E_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad E_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad E_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad E_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{B.2}$$

The 8x8 real matrices:

$$E_0 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \quad E_1 = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{pmatrix} \quad E_5 = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{pmatrix} \quad E_7 = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{pmatrix} \tag{B.3}$$

Important to stress the fact, that all these matrices were found using the structure constant convention A.1. If one changes the convention, it needs to be done by changing the number of the vectors in 3.23, or changing the signs of the vectors and using the Cayley-Dickson multiplication to create new tables of multiplication. The matrices will change very little but the properties 4.16-4.18 will continue being correct.

Appendix C

Left and right action

Using any representation of the octonions, one can perform the left and right action on a general element to see the relation between the octonions and $Cl_{(0,7)}$, see [76]. The left action is:

$$\left[\begin{aligned}
e_1 x &= x^0 e_1 - x^1 + x^2 e_3 - x^3 e_2 + x^4 e_5 - x^5 e_4 + x^6 e_7 + x^7 e_6\\
e_2 x &= x^0 e_2 - x^1 e_3 - x^2 + x^3 e_1 + x^4 e_6 + x^5 e_7 - x^6 e_4 - x^7 e_5\\
e_3 x &= x^0 e_3 + x^1 e_2 - x^2 e_1 - x^3 + x^4 e_7 - x^5 e_6 + x^6 e_5 - x^7 e_4\\
e_4 x &= x^0 e_4 - x^1 e_5 - x^2 e_6 - x^3 e_7 - x^4 + x^5 e_1 + x^6 e_2 + x^7 e_3\\
e_5 x &= x^0 e_5 + x^1 e_4 - x^2 e_7 + x^3 e_6 - x^4 e_1 - x^5 - x^6 e_3 + x^7 e_2\\
e_6 x &= x^0 e_6 + x^1 e_7 + x^2 e_4 - x^3 e_5 - x^4 e_2 + x^5 e_3 - x^6 - x^7 e_1\\
e_7 x &= x^0 e_7 - x^1 e_6 + x^2 e_5 + x^3 e_4 - x^4 e_3 - x^5 e_2 + x^6 e_1 - x^7
\end{aligned}\right. \tag{C.1}$$

putting in matrix form we find:

$$\Gamma_1^L = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \Gamma_2^L = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Gamma_3^L = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_4^L = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_5^L = \begin{pmatrix} 0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}, \quad \Gamma_6^L = \begin{pmatrix} 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}$$

$$\Gamma_7^L = \begin{pmatrix} 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \tag{C.2}$$

These matrices form the Clifford algebra $Cl_{(0,7)}$. However, they can't be written in an alphabetic form because they are not made of tensor products between the fundamental ones. With right action the signs will change, as already said in section 3.3:

$$\left[\begin{array}{l} xe_1 = x^0e_1 - x^1 - x^2e_3 + x^3e_2 - x^4e_5 + x^5e_4 + x^6e_7 - x^7e_6 \\ xe_2 = x^0e_2 + x^1e_3 - x^2 - x^3e_1 - x^4e_6 - x^5e_7 + x^6e_4 + x^7e_5 \\ xe_3 = x^0e_3 - x^1e_2 + x^2e_1 - x^3 - x^4e_7 + x^5e_6 - x^6e_5 + x^7e_4 \\ xe_4 = x^0e_4 + x^1e_5 + x^2e_6 + x^3e_7 - x^4 - x^5e_1 - x^6e_2 - x^7e_3 \\ xe_5 = x^0e_5 - x^1e_4 + x^2e_7 - x^3e_6 + x^4e_1 - x^5 + x^6e_3 - x^7e_2 \\ xe_6 = x^0e_6 - x^1e_7 - x^2e_4 + x^3e_5 + x^4e_2 - x^5e_3 - x^6 + x^7e_1 \\ xe_7 = x^0e_7 + x^1e_6 - x^2e_5 - x^3e_4 + x^4e_3 + x^5e_2 - x^6e_1 - x^7 \end{array} \right. \tag{C.3}$$

the matrices are:

$$\Gamma_1^R = \begin{pmatrix} 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \quad \Gamma_2^R = \begin{pmatrix} 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{pmatrix}$$

$$\Gamma_3^R = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_4^R = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_5^R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_6^R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_7^R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.4}$$

These matrices also form $Cl_{(0,7)}$, but they can be put in alphabetic form:

$$\Gamma_1^R = -ZZA,\ \Gamma_2^R = -ZAI,\ \Gamma_3^R = -ZXA,\ \Gamma_4^R = -AII,\ \Gamma_5^R = -XIA,\ \Gamma_6^R = -XAZ,\ \Gamma_7^R = -XAX \tag{C.5}$$

These matrices C.5 and B.3 are the matrices in 4.19. To consider only left action, but maintaining the alphabetic (re)presentation, then the structure constant convention must change, but to a very specific one, the inverse:

$$C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = -1 \tag{C.6}$$

Inverse conventions:

There are 480 different conventions, 240 with one sign and the other 240 the inverse of them. What the inverse does is change the left action with the right action, meaning that a left action, with a convention, is the same of a right action with the inverse of this convention, they are dual.

There are 7 ways to derive the inverse convention, just change the sign of three specific elements:

$$\bar{e}_1 = -e_1 \quad \bar{e}_2 = -e_2 \quad \bar{e}_3 = -e_3 \tag{C.7}$$

the other six choices are to change the signs of the elements according to the structure constant, in this case, elements 1,4 and 5 or 1,7 and 6 and so on.

Appendix D

Octonionic M-algebra matrices

On section 5.7, the octonionic M-algebra matrices were showed in an alphabetic form. To help visualize these matrices, let's see the usual 4x4 octonionic representation of it. See references [[71],[72],[73],[15]].

$$Cl_0(10,1): \quad \Gamma_i \equiv \begin{pmatrix} 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \\ 0 & e_i & 0 & 0 \\ -e_i & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_8 \equiv \begin{pmatrix} 0 & 0 & 0 & e_0 \\ 0 & 0 & e_0 & 0 \\ 0 & e_0 & 0 & 0 \\ e_0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_9 \equiv \begin{pmatrix} 0 & 0 & e_0 & 0 \\ 0 & 0 & 0 & -e_0 \\ e_0 & 0 & 0 & 0 \\ 0 & -e_0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_{10} \equiv \begin{pmatrix} 0 & 0 & e_0 & 0 \\ 0 & 0 & 0 & e_0 \\ -e_0 & 0 & 0 & 0 \\ 0 & -e_0 & 0 & 0 \end{pmatrix}, \quad \Gamma_{11} \equiv \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & e_0 & 0 & 0 \\ 0 & 0 & -e_0 & 0 \\ 0 & 0 & 0 & -e_0 \end{pmatrix}. \tag{D.1}$$

where $i = 1, 2, ..., 7$

The matrices 4.38 are generated when one substitute the octonionic vectors e_0 and e_i with their 8x8 real representation B.3. One can see that it generates the 32x32 real matrix representation, which in an alphabetic form is much easier to perform calculations with:

$$Cl_0(10,1): \quad \begin{matrix} XAe_i \\ XXe_0 \\ XZe_0 \\ AIe_0 \\ ZIe_0 \end{matrix}$$

The representation above is valid for any representation of the octonionic M-algebra, just change the vector elements for the representations on the appendix B. Let's perform a calculation $e_1 * e_7 = e_6$:

$XAe_1 * XAe_7 = -IIe_6$

It is easy to find the generators of the spin group with the alphabetic (re)presentation:

$$[XAe_i, XAe_j] = -2IIC_{ijk}\delta^{kr} e_r, \quad [XAe_i, XXe_0] = 2IZe_i, \quad [XAe_i, XZe_0] = -2IXe_i,$$
$$[XAe_i, AIe_0] = -2ZAe_i, \quad [XAe_i, ZIe_0] = -2AAe_i, \quad [XXe_0, XZe_0] = -2IAe_0,$$
$$[XXe_0, AIe_0] = -2ZXe_0, \quad [XXe_0, ZIe_0] = -2AXe_0, \quad [XZe_0, AIe_0] = -2ZZe_0,$$
$$[XZe_0, ZIe_0] = -2AZe_0, \quad [AIe_0, ZIe_0] = -2XIe_0. \tag{D.2}$$

Where $j > i$. To be more clear, here is two examples of commutators:

$$[XAe_i, XAe_j] = -2IIC_{ijk}\delta^{kr} e_r = -2C_{ij}{}^k \begin{pmatrix} e_r & 0 & 0 & 0 \\ 0 & e_r & 0 & 0 \\ 0 & 0 & e_r & 0 \\ 0 & 0 & 0 & e_r \end{pmatrix}$$

Second example:

$$[XXe_0, ZIe_0] = -2AXe_0 = -2 \begin{pmatrix} 0 & 0 & 0 & -e_0 \\ 0 & 0 & -e_0 & 0 \\ 0 & e_0 & 0 & 0 \\ e_0 & 0 & 0 & 0 \end{pmatrix}$$

Appendix E

Table of Brackets

Here is all possible table of brackets, produced in different ways, studied in chapter 5 and classified in section 5.12.

Table E.1: (3+S_3):

	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	1	1
011	0	0	1	1	1	0	0	1
010	0	0	1	1	0	1	0	1
100	0	0	1	0	1	1	0	1
101	0	0	0	1	1	1	0	1
111	0	1	1	1	1	1	1	0
110	0	1	0	0	0	0	1	1

Table E.2: (2+S_3):

	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	1	1
011	0	0	0	0	1	0	1	0
010	0	0	0	0	0	1	1	0
100	0	0	1	0	1	1	0	1
101	0	0	0	1	1	1	0	1
111	0	1	1	1	1	1	1	0
110	0	1	1	1	0	0	1	0

Table E.3: (3+3+3+S_3=S_3):

	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	1	1
011	0	0	0	0	1	0	1	0
010	0	0	0	0	0	1	1	0
100	0	0	1	0	0	0	1	0
101	0	0	0	1	0	0	1	0
111	0	1	1	1	1	1	1	1
110	0	1	0	0	0	0	1	0

Table E.4: (4+S_3):

	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	1	1
011	0	0	0	0	0	1	0	1
010	0	0	0	0	1	0	0	1
100	0	0	0	1	0	0	0	1
101	0	0	1	0	0	0	0	1
111	0	1	0	0	0	0	1	1
110	0	1	1	1	1	1	0	1

Table E.5: (4+4+S_3):

	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	1	1	0	0
011	0	0	0	0	1	0	1	0
010	0	0	0	0	1	0	0	1
100	0	1	1	1	0	1	1	1
101	0	1	0	0	1	0	1	0
111	0	0	1	0	1	1	1	1
110	0	0	0	1	1	0	1	0

Table E.6: (0,3):

	$e_{1234567}$	e_{23}	e_{13}	e_{12}	e_{123567}	e_{234}	e_{134}	e_{124}
e_0	0	0	0	0	0	0	0	0
e_1	0	0	1	1	1	1	0	0
e_2	0	1	0	1	1	0	1	0
e_3	0	1	1	0	1	0	0	1
e_4	0	0	0	0	0	0	0	0
e_5	0	0	0	0	1	1	1	1
e_6	0	0	0	0	1	1	1	1
e_7	0	0	0	0	1	1	1	1

Table E.7: (0,1):

	$e_{1234567}$	e_{23}	e_{13}	e_{12}	e_{567}	e_{14}	e_{24}	e_{34}
e_0	0	0	0	0	0	0	0	0
e_1	0	0	1	1	1	1	0	0
e_2	0	1	0	1	1	0	1	0
e_3	0	1	1	0	1	0	0	1
e_4	0	0	0	0	1	1	1	1
e_5	0	0	0	0	0	0	0	0
e_6	0	0	0	0	0	0	0	0
e_7	0	0	0	0	0	0	0	0

Table E.8: (0,5):

	$e_{1234567}$	e_{346}	e_{13}	e_{146}	e_{567}	e_{126}	e_{24}	e_{236}
e_0	0	0	0	0	0	0	0	0
e_1	0	1	1	0	1	0	0	1
e_2	0	1	0	1	1	0	1	0
e_3	0	0	1	1	1	1	0	0
e_4	0	0	0	0	1	1	1	1
e_5	0	1	0	1	0	1	0	1
e_6	0	0	0	0	0	0	0	0
e_7	0	1	0	1	0	1	0	1

Table E.9: (0,3*):

	$e_{1234567}$	e_{123567}	e_{67}	e_{467}	e_{134}	e_{13}	e_{25}	e_{245}
e_0	0	0	0	0	0	0	0	0
e_4	0	0	0	0	0	0	0	0
e_1	0	1	0	1	0	1	0	1
e_5	0	1	0	1	0	1	0	1
e_6	0	1	1	0	1	0	0	1
e_2	0	1	0	1	1	0	1	0
e_7	0	1	1	0	1	0	0	1
e_3	0	1	0	1	0	1	0	1

Table E.10: (0,7):

	$e_{1234567}$	e_{567}	e_{247}	e_{137}	e_{125}	e_{345}	e_{236}	e_{146}
e_0	0	0	0	0	0	0	0	0
e_4	0	1	0	1	1	0	1	0
e_1	0	1	1	0	0	1	1	0
e_5	0	0	1	1	0	0	1	1
e_6	0	0	1	1	1	1	0	0
e_2	0	1	0	1	0	1	0	1
e_7	0	0	0	0	1	1	1	1
e_3	0	1	1	0	1	0	0	1

Table E.11: (4,3):

	$e_{1234567}$	e_{23}	e_{13}	e_{12}	e_4	e_{234}	e_{134}	e_{124}
e_0	0	0	0	0	0	0	0	0
e_1	0	0	1	1	1	1	0	0
e_2	0	1	0	1	1	0	1	0
e_3	0	1	1	0	1	0	0	1
e_{567}	0	0	0	0	1	1	1	1
e_{467}	0	0	0	0	0	0	0	0
e_{457}	0	0	0	0	0	0	0	0
e_{456}	0	0	0	0	0	0	0	0

Table E.12: (4,1):

	$e_{1234567}$	e_{23}	e_{13}	e_{12}	e_{1234}	e_{14}	e_{24}	e_{34}
e_0	0	0	0	0	0	0	0	0
e_1	0	0	1	1	1	1	0	0
e_2	0	1	0	1	1	0	1	0
e_3	0	1	1	0	1	0	0	1
e_{567}	0	0	0	0	0	0	0	0
e_{467}	0	0	0	0	1	1	1	1
e_{457}	0	0	0	0	1	1	1	1
e_{456}	0	0	0	0	1	1	1	1

Table E.13: (4,5):

	$e_{1234567}$	e_{346}	e_{13}	e_{146}	e_{1234}	e_{126}	e_{24}	e_{236}
e_0	0	0	0	0	0	0	0	0
e_1	0	1	1	0	1	0	0	1
e_2	0	1	0	1	1	0	1	0
e_3	0	0	1	1	1	1	0	0
e_{567}	0	0	0	0	0	0	0	0
e_{467}	0	1	0	1	1	0	1	0
e_{457}	0	0	0	0	1	1	1	1
e_{456}	0	1	0	1	1	0	1	0

Table E.14: (4,3*):

	$e_{1234567}$	e_{123567}	e_{67}	e_{1235}	e_{134}	e_{13}	e_{25}	e_{245}
e_0	0	0	0	0	0	0	0	0
e_{567}	0	1	0	1	1	0	1	0
e_1	0	1	0	1	0	1	0	1
e_{467}	0	0	0	0	0	0	0	0
e_{457}	0	0	1	1	0	0	1	1
e_2	0	1	0	1	1	0	1	0
e_{456}	0	0	1	1	0	0	1	1
e_3	0	1	0	1	0	1	0	1

Table E.15: (4,7):

	$e_{1234567}$	e_{1234}	e_{247}	e_{137}	e_{125}	e_{345}	e_{236}	e_{146}
e_0	0	0	0	0	0	0	0	0
e_{567}	0	0	0	0	0	0	0	0
e_1	0	1	1	0	0	1	1	0
e_{467}	0	1	1	0	1	0	0	1
e_{457}	0	1	1	0	0	1	1	0
e_2	0	1	0	1	0	1	0	1
e_{456}	0	1	0	1	0	1	0	1
e_3	0	1	1	0	1	0	0	1

Table E.16: Octonions multiplication

$e_i e_j$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table E.17: Split-octonions multiplication

$e_i e_j$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	1	$-e_1$	$-e_2$	$-e_3$
e_5	e_4	$-e_7$	e_6	e_1	1	e_3	$-e_2$
e_6	e_7	e_4	$-e_5$	e_2	$-e_3$	1	e_1
e_7	$-e_6$	e_5	e_4	e_3	e_2	$-e_1$	1

Table E.18: Bi-quaternions multiplication

$e_i e_j$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_5	$-e_4$	e_7	$-e_6$	$-e_1$	1	$-e_3$	e_2
e_6	$-e_7$	$-e_4$	e_5	$-e_2$	e_3	1	$-e_1$
e_7	e_6	$-e_5$	$-e_4$	$-e_3$	$-e_2$	e_1	1

Table E.19: Tri-complex multiplication

e_ie_j	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	e_3	-1	$-e_1$	e_6	e_7	$-e_4$	$-e_5$
e_3	$-e_2$	$-e_1$	1	e_7	$-e_6$	$-e_5$	e_4
e_4	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_5	$-e_4$	e_7	$-e_6$	$-e_1$	1	$-e_3$	e_2
e_6	e_7	$-e_4$	$-e_5$	$-e_2$	$-e_3$	1	e_1
e_7	$-e_6$	$-e_5$	e_4	$-e_3$	e_2	e_1	-1

Table E.20: Split-bi-quaternions multiplication

e_ie_j	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	e_5	e_6	e_7	1	e_1	e_2	e_3
e_5	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Table E.21: Split-bi-split-quaternions multiplication

e_ie_j	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	1	e_3	e_2	e_5	e_4	e_7	e_6
e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	$-e_2$	$-e_1$	1	e_7	$-e_6$	$-e_5$	e_4
e_4	e_5	e_6	e_7	1	e_1	e_2	e_3
e_5	e_4	e_7	e_6	e_1	1	e_3	e_2
e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_7	$-e_6$	$-e_5$	e_4	e_3	$-e_2$	$-e_1$	1

Table E.22: Split-tri-complex multiplication

e_ie_j	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	1	e_3	e_2	e_5	e_4	e_7	e_6
e_2	e_3	1	e_1	e_6	e_7	e_4	e_5
e_3	e_2	e_1	1	e_7	e_6	e_5	e_4
e_4	e_5	e_6	e_7	1	e_1	e_2	e_3
e_5	e_4	e_7	e_6	e_1	1	e_3	e_2
e_6	e_7	e_4	e_5	e_2	e_3	1	e_1
e_7	e_6	e_5	e_4	e_3	e_2	e_1	1

Table E.23: E.16 -1=0 and 1=1

$e_i e_j$	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	0	1	0	1	0	0	1
011	1	0	0	1	1	1	0	0
010	1	1	0	0	1	0	1	0
100	1	0	0	0	0	1	1	1
101	1	1	0	1	0	0	0	1
111	1	1	1	0	0	1	0	0
110	1	0	1	1	0	0	1	0

Table E.24: E.16: -1=1 and 1=0

$e_i e_j$	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	1	0	1	0	1	1	0
011	0	1	1	0	0	0	1	1
010	0	0	1	1	0	1	0	1
100	0	1	1	1	1	0	0	0
101	0	0	1	0	1	1	1	0
111	0	0	0	1	1	0	1	1
110	0	1	0	0	1	1	0	1

Table E.25: E.17 -1=0 and 1=1

$e_i e_j$	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	0	1	0	1	0	0	1
011	1	0	0	1	1	1	0	0
010	1	1	0	0	1	0	1	0
100	1	0	0	0	1	0	0	0
101	1	1	0	1	1	1	1	0
111	1	1	1	0	1	0	1	1
110	1	0	1	1	1	1	0	1

Table E.26: E.17: -1=1 and 1=0

e_ie_j	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	1	0	1	0	1	1	0
011	0	1	1	0	0	0	1	1
010	0	0	1	1	0	1	0	1
100	0	1	1	1	0	1	1	1
101	0	0	1	0	0	0	0	1
111	0	0	0	1	0	1	0	0
110	0	1	0	0	0	0	1	0

Table E.27: E.18: -1=0 and 1=1

e_ie_j	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	0	1	0	1	0	1	0
011	1	0	0	1	1	0	0	1
010	1	1	0	0	1	1	0	0
100	1	1	1	1	0	0	0	0
101	1	0	1	0	0	1	0	1
111	1	0	0	1	0	1	1	0
110	1	1	0	0	0	0	1	1

Table E.28: E.18: -1=1 and 1=0

e_ie_j	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	1	0	1	0	1	0	1
011	0	1	1	0	0	1	1	0
010	0	0	1	1	0	0	1	1
100	0	0	0	0	1	1	1	1
101	0	1	0	1	1	0	1	0
111	0	1	1	0	1	0	0	1
110	0	0	1	1	1	1	0	0

Table E.29: E.19: -1=0 and 1=1

e_ie_j	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	0	1	0	1	0	1	0
011	1	1	0	0	1	1	0	0
010	1	0	0	1	1	0	0	1
100	1	1	1	1	0	0	0	0
101	1	0	1	0	0	1	0	1
111	1	1	0	0	0	0	1	1
110	1	0	0	1	0	1	1	0

Table E.30: E.19 -1=1 and 1=0

e_ie_j	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	1	0	1	0	1	0	1
011	0	0	1	1	0	0	1	1
010	0	1	1	0	0	1	1	0
100	0	0	0	0	1	1	1	1
101	0	1	0	1	1	0	1	0
111	0	0	1	1	1	1	0	0
110	0	1	1	0	1	0	0	1

Table E.31: E.20: -1=0 and 1=1

e_ie_j	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	0	1	0	1	0	1	0
011	1	0	0	1	1	0	0	1
010	1	1	0	0	1	1	0	0
100	1	1	1	1	1	1	1	1
101	1	0	1	0	1	0	1	0
111	1	0	0	1	1	0	0	1
110	1	1	0	0	1	1	0	0

Table E.32: E.20: -1=1 and 1=0

e_ie_j	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	1	0	1	0	1	0	1
011	0	1	1	0	0	1	1	0
010	0	0	1	1	0	0	1	1
100	0	0	0	0	0	0	0	0
101	0	1	0	1	0	1	0	1
111	0	1	1	0	0	1	1	0
110	0	0	1	1	0	0	1	1

Table E.33: E.21: -1=0 and 1=1

e_ie_j	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	1	1	1	1	1	1	1
011	1	0	0	1	1	0	0	1
010	1	0	0	1	1	0	0	1
100	1	1	1	1	1	1	1	1
101	1	1	1	1	1	1	1	1
111	1	0	0	1	1	0	0	1
110	1	0	0	1	1	0	0	1

Table E.34: E.21: -1=1 and 1=0

e_ie_j	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	0	0
011	0	1	1	0	0	1	1	0
010	0	1	1	0	0	1	1	0
100	0	0	0	0	0	0	0	0
101	0	0	0	0	0	0	0	0
111	0	1	1	0	0	1	1	0
110	0	1	1	0	0	1	1	0

Table E.35: E.22: -1=0 and 1=1

$e_i e_j$	000	001	011	010	100	101	111	110
000	1	1	1	1	1	1	1	1
001	1	1	1	1	1	1	1	1
011	1	1	1	1	1	1	1	1
010	1	1	1	1	1	1	1	1
100	1	1	1	1	1	1	1	1
101	1	1	1	1	1	1	1	1
111	1	1	1	1	1	1	1	1
110	1	1	1	1	1	1	1	1

Table E.36: E.22: -1=1 and 1=0

$e_i e_j$	000	001	011	010	100	101	111	110
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	0	0
011	0	0	0	0	0	0	0	0
010	0	0	0	0	0	0	0	0
100	0	0	0	0	0	0	0	0
101	0	0	0	0	0	0	0	0
111	0	0	0	0	0	0	0	0
110	0	0	0	0	0	0	0	0

Bibliography

[1] I. P. Freitas, "Techniques for cayley-dickson algebras and graded lie (super)algebras for physics," Master's thesis, Centro Brasileiro de Pesquisas Físicas, 2023.

[2] R. D. Schafer, "On the algebras formed by the Cayley-Dickson process," *American Journal of Mathematics*, vol. 76, no. 2, pp. 435–446, 1954.

[3] K. McCrimmon, *A taste of Jordan algebras*. Springer, 2004, vol. 1.

[4] J. Baez, "The octonions," *Bulletin of the american mathematical society*, vol. 39, no. 2, pp. 145–205, 2002.

[5] I. Todorov, "Quantization is a mystery," *Bulg. J. Phys.*, no. 39, pp. 107–149, 2012.

[6] G. M. Dixon, *Division Algebras:: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics*. Springer Science & Business Media, 2013, vol. 290.

[7] C. Furey, "Standard model physics from an algebra?" 2016. [Online]. Available: https://arxiv.org/abs/1611.09182

[8] ——, "Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra," *Physics Letters B*, vol. 785, pp. 84–89, 2018.

[9] A. Lasenby, "Some recent results for SU (3) SU (3) and octonions within the geometric algebra approach to the fundamental forces of nature," *Mathematical Methods in the Applied Sciences*, 2022.

[10] R. da Rocha and J. Vaz Jr, "Clifford algebra-parametrized octonions and generalizations," *Journal of Algebra*, vol. 301, no. 2, pp. 459–473, 2006.

[11] J. Vaz Jr and R. da Rocha Jr, *An introduction to Clifford algebras and spinors*. Oxford University Press, 2016.

[12] M. F. Atiyah, R. Bott, and A. Shapiro, "Clifford modules," *Topology*, vol. 3, pp. 3–38, 1964.

[13] S. Okubo, "Real representations of finite Clifford algebras. i. classification," *Journal of mathematical physics*, vol. 32, no. 7, pp. 1657–1668, 1991.

[14] ——, "Real representations of finite Clifford algebras. ii. explicit construction and pseudo-octonion," *Journal of mathematical physics*, vol. 32, no. 7, pp. 1669–1673, 1991.

[15] H. L. Carrion, M. Rojas, and F. Toppan, "Quaternionic and octonionic spinors. a classification," *Journal of High Energy Physics*, vol. 2003, no. 04, p. 040, 2003.

[16] J. Tits, R. M. Weiss, J. Tits, and R. M. Weiss, *Moufang polygons*. Springer, 2002.

[17] C. Culbert, "Cayley-dickson algebras and loops," *Journal of Forensic Biomechanics*, vol. 1, no. 1, pp. 1–17, 2007.

[18] V. Rittenberg and D. Wyler, "Generalized superalgebras," *Nuclear Physics B*, vol. 139, no. 3, pp. 189–202, 1978.

[19] ——, "Sequences of $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebras and superalgebras," *Journal of Mathematical Physics*, vol. 19, no. 10, pp. 2193–2200, 1978.

[20] M. Scheunert, "Generalized Lie algebras," *Journal of Mathematical Physics*, vol. 20, no. 4, pp. 712–720, 1979.

[21] R. Ree, "Generalized lie elements," *Canadian Journal of Mathematics*, vol. 12, pp. 493–502, 1960.

[22] V. G. Kac, "Lie superalgebras," *Advances in mathema tics*, vol. 26, no. 1, pp. 8–96, 1977.

[23] S. Silvestrov, "On the classification of 3-dimensional coloured lie algebras," *Banach Center Publications*, vol. 40, no. 1, pp. 159–170, 1997.

[24] Y. Su, K. Zhao, and L. Zhu, "Classification of derivation-simple color algebras related to locally finite derivations," *Journal of Mathematical Physics*, vol. 45, no. 1, pp. 525–536, 2004.

[25] X.-W. Chen, S. D. Silvestrov, and F. Van Oystaeyen, "Representations and cocycle twists of color lie algebras," *Algebras and representation theory*, vol. 9, no. 6, pp. 633–650, 2006.

[26] R. Campoamor-Stursberg and M. R. De Traubenberg, "Color lie algebras and lie algebras of order f," *Journal of Generalized Lie Theory and Applications*, vol. 3, no. 2, pp. 113–130, 2009.

[27] N. Aizawa, P. Isaac, and J. Segar, "$\mathbb{Z}_2 \times \mathbb{Z}_2$-generalizations of infinite-dimensional lie superalgebra of conformal type with complete classification of central extensions," *Reports on Mathematical Physics*, vol. 85, no. 3, pp. 351–373, 2020.

[28] P. S. Isaac, N. I. Stoilova, and J. V. der Jeugt, "The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded general linear lie superalgebra," *Journal of Mathematical Physics*, vol. 61, no. 1, p. 011702, jan 2020. [Online]. Available: https://doi.org/10.10632F1.5138597

[29] J. Lukierski and V. Rittenberg, "Color-de sitter and color-conformal superalgebras," *Physical Review D*, vol. 18, no. 2, p. 385, 1978.

[30] M. Vasiliev, "de sitter supergravity with positive cosmological constant and generalised lie superalgebras," *Classical and Quantum Gravity*, vol. 2, no. 5, p. 645, 1985.

[31] P. Jarvis, M. Yang, and B. Wybourne, "Generalized quasispin for supergroups," *Journal of Mathematical Physics*, vol. 28, no. 5, pp. 1192–1197, 1987.

[32] N. Aizawa, Z. Kuznetsova, H. Tanaka, and F. Toppan, "$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetries of the Lévy-Leblond equations," *Progress of Theoretical and Experimental Physics*, vol. 2016, no. 12, p. 123A01, 2016.

[33] ——, "Generalized supersymmetry and the Lévy-Leblond equation," in *Physical and Mathematical Aspects of Symmetries: Proceedings of the 31st International Colloquium in Group Theoretical Methods in Physics.* Springer, 2017, pp. 79–84.

[34] Aizawa, N., Kuznetsova, Z., and Toppan, F., "$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: the classical theory," *Eur. Phys. J. C*, vol. 80, no. 7, p. 668, 2020. [Online]. Available: https://doi.org/10.1140/epjc/s10052-020-8242-x

[35] A. J. Bruce, "$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supersymmetry: 2-d sigma models," *Journal of Physics A: Mathematical and Theoretical*, vol. 53, no. 45, p. 455201, oct 2020. [Online]. Available: https://doi.org/10.10882F1751-81212Fabb47f

[36] Z. Kuznetsova and F. Toppan, "Classification of minimal $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super) algebras and some applications," *Journal of Mathematical Physics*, vol. 62, no. 6, p. 063512, 2021.

[37] A. J. Bruce, "Is the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded sine-Gordon equation integrable?" *Nuclear Physics B*, vol. 971, p. 115514, 2021.

[38] A. J. Bruce and S. Duplij, "Double-graded supersymmetric quantum mechanics," *Journal of Mathematical Physics*, vol. 61, no. 6, p. 063503, 2020.

[39] N. Aizawa, Z. Kuznetsova, and F. Toppan, "$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: the quantization," *Nuclear Physics B*, vol. 967, p. 115426, 2021.

[40] N. Aizawa, K. Amakawa, and S. Doi, "$\mathcal{N}$-extension of double-graded supersymmetric and superconformal quantum mechanics," *Journal of Physics A: Mathematical and Theoretical*, vol. 53, no. 6, p. 065205, 2020.

[41] A. J. Bruce and S. Duplij, "Double-graded quantum superplane," *Reports on Mathematical Physics*, vol. 86, no. 3, pp. 383–400, 2020.

[42] S. Doi and N. Aizawa, "Comments of $\mathbb{Z}_2 \times \mathbb{Z}_2$-supersymmetry in superfield formalism," *Nuclear Physics B*, vol. 974, p. 115641, jan 2022. [Online]. Available: https://doi.org/10.10162Fj.nuclphysb.2021.115641

[43] A. J. Bruce, "On a $\mathbb{Z}_2^n$-graded version of supersymmetry," *Symmetry*, vol. 11, no. 1, p. 116, 2019.

[44] N. Aizawa, K. Amakawa, and S. Doi, "$\mathbb{Z}_2^n$-graded extensions of supersymmetric quantum mechanics via clifford algebras," *Journal of Mathematical Physics*, vol. 61, no. 5, p. 052105, 2020.

[45] C. Quesne, "Minimal bosonization of double-graded quantum mechanics," *Modern Physics Letters A*, vol. 36, no. 33, p. 2150238, 2021.

[46] N. Aizawa and S. Doi, "$\mathbb{Z}_2^3$-graded extensions of lie superalgebras and superconformal quantum mechanics," *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, vol. 17, p. 071, 2021.

[47] A. Pashnev and F. Toppan, "On the classification of n-extended supersymmetric quantum mechanical systems," *Journal of Mathematical Physics*, vol. 42, no. 11, pp. 5257–5271, 2001.

[48] Z. Kuznetsova, M. Rojas, and F. Toppan, "Classification of irreps and invariants of the n-extended supersymmetric quantum mechanics," *Journal of High Energy Physics*, vol. 2006, no. 03, p. 098, 2006.

[49] Z. Kuznetsova and F. Toppan, "D-module representations of n= 2, 4, 8 superconformal algebras and their superconformal mechanics," *Journal of mathematical physics*, vol. 53, no. 4, p. 043513, 2012.

[50] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, "Superconformal mechanics," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 17, p. 173001, 2012.

[51] S. J. Gates Jr and L. Rana, "A theory of spinning particles for large n-extended supersymmetry," *Physics Letters B*, vol. 352, no. 1-2, pp. 50–58, 1995.

[52] ——, "A theory of spinning particles for large n-extended supersymmetry (ii)," *Physics Letters B*, vol. 369, no. 3-4, pp. 262–268, 1996.

[53] S. Bellucci, S. Krivonos, A. Marrani, and E. Orazi, ""root" action for n= 4 supersymmetric mechanics theories," *Physical Review D*, vol. 73, no. 2, p. 025011, 2006.

[54] F. Toppan, "$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parastatistics in multiparticle quantum Hamiltonians," *Journal of Physics A: Mathematical and Theoretical*, vol. 54, no. 115203, p. 37, 2021.

[55] V. N. Tolstoy, "Once more on parastatistics," *Physics of Particles and Nuclei Letters*, vol. 11, pp. 933–937, 2014.

[56] N. Stoilova and J. Van der Jeugt, "The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded lie superalgebra pso (2m+ 1| 2n) and new parastatistics representations," *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 13, 2018.

[57] F. Toppan, "Inequivalent quantizations from gradings and $\mathbb{Z}_2 \times \mathbb{Z}_2$-parabosons," *Journal of Physics A: Mathematical and Theoretical*, vol. 54, no. 35, p. 355202, 2021.

[58] F. J. Dyson, "The threefold way. algebraic structure of symmetry groups and ensembles in quantum mechanics," *Journal of Mathematical Physics*, vol. 3, no. 6, pp. 1199–1215, 1962.

[59] J. C. Baez, "The tenfold way," *Notices of the American Mathematical Society*, vol. 67, no. 10, p. 1599, 2020. [Online]. Available: https://arxiv.org/abs/2011.14234

[60] A. Kitaev, "Periodic table for topological insulators and superconductors," in *AIP conference proceedings*, vol. 1134, no. 1. American Institute of Physics, 2009, pp. 22–30.

[61] M. R. Zirnbauer, "Riemannian symmetric superspaces and their origin in random-matrix theory," *Journal of Mathematical Physics*, vol. 37, no. 10, pp. 4986–5018, 1996.

[62] A. Altland and M. R. Zirnbauer, "Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures," *Physical Review B*, vol. 55, no. 2, p. 1142, 1997.

[63] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. Ludwig, "Topological insulators and superconductors: tenfold way and dimensional hierarchy," *New Journal of Physics*, vol. 12, no. 6, p. 065010, 2010.

[64] D. S. Freed and G. W. Moore, "Twisted equivariant matter," in *Annales Henri Poincaré*, vol. 14, no. 8. Springer, 2013, pp. 1927–2023.

[65] Z. Kuznetsova and F. Toppan, "Beyond the 10-fold way: 13 associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superdivision algebras," *Advances in Applied Clifford Algebras*, vol. 33, no. 24, p. 24, 2023.

[66] F. Toppan and P. W. Verbeek, "On alphabetic presentations of Clifford algebras and their possible applications," *Journal of mathematical physics*, vol. 50, no. 12, p. 123523, 2009.

[67] S. Azam, "A new characterization of Kac–Moody–Malcev superalgebras," *Journal of Algebra and Its Applications*, vol. 16, no. 08, p. 1750144, 2017.

[68] H. Albuquerque and S. Benayadi, "Quadratic Malcev superalgebras," *Journal of Pure and Applied Algebra*, vol. 187, no. 1-3, pp. 19–45, 2004.

[69] I. P. Shestakov, "Prime Malcev superalgebras," *Mathematics of the USSR-Sbornik*, vol. 74, no. 1, p. 101, 1993.

[70] E. Barreiro, "Odd-quadratic Malcev superalgebras," *Algebra and Discrete Mathematics*, vol. 9, no. 2, 2018.

[71] J. Lukierski and F. Toppan, "Generalized space–time supersymmetries, division algebras and octonionic m-theory," *Physics Letters B*, vol. 539, no. 3-4, pp. 266–276, 2002.

[72] F. Toppan, "On the octonionic m-superalgebra," 2003. [Online]. Available: https://arxiv.org/abs/hep-th/0301163

[73] ——, "On the octonionic superconformal m-algebra," *International Journal of Modern Physics A*, vol. 18, no. 12, pp. 2135–2141, 2003.

[74] J. D. Bjorken and S. D. Drell, *Relativistic quantum mechanics.* Mcgraw-Hill College, 1964.

[75] R. da Rocha and M. A. Traesel, "Generalized non-associative structures on the 7-sphere," in *Journal of Physics: Conference Series*, vol. 343, no. 1. IOP Publishing, 2012, p. 012026.

[76] N. Aizawa, Z. Kuznetsova, and F. Toppan, "The quasi-nonassociative exceptional f (4) deformed quantum oscillator," *Journal of Mathematical Physics*, vol. 59, no. 2, p. 022101, 2018.

[77] M. Zorn, "Theorie der alternativen ringe," in *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 8. Springer, 1931, pp. 123–147.

[78] T. Kugo and P. Townsend, "Supersymmetry and the division algebras," *Nuclear Physics B*, vol. 221, no. 2, pp. 357–380, 1983.

[79] M. De Andrade and F. Toppan, "Real structures in Clifford algebras and Majorana conditions in any space–time," *Modern Physics Letters A*, vol. 14, no. 26, pp. 1797–1814, 1999.

[80] N. Bourbaki, *Algebra I: chapters 1-3.* Springer Science & Business Media, 1998, vol. 1.

[81] F. Delduc, F. Gieres, S. Gourmelen, and S. Theisen, "Nonstandard matrix formats of lie superalgebras," *International Journal of Modern Physics A*, vol. 14, no. 25, pp. 4043–4060, 1999.

[82] R. A. Mosna, D. Miralles, and J. Vaz Jr, "$\mathbb{Z}_2$-gradings of clifford algebras and multivector structures," *Journal of Physics A: Mathematical and General*, vol. 36, no. 15, p. 4395, 2003.

[83] A. Salam and J. A. Strathdee, "Supergauge Transformations," *Nucl. Phys. B*, vol. 76, pp. 477–482, 1974.

[84] E. Witten, "Constraints on supersymmetry breaking," *Nuclear Physics B*, vol. 202, no. 2, pp. 253–316, 1982.

[85] L. Baulieu, N. L. Holanda, and F. Toppan, "A world-line framework for 1D topological conformal σ-models," *Journal of Mathematical Physics*, vol. 56, no. 11, p. 113507, nov 2015. [Online]. Available: https://doi.org/10.10632F1.4935851

[86] A. Malcev, "Analytical loops," *Mat. Sb. (N.S.)*, vol. 36(78), pp. 569–576, 1955.

[87] A. A. Sagle, "Malcev algebras," *Transactions of the American Mathematical Society*, vol. 101, no. 3, pp. 426–458, 1961.

[88] H. S. Green, "A generalized method of field quantization," *Physical Review*, vol. 90, no. 2, p. 270, 1953.