CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Quantum Fluctuations and Spontaneous Symmetry Breaking: A Functional Renormalization Group Approach

Philipe de Fabritiis

Supervisor: José Abdalla Helayël-Neto

RIO DE JANEIRO
2019

CBPF 70 ANOS

MINISTÉRIO DA CIÊNCIA, TECNOLOGIA, INOVAÇÕES E COMUNICAÇÕES

"QUANTUM FLUCTUATIONS AND SPONTANEOUS SYMMETRY BREAKING: A FUNCTIONAL RENORMALIZATION GROUP APPROACH"

PHILIPE OSÓRIO DE FABRITIIS

Dissertação de Mestrado em Física, apresentada no Centro Brasileiro de Pesquisas Físicas do Ministério da Ciência, Tecnologia, Inovações e Comunicações. Fazendo parte da banca examinadora os seguintes professores:

José Abdalla Helayel Neto - Presidente/Orientador

Rodrigo Ferreira Sobreiro

Sebastião Alves Dias

Rio de Janeiro, 26 de abril de 2019.

Abstract

In this thesis we investigate the spontaneous symmetry breaking induced by quantum fluctuations in scalar Quantum Electrodynamics. This phenomenon was originally studied by S. Coleman and E. Weinberg using the perturbative approach to calculate the effective potential. Their pioneering work, however, was limited to a specific region of the parameter space, therefore it is interesting to investigate if their results remain valid beyond the perturbative regime. To do this, we follow Wetterich's path and use the functional renormalization group tools to obtain non-perturbative informations about the theory, studying the flow equations for the effective average potential and for the gauge coupling, taking into account the anomalous dimensions corrections, in such a way that we can recover the original results and go further. Solving numerically the complete flow equations with suitable approximations, we discuss whether or not spontaneous symmetry breaking takes place in each regime and, for the range of parameters analyzed, the different phases of the theory as a function of mass.

Key Words: Spontaneous Symmetry Breaking; Functional Renormalization Group; Coleman-Weinberg Mechanism; Scalar Quantum Electrodynamics.

Resumo

Nesta dissertação investigamos a quebra espontânea de simetria causada por correções quânticas na Eletrodinâmica Quântica escalar. Tal fenômeno foi originalmente estudado por S. Coleman e E. Weinberg usando a teoria de perturbação para calcular o potencial efetivo. A abordagem original, entretanto, se limita a uma certa região do espaço de parâmetros, portanto é interessante investigar se os resultados permanecem válidos fora do regime perturbativo. Para isso, seguimos Wetterich e usamos as ferramentas do grupo de renormalização funcional para obter informações não perturbativas, estudando as equações de fluxo para o potencial efetivo médio e para o acoplamento de *gauge*, levando em conta as correções da dimensão anômala, de tal forma que possamos recuperar os resultados originais e ir além deles. Resolvendo numericamente as equações de fluxo completas com algumas aproximações, discutimos se ocorre ou não a quebra espontânea de simetria em cada regime e quais são as diferentes fases da teoria em função da massa, no conjunto de parâmetros analisados.

Palavras-Chave: Quebra Espontânea de Simetria; Grupo de Renormalização Funcional ; Mecanismo de Coleman-Weinberg; Eletrodinâmica Quântica Escalar.

Acknowledgements

• About life, I would like to thank to:

- all scientists and professors, who in their daily struggle contribute to the advance of knowledge and therefore of humanity as a whole;

- my parents, for all;

- my aunt Silvana, for all love and affection ever since;

- my partner Alice, that with her love showed me the Wonderland;

- Prof. Helayël, who taught me how to be a Scientist and is an outstanding example of Researcher, Professor, and Human being. For those who say that he's the Jesus of Physics, I say with pleasure that I'm one of his Apostles, and I'll preach his word wherever I am;

- Prof. Tião, who had a fundamental role in my Master education and for whom I have an enormous affection. Your classes are really excellent, it was a privilege to have been your student. "Ah, of course, this is another of the endless tricks...";

- Gustavo, an older brother that Physics gave me (we were born the same year, but he is much older). Guga has been with me since the *Iniciação Científica* to this day. But I'm sorry, the right one is *biscoito*, and not *bolacha*;

- Prof. Kátia Nunes Pinto and Prof. Almir Paz de Lima who guided me at the beginning of my journey. This initial support was fundamental to me;

- the Department of Applied Mathematics of UFRJ, and to the coordinator of the course, Marco Cabral, for making ABC 116 an oasis in *Fundão*, giving freedom and helping its students to develop their full potential;

- Felipe Rosa, Farina and Reinaldo, for guidance and friendship;

- the guys of *Sala Dirac* for the companionship, to the friends of *Prosperidade* for the friendship since childhood, to Verônica Alencar for the patience, to Marcos Turques for the friendship, and especially to Lousada's family for the help in the hard moments;

- Jiu-Jitsu and Volleyball, for helping to mantain my mental health.

• About this thesis, I would like to thank to:

- Prof. Helayël for guidance, inspiration and teachings in general;

- Gustavo, for the infinite help with this work. He participated in every step and was at my side at every time. This work is ours in its qualities, and only mine in its defects. I am extremely grateful for your dedication;

- all those who have corrected at least one mistake in this work, among which I highlight those who contributed the most: Gustavo, Alice and Judismar;

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

Presentation and General Framework

Quantum Field Theory is the mathematical framework with which we build up the Standard Model of Particle Physics, an extremely successful theory that describes the fundamental interactions of nature and classifies the existing elementary particles.

In the Standard Model and in the Quantum Field Theory as a whole, the idea of symmetry plays a key role, helping us to understand the patterns that are present in nature and build the proper mathematical structure to describe them. One of the main pillars of the Standard Model is gauge symmetry, a hallmark in the description of all fundamental interactions known today. Another building block of the Standard Model is the concept of spontaneous symmetry breaking, in which the fundamental state of a theory does not respect a symmetry present in its dynamics. By bringing together the two ingredients above, that is, when the spontaneous breaking of a gauge symmetry occurs, we are faced with the mechanism by which the particles of the Standard Model acquire mass, making clear the fundamental importance of these concepts for the building of Standard Model.

During the development of Quantum Field Theory, physicists had to find a way to deal with the infinities that resulted from their calculations, modifying the parameters of their theory to take into account the effects of quantum fluctuations, giving an adequate interpretation of their results and ensuring that the theory was meaningful. Further on, in an attempt to better understand what was happening, they realized that the change of the physics with scale was enormously relevant, and that the coupling constants they used were not so constant as they thought. Because of this behavior, in Quantum Chromodynamics for example, the gauge coupling is small in the high energy regime, which made possible a great success of the perturbative approach of Quantum Field Theory. However, even today we do not understand some issues about the non-perturbative regime of Quantum Field Theory, and this subject is still being discussed.

The concepts considered above were essential for building and better understanding of the Standard Model as a whole, therefore, comprehending these ingredients and how they relate is extremely important. This is the context in which this dissertation is developed, since we are studying a model in which all these ideas appear intrinsically related.

The central idea of this work is to study the phenomenon of spontaneous symmetry breaking induced by quantum fluctuations in a gauge theory, beyond the perturbative regime. In fact, working with a simple gauge theory, we part from a situation where the classical analysis would say that there is no spontaneous breaking and then verify the possibility of, if taking into account quantum corrections, the system starts to show the phenomenon of spontaneous breaking. In this way we would have the most spontaneous of all spontaneous symmetry breakings, since the phenomenon would be caused solely by the quantum fluctuation of the fields.

This work is organized as follows:

- In Chapter 1, we present a general introduction to the subject of this thesis;
- In Chapter 2 we introduce the problem from the original point of view, with the purpose of putting the problem well-defined, fixing the notation and helping the less experienced reader. The savvy reader can quickly read or even skip this initial chapter.
- Chapter 3 reviews the ideas of functional renormalization group used in this work, which is essential for understanding what comes next. We introduce the effective average action and derive the Wetterich equation, the principal tool of this approach. At the end we explain how to extract information from the Wetterich equation, for instance, how to compute beta-functions and anomalous dimensions. The reader who knows Wetterich's approach will not learn anything new there.
- Chapter 4 deals with scalar Quantum Electrodynamics and is the heart of the thesis. We use all the tools introduced in previous chapters to investigate the issues raised. We begin by giving a truncation and obtaining the ingredients of the Wetterich equation. Then, we present the relevant flow equations through two different regimes. In sequence, we show that our approach is consistent with the original results, we discuss numerical solutions for the flow equations and interpret the solutions showing what happens in each regime of the theory.
- In Chapter 5 we make the conclusions and, in the Appendix, we compute the anomalous dimensions used in the Chapter 4.

We used the software *Mathematica*[1](and in particular the *Package-X*[2]) to do many computations here.

[1]Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL (2018)
[2]Hiren H. Patel, Comput. Phys. Commun. 197, 276 (2015), ePrint: arXiv:1503.01469

“Physics is like sex: sure, it may give some practical results, but that’s not why we do it.”

Richard P. Feynman

Contents

Chapter 1

Introduction

How spontaneous is the spontaneous symmetry breaking?

In Quantum Field Theory, following the traditional approach in the analysis of spontaneous symmetry breaking, we usually limit ourselves to a classical approximation of the situation, looking at the potential and saying whether or not spontaneous symmetry breaking occurs. We do so because we believe that quantum fluctuations are only capable of modifying the theory by making small quantitative corrections in it. If we are based in the usual paradigm of perturbation theory this approximation will be successful most of the time.

It is very common in textbooks to treat this question by giving as an example a model where the phenomenon of spontaneous symmetry breaking occurs due to a signal exchanged in the Lagrangian mass term [1]. However, such term is artificial, and there is nothing spontaneous and natural about assuming it from the beginning. Another very common example is that of the spontaneous symmetry breaking caused by the change of an external parameter of the theory but, although it is not so artificial, this phenomenon is associated with the explicit change of a parameter that can be performed by someone in a laboratory and therefore it is not so spontaneous. Is there a truly spontaneous symmetry breaking in nature, that does not depend on an artificial term placed in the Lagrangian or on the change of an external parameter? A breaking that occurs because of the intrinsic properties of quantum fields, fundamental objects used to describe everything in nature? The answer is yes, quantum fluctuations may be wild enough to modify the whole theory from a qualitative point of view, making it completely different from that expected by classical analysis.

The spontaneous symmetry breaking due to radiative corrections was originally studied by S.Coleman and E. Weinberg in 1973 in a seminal paper [2], and since then it has obtained several applications, for example, in Condensed Matter in the study of quantum phase transitions [3], and in Cosmology, in cosmological models with inflation [4]. In the original work, the authors adopted the functional approach of Quantum Field Theory, which is extremely suitable for dealing with quantum fluctuations and spontaneous symmetry breaking and after introducing such formalism, proposed to study the effective potential, a fundamental object for the analysis of this phenomenon. The authors used the perturbative expansion to calculate the 1-loop effective potential through the sum of Feynman diagrams, giving as a basic (but instructive) example a simple scalar field theory, and after that studying the case of massless scalar Quantum Electrodynamics, used as a model to understand this rich phenomenon. With this model at hand, they computed the effective potential at 1-loop, with an additional hypothesis of a relation between the orders of magnitude of the couplings of the theory, which is later justified in light of the ideas of renormalization group. They found the region of parameters where the spontaneous symmetry breaking by quantum effects occurs, showing that such phenomenon can in fact take place. Even though from the classical point of view the massless scalar Quantum Electrodynamics does not present spontaneous symmetry breaking, at the quantum level it is induced by the fluctuations, thus causing the Higgs mechanism [5] which gives mass to the photon and to the scalar field, making this theory not as massless as we had thought before. In addition, it was observed the interesting phenomenon of dimensional transmutation and obtained a relation between the masses of the photon and the scalar field. This pioneering work, however, was restricted to the perturbative sector of the theory, and therefore the result is valid only for the small coupling regime. The natural following question is: does this phenomenon also occurs in the non-perturbative sector of theory?[1]

We need a tool that allows us to analyze the non-perturbative regime of the theory, without losing the advantages obtained with the use of functional formalism. For this, we use the functional renormalization group according to Wetterich's approach [6] (there are other approaches, such as Wilson-Polchinski's [7]). Based on Wilson's ideas about renormalization group [8], this approach uses the Quantum Field Theory functional method and is capable to obtain non-perturbative information by studying the flow equations for effective potential and for the couplings of the theory. For this we use the effective average action, a generalization of the usual effective action and obtain for it an exact equation, the Wetterich equation, which tells us how this object change as we change the scale. With this object, we can look at the classical approximation on the one hand and to the full quantum theory on the other hand. However, it is not all flowers. Wetterich equation, although exact, is too complicated to be solved analytically with the currently available methods. Therefore we are forced to make approximations and eventually use numerical methods to extract relevant information from the theory.

Even though working with approximations, however, the method proves to be very effective. With this approach we can obtain, for example, the Coleman-Weinberg expression

[1] It is important to remark that we could have done the path integral explicitly as in [9] in the massless scalar Quantum Electrodynamics case, but we preferred to adopt other approach

for the 1-loop effective potential as well as the already known beta-function of the scalar Quantum Electrodynamics obtained with the usual Quantum Field Theory, and we can find for both a correction coming from the anomalous dimension, which shows that our analysis is tuned with the known results, and allows us to go further. In fact, as we shall see latter, we can solve the flow equations numerically (with suitable approximations) and, therefore, investigate different regimes to see if and where spontaneous symmetry breaking occurs.

The basic physical idea behind the functional renormalization group is to connect microscopic and macroscopic physics. Physics changes with the scale, and the functional renormalization group is a suitable tool to connect effective descriptions on different scales, and derive macroscopic physics from underlying microscopic descriptions, including the effect of quantum fluctuations on all intermediate scales. With this approach, we can access regimes where physics is governed by strong correlations and non-perturbative effects [10],[11].

The functional renormalization group has been applied to an enormous variety of physical phenomena, of which we can cite: i) Quantum Gravity [12]; ii) Finite-temperature Yang-Mills [13]; iii) Higgs sector of SM [14]; iv) Non-equilibrium physics [15]; v) Nuclear physics [16]; vi) QCD phase diagrams [17]; vii) Supersymmetric models [18].

Chapter 2

Quantum Fluctuations and Spontaneous Symmetry Breaking

2.1 Introduction

The concept of spontaneous symmetry breaking [19] is extremely important for many areas of Physics and in particular for the Standard Model. Through this concept it was possible to understand the mass generation for the gauge bosons of the weak interaction without losing the benefits of gauge symmetry, a crucial question to guarantee the renormalizability of the theory and therefore to guarantee the Electroweak unification developed by Salam-Weinberg-Glashow [20]. Moreover, the Higgs mechanism, the expression of the spontaneous breaking of a gauge symmetry, is the way in which all fundamental particles acquire mass and therefore is a crucial aspect of all Particle Physics. The precursor of the Higgs mechanism, however, came from the Condensed Matter through the works of Anderson [21] that, while studying the superconductivity phenomenon, observed the need for the photon to acquire mass to sustain the already known Meissner effect [22] and suggested a breaking mechanism for the mass generation in a Condensed Matter system. This concept is also central to Statistical Mechanics in the study of fluctuation induced quantum phase transitions (see for example [3]) and finds applications even in Cosmology, where a spontaneous symmetry breaking was proposed by Linde for a cosmological model with inflation [4].

Basically, the spontaneous symmetry breaking occurs when the vacuum of a theory does not respect some symmetry present in the Lagrangian. In the traditional treatment of the subject, we assume that by looking at the Lagrangian, we can determine whether or not there is spontaneous symmetry breaking. In fact, given a Lagrangian $\mathcal{L}(\Phi, \partial_\mu \Phi)$, we group the non-derivative terms defining a potential $V(\Phi)$, and then we can already search for symmetries, that is, transformations of the fields that leave $\mathcal{L}$ invariant. To find the minimum energy configuration, we assume that all fields are constant in space-time in the vacuum state (to minimize the kinetic part), and minimize the potential $V(\Phi)$ as a function of the fields, basically by taking $\frac{dV(\Phi)}{d\Phi} = 0$ and defining the solution $\langle\Phi\rangle_0$ as the expected value of the field in the vacuum state.

In general, when we find a nontrivial value for the vacuum expectation value of the field, that is, $\langle\Phi\rangle_0 \neq 0$, we obtain a degenerate vacuum situation, where there are several possible vacua (generally infinite, in the case of a continuous symmetry), connected by a symmetry. In choosing a minimal energy configuration around which we are going to perform perturbations that will be associated with the particles, we choose one of the possible vacua, which is now considered to be the vacuum of theory, and therefore it is no longer invariant under the symmetry in question. Thus, in general, we say that there is spontaneous symmetry breaking if the vacuum expectation value of any of the fields present in the theory is not invariant by some symmetry of the Lagrangian.

In Quantum Field Theory, we define fields as distributions that take values in operators. These do not commute, nor do they have well-defined product at the same point in space-time. This wild nature of quantum fields gives rise to radiative corrections to the interactions of the theory and even to the emergence of interactions that were not present in the original Lagrangian. Therefore, traditional treatment, since it only takes into account the effects present in classical theory, serves at best as an approximation if we are dealing with a Quantum Field Theory. The effects of quantum fluctuations will change the vacuum of our theory. To determine the true nature of the vacuum, we will need a method that takes into account not only the interactions present in the Lagrangian, but also the quantum corrections that will arise. It should be noted that, unlike what is assumed in the traditional approach, the effects of quantum corrections not only make small quantitative changes but can qualitatively change the whole theory, as we shall see later.

An appropriate method to deal with these questions is the functional formalism of Quantum Field Theory, introduced by Schwinger [23] and developed by Jona-Lasinio [24], which permits us to deal with the question of quantum corrections and spontaneous symmetry breaking in a natural way. This method enables us to define a function, the effective potential, which includes all interactions present in the theory (both classical and quantum corrections), and whose minimum determines the true vacuum of the theory, the quantum vacuum, which includes influences of the quantum fluctuations present in nature, and that will allow us to investigate more precisely the question of spontaneous symmetry breaking.

2.2 Functional Methods

Let a theory be described by a Lagrangian $\mathcal{L}(\Phi, \partial_\mu \Phi)$ in a d-dimensional space-time with Euclidean metric. As a starting point, we will define the generating functional $Z[J]$ as the vacuum transition amplitude in the asymptotic states in the presence of an external source $J(x)$, that is,

$$Z[J] = \langle 0|0\rangle_J. \tag{2.1}$$

We can write $Z[J]$ in the functional integral representation

$$Z[J] = \int D\phi \, e^{-S[\phi]+\int d^dx \, J(x)\phi(x)}. \tag{2.2}$$

The functional $Z[J]$ is called generating functional because with it, we can expand on a functional Taylor series and compute the correlation functions, or n-points Green functions through

$$Z[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^dx_1 ... d^dx_n G^{(n)}(x_1, ..., x_n) J(x_1) ... J(x_n), \tag{2.3}$$

where we have

$$G^{(n)}(x_1, ..., x_n) = \langle \phi(x_1)...\phi(x_n)\rangle = \frac{1}{Z[J]} \frac{\delta^{(n)} Z[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0}. \tag{2.4}$$

However, in these Green functions there are still contributions of disconnected events, which do not matter in the calculation of amplitudes in a scattering. Therefore, we can define an object that does not have these terms, namely

$$W[J] = \log Z[J], \tag{2.5}$$

and then we have

$$e^{W[J]} = \langle 0|0\rangle_J = \int D\phi \, e^{-S[\phi]+\int d^dx \, J(x)\phi(x)}. \tag{2.6}$$

Similarly, we can expand $W[J]$, the generating functional of the connected Green functions, by

$$W[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^dx_1 ... d^dx_n G_c^{(n)}(x_1, ..., x_n) J(x_1) ... J(x_n), \tag{2.7}$$

and then,

$$G_c^{(n)}(x_1, ..., x_n) = \langle \phi(x_1)...\phi(x_n) \rangle_c = \left. \frac{\delta^{(n)} W[J]}{\delta J(x_1)...\delta J(x_n)} \right|_{J=0}. \tag{2.8}$$

These correlation functions are the fundamental objects that allows the Quantum Field Theory to make contact with the values measured in the experimental Particle Physics, because these Green functions enter in the so-called LSZ reduction formula [25], which enables us to calculate the coefficients of the S-matrix and obtain the probability amplitudes associated with events in a scattering. In a field theory with Euclidean signature, these are the so-called Schwinger Functions. If we adopt a certain set of axioms for these functions, we can define Wightman's functions [26] in the space with Lorentz signature through an analytical continuation, as Osterwalder and Schrader proved [27].

Let us now define an extremely important physical quantity, the expected value of the quantum field ϕ. The classical field, sometimes also called average field, is defined as

$$\varphi_J(x) = \langle \phi(x) \rangle_J = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)} = \frac{\delta W[J]}{\delta J(x)}. \tag{2.9}$$

From this definition, we can invert the relation to obtain J_φ as a function of φ_J, so we can perform the Legendre transform.

Finally, we define the quantum action as

$$\Gamma[\varphi] = -W[J_\varphi] + \int d^d x J_\varphi(x) \varphi(x). \tag{2.10}$$

Differentiating with respect to φ, and using the definition of classical field, we can obtain immediately the quantum equation of motion

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = J_\varphi(x). \tag{2.11}$$

Substituting these definitions in the expression (2.6), we obtain

$$e^{-\Gamma[\varphi] + \int d^d x J_\varphi(x) \varphi(x)} = \int D\phi \, e^{-S[\phi] + \int d^d x \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} \phi(x)}. \tag{2.12}$$

Let us now make a change of variables through $\phi = \varphi + \xi$, splitting the full quantum field into classical field plus fluctuation field. Now, since we defined $\varphi = \langle \phi \rangle$, we naturally have $\langle \xi \rangle = 0$. Since the φ field does not fluctuate, it does not contribute to the functional measure, and therefore we have $D\phi \to D\xi$. Thus, we obtain an integral-differential equation for the quantum action

$$e^{-\Gamma[\varphi]} = \int D\xi e^{-S[\varphi + \xi] + \int d^d x \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} \xi(x)}. \tag{2.13}$$

This equation is the fundamental representation of the quantum action through the functional integral, and could even serve as an alternative definition of it, being used as a starting point. With this, we could at first get any information desired about our theory, just solving this integral. Surely, this is only a formal solution, as we have no methods available to solve such a complicated equation. In the next chapter we will show a possible way to get information from this object without having to solve this equation.

The functional Γ also admits a Taylor expansion, and can be seen as a functional generator of the proper vertices $\Gamma^{(n)}(x_1, ..., x_n)$ according to

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 ... d^d x_n \Gamma^{(n)}(x_1, ..., x_n)\varphi(x_1)...\varphi(x_n). \tag{2.14}$$

In the perturbative approach, we can show that these $\Gamma^{(n)}$ are given by the sum of all 1PI Feynman diagrams (i.e. connected graphs that can not be disconnected by cutting an inner line) with n external legs, with the usual convention of amputating the external propagators to perform the calculation. It is with these amputated graphs that we usually work, because from the point of view of perturbative field theory, they carry the most important information about our theory.

2.3 The Effective Potential

A convenient way to expand the quantum action Γ, is in powers of momentum around a point at which momenta cancels out, that is, expand in

$$\Gamma[\varphi] = \int d^d x \left(V_{eff}(\varphi) + \frac{1}{2} Z_\varphi (\partial_\mu \varphi)^2 + \ldots \right). \tag{2.15}$$

In this expansion, the term without derivatives V_{eff}, is the so-called effective potential, which we announced at the beginning of the chapter. From the perturbative point of view, we can compare this expression with the Taylor expansion, and see that V_{eff} is given by the sum of all 1PI Feynman graphs with zero external momenta. The effective potential will be the central object of our study as it is fundamental to the determination of the true vacuum of the theory and to the investigation of questions about spontaneous symmetry breaking. We will now see how to describe spontaneous symmetry breaking using functional methods.

Suppose that the Lagrangian has a simple symmetry. We want to know if the vacuum expectation value of our theory respects this symmetry. Note that

$$\varphi_J(x) = \frac{\delta W[J]}{\delta J(x)} = \langle \phi(x) \rangle_J = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}, \tag{2.16}$$

then, taking $J \to 0$, we get the vacuum expectation value of the field

$$\lim_{J\to 0} \varphi_J(x) = \langle 0|\phi(x)|0\rangle = \langle 0|\phi(0)|0\rangle = \langle \phi \rangle_0, \tag{2.17}$$

where we are supposing Poincaré invariance for the vacuum and the usual normalization.

A non-trivial outcome here ($\langle \phi \rangle_0 \neq 0$) can result in non-invariance of the vacuum under the studied symmetry. Putting this same condition $J \to 0$ in the equation of motion, we obtain that

$$\lim_{J\to 0} \left(\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = J_\varphi \right) \quad \to \quad \left. \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} \right|_{\varphi^*} = 0. \tag{2.18}$$

Therefore, we can say that there is spontaneous symmetry breaking in a given theory, if and only if there is a field configuration with nonzero vacuum expected value that solves the quantum equation of motion, that is,

$$\exists \varphi^* \neq 0 \quad \text{such that} \quad \left. \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} \right|_{\varphi^*} = 0. \tag{2.19}$$

Now, using the momentum expansion and assuming that our theory has Poincaré invariance, as well as our vacuum state, we can simplify the above condition and get the main result of this section:

$$\text{Spontaneous Symmetry Breaking} \quad \equiv \quad \left(\exists \varphi^* \neq 0 \quad \text{such that} \quad \left. \frac{\delta V_{eff}(\varphi)}{\delta\varphi(x)} \right|_{\varphi^*} = 0 \right) \tag{2.20}$$

Therefore, the investigation of the true vacuum of a theory and the existence or not of spontaneous symmetry breaking is summarized in the calculation of the effective potential V_{eff} and in the calculation of the configurations that minimize it.

2.4 Massless Scalar Quantum Electrodynamics

Let us consider here the massless scalar Quantum Electrodynamics, the theory of a complex scalar field minimally coupled to the electromagnetic field, whose classical Lagrangian is given by

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 + (D_\mu\phi)^*(D_\mu\phi) + \frac{\lambda}{4!}\phi^4. \tag{2.21}$$

Here, we define the field strength $F_{\mu\nu}$ as usual ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) and the covariant derivative as $D_\mu = \partial_\mu + ieA_\mu$.

Analyzing classically the potential, this theory seems to have a symmetric vacuum.

Let us investigate what happens when we take into account quantum fluctuations. For this, as discussed in the previous section, we need to calculate the effective potential. Following the results of the original paper [2], the 1-loop effective potential for this theory can be obtained in the Landau gauge and is of the form

$$V_{eff}^{1-loop} = \frac{\lambda}{4!}\,\varphi^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\varphi^4\left(\log\frac{\varphi^2}{M^2} - \frac{25}{6}\right), \tag{2.22}$$

where M is an arbitrary mass parameter coming from the renormalization conditions.

Now, analyzing this expression for the effective potential, we have indications that we can have a non-trivial minimum in this approximation. Assuming that we have $\lambda \approx e^4$, which can be done without loss of generality, according to arguments from the renormalization group of Gell-Mann and Low [28], we can consider the contribution of order λ^2 irrelevant compared to others and simplify the above expression, getting

$$V_{eff}^{1-loop} = \frac{\lambda}{4!}\,\varphi^4 + \frac{3e^4}{64\pi^2}\varphi^4\left(\log\frac{\varphi^2}{\langle\varphi\rangle^2} - \frac{25}{6}\right), \tag{2.23}$$

where we chose the arbitrary mass parameter M (the point where we define the renormalization conditions) as the minimum of the potential, $\langle\varphi\rangle$, to simplify the computations.

In fact, if $\langle\varphi\rangle$ is the minimum of the effective potential, we can easily deduce that

$$V'(\langle\varphi\rangle) = 0 \quad \rightarrow \quad \left(\frac{\lambda}{6} - \frac{11e^4}{16\pi^2}\right)\langle\varphi\rangle^3 = 0. \tag{2.24}$$

Therefore, we obtain the relation between the couplings:

$$\lambda = \frac{33}{8\pi^2}\,e^4. \tag{2.25}$$

Thus, we begin the description of our theory through two dimensionless parameters, e and λ, and we end up describing the theory with e and $\langle\varphi\rangle$. This shift from a dimensionless parameter to one with mass dimension is what Coleman and Weinberg called dimensional transmutation, and it is a common phenomenon in massless gauge theories that undergo spontaneous symmetry breaking.

Using the relation between the coupling constants, we obtain a final expression for the effective potential at 1-loop

$$V_{eff}^{1-loop} = \frac{3e^4}{64\pi^2}\varphi^4\left(\log\frac{\varphi^2}{\langle\varphi\rangle^2} - \frac{1}{2}\right). \tag{2.26}$$

From this expression, it is clear that we have a non-trivial minimum for the effective potential, at least at 1-loop order. (Note that we have $V(0) = 0$, and $V(\langle\varphi\rangle) = -\frac{3\,e^4\,\langle\varphi\rangle^4}{128\pi^2}$).

Therefore, although classical analysis indicates a symmetric vacuum, 1-loop quantum corrections indicate that there is spontaneous symmetry breaking. In this way we conclude that it is possible for a theory to have the spontaneous symmetry breaking induced by quantum corrections, even when classical analysis says otherwise, giving an example of a simple theory where this occurs. However, the result obtained above is only valid in the perturbative regime, where we are assuming small couplings. The natural question that we could ask next is: does this phenomenon occur if we consider the non-perturbative regime?

In order to face this issue, we need tools that access the non-perturbative sector of Quantum Field Theory and allow us to reproduce the above computations and verify whether spontaneous breaking induced by quantum fluctuations occurs or not. To do this, we will review the ideas of the functional renormalization group in the next chapter according to Wetterich's approach, and we will use this tool in Chapter 4 to re-study the scalar Quantum Electrodynamics, now from a broader point of view.

Chapter 3

The Functional Renormalization Group

3.1 Introduction

The heart of a Quantum Field Theory is its quantum action Γ, the generating functional of the 1PI Green functions. This object includes all the effects of quantum fluctuations of the theory and with Γ we can calculate the correlations that give us scattering amplitudes, cross sections, among other things. If we could find the quantum action Γ and therefore the 1PI Green functions, we could say that we solved the quantum theory.

We want to understand how a theory changes as we change the scale, so we will modify a fundamental object, the generating functional, implementing a scale dependence directly on the functional integral, so that we will have a non-perturbative character in the regularization of the theory. Based on the Wilsonian approach of the renormalization group, instead of integrating all quantum fluctuations at once taking into account all modes of the field to calculate the functional integral, we will restrict ourselves to integrating only the modes with momenta above a specific cutoff scale by suppressing the contribution of the modes with momenta below this cutoff scale in the functional integral.

In this sense, the effective average action Γ_k is defined as a generalization of the effective action, obtained by restricting the integration of quantum fluctuations through a cutoff scale. This object, besides carrying a scale dependence, will have as asymptotic limits on the one hand the bare action S when we suppress all the quantum fluctuations ($k \to \infty$), and on the other hand the quantum action Γ when we do not suppress any mode ($k \to 0$). The effective average action Γ_k will be seen as a renormalization group trajectory in a "theory space", linking classical theory to the full quantum theory whose scale dependence will be given by a non-perturbative flow equation , the Wetterich equation. From it, we can obtain a hierarchy of flow equations for their proper vertices, forming an infinite system of equations analogous to the Dyson-Schwinger equations. Such equations will be useful for calculating the so-called anomalous dimension, for example. It should be noted that the equations studied here admit a schematic representation very similar to Feynman diagrams in perturbation theory, which can be very useful when dealing with more complicated models.

3.2 Wilsonian Approach to RG

The Quantum Field Theory since its beginning has been struggling with the question of the infinities that arose naturally from its computations. At one point, the question of the infinities caused a certain skepticism and even the abandonment of the theory by some scientists. However, with the works of Feynman, Schwinger, and Tomonaga [29], the Quantum Electrodynamics became an example to be followed, and there was an attempt to describe everything that was known at the time with Field Theory. In this process it was found that to be predictive, a theory should be renormalizable. But at this time of development, questions about the renormalization procedure as a whole were still unclear, and scientists still did not trust the method and did not understand the issues involved. It was with the introduction of the renormalization group concept in the context of perturbative renormalization by Gell-Mann and Low [28] that things became gradually clearer. The non-perturbative vision of the renormalization group came only later, created by Wilson [8], in the context of Statistical Mechanics in Condensed Matter, helping in the general understanding of the renormalization procedure. The methods of functional renormalization group used in this thesis are all based on the central ideas of Wilson's approach, and so it is worth briefly commenting on this conception here.

In the functional approach of Quantum Field Theory that we presented in the previous chapter, we integrate over all quantum fluctuations at once. Naturally, the expression of Z for example, has only a formal character since we are not able to realize that integral and not even to define it rigorously mathematically speaking. We also do not have a guarantee that our theories apply to any energy scale, so let us work from a point of view of effective theories, and define the theory only up to a certain UV scale. The central question that arises is: what happens when we change the cutoff scale, that is, how does the theory changes with scale?

The central idea of the Wilsonian approach is to perform the functional integral in an iterative way, a finite momentum shell at a time, that is, to perform the integral of modes with momentum in a finite interval to guarantee finitude at each step, as shown below.[1]

Consider a theory with bare S action on a UV scale Λ. Separate the slow modes ϕ_s, with momenta $|q| < \Lambda_1 = b\Lambda$, and the fast modes ϕ_f with momenta $\Lambda_1 < |q| < \Lambda$ (here,

[1]note that we discuss here in a simplified way, since this section serves only as a motivation for what follows.

$0 < b < 1$). The generating functional of the theory is given by

$$Z = \int d\phi\, e^{-S[\phi]} = \int d\phi_s \int d\phi_f\, e^{-S[\phi]} = \int d\phi_s\, e^{-S_1[\phi_s]}, \tag{3.1}$$

where in the above expression, we define the Wilsonian Action in the scale Λ_1 integrating only the fast modes. That is, we can write formally

$$e^{-S_1[\phi_s]} = \int d\phi_f\, e^{-S[\phi]}. \tag{3.2}$$

The Wilsonian action $S_1[\phi_s]$, obtained by integrating the fast modes of the original action, is the action that describes the slow modes, that is, the theory in the $|q| < \Lambda_1 = b\Lambda$ scale. Certainly, by integrating the other modes, we get the same generating functional, since we are still talking about the same theory. We can now repeat the procedure and perform the integration in the finite momentum shell $b\Lambda_1 = \Lambda_2 < |q| < \Lambda_1$, getting the Wilsonian action S_2, defined for modes with momenta $|q| < \Lambda_2$. Iterating this process, we will obtain a sequence of Wilsonian actions $S_1, S_2, ..., S_n$ valid on scales each time smaller $\Lambda_1, \Lambda_2, ..., \Lambda_n$, such that all of them reproduce the same physics if we integrate the remaining modes. To reach the zero momentum, we would need to do infinite iterations. However, in general we can content ourselves iterating up to a given momentum L^{-1}, where L is the size of the system.

The above process is only explained in a qualitative way, since these integrals are very hard to execute. It should be noted that by performing the first integral, even if we start from a very simple and local bare action, we obtain a very complicated Wilsonian action, typically involving all the terms that are allowed by the symmetries of the original action. Thus, the Wilsonian renormalization group naturally introduces the space of all possible actions compatible with symmetries, the so-called theory space, where the flow of the renormalization group takes place, described by the flows of the couplings that change as we change the scale.

We can also take this b arbitrarily close to 1, such that $\Lambda = b\,\Lambda_{UV}$ is infinitesimally close to Λ_{UV}, and observe what is the infinitesimal change that occurs in the Wilsonian action, to obtain a continuous formulation of the method. In fact, integrating all modes with $\Lambda < |q| < \Lambda_{UV}$

$$Z = \int_{0<|q|<\Lambda_{UV}} d\phi\, e^{-S[\phi]} = \int_{0<|q|<\Lambda} d\phi_s \int_{\Lambda<|q|<\Lambda_{UV}} d\phi_f\, e^{-S[\phi]} = \int_{0<|q|<\Lambda} d\phi_s\, e^{-S_\Lambda}. \tag{3.3}$$

Thus, we can obtain a continuous family of Wilsonian actions whose derivative will define the so-called beta-functionals that will describe the flow of the full action through simple differential equations.

Motivated by the Wilsonian renormalization group, we will next introduce the functional renormalization group. We follow the approach presented in Percacci's book [30].

3.3 The Effective Average Action Γ_k

Let be a scalar theory in d Euclidean dimensions, whose classical dynamics is given by $S[\phi]$. We define its generating functional in the functional integral representation, as we have already seen, by

$$Z[J] = \int D\phi\, e^{-S[\phi]+\int d^dx\, J(x)\phi(x)}. \tag{3.4}$$

In order to study the behavior of our theory with the change of scale, we will implement a procedure in the Wilsonian spirit to suppress quantum fluctuations in the functional integral, adding to the classical action a cutoff action of the form

$$\Delta S_k[\phi] = \frac{1}{2}\int d^dx\, \phi(x)\, \mathcal{R}_k\, \phi(x), \tag{3.5}$$

where $\mathcal{R}_k$ is the cutoff operator, object responsible for suppressing the modes with momenta smaller than the given cutoff scale k, about which we will give detailed information later, in section 3.5. It is important to use a quadratic cutoff action in the fluctuations, because we do not want to change the vertices, the interactions of the theory, we just want to regulate its propagator. With this strategy, in addition to achieving the dependence on the scale that we want, we are able to avoid infrared divergences and also maintain a simple structure for the flow equation that we will see later.

Taking $S \to S_k = S + \Delta S_k$, we obtain a scale dependence as we wish. Define

$$Z_k[J] = \int D\phi\, e^{-S[\phi]-\Delta S_k[\phi]+\int d^dx\, J(x)\phi(x)} = e^{W_k[J]}. \tag{3.6}$$

With this object at hand, we can follow a path analogous to that already followed in the usual functional approach of Quantum Field Theory, now carrying a scale dependence.

Define the classical field

$$\varphi(x) = \frac{\delta W_k[J]}{\delta J(x)} = \langle \phi(x) \rangle_J, \tag{3.7}$$

and with it, propose the change of variables $\phi = \varphi + \xi$ in the functional integral. Now, as the field φ determined by the equation above does not fluctuate, it does not contribute to the functional measure and therefore $D\phi \to D\xi$.

Using the result

$$S[\phi+\xi]+\Delta S_k[\phi+\xi]=S[\phi+\xi]+\Delta S_k[\phi]+\Delta S_k[\xi]+\int d^dx\,\xi(x)\,\frac{\delta}{\delta\phi(x)}\Delta S_k[\phi], \quad (3.8)$$

and substituting this expression in the integral after doing the split in the quantum field

$$e^{W_k}=\int D\xi\; e^{-S[\phi+\xi]-\Delta S_k[\phi]-\Delta S_k[\xi]-\int d^dx\,\xi(x)\,\frac{\delta}{\delta\phi(x)}\Delta S_k[\phi]+\int d^dx\,(\varphi(x)+\xi(x))\,J(x)}. \quad (3.9)$$

Removing from the integrals the terms that do not contribute, we obtain

$$e^{W_k[J]-\int d^dx\,J(x)\,\varphi(x)\,+\,\Delta S_k[\varphi]}=\int D\xi\; e^{-S[\varphi+\xi]-\Delta S_k[\xi]+\int d^dx\,\xi(x)\,(J(x)-\frac{\delta}{\delta\phi(x)}\Delta S_k[\varphi])}. \quad (3.10)$$

Looking at the expression that appeared naturally on the left side, we will define the effective average action as the usual Legendre transform corrected by the cutoff action

$$\Gamma_k[\varphi]=-W_k[J]+\int d^dx\,J(x)\varphi(x)-\Delta S_k[\varphi], \quad (3.11)$$

where as before, we obtain $J(x)$ by inverting the relation $\varphi(x)=\frac{\delta W_k[J]}{\delta J(x)}$. From this definition of Γ_k, we can derive functionally with respect to φ, to obtain the quantum equation of motion

$$\frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)}=J(x)-\frac{\delta}{\delta\varphi(x)}\Delta S_k[\varphi]. \quad (3.12)$$

Therefore, replacing $J(x)$ in the expression (3.9) by the value given in the equation (3.12), we obtain an alternative definition for the effective average action in the functional integral representation

$$e^{-\Gamma_k[\varphi]}=\int D\xi\; e^{-S[\varphi+\xi]-\Delta S_k[\xi]+\int d^dx\,\frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)}\,\xi(x)}. \quad (3.13)$$

The above equation could be used to define the effective average action, and we could start from here and walk in the opposite way. With this expression, we can verify that Γ_k satisfies the desired asymptotic behavior, namely

- $(\lim_{k\to 0}\Gamma_k=\Gamma)$;
- $(\lim_{k\to\Lambda}\Gamma_k=S_\Lambda)$.

Therefore, we have obtained an exact expression for a scale-dependent object that interpolates between the bare action (used as input from the theory we are interested in) and the quantum action that has all the information of the full theory, taking into account all quantum fluctuations.

3.4 The Wetterich Equation

Let us now study the behavior of the effective average action by a change of scale. For this we define for convenience the variable $t = \log(\frac{k}{\Lambda})$ such that we have $\frac{\partial}{\partial t} = k\frac{\partial}{\partial k}$. Define as starting point the generating functional with the addition of a cutoff

$$Z_k[J] = \int D\phi \, e^{-S[\phi]-\Delta S_k[\phi]+\int d^dx \, J(x)\,\phi(x)} = e^{W_k[J]}. \tag{3.14}$$

Differentiating the above expression with respect to the renormalization group parameter t, we obtain

$$\partial_t W_k \, e^{W_k} = \int D\phi \, (-\partial_t \Delta S_k) e^{-S-\Delta S_k+\int d^dx \, J(x)\,\phi(x)}. \tag{3.15}$$

Recalling the definition of the expected value of an observable, and remembering that $Z_k[J] = e^{W_k[J]}$, we have

$$\partial_t W_k = \langle -\partial_t \Delta S_k \rangle_J. \tag{3.16}$$

Now, recalling the definition of ΔS_k we see that the only scale dependence is in the cutoff operator $\mathcal{R}_k$. Introducing a delta function and an integral to replace the argument of one of the fields,

$$-\partial_t W_k = \frac{1}{2} \int d^dx \, d^dy \, \delta(x-y) \, \partial_t \mathcal{R}_k \, \langle \phi(x)\phi(y) \rangle_J. \tag{3.17}$$

From the definition of the effective average action $\Gamma_k[\varphi] = -W_k[J] + \int d^dx J(x)\varphi(x) - \Delta S_k[\varphi]$, differentiating with respect to t and using the above result

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \int d^dx \, d^dy \, \delta(x-y) \, \partial_t \mathcal{R}_k \, \langle \phi(x)\phi(y) \rangle_J - \frac{1}{2} \int d^dx \, d^dy \, \delta(x-y) \, \partial_t \mathcal{R}_k \, \varphi(x)\varphi(y). \tag{3.18}$$

Note that we have defined $\varphi(x) = \langle \phi(x) \rangle_J$, then we can group the two parts and write

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \int d^dx \, d^dy \, \delta(x-y) \, \partial_t \mathcal{R}_k \, (\langle \phi(x)\phi(y) \rangle_J - \langle \phi(x) \rangle_J \langle \phi(y) \rangle_J). \tag{3.19}$$

Now, the expression in parentheses is precisely the already known connected 2-point function, or the propagator. Defining the integrals with the delta as the functional trace in the position representation, we can write

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} Tr \left[\left(\frac{\delta^2 W_k[J]}{\delta J(x) J(y)} \right) \partial_t \mathcal{R}_k \right]. \tag{3.20}$$

We have $\varphi(y) = \frac{\delta W_k[J]}{\delta J(y)}$, so differentiating this expression

$$\frac{\delta\varphi(y)}{\delta J(x)} = \frac{\delta^2 W_k[J]}{\delta J(x)\delta J(y)}. \tag{3.21}$$

On the other hand we have $\frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)} = J(x) - \frac{\delta}{\delta\varphi(x)}\Delta S_k[\varphi]$, so differentiating again

$$\frac{\delta J(x)}{\delta\varphi(y)} = \frac{\delta^2\Gamma_k}{\delta\varphi(x)\delta\varphi(y)} + \frac{\delta^2\Delta S_k}{\delta\varphi(x)\delta\varphi(y)} = (\Gamma_k^2 + \mathcal{R}_k)_{x,y}\,, \tag{3.22}$$

where we express compactly the dependence of the expression on x and y, typically a $\delta(x-y)$.

Looking at the equations (3.21) and (3.22), we clearly see an inversion relationship here. Therefore, we have

$$\frac{\delta^2 W_k[J]}{\delta J(x)\delta J(y)} = (\Gamma_k^2 + \mathcal{R}_k)^{-1}_{x,y}. \tag{3.23}$$

Substituting this result in the equation (3.20), we obtain the main result of this chapter, the famous Wetterich Equation

$$\partial_t\Gamma_k[\varphi] = \frac{1}{2}\,Tr\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right]. \tag{3.24}$$

If we can solve this equation, we obtain the effective average action, and with it, having as a boundary condition in $k = \Lambda$ our bare action S_Λ, we can take the limit $k \to 0$ to get the quantum action Γ. However, this equation is too complicated to be solved exactly, so we will have to call for approximate solutions. We will see that even though we are forced to content ourselves with approximations, this equation can give us an enormous amount of information, allowing us to study the evolution of the couplings of our theory through the beta-functions and look for possible fixed points, for example. We can also obtain the flow equation for the effective average potential and thus study the true vacuum of the theory in question taking into account all quantum fluctuations, and without the usual weak coupling restriction present in the usual perturbative Quantum Field Theory. With this we can analyze questions about the spontaneous symmetry breaking, among other things.

Let us now make some comments about the Wetterich Equation:

- It is an exact equation, since we did not make any approximation in its derivation. It is an all loop order result;
- It is a closed equation, since the evolution of Γ_k on the left side is given as a function of the Γ_k itself on the right side ;

- It is a non-perturbative equation, since there are no restrictions in the values of the couplings, nor in the regime in which we work;

- It is a finite equation for any $k \neq 0$. According to the properties of the cutoff operator, it is possible to show that the presence of $\partial_t \mathcal{R}_k$ in the numerator guarantees finitude in the UV regime and that the presence of $\mathcal{R}_k$ guarantees finiteness in the IR regime;

- It is an equation with 1-loop structure. By this we mean that the Wetterich equation has the structure very similar to that of the equation for the quantum action at 1-loop order.

  In fact, looking at the quantum action at 1-loop order, obtained through perturbation theory, we have

  $$\Gamma^{(1-loop)}[\varphi] = S[\varphi] + \frac{1}{2} Tr \, \log \, S^{(2)}. \tag{3.25}$$

  Taking $S \to S + \Delta S_k$ and removing ΔS_k from the Legendre transform, we get

  $$\Gamma_k^{(1-loop)}[\varphi] = S + \Delta S_k + \frac{1}{2} Tr \, \log \, (S + \Delta S_k)^{(2)} - \Delta S_k, \tag{3.26}$$

  that is, $\Gamma_k^{(1-loop)} = S + \frac{1}{2} Tr \, \log(S^{(2)} + \mathcal{R}_k)$. Taking the derivative

  $$\partial_t \Gamma_k^{(1-loop)} = \frac{1}{2} Tr \left[\left(S^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right]. \tag{3.27}$$

  Therefore, with our definition of effective average action, to pass from the result at 1-loop to the exact result at any order in the loop expansion, we just have to make a renormalization group improvement $S^{(2)} \to \Gamma_k^{(2)}$, while maintaining the same structure of the equation.

3.5 More About the Cutoff $\mathcal{R}_k$

In this section, we will discuss about what we expect from a good cutoff operator. Remember that it appears when we make the transformation $S \to S + \Delta S_k$, where we define $\Delta S_k = \int d^d x \, \phi(x) \, \mathcal{R}_k \, \phi(x)$. We have already talked about the need to keep the cutoff action quadratic in the fluctuations when we introduced the cutoff, so I will not repeat myself in this matter.

The cutoff operator $\mathcal{R}_k$ has the function of suppressing the slow modes in the functional integral. As we will do this by modifying the propagator structure, the cutoff operator will depend fundamentally on the dynamic operator of our theory, that is, the operator that appears in the kinetic part. More explicitly, given a theory with a kinetic term of the form $\frac{1}{2}\phi(\Delta)\phi$, the role of the cutoff operator $\mathcal{R}_k(\Delta)$ will be to suppress in the functional integral eigenvectors ϕ_n of the operator Δ whose eigenvalues λ_n are smaller than a given cutoff scale k^2 ($\Delta\phi_n = \lambda_n\phi_n$).

The functional form of the cutoff operator is arbitrary at first, but to be able to fulfill its role, it is interesting that $\mathcal{R}_k(z)$ has the following properties:

1. For fixed k, $\mathcal{R}_k(z)$ must be monotonically decreasing with z;
2. For fixed z, $\mathcal{R}_k(z)$ must be monotonically increasing with k;
3. $\lim_{k\to 0}\mathcal{R}_k(z) = 0$, for any z;
4. if, $z >> k^2$, then $\mathcal{R}_k(z) \to 0$ fast enough;
5. $\mathcal{R}_k(0) = k^2$.

Properties 1 and 2 are expected for a cutoff operator and are those that will determine the suppression of the slow modes without changing the other sector. The 3 guarantees that when we remove the cutoff we recover the original theory; 4 emphasizes that fast modes must be integrated without suppression; the 5 act as a sort of normalization. With all this, we see that the role of the cutoff operator is basically to replace the original propagator $G(z) = \frac{1}{z}$ by a regularized propagator $G_k(z) = \frac{1}{P_k(z)}$, where $P_k(z) = z + \mathcal{R}_k(z)$. Note that the cutoff scale k acts as an infra-red cutoff since it does not affect the fast modes, and it increasingly suppresses the modes below the k-scale as if it were giving a mass of order k for these modes. Therefore, besides implementing scale dependence, this procedure helps to avoid divergences in the IR sector of theory. We mention here the most used cutoff operators as examples:

- the Litim cutoff [31]

$$\mathcal{R}_k(z) = (k^2 - z)\theta(k^2 - z); \tag{3.28}$$

- the exponential cutoff

$$\mathcal{R}_k(z) = \frac{z}{e^{\frac{z}{k^2}} - 1}. \tag{3.29}$$

3.6 Working with the Wetterich Equation

Our goal here is to understand the behavior of a theory when we change the scale. For this, we defined an object that interpolates between the classical action and the quantum action and that satisfies a flow equation of the functional renormalization group. The Wetterich equation is finite at all scales, thanks to the properties of the cuttoff operator and the way it was defined. In this way, we change the task of solving a very complicated integro-differential equation, by solving a complicated differential equation, which at first sight is a good exchange. However, as we have already said, we are not able to solve this equation exactly, so let us adopt a strategy of approximation to extract as much information as we can from it. The path we are going to follow is to perform a systematic expansion in the effective average action and truncate the result in a given finite order.

Based on the Wilsonian approach, we know that the effective action can be expanded in a series where all terms compatible with the symmetries of a given theory appear. What we can do is adopt a finite set of operators compatible with the symmetries of theory and carry out an expansion on that basis, so that we put all the terms that we deem relevant, but in a form that we can operate with this expansion. With this procedure, we will naturally get an approximate answer, but we can improve it with the addition of new terms, and even being approximate, it is a non-perturbative approach, since we are not specifying the region in which we take our couplings.

Therefore, the procedure is: we expand Γ_k in a systematic way, truncate the sum with a finite number of terms and place that *ansatz* in the Wetterich equation. Hence, we can use projections to extract information about a given term, and obtain for example its beta-function. Doing this, we are trying to understand the effective average action as an RG trajectory in the theory space, linking the classical action with the full quantum action. The directions in this theory space are given by the operators used as the basis for our truncation, and the coupling constants works as the coordinates of the space; the beta-functions will be like velocities there. In this way, we can take a geometric point of view for the flows of the functional renormalization group.

Let the truncation for the effective average action be given by

$$\Gamma_k = \sum_{i=1}^{N} g_i(k)\, k^{-\Delta_i}\, O_i[\phi], \tag{3.30}$$

where $O_i[\phi]$ is a finite basis of operators compatible with the symmetries of the theory, $g_i(k)$ are the associated coupling constants, and Δ_i is the associated canonical dimension, which we explicitly put into the sum to ensure that the couplings are dimensionless and avoid having to deal with readjustments in the future.

Putting this *ansatz* in the left hand side of Wetterich equation, we have

$$\partial_t \Gamma_k = \sum_{i=1}^{N} (\partial_t g_i - \Delta_i\, g_i)\, k^{-\Delta_i}\, O_i. \tag{3.31}$$

On the right hand side, by expanding the result of the trace on a basis containing these operators O_i, with coefficients $\mathcal{A}_i$ (apart from the $k^{-\Delta_i}$), and throwing out any term beyond the N of the basis considered for our *ansatz*, we obtain

$$\frac{1}{2}\,Tr(...) = \sum_{i=1}^{N} k^{-\Delta_i}\,\mathcal{A}_i\,O_i. \tag{3.32}$$

Comparing both sides, we obtain $\partial_t g_i - \Delta_i\, g_i = \mathcal{A}_i$, and with the usual definition of the beta-function

$$\beta_i(g) = \partial_t g_i = \Delta_i\, g_i + \mathcal{A}_i(g). \tag{3.33}$$

Since we are not able to solve the Wetterich equation exactly, we need to use approximations. We want to maintain the non-perturbative character of our analysis, so we do not want to use expansions in terms of small parameters. Instead, our strategy is to restrict the functional space in which we will solve the renormalization group equation by proposing a systematic way of expanding effective action and truncating the series in a finite number of terms. Naturally, we have to choose well the operators present in the truncation, among all the infinite operators compatible with the symmetries, and in that choice we can end up leaving out operators (and therefore couplings) that are very important for the theory. There is nothing we can do about it, and we must always be careful to refine the results obtained with the already known data, and eventually improve our truncation including new terms in order to capture relevant information.

3.7 Flow Equation for the Proper Vertices

With the Wetterich equation at hand, we can derive a hierarchy of new flow equations for the proper vertices $\left(\Gamma^{(n)}_{k;x_1,\ldots,x_n} = \frac{\delta^{(n)}\Gamma_k[\varphi]}{\delta\varphi(x_n)\ldots\delta\varphi(x_1)}\right)$, deriving the Wetterich equation with respect to the fields. Such equations give us information about the flow of a given vertex, and may be useful for obtaining some more specific information about a certain part of our truncation. We will see an example of this in the next section by using the equation for $\Gamma^{(2)}_k$ to compute the anomalous dimension.

Define the regularized propagator

$$G_k(x,y) = (\Gamma^{(2)}_k + \mathcal{R}_k)^{-1}_{x,y}. \tag{3.34}$$

Then, we can rewrite the Wetterich equation as

$$\begin{aligned}\partial_t\Gamma_k[\varphi] &= \frac{1}{2}\,Tr\left[\left(\Gamma^{(2)}_k + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right] = \frac{1}{2}\,Tr\left[G_k\,\partial_t\mathcal{R}_k\right]\\ &= \frac{1}{2}\int_{x,y}\delta(x-y)\,\partial_t\mathcal{R}_k\,G_k(x,y).\end{aligned} \tag{3.35}$$

With this definition for the $G_k(x,y)$, we obtain

$$\frac{\delta G_k(x,y)}{\delta\varphi(x_1)} = -\int_{w,z} G_k(x,w)\,\Gamma^{(3)}_{k;wzx_1}\,G_k(z,y). \tag{3.36}$$

Now, deriving the Wetterich equation we have

$$\partial_t\Gamma^{(1)}_{k,x_1}[\varphi] = -\frac{1}{2}\int_{x,y,z,w}\delta(x-y)\,\partial_t\mathcal{R}_k\,G_k(x,w)\,\Gamma^{(3)}_{k;wzx_1}\,G_k(z,y) = -\frac{1}{2}\,Tr\left[\partial_t\mathcal{R}_k\,G_k\,\Gamma^{(3)}_{k;x_1}\,G_k\right].$$

Deriving again, we will obtain two parts because of the chain rule

$$\begin{aligned}\partial_t\Gamma^{(2)}_{k,x_1,x_2}[\varphi] = &-\frac{1}{2}\int_{x,y,z,w}\delta(x-y)\,\partial_t\mathcal{R}_k\,G_k(x,w)\,\Gamma^{(4)}_{k;wzx_1x_2}\,G_k(z,y)\\ &+\int_{x,y,z,w,z',w'}\delta(x-y)\,\partial_t\mathcal{R}_k\,G_k(x,w)\,\Gamma^{(3)}_{k;wzx_1}\,G_k(z,w')\,\Gamma^{(3)}_{k;w'z'x_2}\,G_k(z',y).\end{aligned}$$

So, we have in a simple form

$$\partial_t\Gamma^{(2)}_{k,x_1,x_2}[\varphi] = -\frac{1}{2}\,Tr\left[\partial_t\mathcal{R}_k\,G_k\,\Gamma^{(4)}_{k;x_1x_2}\,G_k\right] + Tr\left[\partial_t\mathcal{R}_k\,G_k\,\Gamma^{(3)}_{k;x_1}\,G_k\,\Gamma^{(3)}_{k;x_2}\,G_k\right]. \tag{3.37}$$

We could go on and get the flow equations for the other vertices in the same way. In dealing with a truncation in the form of a expansion in $\Gamma^{(n)}_k$, these equations will serve to tie the vertices, they will be like constraints.

3.8 Anomalous Dimension

When calculating the beta-functions for the couplings, taking into account the wave-function renormalization Z_ϕ, it is common to occur a contribution due to the evolution of Z_ϕ, expressed through the anomalous dimension, defined as $\eta_\varphi = -\frac{1}{Z_\varphi}\,\partial_t Z_\varphi$. To obtain a closed expression for the beta-function, we must be able to express the anomalous dimension in terms of the couplings present in the theory. Let us now describe a simple procedure for calculating the anomalous dimension.

The simplest term in which Z_φ appears is the kinetic term, for example, $\frac{1}{2}Z_\varphi(\partial_\mu\varphi)^2$. Note that we would not be able to extract information about Z_φ using the Wetterich equation in the scheme of constant field configurations, since in these circumstances the kinetic term cancels out. However, using the flow equation for the vertex $\Gamma^{(2)}_k$, derived from the previous section, we can easily isolate the Z_φ, as follows.

Consider a truncation for the effective average action given by a simple kinetic term plus other terms involving other fields, interactions, etc.

$$\Gamma_k = \int_x \frac{1}{2}Z_\varphi(\partial_\mu\varphi)^2 + (...) = \int_p \frac{1}{2}\,Z_\varphi\,p^2\,\varphi(p)\,\varphi(-p) + (...). \tag{3.38}$$

Deriving this Fourier expression with respect to the fields, we have

$$\frac{\delta^2 \Gamma_k}{\delta\varphi(p)\,\delta\varphi(-p)} = Z_\varphi\, p^2 + (...). \tag{3.39}$$

Therefore, we can isolate the Z_φ taking the derivative with respect to p^2 in the configuration in which we have all the fields zero ($\Phi = 0$) and also the moments zero ($p^2 = 0$) to ensure that any other undesirable part is canceled. That is,

$$Z_\varphi = \frac{\partial}{\partial p^2} \frac{\delta^2 \Gamma_k}{\delta\varphi(p)\delta\varphi(-p)}\bigg|_{p^2=0;\Phi=0}. \tag{3.40}$$

To compute the anomalous dimension, we only have to take the derivative in the renormalization group parameter t

$$\eta_\varphi = -\frac{1}{Z_\varphi}\,\partial_t Z_\varphi = -\frac{1}{Z_\varphi}\,\frac{\partial}{\partial p^2}\,\partial_t \Gamma_k^{(2)}\bigg|_{p^2=0;\Phi=0}. \tag{3.41}$$

Using the result obtained in the last section, (3.37), we obtain

$$\eta_\varphi = -\frac{1}{Z_\varphi}\,\frac{\partial}{\partial p^2}\,\left[-\frac{1}{2}\,Tr\left(\partial_t \mathcal{R}_k\, G_k\, \Gamma_k^{(4)}\, G_k\right) + Tr\left(\partial_t \mathcal{R}_k\, G_k\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)}\, G_k\right)\right]\bigg|_{p^2=0;\Phi=0}. \tag{3.42}$$

3.9 Getting the Hands Dirty

In this section, we will perform some calculations in a simple scalar model to illustrate the tools developed throughout this chapter [32]. As we discussed earlier, we are not able to solve the Wetterich equation exactly, so we need to use approximations to work. We do this by means of truncations of the effective average action, restricting the functional space in which we describe it, using only a subset of the operators compatible with the symmetries of the theory. Note that this is a truncation in the sense that we are neglecting couplings that may be non-zero in our description, but we will see that we can still get good results with this method.

3.9.1 The Local Potential Approximation

Consider the local potential approximation for a scalar model in a space without curvature in d Euclidean dimensions, a truncation of the effective average action given by

$$\Gamma_k[\varphi] = \int d^d x\left(\frac{Z_k}{2}(\partial_\mu\varphi)^2 + V_k(\varphi^2)\right), \tag{3.43}$$

where the above potential has the symmetry $(\varphi \to -\varphi)$, and its form is generic. The Z_k factor is the wavefunction renormalization of the φ field.

In the following, we will use the conventions

$$\int_x = \int d^d x \quad \text{and} \quad \int_p = \int \frac{d^d p}{(2\pi)^d}. \tag{3.44}$$

We will insert this *ansatz* into the Wetterich equation to extract information. For this, we have to calculate the Hessian, impose the regularization and calculate the functional trace. Deriving the truncation above, we get

$$\frac{\delta \Gamma_k[\varphi]}{\delta \varphi(x_1)} = \int_x \left(Z_k \Delta_x \varphi(x) + \frac{\delta V_k}{\delta \varphi} \right) \delta(x - x_1). \tag{3.45}$$

Deriving again and integrating the delta function

$$\frac{\delta^2 \Gamma_k[\varphi]}{\delta \varphi(x_1)\, \delta \varphi(x_2)} = \left(Z_k \Delta_{x_1} + \frac{\delta^2 V_k}{\delta \varphi\, \delta \varphi} \right) \delta(x_2 - x_1). \tag{3.46}$$

So we can write our Hessian in the position space

$$\Gamma^{(2)}(x_2 - x_1) = (Z_k \Delta_{x_1} + V_k'')\, \delta(x_2 - x_1). \tag{3.47}$$

To facilitate the calculation, let us consider the particular case in which we have constant field configuration $\varphi(x) = \varphi_0$, so that the potential will no longer depend on the position and hence we can integrate more easily (we will use the notation $\Gamma_0^{(2)}(x_2 - x_1)$). With this simplification, we can easily write the Hessian operator in the Fourier representation

$$\Gamma_0^{(2)}(p) = Z_k\, p^2 + V_k''. \tag{3.48}$$

Using the Fourier transform, we can write the cutoff action in the form

$$\Delta S_k(\varphi) = \frac{1}{2} \int_q \varphi(q)\, \mathcal{R}_k(q^2)\, \varphi(-q). \tag{3.49}$$

Then, the Wetterich equation is

$$\partial_t \Gamma_k = \frac{1}{2}\, Tr \left(\frac{\partial_t \mathcal{R}_k}{\Gamma_0^{(2)} + \mathcal{R}_k} \right). \tag{3.50}$$

3.9.2 Computing the Trace

To calculate this functional trace, we will express it more directly, and take advantage of the dependence in p^2 to express the integral in the Fourier space in generalized spherical coordinates

$$Tr\, W(\Delta) = \int d^d x\, d^d y\, \delta(x-y)\, W(\Delta) = \int d^d x \int \frac{d^d q}{(2\pi)^d}\, W(q^2)$$
$$= \int_x \int \frac{d\Omega_d}{(2\pi)^d} \int_{|q|} |q|^{d-1}\, d|q|\; W(|q|^2)$$
$$= \int_x \frac{1}{(4\pi)^{d/2}} \frac{2}{\Gamma(d/2)} \int \frac{dz}{2} z^{\frac{d}{2}-1} W(z), \tag{3.51}$$

where we used the already known result of the integration over the angular part

$$\int \frac{d\Omega_d}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2}} \frac{2}{\Gamma(d/2)}. \tag{3.52}$$

Then, we can write for the functional trace of an arbitrary functional

$$Tr\, W(\Delta) = \int_x \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \int dz\, z^{(\frac{d}{2}-1)} W(z) = \frac{1}{(4\pi)^{d/2}} \int_x Q_{d/2}[W], \tag{3.53}$$

where we defined the Q-functionals as

$$Q_n[W] = \frac{1}{\Gamma(n)} \int_0^\infty dz\; z^{(n-1)}\, W(z). \tag{3.54}$$

The left hand side of Wetterich equation will give us, on the other hand

$$\partial_t \Gamma_k = \int_x (\partial_t Z_k) \frac{1}{2} (\partial_\mu \varphi)^2 + \partial_t V_k. \tag{3.55}$$

Therefore, by restricting us to constant field configurations in addition to canceling the kinetic term, we can factorize the volume element on both sides and obtain the flow equation for the effective average potential

$$\partial_t V_k = \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \int_0^\infty dz\; z^{(d/2-1)}\, \frac{\partial_t \mathcal{R}_k(z)}{\Gamma_0^{(2)}(z) + \mathcal{R}_k(z)}. \tag{3.56}$$

A convenient way to introduce the cutoff operator is by the following formula

$$\mathcal{R}_k(z) = \Gamma_0^{(2)}(P_k(z)) - \Gamma_0^{(2)}(z), \tag{3.57}$$

where $P_k(z) = z + r_k(z)$, with r_k a function with the same properties as $\mathcal{R}_k$. With this prescription, we have for our model

$$\mathcal{R}_k(z) = [Z_k\,(P_k(z)) + V_k''] - [Z_k\,(z) + V_k''], \tag{3.58}$$

that is, we will use a cutoff function given by

$$\mathcal{R}_k(z) = Z_k\, r_k(z). \tag{3.59}$$

Therefore, taking this cutoff operator, the quotient present in the flow equation takes the form

$$\frac{\partial_t \mathcal{R}_k(z)}{\Gamma_0^{(2)}(z) + \mathcal{R}_k(z)} = \frac{\partial_t (Z_k\, r_k(z))}{[Z_k\, z + V_k''] + Z_k\, r_k(z)} = \frac{(\partial_t Z_k) r_k + Z_k(\partial_t r_k)}{[Z_k P_k + V_k'']}. \tag{3.60}$$

Using $\partial_t P_k = \partial_t r_k$, we obtain

$$\frac{\partial_t \mathcal{R}_k(z)}{\Gamma_0^{(2)}(z) + \mathcal{R}_k(z)} = \frac{\partial_t P_k - \eta_k\, r_k}{P_k + Z_k^{-1}\, V_k''}, \tag{3.61}$$

where naturally we defined the anomalous dimension as

$$\eta_k = -\frac{\partial_t Z_k}{Z_k}. \tag{3.62}$$

To proceed further, let us specify a cutoff function to perform the calculations more explicitly. We will use here the Litim cutoff [31], also called optimized cutoff

$$r_k(z) = (k^2 - z)\, \theta(k^2 - z), \tag{3.63}$$

where the $\theta(k^2 - z)$ is a Heaviside function, responsible for making a sharp cutoff at $z = k^2$. Deriving in $\frac{\partial}{\partial t} = k\frac{\partial}{\partial k}$, we obtain

$$\partial_t P_k(z) = \partial_t r_k(z) = 2k^2 \theta(k^2 - z). \tag{3.64}$$

Note that we ignored the part with the delta, since being inside an integral in z, it cancels out due to the factor $(k^2 - z)$. We can write

$$\int_0^\infty dz\, z^{(d/2-1)}\, \frac{\partial_t P_k - \eta_k\, r_k}{P_k + Z_k^{-1}\, V_k''} = \int_0^\infty dz\, z^{(d/2-1)}\, \frac{[2k^2 - \eta_k\, (k^2 - z)]\, \theta(k^2 - z)}{z + (k^2 - z)\, \theta(k^2 - z) + Z_k^{-1}\, V_k''}$$
$$= \int_0^{k^2} dz\, z^{(d/2-1)}\, \frac{(2 - \eta_k) k^2}{k^2 + Z_k^{-1}\, V_k''} + \int_0^{k^2} dz\, z^{d/2} \frac{\eta_k}{k^2 + Z_k^{-1}\, V_k''}. \tag{3.65}$$

3.9.3 Flow Equations

Performing the above integrals and putting this result in the flow equation, we finally obtain

$$\partial_t V_k = \frac{1}{2} \frac{1}{(4\pi)^{d/2}}\, \frac{1}{\Gamma(d/2 + 2)}\, \frac{(d + 2 - \eta_k)\, k^{d+2}}{k^2 + Z_k^{-1}\, V_k''}. \tag{3.66}$$

With the flow equation at hand, we can begin to extract information about the evolution of couplings. From now on, for simplicity, let us restrict ourselves to the 4-dimensional case and ignore the anomalous dimension, taking ($\eta_\phi = 0\,;\ Z_\phi = 1$). Under these conditions, the equation above is

$$\partial_t V_k = \frac{1}{32\pi^2}\frac{k^6}{k^2 + V_k''}. \tag{3.67}$$

Expanding the potential in powers, we obtain

$$V_k = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!}\,\lambda_m\,\varphi^m, \tag{3.68}$$

where $\Delta_m = m - 4$ is the canonical dimension associated with the monomial φ^m. Since we have the symmetry $V_k(-\varphi) = V_k(\varphi)$, *only the even powers will appear here.* It follows that

$$\partial_t V_k = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!}\,\left(\partial_t\lambda_m - \Delta_m\lambda_m\right)\,\varphi^m. \tag{3.69}$$

We can expand the right hand side (RHS) of the flow equation on the same basis that we expanded the left hand side, as

$$RHS = \frac{1}{32\pi^2}\frac{k^6}{k^2 + V_k''} = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!}\,\mathcal{A}_m(\lambda)\,\varphi_0^m, \tag{3.70}$$

where we defined the coefficients $\mathcal{A}_m(\lambda)$ as

$$\mathcal{A}_m(\lambda) = \left[\frac{\partial^m}{\partial\varphi_0^m}\left(k^{\Delta_m}\,RHS\right)\right]_{\varphi_0=0}. \tag{3.71}$$

From the equations (3.69) and (3.70), comparing the coefficients we obtain

$$\beta_m(\lambda) = \partial_t\lambda_m = \Delta_m\,\lambda_m + \mathcal{A}_m. \tag{3.72}$$

With the expansion (3.68) for the V_k, we have explicitly

$$V_k = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!}\,\lambda_m\,\varphi^m = \tag{3.73}$$

$$= \frac{k^4}{0!}\,\lambda_0 + \frac{k^2}{2!}\,\lambda_2\,\varphi_0^2 + \frac{k^0}{4!}\,\lambda_4\,\varphi_0^4 + \frac{k^{-2}}{6!}\,\lambda_6\,\varphi_0^6 + \frac{k^{-4}}{8!}\,\lambda_8\,\varphi_0^8 + (...). \tag{3.74}$$

Then, it follows directly

$$V_k'' = k^2\,\lambda_2 + \frac{k^0}{2!}\,\lambda_4\,\varphi_0^2 + \frac{k^{-2}}{4!}\,\lambda_6\,\varphi_0^4 + \frac{k^{-4}}{6!}\,\lambda_8\,\varphi_0^6 + (...). \tag{3.75}$$

With this expression, we can calculate $\mathcal{A}_m$

$$\mathcal{A}_0 = \frac{1}{32\pi^2}\left[\frac{\partial^0}{\partial\varphi_0^0}\left(k^{\Delta_0}\,\frac{k^6}{k^2+V_k''}\right)\right]_{\varphi_0=0} = \frac{1}{32\pi^2}\frac{1}{(1+\lambda_2)}; \tag{3.76}$$

$$\mathcal{A}_2 = \frac{1}{32\pi^2}\left[\frac{\partial^2}{\partial\varphi_0^2}\left(k^{\Delta_2}\,\frac{k^6}{k^2+V_k''}\right)\right]_{\varphi_0=0} = \frac{1}{32\pi^2}\frac{-\lambda_4}{(1+\lambda^2)^2}; \tag{3.77}$$

$$\mathcal{A}_4 = \frac{1}{32\pi^2}\left[\frac{\partial^4}{\partial\varphi_0^4}\left(k^{\Delta_4}\,\frac{k^6}{k^2+V_k''}\right)\right]_{\varphi_0=0} = \frac{1}{32\pi^2}\left[\frac{6\,\lambda_4^2}{(1+\lambda_2)^3} - \frac{\lambda_6}{(1+\lambda_2)^2}\right]. \tag{3.78}$$

Therefore, we obtain the beta-functions

$$\beta_0(\lambda) = -4\,\lambda_0 + \frac{1}{32\pi^2}\frac{1}{(1+\lambda_2)}; \tag{3.79}$$

$$\beta_2(\lambda) = -2\lambda_2 - \frac{1}{32\pi^2}\frac{\lambda_4}{(1+\lambda^2)^2}; \tag{3.80}$$

$$\beta_4(\lambda) = 0 + \frac{1}{32\pi^2}\left[\frac{6\,\lambda_4^2}{(1+\lambda_2)^3} - \frac{\lambda_6}{(1+\lambda_2)^2}\right]. \tag{3.81}$$

It is important to note that the beta-functions of a given coupling are related not only with the coupling in question but also with other couplings of the theory. Thus, even if we begin with a simplification by considering a certain coupling zero, the flow of the renormalization group may turn on such a coupling, showing us that the initial *ansatz* should include it for a more precise description of our theory.

Once these examples of beta-functions are obtained explicitly for the 4-dimensional case, we write here a general formula for obtaining beta-functions in the most general case

$$\beta_{2n}(\lambda) = [(d-2)n - d]\,\lambda_{2n} + \frac{k^{(d-2)n}}{(4\pi)^{d/2}\,\Gamma(d/2+1)}\tilde{\mathcal{A}}_{2n}(\lambda), \tag{3.82}$$

where we defined the modified coefficients

$$\tilde{\mathcal{A}}_{2n}(\lambda) = \left[\frac{\partial^{2n}}{\partial\varphi_0^{2n}}\left(1 + \sum_{m=2}^{\infty}\frac{k^{-(d-2)(m-1)}}{(2m-2)!}\lambda_{2m}\,\varphi_0^{(2m-2)}\right)^{-1}\right]_{\varphi_0=0}. \tag{3.83}$$

Next we will include the evolution of Z_k in the equations to observe the changes caused by taking into account the anomalous dimension.

3.9.4 The Evolution of Z_k

In the previous section, we calculated the flow equations ignoring the contribution of the evolution of Z_k, which we now consider. Again, we start with the truncation

$$\Gamma_k[\varphi] = \int_x \left(\frac{Z_k}{2} \partial_\mu \varphi \partial_\mu \varphi + V_k(\varphi) \right). \tag{3.84}$$

And as we have already obtained in the last section, we have the flow equation

$$\partial_t V_k = \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2+2)} \frac{(d+2-\eta_k)\, k^{d+2}}{k^2 + Z_k^{-1}\, V_k''}. \tag{3.85}$$

Performing the expansion of V_k in powers again, now taking into account the wavefunction renormalization

$$V_k(\varphi_0) = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!}\, Z_k^{m/2}\, \lambda_m\, \varphi_0^m. \tag{3.86}$$

Taking the derivative of this new expression, we get one more contribution, according to

$$\partial_t V_k = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!} \left(\partial_t \lambda_m - \Delta_m \lambda_m + \frac{m}{2} \eta_k \lambda_m \right) Z_k^{m/2}\, \varphi_0^m. \tag{3.87}$$

In four dimensions, we can expand the right hand side according to

$$\frac{1}{12} \frac{1}{(4\pi)^2} \frac{(6-\eta_k)\, k^6}{k^2 + Z_k^{-1}\, V_k''} = \sum_{m=0}^{\infty} \frac{k^{-\Delta_m}}{m!} \mathcal{A}_m\, Z_k^{m/2}\, \lambda_m\, \varphi_0^m, \tag{3.88}$$

where we defined the coefficients

$$\mathcal{A}_m = \left[\frac{\partial^m}{\partial \varphi_0^m} \left(\frac{1}{12} \frac{k^{\Delta_m}\, Z_k^{-m/2}}{16\pi^2} \frac{(6-\eta_k)\, k^6}{k^2 + Z_k^{-1}\, V_k''} \right) \right]_{\varphi_0 = 0}. \tag{3.89}$$

Comparing the coefficients, we obtain the relation

$$\beta_m(\lambda) = \partial_t \lambda_m(k) = \Delta_m\, \lambda_m + \frac{m}{2}\, \eta_k\, \lambda_m + \mathcal{A}_m. \tag{3.90}$$

Therefore, taking into account the anomalous dimension will change the beta-functions directly through the part proportional to η_k in the above equation and indirectly through the coefficients $\mathcal{A}_m$. Following the same steps of the calculation previously made, we need to get the $\mathcal{A}_m$ and then calculate the beta-function with the above expression. We get

$$\beta_m(\lambda) = \left(m - 4 + \eta_k\, \frac{m}{2} \right) \lambda_m + \frac{(6-\eta_k)}{192\pi^2} \left[\frac{\partial^m}{\partial \varphi_0^m} \left(\frac{k^m\, Z_k^{-m/2}}{1 + k^{-2}\, Z_k^{-1}\, V_k''(\varphi_0)} \right) \right]_{\varphi_0 = 0}. \tag{3.91}$$

And from there, we can extract the flow equations

$$\beta_0(\lambda) = -4\lambda_0 + \frac{(6-\eta_k)}{192\pi^2}\,\frac{1}{1+\lambda_2}; \tag{3.92}$$

$$\beta_2(\lambda) = (\eta_k - 2)\lambda_2 - \frac{(6-\eta_k)}{192\pi^2}\,\frac{\lambda_4}{(1+\lambda_2)^2}; \tag{3.93}$$

$$\beta_4(\lambda) = 2\eta_k\,\lambda_4 + \frac{(6-\eta_k)}{192\pi^2}\left(\frac{6\,\lambda_4^2}{(1+\lambda_2)^3} - \frac{\lambda_6}{(1+\lambda_2)^2}\right). \tag{3.94}$$

Therefore, the comment made on the relation between the different couplings of the theory still holds true. It should be noted that in the case where we do not consider the evolution of Z_k, the first portion of a generic beta-function is completely determined by the canonical dimension of the coupling in question $\beta_{2n} = [(d-2)n - d]\,\lambda_{2n} + (...)$. When we consider the anomalous dimension, these terms receive a contribution of η_k, as we can observe in the general expression

$$\beta_{2n}(\lambda) = [(d-2+\eta_k)n - d]\lambda_{2n} + \frac{(d+2-\eta_k)\,k^{(d-2)n}\,Z_k^{-n}}{2\,(4\pi)^{d/2}\,\Gamma(d/2+2)}\,\tilde{\mathcal{A}}_{2n}, \tag{3.95}$$

where we defined the modified coefficients

$$\tilde{\mathcal{A}}_{2n}(\lambda) = \left[\frac{\partial^{2n}}{\partial\varphi_0^{2n}}\left(1 + \sum_{m=2}^{\infty}\frac{k^{-(d-2)(m-1)}\,Z_k^{m-1}}{(2m-2)!}\,\lambda_{2m}\,\varphi_0^{2m-2}\right)^{-1}\right]_{\varphi_0=0}. \tag{3.96}$$

We will not compute explicitly the anomalous dimension here, but we remark that in this simple truncation example, the result for the anomalous dimension is zero.

Chapter 4

Functional Renormalization Group - Scalar Quantum Electrodynamics

4.1 Introduction

Gauge theories occupy a central place among the Quantum Field Theories. In fact, they are the building blocks of the Standard Model, the theory that describes the fundamental interactions between the elementary particles with extraordinary precision between theory and experiment. By definition, a gauge theory is a field theory that has a Lagrangian invariant by a Lie group of local transformations of the fields. The gauge symmetry is, at heart, a redundancy in our description of the world[1], but nevertheless, it is the sort of thing that brings joy to the heart of an elementary particle theorist.[2]

The first field theory to present a gauge symmetry was Maxwell's Electromagnetism [33], where we notice a redundancy in the definition of the electromagnetic fields. The importance of this, however, was only clear in the last century, when the gauge symmetry became one of the pillars of the construction of Particle Physics. In fact, in addition to the extremely successful Quantum Electrodynamics of Feynman, Schwinger, and Tomonaga [29], the generalization of the gauge principle to non-abelian symmetry groups developed by Yang and Mills [34] was used for the construction of Quantum Chromodynamics, the Quantum Field Theory used to describe the strong interactions between quarks and gluons, according to the Gell-Mann and Zweig Quarks Model [35], and which naturally describes the important property of asymptotic freedom present in this interaction. Gauge symmetry is also at the heart of Glashow-Weinberg-Salam Electroweak unification, which, in addition to adequately describing the weak interaction, shows a common origin to electromagnetic and weak interactions. To be possible that two interactions with such distinct characteristics can be described by the same model, it is necessary to break this symmetry in some scale. The spontaneous breaking of a gauge symmetry is described

[1] "Gauge Theories" - *David Tong*
[2] "The Making of the Standard Model"- *Steven Weinberg*

by the Higgs mechanism, which shows a loophole in Goldstone theorem [36] and gives a mechanism of mass generation for the mediators of the weak interaction, absorbing what would be Goldstone bosons, without losing the renormalizability of the theory as would occur if the mass of the gauge bosons had been put by hand.

Despite the extraordinary success achieved by gauge theories using perturbation theory, we still do not fully understand the non-perturbative sector of the theory, where strong couplings do not allow the expansions usually made. So, to get a more general understanding of these theories, we need to develop tools that access information beyond the perturbative regime. The functional renormalization group, as we have already discussed, is a path. This approach allows us to obtain non-perturbative information through the flow equations for the coupling constants present in the theory. However, implementing the procedure described in the previous chapter in a gauge theory is a slightly more complicated task, and to do so, it will be convenient to use the background field method [38] to deal with the gauge symmetry and to get the flow equations efficiently.

In this chapter, we will study the scalar Quantum Electrodynamics from the point of view of the functional renormalization group. In this way, we can study the question of spontaneous symmetry breaking in the Coleman-Weinberg model discussed at the beginning of this thesis, with the tools developed in the previous chapter to understand what happens when we take into account the non-perturbative sector of the theory. An excellent first study in this direction was made in [37]. Here we follow their work to get the flow equations, and extend slightly their results, including the anomalous dimensions corrections and using a more general potential.

4.2 Development

Let be a theory of a complex scalar field ϕ minimally coupled to a gauge field A_μ. Consider the following truncation for the effective average action (from now on we restrict ourselves to four space-time dimensions with Euclidean signature)

$$\Gamma_k = \int_x \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* (D_\mu \phi) + U_k(\phi^* \phi) + \Gamma_{gf}, \tag{4.1}$$

where $U_k(\phi^*\phi)$ is a generic potential, and the last term is a gauge-fixing term that will be written later. Our convention for the covariant derivative is

$$D_\mu = \partial_\mu + ieA_\mu \phi. \tag{4.2}$$

Inspired by the background field method [38], consider a splitting of the gauge field in a classical background field $\bar{A}_\mu$ and quantum fluctuations a_μ according to

$$A_\mu = \bar{A}_\mu + a_\mu. \tag{4.3}$$

Thus, we will have

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + f_{\mu\nu}, \tag{4.4}$$

and we will also have the splitting in the covariant derivative

$$D_\mu\phi = (\partial_\mu\phi + ie\bar{A}_\mu\phi) + iea_\mu\phi = \bar{D}_\mu\phi + iea_\mu\phi, \tag{4.5}$$

where we defined the background covariant derivative

$$\bar{D}_\mu\phi = \partial_\mu\phi + ie\bar{A}_\mu\phi. \tag{4.6}$$

Since we want to quantize only the fluctuations, we will follow the prescription of the background field method and adopt the gauge fixing only in this sector, writing

$$\Gamma_{gf} = \frac{1}{2\xi}\int_x (\partial_\mu a_\mu)^2. \tag{4.7}$$

Therefore, by rewriting the action using this decomposition, we obtain

$$\begin{aligned}\Gamma_k = \int_x \frac{1}{4}\bar{F}^2_{\mu\nu} + \frac{1}{2}\bar{F}_{\mu\nu}f_{\mu\nu} + \frac{1}{4}f^2_{\mu\nu} + \bar{D}_\mu\phi^*\bar{D}_\mu\phi +& \\ + iea_\mu(\phi\,\bar{D}_\mu\phi^* - \phi^*\,\bar{D}_\mu\phi) + e^2a^2_\mu\,\phi^*\phi + U_k(\phi^*\phi) + \Gamma_{gf}.&\end{aligned} \tag{4.8}$$

Let us now consider the wavefunction renormalization

$$\begin{aligned}\bar{A}_\mu \to Z_A^{1/2}\bar{A}_\mu\,; \qquad \phi \to Z_\phi^{1/2}\phi\,;\\ a_\mu \to Z_a^{1/2}a_\mu\,; \qquad \phi^* \to Z_\phi^{1/2}\phi^*.\end{aligned} \tag{4.9}$$

We will also impose the condition

$$U_k(Z_\phi\,\phi^*\phi) \to U_k(\phi^*\phi), \tag{4.10}$$

absorbing the scale dependence in the functional form of the potential, that is, in the coupling constants inside it.

Following this approach, the background field has no dynamics, so we need to ensure that the background covariant derivative does not renormalize. That is,

$$\partial_t \bar{D}_\mu = 0 \quad \to \quad \partial_t(\partial_\mu + ieZ_A^{1/2}\bar{A}_\mu) = 0 \quad \to \quad \partial_t(Z_A^{1/2}e) = 0. \tag{4.11}$$

Therefore, without loss of generality, we have

$$(Z_A^{1/2}e) = 1 \quad \to \quad Z_A^{1/2} = e^{-1}. \tag{4.12}$$

The non-renormalization condition in the background derivative (4.11) is a fundamental condition, since it creates a relation between the gauge coupling and the wavefunction renormalization (4.12), whereby we will calculate the flow equation for the coupling

$$Z_A^{1/2} = e^{-1} \quad \to \quad \beta_e = \partial_t e = -\frac{1}{2}e^3\partial_t Z_A. \tag{4.13}$$

Taking into account the wavefunction renormalization, we obtain

$$\begin{aligned}\Gamma_k = \int_x \frac{Z_A}{4}\,\bar{F}_{\mu\nu}^2 + \frac{1}{2}Z_A^{1/2}Z_a^{1/2}\,\bar{F}_{\mu\nu}f_{\mu\nu} + \frac{Z_a}{4}f_{\mu\nu}^2 + \frac{Z_a}{2\xi}\,(\partial_\mu a_\mu)^2 + Z_\phi\,\bar{D}_\mu\phi^*\bar{D}_\mu\phi + \\ + ieZ_a^{1/2}Z_\phi\,a_\mu(\phi\bar{D}_\mu\phi^* - \phi^*\bar{D}_\mu\phi) + e^2 Z_a\,Z_\phi\,a_\mu^2\phi^*\phi + U_k(\phi^*\phi).\end{aligned} \tag{4.14}$$

In what follows, we need to look at the quadratic sector in the fluctuations. For this, we will consider an expansion of the scalar field around a constant configuration according to $\phi \to \phi_0 + \phi$. In the gauge sector, we get

$$\Gamma_{AA}^{quad} = \frac{Z_a}{4}f_{\mu\nu}^2 + \frac{Z_a}{2\xi}\,(\partial_\mu a_\mu)^2 = \frac{Z_a}{2}\,a_\mu\left(\Delta\,\theta_{\mu\nu} + \frac{\Delta}{\xi}\,\omega_{\mu\nu}\right)a_\nu, \tag{4.15}$$

where we defined ($\Delta = -\partial^2$) and used the transversal and longitudinal projectors, given by

$$\theta_{\mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2} \quad ; \quad \omega_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\partial^2}. \tag{4.16}$$

In the sector involving the scalar field, restricting us to the quadratic part in the fluctuations, we obtain

$$\begin{aligned}\Gamma_{\phi\phi}^{quad} + \Gamma_{A\phi}^{quad} = +\; Z_\phi\,\bar{D}_\mu\phi^*\bar{D}_\mu\phi \;+\; U^{(2)} \;+\; e^2 Z_a\,a_\mu^2\,\phi_0^*\phi_0 + \\ +\; ieZ_a^{1/2}Z_\phi\,a_\mu\left(\phi\,\bar{D}_\mu\phi_0^* + \phi_0\,\bar{D}_\mu\phi^* - \phi_0^*\,\bar{D}_\mu\phi - \phi^*\,\bar{D}_\mu\phi_0\right).\end{aligned} \tag{4.17}$$

To obtain in the above expression the contribution $U^{(2)}$, that is, the terms from the potential, we Taylor expand and keep only those terms which are relevant for our computation

(here we use $\rho = \phi^*\phi$)

$$\phi^*\phi \to (\phi_0^* + \phi^*)(\phi_0 + \phi) = \phi_0^*\phi_0 + (\phi_0^*\phi + \phi_0\phi^* + \phi^*\phi) = \rho_0 + \delta\rho, \tag{4.18}$$

$$U(\rho) = U(\rho_0) + U'(\rho_0)\,\delta\rho + \frac{1}{2}U''(\rho_0)\,(\delta\rho)^2. \tag{4.19}$$

Therefore, using the above expansion for the potential in terms of the fields defined around the constant configuration ϕ_0, with the shorthand notation $U(\rho_0) = U_0$ and retaining only the quadratic contributions in the fluctuations, we obtain

$$U^{(2)} = U_0'\,\phi^*\phi + \frac{1}{2}\,U_0''\,(\phi\,\phi\,\phi_0^*\,\phi_0^* + \phi^*\phi^*\phi_0\,\phi_0 + 2\,\phi^*\phi\,\phi_0^*\,\phi_0)\,. \tag{4.20}$$

With these partial results at hand, let us now combine all quadratic contributions into one expression. To simplify, let us restrict ourselves to a scheme in which we do not consider effects of scalar field fluctuations with respect to the background derivative, that is, let us adopt the simplification $\left(\bar{D}_\mu\phi_0 = \bar{D}_\mu\phi_0^* = 0\right)$. Note that we have

$$\begin{aligned}\Gamma_k^{(2)} = &\int_x \frac{Z_a}{2}\,a_\mu\left(\Delta\,\theta_{\mu\nu} + \frac{\Delta}{\xi}\,\omega_{\mu\nu}\right)a_\nu + e^2 Z_a\,Z_\phi\,\rho_0\,a_\mu\,(\theta_{\mu\nu} + \omega_{\mu\nu})\,a_\nu\\ &+\ ieZ_a^{1/2}Z_\phi\,a_\mu\left(\phi_0\,\bar{D}_\mu\phi^* - \phi_0^*\,\bar{D}_\mu\phi\right) + \\ &+\ Z_\phi\,\phi\left(-\bar{D}_\mu\bar{D}^\mu\right)\phi^*\ +\ \phi\,(U_0')\,\phi^* + \phi\,(U_0''\rho_0)\,\phi^*\\ &+ \frac{1}{2}\,U_0''\,(\phi\,\phi\,\phi_0^*\,\phi_0^* + \phi^*\phi^*\phi_0\,\phi_0)\,.\end{aligned} \tag{4.21}$$

In this way, we can easily set up the Hessian

$$\Gamma_k^{(2)} = \begin{pmatrix}\Gamma_{\mu\nu} & \Gamma_{AS}\,\bar{D}_\mu\\ \Gamma_{SA}\,\partial_\nu & \Gamma_{SS}\end{pmatrix}, \tag{4.22}$$

where we have the following definitions

$$\Gamma_{\mu\nu} = Z_a\left(\Delta + 2Z_\phi e^2\rho_0\right)\theta_{\mu\nu} + \frac{Z_a}{\xi}\left(\Delta + 2\xi Z_\phi e^2\rho_0\right)\omega_{\mu\nu}; \tag{4.23}$$

$$\Gamma_{AS} = \left(-ieZ_\phi Z_a^{1/2}\phi_0^*\quad;\quad ieZ_\phi Z_a^{1/2}\phi_0\right); \tag{4.24}$$

$$\Gamma_{SA} = \left(ieZ_\phi Z_a^{1/2}\phi_0^*\quad;\quad -ieZ_\phi Z_a^{1/2}\phi_0\right)^t; \tag{4.25}$$

$$\Gamma_{SS} = \begin{pmatrix}\phi_0^*\,\phi_0^*\,U_0'' & -Z_\phi\,\bar{D}^2 + U_0' + \rho_0\,U_0''\\ -Z_\phi\,\bar{D}^2 + U_0' + \rho_0\,U_0'' & \phi_0\,\phi_0\,U_0''.\end{pmatrix} \tag{4.26}$$

Let us now use the Wetterich equation to obtain the flow equations for the gauge coupling and for the effective average potential. We have already found the Hessian, so it is enough to propose a cutoff function $\mathcal{R}_k$ to proceed. Note that with this Hessian structure, the cutoff must have a matrix structure in order to achieve its goal.

Take a generic prescription for the cutoff operator: $\mathcal{R}_k(z) = \Gamma_k^{(2)}(P_k(z)) - \Gamma_k^{(2)}(z)$, where we defined as before $P_k(z) = z + r_k(z)$, and used as a profile r_k the Litim cutoff, given by

$$r_k(z) = (k^2 - z)\,\theta(k^2 - z). \tag{4.27}$$

This profile will cutoff the limit of the integral, helping a lot in the computations. Note that we ignored the part with the delta, since being inside an integral in z, it cancels out due to the factor $(k^2 - z)$. In addition, $\partial_t P_k = \partial_t r_k = 2k^2\theta(k^2 - z)$, so if a $\theta(k^2 - z)$ has already been used, transforms terms like $\partial_t P_k$ in $2k^2$, which greatly simplifies the integrals that we have to calculate.

The Wetterich equation here in this model, even with the usual simplifications becomes difficult to work because it presents a complicated matrix structure, with two types of operators to work with, $\bar{D}_\mu$ and ∂_μ, hindering the implementation of the cutoff and making it hard to invert the regularized Hessian and take the functional trace of the operator. To obtain the information that we want in a simpler way, the strategy we will adopt here is to consider the Wetterich equation in two different regimes, to extract information from disjoint sectors, and then put them together.

- **Regime A** : $(\boldsymbol{\phi_0 = 0})$

  In this regime, considering scalar field fluctuations around the trivial field configuration, we obtain a rather simplified situation, where only few terms survive in $\Gamma_k^{(2)}$, so it is easy to calculate the operator trace that appears on the right side of Wetterich equation. There are still two types of operator in the game, however, because of the simpler Hessian structure in this scheme, we can conveniently regulate it and calculate the functional trace with the help of the Heat Kernel expansion method [39].

  From here, we will extract information about the gauge coupling to calculate its beta-function.

- **Regime B**: $\left(\bar{\mathbf{A}}_\mu = \mathbf{0}\right)$

  By taking the background field equal to zero, we will have the simplification $\bar{D}_\mu \to \partial_\mu$, which gives us only one type of operator to work with, the Laplacian $\Delta = -\partial^2$, and facilitates the implementation of the cutoff. We still have a complicated matrix structure, which allows us to use a block structure to start the account by hand, but it will require the use of a software to obtain the expressions of the products and the inverse of the matrices.

  From here, we will obtain the flow equation for the effective potential.

With these expressions at hand, we can recover in a certain limit the expression for the effective potential found in the original paper of S.Coleman and E.Weinberg and the usual expression for the beta-function of the gauge coupling in scalar Quantum Electrodynamics. The equations that we are going to obtain are, therefore, an extension of Coleman-Weinberg's results, with the correction of the anomalous dimension and with validity in a non-perturbative regime, different from the results of those authors, that was restricted to the perturbative regime of the theory. With these equations, we can investigate the occurrence or not of spontaneous symmetry breaking, understanding the true nature of the quantum vacuum in scalar Quantum Electrodynamics.

4.3 Regime A : $(\phi_0 = 0)$

4.3.1 Setting the Stage

In this regime, we write the Wetterich equation as

$$\partial_t \Gamma_k \left[\varphi = \phi_0 = 0\,; \bar{A}\right] = \frac{1}{2} Tr \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right]_{\varphi=0\,;\,\phi_0=0} . \tag{4.28}$$

Taking the condition $\phi_0 = 0$ in the expression of the Hessian $\Gamma_k^{(2)}$, we get

$$\Gamma_k^{(2)} = \begin{pmatrix} Z_a\, \Delta \left(\theta_{\mu\nu} + \frac{1}{\xi}\omega_{\mu\nu}\right) & 0 & 0 \\ 0 & 0 & -Z_\phi\, \bar{D}^2 + U_0' \\ 0 & -Z_\phi\, \bar{D}^2 + U_0' & 0 \end{pmatrix} . \tag{4.29}$$

This way, following the prescription $\mathcal{R}_k(z) = \Gamma_k^{(2)}(P_k(z)) - \Gamma_k^{(2)}(z)$ to implement the cutoff operator, it is easy to obtain

$$\mathcal{R}_k = \begin{pmatrix} Z_a\, r_k \left(\theta_{\mu\nu} + \frac{1}{\xi}\omega_{\mu\nu}\right) & 0 & 0 \\ 0 & 0 & Z_\phi\, \tilde{r}_k \\ 0 & Z_\phi\, \tilde{r}_k & 0 \end{pmatrix} , \tag{4.30}$$

here, the convention is: when the cutoff dependence is on the usual Laplacian, we write $r_k = r_k(\Delta) = r_k(-\partial^2)$, and when the dependence is on the gauge covariant Laplacian, we write $\tilde{r}_k = \tilde{r}_k(-\bar{D}^2)$.

Thus, recalling the definition of $P_k(z) = z + r_k(z)$ (and similarly for the $\tilde{P}_k$), it is easy to obtain the regularized Hessian (basically, what occurs is the substitution $z \to P_k(z)$)

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k\right) = \begin{pmatrix} Z_a\, P_k \left(\theta_{\mu\nu} + \frac{1}{\xi}\omega_{\mu\nu}\right) & 0 & 0 \\ 0 & 0 & Z_\phi\, \tilde{P}_k + U_0' \\ 0 & Z_\phi\, \tilde{P}_k + U_0' & 0 \end{pmatrix}. \tag{4.31}$$

Now, with such a simple structure it is trivial to invert this matrix to get the regularized propagator G_k, as below

$$G_k = \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} = \begin{pmatrix} \frac{1}{Z_a\, P_k}\left(\theta_{\mu\nu} + \xi\omega_{\mu\nu}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{Z_\phi\, \tilde{P}_k + U_0'} \\ 0 & \frac{1}{Z_\phi\, \tilde{P}_k + U_0'} & 0 \end{pmatrix}. \tag{4.32}$$

Deriving the cutoff operator matrix, we obtain

$$\partial_t \mathcal{R}_k = \begin{pmatrix} (\partial_t Z_a\, r_k + Z_a\, \partial_t r_k)(\theta_{\mu\nu} + \frac{1}{\xi}\omega_{\mu\nu}) & 0 & 0 \\ 0 & 0 & \partial_t Z_\phi\, \tilde{r}_k + Z_\phi\, \partial_t \tilde{r}_k \\ 0 & \partial_t Z_\phi\, \tilde{r}_k + Z_\phi\, \partial_t \tilde{r}_k & 0 \end{pmatrix}. \tag{4.33}$$

Now, we have to multiply the expressions above to get the functional $W = [G_k\, \partial_t \mathcal{R}_k]$, from which we will compute the trace. Then,

$$W = \begin{pmatrix} \frac{Z_a'\, r_k + Z_a P_k'}{Z_a\, P_k}\left(\theta_{\mu\nu} + \omega_{\mu\nu}\right) & 0 & 0 \\ 0 & \frac{Z_\phi' \tilde{r}_k + Z_\phi \tilde{P}_k'}{Z_\phi \tilde{P}_k + U_0'} & 0 \\ 0 & 0 & \frac{Z_\phi' \tilde{r}_k + Z_\phi \tilde{P}_k'}{Z_\phi \tilde{P}_k + U_0'} \end{pmatrix}. \tag{4.34}$$

In fact, it is from this object that we will take the trace. Recalling first that in 4 dimensions the trace of the projectors is given by $Tr(\theta_{\mu\nu}) = 3$ and also $Tr(\omega_{\mu\nu}) = 1$, taking the trace of the operators that appear in the diagonal, we have

$$Tr[W] = 4\, Tr\left[\frac{Z_a'\, r_k + Z_a P_k'}{Z_a\, P_k}\right] + 2\, Tr\left[\frac{Z_\phi' \tilde{r}_k + Z_\phi \tilde{P}_k'}{Z_\phi \tilde{P}_k + U_0'}\right]. \tag{4.35}$$

Defining the operators W_a and W_s using the anomalous dimensions $\left(\eta_\Phi = -\frac{1}{Z_\Phi}\partial_t Z_\Phi\right)$

$$W_a = \frac{P_k' - \eta_a r_k}{P_k}; \tag{4.36}$$

$$W_s = \frac{\tilde{P}_k' - \eta_\phi \tilde{r}_k}{\tilde{P}_k + Z_\phi^{-1} U_0'}. \tag{4.37}$$

Therefore, we obtain the equation

$$\partial_t \Gamma_k = \frac{1}{2} Tr\,[G_k\,\partial_t \mathcal{R}_k] = \frac{1}{2}\,Tr\,[W] = 2\,Tr\,[W_a] + Tr\,[W_s]. \tag{4.38}$$

4.3.2 Computing Traces

To calculate these functional traces, we will use the Heat Kernel expansion [39].[3] With this method, in a space-time with four dimensions and without curvature, we can carry out the following expansion for the trace

$$Tr\,W[\Delta] = \frac{1}{(4\pi)^2}\,[\,Q_2[W]B_0[\Delta] + Q_1[W]B_2[\Delta] + Q_0[W]B_4[\Delta]\,], \tag{4.39}$$

where we have $B_n[\Delta] = \int d^4x\,Tr\,b_n[\Delta]$, and the traces of the coefficients $b_n[\Delta]$ can be found in the literature for the most usual operators [39],[30]. For our needs, we need just look at the coefficients for the operators $\Delta = -\partial^2$ and $\Delta = -\bar{D}^2$. In the first case the expansion is trivial, that is, we have $b_0[-\partial^2] = \mathbb{I}$, and the other coefficients are zero. In the case of the gauge covariant Laplacian, we have

$$Tr\,[b_0] = \mathbb{I}; \qquad Tr\,[b_2] = 0; \qquad Tr\,[b_4] = -\frac{1}{12}\bar{F}^2_{\mu\nu}. \tag{4.40}$$

In our model, the only Q-functionals that matter are those with integer indices, so we will use the definition

$$Q_n[W] = \frac{1}{\Gamma(n)} \int dz\, z^{n-1}\, W(z). \tag{4.41}$$

Computing $Tr\,[W_a]$

To perform this calculation, let us remember that the dependence of the functional W_a is on the operator $\Delta = -\partial^2$. Therefore, its Heat Kernel expansion is trivial and we only have to calculate the Q_2, because the B_0 only gives a volume factor $\int d^4x$ and the other coefficients are zero (we will see in the other section a more direct way of calculating the Q-functionals in this simpler case). Therefore,

$$\begin{aligned} Tr\,[W_a(\Delta)] =& Tr\left[\frac{P_k' - \eta_a r_k}{P_k}\right] = \frac{1}{16\pi^2}(Q_2[W_a]\,B_0) \\ &= \frac{1}{16\pi^2}\frac{1}{\Gamma(2)}\int dz\, z^{2-1}\left[\frac{P_k' - \eta_a r_k}{P_k}\right]\int d^4x. \end{aligned} \tag{4.42}$$

[3] A good reference for this technique is also the Appendix A of Codello's thesis, in the reference [11]

Taking the Litim cutoff $r_k(z) = (k^2 - z)\theta(k^2 - z)$, with the hints already given before, we have

$$\begin{aligned} Tr\,[W_a] &= \frac{1}{16\pi^2}\frac{1}{\Gamma(2)}\int dz\, z\left[\frac{2k^2 - \eta_a(k^2 - z)}{k^2}\theta(k^2 - z)\right]\int d^4x \\ &= \frac{1}{16\pi^2}\int_0^{k^2} dz\, z\left[2 - \eta_a + \frac{\eta_a}{k^2}z\right]\int d^4x \\ &= \frac{1}{16\pi^2}\int d^4x\left(k^4 - \frac{\eta_a}{2}k^4 + \frac{\eta_a}{3}k^4\right). \end{aligned} \tag{4.43}$$

Therefore, we obtain for this first functional trace, the following result

$$Tr\,[W_a] = \frac{(6 - \eta_a)}{96\pi^2}k^4\int d^4x. \tag{4.44}$$

Computing $Tr\,[W_s]$

Now since our functional depends on the gauge covariant Laplacian $\Delta = -\bar{D}^2$, we have to take into account the coefficients given above for the traces, and therefore we have to compute

$$\begin{aligned} Tr\,[W_s] &= \frac{1}{16\pi^2}[Q_2[W_s]B_0[\bar{D}^2] + Q_1[W_s]B_2[\bar{D}^2] + Q_0[W_s]B_4[\bar{D}^2]] \\ &= \frac{1}{16\pi^2}\left[Q_2[W_s]\int d^4x + Q_0[W_s]\left(\frac{-1}{12}\right)\int d^4x\, F^2_{\mu\nu}\right]. \end{aligned} \tag{4.45}$$

Let us now calculate the Q-functionals to replace in the above expression

$$\begin{aligned} Q_n[W_s] &= \frac{1}{\Gamma(n)}\int dz\, z^{n-1}\left(\frac{\tilde{P}'_k - \eta_\phi\tilde{r}_k}{\tilde{P}_k + Z_\phi^{-1}U'_0}\right) \\ &= \frac{1}{\Gamma(n)}\int dz\, z^{n-1}\left(\frac{2k^2 - \eta_\phi(k^2 - z)}{k^2 + Z_\phi^{-1}U'_0}\right)\theta(k^2 - z) \\ &= \frac{1}{\Gamma(n)}\frac{1}{(k^2 + Z_\phi^{-1}U'_0)}\int_0^{k^2} dz\, z^{n-1}\,\left[(2 - \eta_\phi)k^2 + \eta_\phi z\right] \\ &= \frac{1}{\Gamma(n)}\frac{1}{(k^2 + Z_\phi^{-1}U'_0)}\left[(2 - \eta_\phi)\frac{k^{2n+2}}{n} + \eta_\phi\frac{k^{2n+2}}{n+1}\right]. \end{aligned} \tag{4.46}$$

Then, we obtain for the Q-functionals

$$Q_n[W_s] = \frac{1}{(k^2 + Z_\phi^{-1}U'_0)}\left[\frac{(2 - \eta_\phi)}{\Gamma(n+1)}k^{2n+2} + \frac{n\,\eta_\phi}{\Gamma(n+2)}k^{2n+2}\right]. \tag{4.47}$$

From the above expression we get immediately

$$Q_0[W_s] = \frac{(2-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')}k^2 \quad ; \quad Q_2[W_s] = \frac{(6-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')}\frac{k^6}{6}. \tag{4.48}$$

Now, substituting these values in the expression of the trace, we obtain finally the result

$$Tr\,[W_s] = \int d^4x \left[\left(\frac{k^6}{96\pi^2}\frac{(6-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')} \right) - \left(\frac{k^2}{192\pi^2}\frac{(2-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')} \right) \bar{F}_{\mu\nu}^2 \right]. \tag{4.49}$$

4.3.3 The Flow Equation

So, putting it all together, we obtain the equation

$$\partial_t \Gamma_k \Big|_{\phi_0 = 0} = \int d^4x \left[\left(\frac{k^4}{48\pi^2}(6-\eta_a) + \frac{k^6}{96\pi^2}\frac{(6-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')} \right) - \left(\frac{k^2}{192\pi^2}\frac{(2-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')} \right) \bar{F}_{\mu\nu}^2 \right]. \tag{4.50}$$

Now taking Γ_k in our truncation and imposing the condition (fluctuations = 0) and also ($\phi_0 = 0$), we get

$$\Gamma_k = \int_x \frac{1}{4} Z_A \bar{F}_{\mu\nu}^2 + U_k(0). \tag{4.51}$$

Therefore,

$$\partial_t \Gamma_k = \int_x \frac{1}{4}\, \partial_t Z_A\, \bar{F}_{\mu\nu}^2 + \partial_t U_k(0). \tag{4.52}$$

Comparing both sides of the Wetterich equation, we obtain the equation

$$\frac{1}{4}\,\partial_t Z_A = \frac{-k^2}{192\pi^2}\frac{(2-\eta_\phi)}{(k^2 + Z_\phi^{-1}U_0')}. \tag{4.53}$$

Now, we have already stated the non-renormalization condition for the background covariant derivative (4.11), which makes the constraint (4.12),

$$Z_A^{1/2} = e^{-1} \quad \rightarrow \quad \partial_t Z_A = -2e^{-3}\,\partial_t e. \tag{4.54}$$

So, replacing this expression in the (4.53) we obtain the flow equation for the gauge coupling, expressed by its beta-function

$$\beta_e = \partial_t e = \frac{e^3\,(2-\eta_\phi)}{96\pi^2}\frac{k^2}{(k^2 + Z_\phi^{-1}U_0')}. \tag{4.55}$$

It should be noted that this result is consistent with the known result obtained for scalar

Quantum Electrodynamics in the perturbative approach of Quantum Field Theory, since by turning off the anomalous dimension contribution and ignoring the potential contribution, we obtain exactly

$$\beta_e = \frac{e^3}{48\pi^2}. \tag{4.56}$$

4.4 Regime B: $\left(\bar{A}_\mu = 0\right)$

4.4.1 Setting the Stage

In this regime, we write the Wetterich equation according to

$$\partial_t \Gamma_k \left[\varphi = 0\,; \bar{A} = 0\right] = \frac{1}{2} Tr \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right]_{\varphi = 0\,;\, \bar{A}_\mu = 0}. \tag{4.57}$$

Taking the condition $\bar{A}_\mu = 0$ in the expression of the Hessian $\Gamma_k^{(2)}$, we obtain

$$\Gamma_k^{(2)} = \begin{pmatrix} \Gamma_{\mu\nu} & \Gamma_{AS}\, \partial_\mu \\ \Gamma_{SA}\, \partial_\nu & \Gamma_{SS} \end{pmatrix}, \tag{4.58}$$

where we have the same initial structure, but with $\bar{D}_\mu \to \partial_\mu$. Since now we only have one type of operator to regularize (Δ), propose the regulator so that it just turns in P_k, as in $(\Delta \to P_k(\Delta) = \Delta + r_k(\Delta))$, that is

$$\mathcal{R}_k = \begin{pmatrix} Z_a\, r_k \left(\theta_{\mu\nu} + \frac{1}{\xi}\omega_{\mu\nu}\right) & 0 & 0 \\ 0 & 0 & Z_\phi\, r_k \\ 0 & Z_\phi\, r_k & 0 \end{pmatrix}. \tag{4.59}$$

Then, we have to invert the regularized Hessian

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k\right) = \begin{pmatrix} \Gamma^\theta_{AA}\theta_{\mu\nu} + \Gamma^\omega_{AA}\omega_{\mu\nu} & \Gamma_{AS}\, \partial_\mu \\ \Gamma_{SA}\, \partial_\nu & \Gamma_{SS} \end{pmatrix}, \tag{4.60}$$

where we have the same structure as before, but with $\Delta \to P_k(\Delta)$, as

$$\Gamma_{\mu\nu} = Z_a \left(P_k + 2 Z_\phi e^2 \rho_0\right) \theta_{\mu\nu} + \frac{Z_a}{\xi} \left(P_k + 2\xi Z_\phi e^2 \rho_0\right) \omega_{\mu\nu}; \tag{4.61}$$

$$\Gamma_{AS} = \left(-ie Z_\phi Z_a^{1/2} \phi_0^* \quad ; \quad ie Z_\phi Z_a^{1/2} \phi_0\right); \tag{4.62}$$

$$\Gamma_{SA} = \left(ie Z_\phi Z_a^{1/2} \phi_0^* \quad ; \quad -ie Z_\phi Z_a^{1/2} \phi_0\right)^t; \tag{4.63}$$

$$\Gamma_{SS} = \begin{pmatrix} \phi_0^*\, \phi_0^*\, U_0'' & -Z_\phi\, P_k + U_0' + \rho_0\, U_0'' \\ -Z_\phi\, P_k + U_0' + \rho_0\, U_0'' & \phi_0\, \phi_0\, U_0'' \end{pmatrix}. \tag{4.64}$$

That is, we need to obtain a formal matrix such that we have

$$\begin{pmatrix} A^\theta \theta_{\mu\alpha} + A^\omega \omega_{\mu\alpha} & B\partial_\mu \\ C\partial_\alpha & D \end{pmatrix} \begin{pmatrix} \Gamma^\theta_{AA}\theta_{\alpha\nu} + \Gamma^\omega_{AA}\omega_{\alpha\nu} & \Gamma_{AS}\,\partial_\alpha \\ \Gamma_{SA}\,\partial_\nu & \Gamma_{SS} \end{pmatrix} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \tag{4.65}$$

From here, using simple properties of the projectors, we have to solve the system

$$A^\theta\,\Gamma^\theta_{AA}\,\theta_{\mu\nu} + (A^\omega\,\Gamma^\omega_{AA} + B\,\Gamma_{SA}\,\partial^2)\omega_{\mu\nu} = \delta_{\mu\nu}; \tag{4.66}$$

$$(A^\omega\,\Gamma_{AS} + B\,\Gamma_{SS})\,\partial_\mu = 0; \tag{4.67}$$

$$(C\,\Gamma^\omega_{AA} + D\,\Gamma_{SA})\,\partial_\nu = 0; \tag{4.68}$$

$$C\,\Gamma_{AS}\,\partial^2 + D\,\Gamma_{SS} = \mathbb{I}. \tag{4.69}$$

Using the homogeneous equations to substitute, it is easy to obtain the entries of the inverse of the regularized Hessian

$$A^\theta = (\Gamma^\theta_{AA})^{-1}; \tag{4.70}$$

$$A^\omega = (\Gamma^\omega_{AA} + \Gamma_{AS}\,(\Gamma_{SS})^{-1}\,\Gamma_{SA}\,\Delta)^{-1}; \tag{4.71}$$

$$B = -(\Gamma^\omega_{AA} + \Gamma_{AS}\,(\Gamma_{SS})^{-1}\,\Gamma_{SA}\,\Delta)^{-1}\,\Gamma_{AS}\,(\Gamma_{SS})^{-1}; \tag{4.72}$$

$$C = -(\Gamma_{SS} + \Gamma_{SA}\,(\Gamma^\omega_{AA})^{-1}\,\Gamma_{AS}\,\Delta)^{-1}\,\Gamma_{SA}\,(\Gamma^\omega_{AA})^{-1}; \tag{4.73}$$

$$D = (\Gamma_{SS} + \Gamma_{SA}\,(\Gamma^\omega_{AA})^{-1}\,\Gamma_{AS}\,\Delta)^{-1}. \tag{4.74}$$

Then, the inverse of the regularized Hessian $\left(\Gamma^{(2)}_k + \mathcal{R}_k\right)^{-1}$ is of the form

$$\begin{pmatrix} (\Gamma^\theta_{AA})^{-1}\theta_{\mu\nu} + (\Gamma^\omega_{AA} + \Gamma_{AS}\,(\Gamma_{SS})^{-1}\,\Gamma_{SA}\,\Delta)^{-1}\omega_{\mu\nu} & (...) \\ (...) & (\Gamma_{SS} + \Gamma_{SA}\,(\Gamma^\omega_{AA})^{-1}\,\Gamma_{AS}\,\Delta)^{-1} \end{pmatrix}.$$

From this expression we can obtain the regularized propagators, simply by carrying out the above products. As an example, we mention here the propagator of the gauge field

$$G^A_k = \frac{1}{(Z_A\,(P_k + 2Z_\phi \rho e^2))}\,\theta_{\mu\nu} + \frac{1}{\left(Z_A\left[\frac{P_k}{\xi} + 2Z_\phi\rho\left(e^2 - \frac{\Delta e^2}{P_k + U'}\right)\right]\right)}\,\omega_{\mu\nu}. \tag{4.75}$$

From the prescription used for $\mathcal{R}_k$, we get immediately

$$\partial_t \mathcal{R}_k = \begin{pmatrix} Z_A\,\partial_t P_k + r_k\,\partial_t Z_A & 0 & 0 \\ 0 & 0 & Z_\phi\,\partial_t P_k + r_k\,\partial_t Z_\phi \\ 0 & Z_\phi\,\partial_t P_k + r_k\,\partial_t Z_\phi & 0 \end{pmatrix}. \tag{4.76}$$

4.4.2 Computing Traces

To move on, we must take the product of the regularized Hessian with the above matrix obtaining the matrix W, and then take the functional trace as prescribed by the Wetterich equation. Then, to simplify the calculations, soon after we make the product of these two matrices we will restrict ourselves to the Landau gauge ($\xi \to 0$). Therefore, remembering that $\eta_\Phi = -Z_\Phi^{-1}\,\partial_t Z_\Phi$, we have

$$W^{\theta}_{AA} = \frac{\partial_t P_k - r_k\,\eta_A}{P_k + 2\rho\, e^2\, Z_\phi}; \tag{4.77}$$

$$W^{\omega}_{AA} = \frac{\partial_t P_k - r_k\,\eta_A}{P_k}. \tag{4.78}$$

Then, using $Tr\,\theta_{\mu\nu} = 3$ and $Tr\,\omega_{\mu\nu} = 1$, in four dimensions, we get

$$\begin{aligned} Tr\,[W_{AA}] &= 3\,Tr[W^{\theta}_{AA}] + Tr[W^{\omega}_{AA}] \\ &= Tr\left[\left(\frac{3}{P_k + 2\rho\, e^2\, Z_\phi} + \frac{1}{P_k}\right)(\partial_t P_k - r_k\,\eta_A)\right]. \end{aligned} \tag{4.79}$$

For the scalar sector, we have

$$Tr\,[W_{SS}] = 2\,Tr\left[\frac{(\partial_t P_k - r_k \eta_\phi)\,(P_k + Z_\phi^{-1}U' + \rho\, Z_\phi^{-1}\,U'')}{(P_k + Z_\phi^{-1}U')\,(P_k + Z_\phi^{-1}U' + 2\,\rho\, Z_\phi^{-1}\,U'')}\right] \tag{4.80}$$

As we have already discussed, since here there is only dependence on type of operator Δ, the procedure of calculating the functional trace becomes quite simple. In fact, the task is basically reduced to performing an integral of the type

$$Tr\,[\mathcal{O}] = \int_x \frac{1}{16\pi^2}\int_0^\infty dz\, z\,\mathcal{O}(z). \tag{4.81}$$

Therefore, performing the integrals, we obtain

$$Tr\,[W_{AA}] = \int_x \frac{k^6}{96\pi^2}\,(6-\eta_A)\left(\frac{1}{k^2} + \frac{3}{k^2 + 2\,\rho\, e^2\, Z_\phi}\right); \tag{4.82}$$

$$Tr\,[W_{SS}] = \int_x \frac{k^6}{48\pi^2}\,(6-\eta_\phi)\left[\frac{(k^2 + Z_\phi^{-1}U' + \rho\, Z_\phi^{-1}\,U'')}{(k^2 + Z_\phi^{-1}U')(k^2 + Z_\phi^{-1}U' + 2\,\rho\, Z_\phi^{-1}\,U'')}\right]. \tag{4.83}$$

4.4.3 The Flow Equation

Therefore, on the right hand side of the Wetterich equation, we have

$$\frac{1}{2}Tr\,[W] = \frac{1}{2}\int_x \frac{k^6}{96\pi^2}\Bigg[(6-\eta_A)\left(\frac{1}{k^2}+\frac{3}{k^2+2\,\rho\,e^2\,Z_\phi}\right)+ \\ +(6-\eta_\phi)\frac{2\,(k^2+Z_\phi^{-1}U'+\rho\,Z_\phi^{-1}\,U'')}{(k^2+Z_\phi^{-1}U')(k^2+Z_\phi^{-1}U'+2\,\rho\,Z_\phi^{-1}\,U'')}\Bigg]. \tag{4.84}$$

On the left hand side of the Wetterich equation, in the regime in which we deal, by taking the background field and the fluctuations zero, the only term that remains is the derivative of the effective potential. Therefore, factoring the volume of both sides, we obtain the flow equation for the effective average potential

$$\partial_t U_k = \frac{k^6}{192\pi^2}\Bigg[(6-\eta_A)\left(\frac{1}{k^2}+\frac{3}{k^2+2\,\rho\,e^2\,Z_\phi}\right)+ \\ +(6-\eta_\phi)\left(\frac{1}{k^2+Z_\phi^{-1}\,U'}+\frac{1}{k^2+Z_\phi^{-1}\,U'+2\,\rho\,Z_\phi^{-1}\,U''}\right)\Bigg]. \tag{4.85}$$

4.5 Tying Up the Loose Ends

In the previous section, we obtained the important flow equations for the gauge coupling and for the effective average potential. In this section, we restrict ourselves to a certain approximation to show that we can retrieve the results of Coleman-Weinberg's original paper with our approach to the subject, showing that our results are tuned with the already known result, and extend the validity regime of it.

Given the flow equation for the effective potential obtained above, we will consider only the contribution of the gauge sector, ignoring the effects of scalar fluctuations. In a first approximation, we will also ignore the effects of anomalous dimension, as well as the evolution of the gauge coupling. Therefore,

$$\partial_t U_k = \frac{k^6}{32\pi^2}\left(\frac{1}{k^2}+\frac{3}{k^2+2\,\rho\,e^2}\right) = \frac{k^4}{32\pi^2} + \frac{3}{32\pi^2}\left(\frac{k^6}{k^2+2\,\rho\,e^2}\right). \tag{4.86}$$

Recalling that $\partial_t = k\,\partial_k$, we can integrate the above equation in k from 0 to a certain scale Λ, obtaining

$$U_\Lambda - U_0 = \int_0^\Lambda \partial_k U_k = \int_0^\Lambda \left[\frac{k^3}{32\pi^2} + \frac{3}{32\pi^2}\left(\frac{k^5}{k^2+2\,\rho\,e^2}\right)\right]. \tag{4.87}$$

The effective potential is the limit of U_k when $k \to 0$, so we can perform the integral above, obtaining

$$V_{eff} = U_\Lambda - \frac{\Lambda^4}{128\pi^2} - \frac{3}{32\pi^2}\left(\frac{\Lambda^4}{4} - e^2\Lambda^2\,\rho + 2\,e^4\,\rho^2\,\log\left(\frac{\Lambda^2 + 2\,e^2\,\rho}{2\,e^2\,\rho}\right)\right). \tag{4.88}$$

Suppose that this scale Λ is large enough so that U_Λ is the bare potential, which we will define as

$$U_\Lambda = E_\Lambda + m_\Lambda^2\,\rho + \lambda_\Lambda\,\rho^2. \tag{4.89}$$

Therefore, we obtain the expression for the effective potential in terms of bare couplings

$$V_{eff} = \left(E_\Lambda + m_\Lambda^2\,\rho + \lambda_\Lambda\,\rho^2\right) - \frac{\Lambda^4}{32\pi^2} + \frac{3}{32\pi^2}\left(e^2\Lambda^2\,\rho - 2\,e^4\,\rho^2\,\log\left(\frac{\Lambda^2 + 2\,e^2\,\rho}{2\,e^2\,\rho}\right)\right). \tag{4.90}$$

Let us rewrite the logarithmic term using the following trick

$$\begin{aligned}\log\left(\frac{\Lambda^2 + y}{y}\right) &= \log(\Lambda^2 + y) - \log(y) = \log\left[\Lambda^2\left(1 + \frac{y}{\Lambda^2}\right)\right] - \log(y) = \\ &= \log(\Lambda^2) + \log\left(1 + \frac{y}{\Lambda^2}\right) - \log(y) = \\ &= \log\left(\frac{\Lambda^2}{y}\right) + \left[\frac{y}{\Lambda^2} + \left(\frac{y}{\Lambda^2}\right)^2 + (...)\right].\end{aligned} \tag{4.91}$$

where we used the expansion of $\log\left(1 + \frac{y}{\Lambda^2}\right)$ in the second part of the above expression. Therefore, we have

$$\log\left(\frac{\Lambda^2 + 2\,e^2\,\rho}{2\,e^2\,\rho}\right) = \log\left(\frac{\Lambda^2}{2\,e^2\,\rho}\right) + \left[\frac{2\,e^2\,\rho}{\Lambda^2} - \left(\frac{2\,e^2\,\rho}{\Lambda^2}\right)^2 + (...)\right]. \tag{4.92}$$

Note that all terms of the expansion have increasing powers of Λ in the denominator, so, remembering that Λ is a very large scale, we can neglect these contributions. So let us rewrite our effective potential as

$$V_{eff} = \left(E_\Lambda - \frac{\Lambda^4}{32\pi^2}\right) + \left(m_\Lambda^2 + \frac{3\,e^2\,\Lambda^2}{32\pi^2}\right)\rho + \left(\lambda_\Lambda - \frac{6e^4}{32\pi^2}\log\left(\frac{\Lambda^2}{2\,e^2\,\rho}\right)\right)\rho^2. \tag{4.93}$$

With this expression at hand, we will now impose the renormalization conditions. To begin with, we will impose the normalization of the potential

$$\bullet\ \left(V_{eff}(0) = 0\right). \tag{4.94}$$

which ensures that

$$E_\Lambda = \frac{\Lambda^4}{32\pi^2}. \tag{4.95}$$

Now, we impose the mass renormalization condition

$$\bullet \left(\frac{dV_{eff}}{d\rho}(0) = m_R^2 \right). \tag{4.96}$$

Which gives us the relation

$$m_\Lambda^2 = m_R^2 - \frac{3\, e^2\, \Lambda^2}{32\pi^2}. \tag{4.97}$$

Finally, we renormalize the coupling according to

$$\bullet \left(\frac{d^2 V_{eff}}{d\rho^2}(M^2) = \lambda_R \right). \tag{4.98}$$

Which gives us the last relation between the parameters

$$\lambda_\Lambda = \frac{\lambda_R}{2} - \frac{9\, e^4}{32\pi^2} + \frac{6\, e^4}{32\pi^2} \log\left(\frac{\Lambda^2}{2\, e^2\, M^2} \right). \tag{4.99}$$

Substituting these results into the expression of the effective potential, we obtain an expression that is a function of renormalized couplings, as

$$V_{eff} = m_R^2\, \rho + \left[\frac{\lambda_R}{2} - \frac{9\, e^4}{32\pi^2} + \frac{6\, e^4}{32\pi^2} \log\left(\frac{\rho}{M^2} \right) \right] \rho^2. \tag{4.100}$$

Now, as M is an arbitrary mass parameter at our disposal (the point where we define the renormalization condition (4.98)), let us choose for convenience exactly the value that minimizes the potential, $M^2 = \langle \rho \rangle$, such that we have $V'_{eff}(\langle \rho \rangle) = 0$, that is

$$\left. \frac{dV_{eff}}{d\rho} \right|_{\langle \rho \rangle} = m_R^2 + \left(\lambda_R - \frac{9\, e^4}{16\pi^2} + \frac{3\, e^4}{16\pi^2} \right) \langle \rho \rangle = 0. \tag{4.101}$$

Solving this equation and assuming $\langle \rho \rangle \neq 0$, the choice of the renormalization scale gives us a constraint through the relation

$$\lambda_R = -\frac{m_R^2}{\langle \rho \rangle} + \frac{6\, e^4}{16\pi^2}. \tag{4.102}$$

Turning now to the expression for the effective potential, we obtain

$$V_{eff} = \left(m_R^2\, \rho - \frac{m_R^2}{2\langle \rho \rangle}\, \rho^2 \right) + \frac{3\, e^4}{16\pi^2}\, \rho^2 \left[\log\left(\frac{\rho}{\langle \rho \rangle} \right) - \frac{1}{2} \right]. \tag{4.103}$$

Defining the renormalized mass using a parameter β

$$m_R^2 = \beta \, \frac{3\,e^4}{16\pi^2} \, \langle\rho\rangle, \tag{4.104}$$

we can write the final expression for the effective potential in the form

$$V_{eff} = \frac{3\,e^4}{16\pi^2} \left[\beta\langle\rho\rangle\rho + \rho^2 \left(\log\left(\frac{\rho}{\langle\rho\rangle}\right) - \frac{(1+\beta)}{2}\right)\right]. \tag{4.105}$$

With this expression at hand, we can do an analysis of our theory according to β, a dimensionless parameter, which gives the behavior of the theory as we change the mass. Therefore, taking the particular case of a massless theory ($m_R^2 = 0$), we obtain

$$V_{eff} = \frac{3\,e^4}{16\pi^2} \, \rho^2 \left[\log\left(\frac{\rho}{\langle\rho\rangle}\right) - \frac{1}{2}\right], \tag{4.106}$$

which is exactly the original result obtained by Coleman and Weinberg in their seminal paper [2].

Thus, we see that our flow equation gives us the same result of perturbation theory, if we restrict ourselves to a certain approximation, that is, considering only the contributions of the gauge sector of theory, and disregarding the evolution of the gauge coupling and the effects of anomalous dimension, showing that our result is consistent with the original results.

4.6 Solving Numerically the Flow Equations

Let us briefly summarize what we have already achieved so far and then attack the complete flow equations.

Starting from a truncation for the Γ_k, and using the Wetterich equation we were able to extract the flow equations for the effective average potential and for the gauge coupling, which we rewrite here for convenience of the reader

- **Flow equation for the potential**

$$\begin{aligned}\partial_t U(\rho,t) = \frac{k^6}{192\pi^2} \Bigg[&(6-\eta_A)\left(\frac{1}{k^2} + \frac{3}{k^2 + 2\,\rho\,e^2\,Z_\phi}\right) + \\ &+ Z_\phi\,(6-\eta_\phi)\left(\frac{1}{k^2\,Z_\phi + U'(\rho,t)} + \frac{1}{k^2\,Z_\phi + U'(\rho,t) + 2\,\rho\,U''(\rho,t)}\right)\Bigg].\end{aligned} \tag{4.107}$$

- **Flow equation for the gauge coupling**

$$\partial_t e = \frac{e^3\,(2-\eta_\phi)}{96\pi^2}, \tag{4.108}$$

where we have adopted a simplification in the $\partial_t e$ equation, by neglecting $U'(0,t)$.

The flow equations for the wavefunction renormalizations can be obtained from the definition of anomalous dimension, $\eta_\Phi = -\frac{1}{Z_\Phi}\partial_t Z_\Phi$. Therefore, we obtain the auxiliary flow equations for Z_ϕ and Z_A

- **Flow equation for $\mathbf{Z_\Phi}$**

$$\partial_t Z_\Phi + \eta_\Phi \, Z_\Phi = 0. \tag{4.109}$$

Using the same techniques mentioned in the section 3.8, we can compute the anomalous dimensions η_ϕ e η_A. However, as it is a long computation, to not disturb the line of reasoning, we put this computation in the Appendix. The result obtained in the Landau gauge with the same simplifications as before is given by

- **Anomalous dimensions**

$$\eta_\phi = \frac{288\pi^2 \, e^2 + e^4}{768\pi^4 - 24\pi^2 \, e^2} = -\frac{e^2}{24\pi^2}\frac{(e^2 - 288\pi^2)}{(e^2 - 32\pi^2)}, \tag{4.110}$$

$$\eta_A = \frac{e^2}{24\pi^2}. \tag{4.111}$$

It is important to note that both are determined in terms of e^2, and that with η_ϕ at hand we can write the flow equation for the $\partial_t e$ as a function of e^2 only, as

$$\partial_t e = \frac{e^3}{96\pi^2}\left[\frac{e^4 - 240\pi^2 \, e^2 - 1536\pi^4}{24\pi^2(e^2 - 32\pi^2)}\right]. \tag{4.112}$$

In this way, we have a system of coupled differential equations to solve. However, the flow equation for the effective average action is too complicated to be solved analytically, so from now on we will use numerical methods.

The strategy here is: we solve the flow equation for the gauge coupling (4.112) numerically. In this way, we obtain a numerical solution for the gauge coupling $e[t]$, which we will use to feed the other equations. Using this numerical result of the gauge coupling in the auxiliary flow equations for the wavefunction renormalizations (4.109), we can now solve them numerically using $Z_\phi[0] = Z_A[0] = 1$ as initial conditions to the Z_Φ in the reference scale.

With the numerical solutions obtained for the gauge coupling $e[t]$ and the wavefunction renormalizations $Z_\phi[t], Z_A[t]$, we can now feed the flow equation for the effective average potential, and then obtain a somewhat simpler differential equation for the function $U(\rho,t)$. We solve this flow equation using as boundary condition for the $U(\rho,t)$ the prescription $U(\rho,0) = m_\Lambda^2\,\rho + \lambda_\Lambda\,\rho^2$ (with $\lambda_\Lambda > 0$). We know the results for the effective average potential in $t = 0$, which corresponds to the bare potential used as boundary condition, and we are interested in the $t \to -\infty$ limit, which corresponds to the full quantum effective potential. In practice, we will take $t \to -100$, because this is enough for our purposes since we observed that below this scale we do not have any appreciable change in the results.

It must be stressed that when we turn on the flow, the potential will not be quadratic in ρ anymore, developing other non-zero couplings and taking a generic form. The results obviously depend on our choice for the couplings in the reference scale.

We analyze below the behavior of the effective potential for different coupling regimes characterized by the values of the couplings in the reference scale $(m_\Lambda^2, \lambda_\Lambda, e_\Lambda^2)$.

4.6.1 $m_\Lambda^2 = 0$

Following the strategy announced above, we solved the complete flow equation (4.107) with the initial conditions $m_\Lambda^2 = 0$, $e_\Lambda^2 = 0.1$, $\lambda_\Lambda = 0.1$, as showed in figure 4.1.

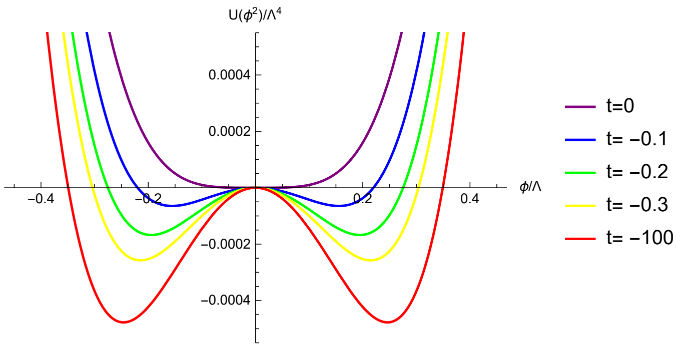

Figure 4.1: Effective Average Potential for different scales $(m_\Lambda^2 = 0,\ e_\Lambda^2 = 0.1,\ \lambda_\Lambda = 0.1)$.

Observing figure 4.1, we note that even starting from a situation where classically there is no spontaneous symmetry breaking, due to quantum corrections spontaneous symmetry breaking takes place. In fact, we have symmetric potential as a boundary condition on the reference scale Λ $(t = 0)$, and as we decrease the parameter t, taking into account the effect of quantum fluctuations, we observe the change in the profile of the effective average potential, until that in the limit where $t \to -\infty$ we have the full quantum effective potential clearly indicating the spontaneous symmetry breaking due to quantum corrections.

The above result goes beyond the Coleman-Weinberg original result, since it considers the contributions of the scalar sector, the running of the gauge coupling and the anomalous dimensions correction, but still does not answer the question initially posed. To verify if the phenomena observed above actually occurs in the non-perturbative regime, we must solve the full flow equation for large couplings.

In fact, by studying the massless case $m_\Lambda^2 = 0$, let us now consider the initial conditions $e_\Lambda^2 = 1$, $\lambda_\Lambda = 1$, where the perturbation theory is no longer valid and see if the phenomenon of spontaneous breaking by quantum fluctuations still occurs in a regime where the perturbative results can no longer help. By solving the flow equations with these specifications, we obtain the result for the effective potential as we can see in figure 4.2.

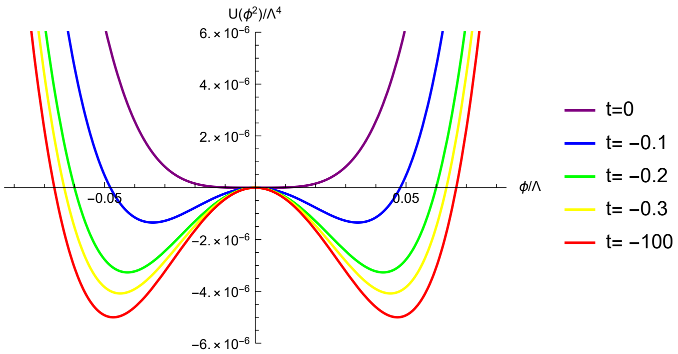

Figure 4.2: Effective Average Potential for different scales ($m_\Lambda^2 = 0,\ e_\Lambda^2 = 1,\ \lambda_\Lambda = 1$).

Therefore, even considering the non-perturbative regime where the couplings are not small and the perturbation theory is not valid, the phenomenon of the spontaneous symmetry breaking due to quantum fluctuations still occurs, as it is clear by observing the profile of the effective potential above, responding in a positive way to the question posed at the beginning of this work.

We observed that taking one coupling small and the other large, we still have the same phenomenon, as for example in the regimes $e_\Lambda^2 = 0.1,\ \lambda_\Lambda = 10$ and $e_\Lambda^2 = 4,\ \lambda_\Lambda = 0.1$, respectively looking at figure 4.3.

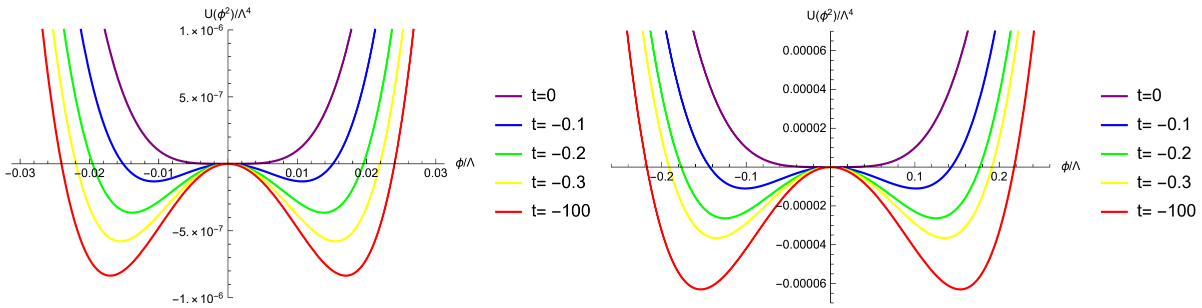

Figure 4.3: Effective Average Potential for different scales ($m_\Lambda^2 = 0,\ e_\Lambda^2 = 1,\ \lambda_\Lambda = 10$) and ($m_\Lambda^2 = 0,\ e_\Lambda^2 = 4,\ \lambda_\Lambda = 0.1$), respectively.

It should be noted, however, that if we have both coupling constants big enough, in a really strongly coupled regime, such as $e_\Lambda^2 = 4,\ \lambda_\Lambda = 10$, we observe the sign of spontaneous breaking when we turn on the quantum fluctuations but to the extent that we take into account all quantum fluctuations, going to full quantum effective potential, we obtain a symmetric vacuum solution evidencing a restoration of the original symmetry by quantum fluctuations, as shown in figure 4.4.

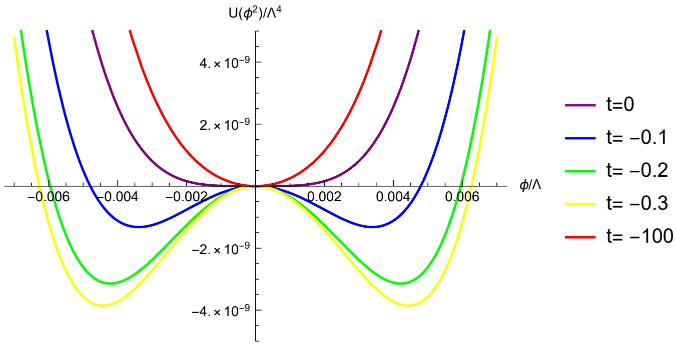

Figure 4.4: Effective Average Potential for different scales $\left(m_\Lambda^2 = 0,\, e_\Lambda^2 = 4,\, \lambda_\Lambda = 10\right)$.

Therefore, we can conclude that for the case $m_\Lambda^2 = 0$, the phenomenon of spontaneous breaking by quantum fluctuations also occurs in the non-perturbative sector of the theory, considering the contributions of the scalar sector, the running of the gauge coupling and the anomalous dimensions correction, but when we consider a regime very strongly coupled the symmetry is restored. The region of couplings where the phenomenon occurs can be seen in the figure 4.5 below, where the green region represents the set of couplings for which the theory exhibits the phenomenon.

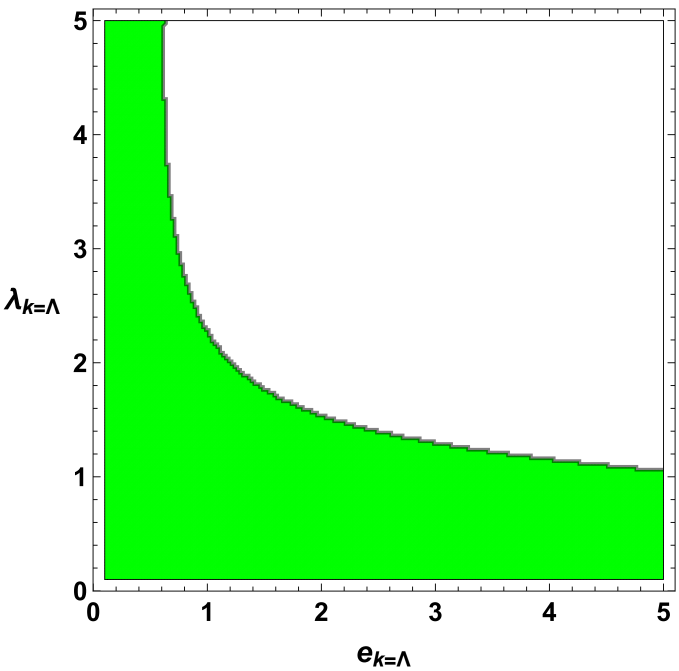

Figure 4.5: Region of couplings where spontaneous symmetry breaking occurs for $m_\Lambda^2 = 0$.

4.6.2 $m_\Lambda^2 > 0$; $m_\Lambda^2 < 0$

It is interesting to investigate the solution of the complete flow equations in the case where we have a non-zero mass parameter in the reference scale Λ. Let us analyze in the sequence for different non-zero values of the mass parameter which are the coupling regimes that show the spontaneous symmetry breaking and which have symmetric vacuum. In this way, we can compare with the results obtained for the massless case, and with the results expected by the traditional classical analysis.

Considering the case of positive mass, with m_Λ^2 large enough, we obtained unbroken symmetry for any configurations of e_Λ^2 and λ_Λ analyzed. Similarly, considering a large enough negative mass, we obtained spontaneous symmetry breaking for all coupling regimes analyzed. In fact, this is the behavior expected by classical analysis and for large values of the mass parameter is the result that dominates. However, by taking very small positive mass values, although the classical analysis states that we have unbroken symmetry, we find coupling regimes in which spontaneous symmetry breaking occur. That is, taking into account the non-perturbative sector, quantum fluctuations may be strong enough to qualitatively change the theory, even with the presence of a positive mass term. Similarly, for the case of very small negative mass, although the classical analysis indicates that spontaneous breaking occurs for any coupling regimes, there are regimes in which the effective potential indicates that there is no spontaneous breaking. Thus, for a sufficiently small neighborhood of the mass parameter, the occurrence or not of the spontaneous breaking will depend on the values of the couplings in the reference scale, and may agree with the classical prediction or present a qualitative change due to the quantum fluctuations. We conclude summarizing the results obtained. There are, in the range of parameters used here, three different regions according to the mass parameter of the theory: the region of large positive masses, where the vacuum is symmetric independent of the values of the couplings; the region of large negative masses, where the spontaneous breaking of symmetry occurs independently of the values of the couplings; and finally the transition region characterized by mass values in a small neighborhood around zero, where the occurrence or not of the spontaneous breaking depends on the values of the couplings of the theory. We next illustrate (figure 4.6) the evolution of regimes according to the mass parameter divided by the reference scale, making clear the behavior of the theory as we move from the regime of negative masses (broken), through the massless neighborhood (transition) to the region of positive masses (unbroken), where the green region indicates the region of couplings where there is spontaneous symmetry breaking.

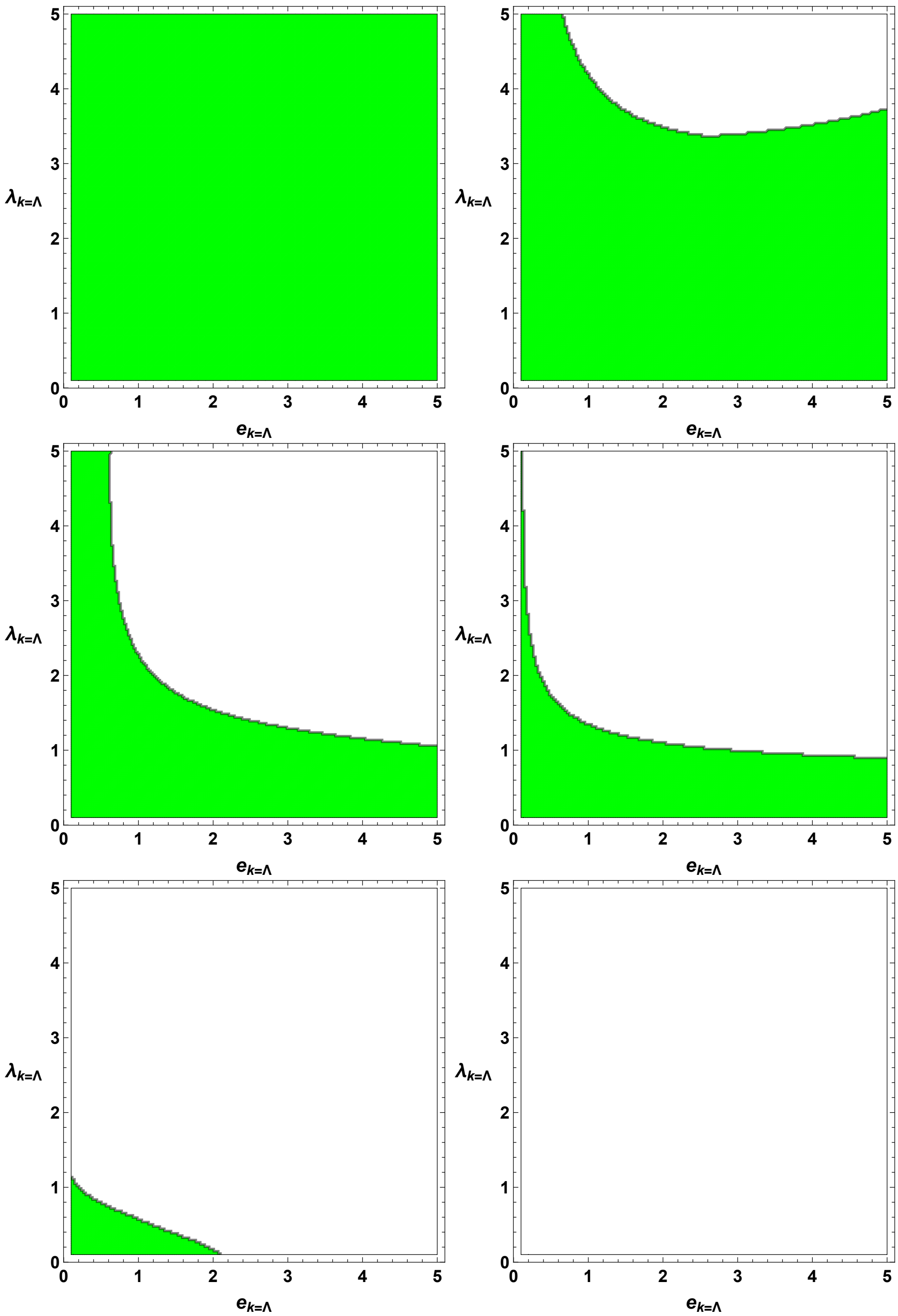

Figure 4.6: The green region indicates the regime in which spontaneous breaking occurs. In each figure, we have a different initial value for the mass parameter, in units of the reference scale Λ^2 $(m_\Lambda^2 = -0.001;\ m_\Lambda^2 = -0.0004;\ m_\Lambda^2 = 0;\ m_\Lambda^2 = 0.003;\ m_\Lambda^2 = 0.01;\ m_\Lambda^2 = 0.02)$.

Chapter 5

Concluding Remarks

We studied here the possibility of spontaneous symmetry breaking, even when the classical analysis says that there is a symmetrical vacuum, that is, we studied the spontaneous symmetry breaking induced by quantum fluctuations. Such phenomenon had already been studied by Coleman and Weinberg in 1973 [2] using perturbation theory to calculate 1-loop effective potential. Such an approach, however, does not allow us to take conclusions about the non-perturbative sector of the theory, since the results obtained by the authors are only valid for small couplings. We then reviewed the functional renormalization group techniques and studied again the problem with these tools to access the non-perturbative sector of the theory. Using the Wetterich equation, we obtained flow equations for the potential and for the gauge coupling and from them we were able to recover the original results in the appropriate limit. However, even with small couplings, we were able to obtain a more complete result than the original, since we take into account the contributions of the scalar sector, the running of the gauge coupling and the anomalous dimensions correction. The strength of this approach lies in considering the more general case, since we need not restrict ourselves to small couplings. In fact, we were able to solve numerically the complete flow equations in the non-perturbative regime with some approximations, and we conclude that in the massless case, there is still the spontaneous breaking by quantum fluctuations. We also note as a limiting case that if we have the couplings large enough, there can be symmetry restoration and we obtain a symmetric potential. We analyzed non-zero mass situations, and note that for large masses the classical result holds, but for small masses we have a situation analogous to the massless case, where fluctuations can alter the character of the theory. Therefore, in the studied range, the theory has 3 regimes: for large positive masses the system maintains unbroken symmetry for any couplings; for large negative masses spontaneous symmetry breaking occurs for any couplings; for masses in a neighborhood around zero, there is spontaneous symmetry breaking or not according to the values of the couplings. We analyzed the neighborhood where transition occurs and we plotted for some values of mass of this region, the regimes where the system presents spontaneous symmetry breaking.

- In this work, we limit ourselves to a certain range in the parameter region and use some approximations. A first improvement would be to broaden the region of interest and remove the approximations, increasing the scope of results and their precision.

- One way to continue this work is to study the modified Ward identities and their influence in the flow equations. In a quantum gauge theory, we lose the explicit gauge symmetry because of gauge-fixing term. The gauge symmetry however remains encoded in constraint equations for quantum action called Ward-Takahashi Identities (WTIs). Such identities in Quantum Electrodynamics, for example, prevents the photon from acquiring mass by quantum corrections. In our approach, in addition to gauge-fixing term, we have yet another term that breaks the gauge symmetry during the quantization process, the cutoff action term ΔS_k. Therefore, we will have a further contribution to the WTIs, altering the constraint equations, in such a way that we call these new identities of *modified WTI*. In the limit $k \to 0$, we have $\Delta S_k \to 0$, so we get the original WTIs. However, for $k \neq 0$, the modified WTIs provide a non-trivial constraint that can give relevant information, changing the possible terms present in the truncation and possibly changing the flow equations. The modified WTIs therefore encode a kind of modified gauge symmetry, which give us the original WTIs and thus the original gauge symmetry, at the limit $k \to 0$. It is interesting therefore to study what are the influences of this kind of modified gauge symmetry in our results.

- An interesting idea would be to generalize this work to a Yang-Mills theory. Coupling the scalar field with a non-abelian gauge field, we could investigate if the studied phenomenon also occurs. Yang-Mills theories are very rich, with more gauge bosons, self-interaction, ghosts, and a beta-function with a completely different behavior. Therefore, it would be extremely interesting to analyze the possibility of spontaneous symmetry breaking induced by quantum fluctuations in this more general context.

- A more ambitious idea would be to study the extension of the phenomenon presented in this thesis to a supersymmetric context, that is, investigate the spontaneous supersymmetry breaking induced by radiative corrections in the context of functional renormalization group.

- We mention here some subjects somehow related to the subject of this thesis, that were investigated by C. Wetterich and his collaborators, and could be used to inspire new paths. Finite temperature phase transition [40] ; critical behavior in 3 dimensions [41] ; Coleman-Weinberg phase transitions with 2 scalars [42] ; phase diagrams of superconductors [43].

Appendix A

The Anomalous Dimensions

Here, we will outline the computation of anomalous dimensions η_ϕ e η_A, using the techniques showed in section (3.8), with the help of *Mathematica*[1], using the *Package-X*[2].

Let the truncation be given by

$$\Gamma_k = \int_x \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* (D_\mu \phi) + U_k(\phi^* \phi) + \frac{1}{2\xi} \int_x (\partial_\mu a_\mu)^2. \tag{A.1}$$

Considering the regime in which we have zero background, taking into account the wave-function renormalizations, we have as before

$$\begin{aligned}\Gamma_k = \int_x & \frac{Z_A}{4} F_{\mu\nu} F_{\mu\nu} + \frac{Z_A}{2\xi} (\partial_\mu A_\mu)^2 + Z_\phi \, \partial_\mu \phi^* \partial_\mu \phi + \\ & + i \, e \, Z_\phi \, Z_A^{1/2} \, A_\mu \, (\phi \, \partial_\mu \phi^* - \phi^* \, \partial_\mu \phi) + e^2 \, Z_A Z_\phi \, A_\mu \, A_\mu \, \phi^* \phi + U_k(\phi^* \phi).\end{aligned} \tag{A.2}$$

Using the following conventions for the Fourier transform

$$A_\mu(x) = \int_p A_\mu(p) \, e^{-i\,p\,x}; \quad \phi(x) = \int_p \phi(p) \, e^{-i\,p\,x}; \quad \phi^*(x) = \int_p \phi^*(p) \, e^{+i\,p\,x}, \tag{A.3}$$

we can write our truncation in the Fourier space as

$$\begin{aligned}\Gamma_k = \int_p & \frac{1}{2} \, Z_A \, A_\mu(-p) \, \left(p^2 \, \theta_{\mu\nu} + \frac{1}{\xi} \, p^2 \, \omega_{\mu\nu} \right) \, A_\nu(p) + Z_\phi \, \phi^*(p) \, p^2 \, \phi(p) + \\ & - Z_\phi \, Z_A^{1/2} \, e_k \, (p_\mu + q_\mu) \; \phi^*(p) \, \phi(q) \, A_\mu(k) (2\pi)^4 \, \delta(-p + q + k) + \\ & + Z_\phi \, Z_A \, e_k^2 \delta_{\mu\nu} \, \phi^*(p) \, \phi(q) \, A_\mu(k) \, A_\nu(l) (2\pi)^4 \, \delta(-p + q + k + l) + \\ & + U_k(0) + \phi^*(p) U_k'(0) \phi(p) + \frac{1}{2} U_k''(0) \, \phi^*(p) \, \phi(q) \, \phi^*(k) \, \phi(l) \, (2\pi)^4 \, \delta(-p + q - k + l).\end{aligned}$$

[1]Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL (2018)
[2]Hiren H. Patel, Comput. Phys. Commun. 197, 276 (2015), ePrint: arXiv:1503.01469

From the expression above, we can easily calculate the Hessians and the vertices. In fact, taking into account the conventions that we have been using, we obtain for the Hessians

$$\Gamma^{(2)}_{A} = Z_A \left(p^2\, \theta_{\mu\nu} + \frac{1}{\xi}\, p^2\, \omega_{\mu\nu} \right); \tag{A.4}$$

$$\Gamma^{(2)}_{\phi} = Z_\phi\, p^2. \tag{A.5}$$

And for the vertices

$$V^{(3)}_{A\phi^*\phi} = -Z_\phi\, Z_A^{1/2}\, e_k\, (p_\mu + q_\mu)\ (2\pi)^4\, \delta(-p+q+k)\,; \tag{A.6}$$

$$V^{(4)}_{AA\phi^*\phi} = +2\, Z_\phi\, Z_A\, e_k^2\, \delta_{\mu\nu}\ (2\pi)^4\, \delta(-p+q+k+l); \tag{A.7}$$

$$V^{(4)}_{\phi^*\phi\phi^*\phi} = +\,2\, U_k''(0)\ (2\pi)^4\, \delta(-p+q-k+l). \tag{A.8}$$

Taking a simple regulator, prescribed by the expression $\mathcal{R}_k(z) = \Gamma^{(2)}(P_k(z)) - \Gamma^{(2)}(z)$, it does the substitution $z \to P_k(z) = z + r_k(z)$, and we can obtain the expression for the regularized propagators

$$G_A = \frac{1}{Z_A\, P_k}\, \theta_{\mu\nu} + \frac{\xi}{Z_A\, P_k}\, \omega_{\mu\nu}; \tag{A.9}$$

$$G_\phi = \frac{1}{Z_\phi\, P_k}. \tag{A.10}$$

With these objects at hand, now we can begin the computation of anomalous dimensions properly.

As we saw in the section 3.8, we can compute the anomalous dimension with the Fourier expression of the truncation with

$$\eta_\varphi = -\frac{1}{Z_\varphi}\, \partial_t Z_\varphi = -\frac{1}{Z_\varphi}\, \frac{\partial}{\partial p^2}\, \partial_t \Gamma^{(2)}_k \bigg|_{p^2=0;\Phi=0}. \tag{A.11}$$

Using the flow equation for the $\Gamma^{(2)}_k$ obtained in the section 3.7, we can write the formal expression for the anomalous dimension

$$\eta_\varphi = -\frac{1}{Z_\varphi}\, \frac{\partial}{\partial p^2} \left[-\frac{1}{2}\, Tr \left(\Gamma^{(4)}_k\, G_k\, \partial_t \mathcal{R}_k\, G_k \right) + Tr \left(\Gamma^{(3)}_k\, G_k\, \partial_t \mathcal{R}_k\, G_k\, \Gamma^{(3)}_k\, G_k \right) \right]_{p^2=0;\Phi=0}. \tag{A.12}$$

Naturally, here the objects have matrix structure, as we are working with more than one field.

Scalar Anomalous dimension η_ϕ

To obtain the anomalous dimension of the scalar field η_ϕ, we have to take the derivatives with respect to the scalar field, then in this case

$$\Gamma^{(4)}_{\phi^*\phi} = \frac{\delta^2\,\Gamma^{(2)}_k}{\delta\phi^*\,\delta\phi}, \qquad \Gamma^{(3)}_{\phi^*} = \frac{\delta\,\Gamma^{(2)}_k}{\delta\phi^*}, \qquad \Gamma^{(3)}_{\phi} = \frac{\delta\,\Gamma^{(2)}_k}{\delta\phi}. \tag{A.13}$$

Given the expressions obtained for the truncation in question, we can easily write the regularized propagator of this theory, as

$$G_k = \left(\Gamma^{(2)}_k + \mathcal{R}_k\right)^{-1} = \begin{pmatrix} G_A & 0 & 0 \\ 0 & 0 & G_\phi \\ 0 & G_\phi & 0 \end{pmatrix}, \tag{A.14}$$

We can also write the 4-vertex matrix

$$\Gamma^{(4)}_{\phi^*\phi} = \begin{pmatrix} V^{(4)}_{AA\phi^*\phi} & 0 & 0 \\ 0 & 0 & V^{(4)}_{\phi^*\phi\phi^*\phi} \\ 0 & V^{(4)}_{\phi^*\phi\phi^*\phi} & 0 \end{pmatrix}, \tag{A.15}$$

and the 3-vertex matrices

$$\Gamma^{(3)}_{\phi^*} = \begin{pmatrix} 0 & V^{(3)}_{A\phi^*\phi} & 0 \\ V^{(3)}_{A\phi^*\phi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma^{(3)}_{\phi} = \begin{pmatrix} 0 & 0 & V^{(3)}_{A\phi^*\phi} \\ 0 & 0 & 0 \\ V^{(3)}_{A\phi^*\phi} & 0 & 0 \end{pmatrix}, \tag{A.16}$$

where the expressions for the entries are given by the vertex expressions (A.6),(A.7),(A.8).

Therefore, we can obtain the matrix

$$\mathcal{M}_k = G_k\,\partial_t\,\mathcal{R}_k\,G_k = \begin{pmatrix} G_A\,\partial_t\,\mathcal{R}^A_k\,G_A & 0 & 0 \\ 0 & 0 & G_\phi\,\partial_t\,\mathcal{R}^\phi_k\,G_\phi \\ 0 & G_\phi\,\partial_t\,\mathcal{R}^\phi_k\,G_\phi & 0 \end{pmatrix}, \tag{A.17}$$

where the entries here are given by

$$\mathcal{M}^{\mu\nu}_A = G^{\mu\alpha}_A\,\partial_t\mathcal{R}^{\alpha\beta}_A\,G^{\beta\nu}_A = (\theta_{\mu\nu} + \xi\,\omega_{\mu\nu})\left(\frac{Z_A\,\partial_t R_k + R_k\,\partial_t Z_A}{P^2_k\,Z^2_A}\right); \tag{A.18}$$

$$\mathcal{M}_\phi = G_\phi\,\partial_t\mathcal{R}_\phi\,G_\phi = \frac{(Z_\phi\,\partial_t R_k + R_k\,\partial_t Z_\phi)}{Z^2_\phi\,P^2_k}. \tag{A.19}$$

And then we can compute the product in the sector with the four-vertex

$$\left(\Gamma_k^{(4)}\,G_k\,\partial_t\mathcal{R}_k\,G_k\right)=\begin{pmatrix}V^{(4)}_{AA\phi^*\phi}\,\mathcal{M}_A & 0 & 0\\ 0 & V^{(4)}_{\phi^*\phi\phi^*\phi}\,\mathcal{M}_\phi & 0\\ 0 & 0 & V^{(4)}_{\phi^*\phi\phi^*\phi}\,\mathcal{M}_\phi\end{pmatrix}. \tag{A.20}$$

Therefore, taking the trace we have

$$-\frac{1}{2}\,Tr\,\left[\Gamma_k^{(4)}G_k\partial_t\mathcal{R}_kG_k\right]=-\frac{1}{2}Tr\left[V^{(4)}_{AA\phi^*\phi}\,\mathcal{M}_A\,\right]-Tr\left[V^{(4)}_{\phi^*\phi\phi^*\phi}\,\mathcal{M}_\phi\right]. \tag{A.21}$$

We also can do the product in the other sector and obtain

$$\left(\Gamma_k^{(3)}\,G_k\,\partial_t\mathcal{R}_k\,G_k\,\Gamma_k^{(3)}\,G_k\right)=\begin{pmatrix}V^{(3)}_{A\phi^*\phi}\,\mathcal{M}_\phi\,V^{(3)}_{A\phi^*\phi}\,G_A & 0 & 0\\ 0 & V^{(3)}_{A\phi^*\phi}\,\mathcal{M}_A\,V^{(3)}_{A\phi^*\phi}\,G_\phi & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{A.22}$$

And then, we obtain

$$Tr\left[\Gamma_k^{(3)}G_k\,\partial_t\mathcal{R}_k\,G_k\Gamma_k^{(3)}G_k\right]=Tr\left[V^{(3)}_{A\phi^*\phi}\,\mathcal{M}_\phi V^{(3)}_{A\phi^*\phi}G_A\right]+Tr\left[V^{(3)}_{A\phi^*\phi}\,\mathcal{M}_A V^{(3)}_{A\phi^*\phi}G_\phi\right]. \tag{A.23}$$

Remember that the expression for the scalar field anomalous dimension η_ϕ is (A.12),

$$\eta_\varphi=-\frac{1}{Z_\varphi}\frac{\partial}{\partial p^2}\left[-\frac{1}{2}\,Tr\left(\Gamma_k^{(4)}\,G_k\,\partial_t\mathcal{R}_k\,G_k\right)+Tr\left(\Gamma_k^{(3)}\,G_k\,\partial_t\mathcal{R}_k\,G_k\,\Gamma_k^{(3)}\,G_k\right)\right]_{p^2=0;\Phi=0},$$

and we have just obtained the expression for these traces in (A.21), (A.23).

Putting these expressions in *Mathematica*, what we obtain is

$$-\frac{1}{Z_\varphi}\frac{\partial}{\partial p^2}\left[-\frac{1}{2}\,Tr\left(\Gamma_k^{(4)}\,G_k\,\partial_t\mathcal{R}_k\,G_k\right)\right]_{p^2=0;\Phi=0}=0, \tag{A.24}$$

therefore, the sector with four-vertex does not contribute for the scalar field anomalous dimension. For the other part we have

$$\begin{aligned}-\frac{1}{Z_\varphi}\frac{\partial}{\partial p^2}&\left[Tr\left(\Gamma_k^{(3)}\,G_k\,\partial_t\mathcal{R}_k\,G_k\,\Gamma_k^{(3)}\,G_k\right)\right]_{p^2=0;\Phi=0}=\\&=-\frac{e^2(-3+2\xi)}{96\pi^2}\,\eta_\phi+\frac{e^2(3+\xi)}{96\pi^2}\,\eta_A+\frac{e^2(-3+\xi)}{8\pi^2}.\end{aligned} \tag{A.25}$$

Therefore, in the Landau gauge $\xi\to 0$, the equation obtained is

$$\left(1-\frac{e^2}{32\pi^2}\right)\,\eta_\phi-\frac{e^2}{32\pi^2}\,\eta_A+\frac{3e^2}{8\pi^2}=0. \tag{A.26}$$

Gauge Anomalous Dimension η_A

Following the same line of reasoning used to calculate the scalar anomalous dimension, we can obtain

$$Z_A\,(\theta_{\mu\nu} + \frac{1}{\xi}\,\omega_{\mu\nu}) = \frac{\partial^2}{\partial p^2}\,\frac{\delta^2\,\Gamma_k}{\delta A_\mu(-p)\,\delta A_\nu(p)}\bigg|_{\Phi=0;p^2=0}. \tag{A.27}$$

Then, contracting this expression with $\theta_{\mu\nu}$ (since we are using the Landau gauge) and dividing by 3 to take into account the trace factor, we can write for the anomalous dimension

$$\eta_A = -\frac{1}{Z_A}\,\frac{1}{3}\,\theta_{\mu\nu}\,\left[\frac{\partial^2}{\partial p^2}\,\left(\frac{\delta^2\,\partial_t\Gamma_k}{\delta A_\mu(-p)\,\delta A_\nu(p)}\right)\right]_{\Phi=0;p^2=0}. \tag{A.28}$$

The formal structure of the flow equation is the same, but taking derivatives with respect to the gauge field, which change the vertices. In fact

$$\Gamma^{(4)}_{AA\phi^*\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & V^{(4)}_{AA\phi^*\phi} \\ 0 & V^{(4)}_{AA\phi^*\phi} & 0 \end{pmatrix}; \tag{A.29}$$

$$\Gamma^{(3)}_{A\phi^*\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & V^{(3)}_{A\phi^*\phi} \\ 0 & V^{(3)}_{A\phi^*\phi} & 0 \end{pmatrix}. \tag{A.30}$$

We also have the matrix

$$\mathcal{M}_k = G_k\,\partial_t\,\mathcal{R}_k\,G_k = \begin{pmatrix} G_A\,\partial_t\,\mathcal{R}^A_k\,G_A & 0 & 0 \\ 0 & 0 & G_\phi\,\partial_t\,\mathcal{R}^\phi_k\,G_\phi \\ 0 & G_\phi\,\partial_t\,\mathcal{R}^\phi_k\,G_\phi & 0 \end{pmatrix}, \tag{A.31}$$

with entries that we will call $\mathcal{M}_A$ and $\mathcal{M}_\phi$, as before. Then, we can compute the product in this sector

$$\Gamma^{(4)}_{AA\phi^*\phi}\,G_k\,\partial_t\,\mathcal{R}_k\,G_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & V^{(4)}_{AA\phi^*\phi}\,\mathcal{M}_\phi & 0 \\ 0 & 0 & V^{(4)}_{AA\phi^*\phi}\,\mathcal{M}_\phi \end{pmatrix}. \tag{A.32}$$

And therefore

$$-\frac{1}{2}\,Tr\,[\Gamma^{(4)}_{AA\phi^*\phi}\,G_k\,\partial_t\,\mathcal{R}_k\,G_k] = -Tr\,[V^{(4)}_{AA\phi^*\phi}\,\mathcal{M}_\phi]. \tag{A.33}$$

In the same way, we can compute

$$\left(\Gamma_k^{(3)}\, G_k\, \partial_t \mathcal{R}_k\, G_k\, \Gamma_k^{(3)}\, G_k\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & V^{(3)}_{A\phi^*\phi}\, \mathcal{M}_\phi\, V^{(3)}_{A\phi^*\phi}\, G_\phi & 0 \\ 0 & 0 & V^{(3)}_{A\phi^*\phi}\, \mathcal{M}_\phi\, V^{(3)}_{A\phi^*\phi}\, G_\phi \end{pmatrix}. \tag{A.34}$$

And therefore

$$Tr\,[\Gamma^{(3)}_{A\phi^*\phi}\, G_k\, \partial_t \mathcal{R}_k\, G_k\, \Gamma^{(3)}_{A\phi^*\phi}\, G_k] = 2\, Tr\,[V^{(3)}_{A\phi^*\phi}\, \mathcal{M}_\phi\, V^{(3)}_{A\phi^*\phi}\, G_\phi]. \tag{A.35}$$

As before, what we have to compute is

$$\eta_A = -\frac{1}{Z_A}\frac{1}{3}\,\theta_{\mu\nu}\,\frac{\partial^2}{\partial p^2}\left[-\frac{1}{2}\,Tr\left(\Gamma_k^{(4)}\, G_k\, \partial_t \mathcal{R}_k\, G_k\right) + Tr\left(\Gamma_k^{(3)}\, G_k\, \partial_t \mathcal{R}_k\, G_k\, \Gamma_k^{(3)}\, G_k\right)\right]_{\Phi=0;p^2=0}, \tag{A.36}$$

and we have already written the expression for the traces in terms of objects that we know.

Putting these expressions in *Mathematica*, we obtain

$$-\frac{1}{Z_A}\frac{1}{3}\,\theta_{\mu\nu}\,\frac{\partial^2}{\partial p^2}\left[-\frac{1}{2}\,Tr\left(\Gamma_k^{(4)}\, G_k\, \partial_t \mathcal{R}_k\, G_k\right)\right]_{\Phi=0;p^2=0} = 0, \tag{A.37}$$

therefore, the sector with four-vertex does not contribute for the gauge field anomalous dimension. For the other part we have

$$-\frac{1}{Z_A}\frac{1}{3}\,\theta_{\mu\nu}\,\frac{\partial^2}{\partial p^2}\left[Tr\left(\Gamma_k^{(3)}\, G_k\, \partial_t \mathcal{R}_k\, G_k\, \Gamma_k^{(3)}\, G_k\right)\right]_{\Phi=0;p^2=0} = \frac{e^2}{24\pi^2}. \tag{A.38}$$

Therefore, in the Landau gauge $\xi \to 0$, the equation obtained is

$$\eta_A - \frac{e^2}{24\pi^2} = 0. \tag{A.39}$$

The results

With the computations made above, we can solve the system and conclude that, in the Landau gauge, the anomalous dimensions are given by

$$\eta_\phi = \frac{e^4 - 288\pi^2 e^2}{768\pi^4 - 24\pi^2 e^2}; \tag{A.40}$$

$$\eta_A = \frac{e^2}{24\pi^2}. \tag{A.41}$$

Bibliography

[1] M. Schwartz (2014), "Quantum field theory and the standard model", Cambridge University Press.

M. Srednicki (2007), "Quantum Field Theory", Cambridge University Press.

M. Peskin and D. Schroeder (1995), "An introduction to quantum field theory" Westview, Boulder, CO.

T.-P. Cheng and L.-F. Li (1984), "Gauge Theory of Elementary Particle Physics," Oxford University Press, Oxford.

[2] S. Coleman and E. Weinberg (1973), "Radiative corrections as the origin of spontaneous symmetry breaking", Phys. Rev., D7, 1888.

[3] M. Continentino (2017), "Quantum Scaling in Many-Body Sistems" 2nd edition, Cambridge University Press.

[4] Linde, A D (1982). "A new Inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems". Phys. Lett. B 108: 389.

[5] P. Higgs (1964), "Broken symmetries and the masses of gauge bosons", Phys. Rev. Lett. 13 (16), 508.

[6] Wetterich, C. (1993), "Exact evolution equation for the effective potential", Phys. Lett. B, 301 (1): 90.

M. Reuter and C. Wetterich (1994), "Exact evolution equation for scalar electrodynamics", Nucl. Phys. B427, 291.

M. Reuter and C. Wetterich (1994), "Effective average action for gauge theories and exact evolution equations", Nucl. Phys. B417, 181.

M. Reuter and C. Wetterich (1993), "Average action for the Higgs model with abelian gauge symmetry", Nucl. Phys. B391, 147.

M. Reuter and C. Wetterich (1993), "Running gauge coupling in three-dimensions and the electroweak phase transition", Nucl. Phys. B408, 91.

[7] Polchinski, J. (1984), "Renormalization and Effective Lagrangians", Nucl. Phys. B, 231 (2): 269.

[8] Wilson, K. (1971). "Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture". Phys. Rev. B. 4 (9): 3174–3183.

Wilson, K. (1971). "Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior". Phys. Rev. B. 4 (9): 3184–3205.

[9] A. P. C. Malbouisson, F.S. Nogueira, N. F. Svaiter (1996). "A new derivation of the Coleman-Weinberg mechanism in massless scalar QED", Mod. Phys. Lett. A, Vol. 11, No. 9, 749-754.

[10] J. Berges; N. Tetradis; C. Wetterich (2002), "Non-perturbative renormalization flow in quantum field theory and statistical mechanics", Phys. Rep., 363 (4–6): 223–386.

H.Gies (2006). "Introduction to the functional RG and applications to gauge theories". Lecture Notes in Physics. 852. pp. 287–348.

B. Delamotte (2007). "An introduction to the nonperturbative renormalization group". Lecture Notes in Physics. 852. pp. 49–132.

Jan M. Pawlowski (2007). "Aspects of the functional renormalization group". Annals Phys. 322, 2831-2915.

[11] A. Codello (2010). "A novel functional Renormalization group framework for gauge theories and gravity", PhD thesis at the University of Mainz.

A. Eichhorn (2011) "Quantum fields in the non-perturbative regime – Yang-Mills theory and gravity", PhD thesis at the Friedrich-Schiller-Universität Jena, arXiv:1111.1237v1 [hep-th].

R. Sondenheimer (2016) "Renormalization group flow of the Higgs sector", PhD thesis at the Friedrich-Schiller-Universität Jena.

[12] A. Eichhorn (2019), "An asymptotically safe guide to quantum gravity and matter", arXiv:1810.07615 [hep-th].

M. Reuter (1998), "Nonperturbative evolution equation for quantum gravity", Phys.Rev., D57:971–985.

[13] A. Cyrol et al. (2018), "Nonperturbative finite-temperature Yang-Mills theory", Phys. Rev. D 97, 054015.

[14] H. Gies, R. Sondenheimer (2018), "Renormalization group flow of the Higgs potential", Phil. Trans. R. Soc. A 376: 20170120.

A. Eichhorn, H. Gies, J. Jaeckel, T. Plehn, M. Schererd and R. Sondenheimer (2015), "The Higgs mass and the scale of new physics", JHEP04(2015)022.

[15] J. Berges and D. Mesterhazy (2012), "Introduction to the nonequilibrium functional renormalization group", Nucl.Phys. Proc.Suppl., 228:37–60, 2012.

[16] R.J. Furnstahl (2012), "The Renormalization Group in Nucl. Phys.", Nucl.Phys. Proc.Suppl., 228:139–175.

[17] J. Pawlowski (2011), "The QCD phase diagram: Results and challenges", AIP Conf. Proc. 1343, 75-80.

J. Braun, L. Haas, F. Marhauser, and J. Pawlowski (2011). "Phase Structure of Two-Flavor QCD at Finite Chemical Potential", Phys.Rev.Lett., 106:022002.

[18] M. Mastaler, F. SynatschkeCzerwonka, and A. Wipf (2012), "Supersymmetric Renormalization Group Flows", Physics of Particles and Nuclei, Vol. 43, No. 5, pp. 593–599.

H. Gies, F. Synatschke, and A. Wipf (2009), "Supersymmetry breaking as a quantum phase transition", Phys.Rev., D80:101701.

F. Synatschke, G. Bergner, H. Gies, and A. Wipf (2009). "Flow Equation for Supersymmetric Quantum Mechanics", JHEP, 0903:028.

[19] G. S. Guralnik; C. R. Hagen; T. W. B. Kibble (1964). "Global conservation laws and massless particles". Phys. Rev. Lett. 13 (20): 585–587.

[20] Salam, A.; Ward, J. C. (1959). "Weak and electromagnetic interactions". Il Nuovo Cimento. 11 (4): 568–577.

Glashow, S. (1959). "The renormalizability of vector meson interactions." Nucl. Phys. 10, 107.

Weinberg, S (1967). "A Model of Leptons" . Phys. Rev. Lett. 19: 1264–66.

[21] P. W. Anderson (1962). "Plasmons, gauge invariance, and mass". Phys. Rev. 130 (1): 439–442.

[22] W. Meissner, R. Ochsenfeld (1933). "Ein neuer Effekt bei Eintritt der Supraleitfähigkeit", Naturwissenschaften, 21 (44): 787–788.

[23] J. Schwinger (1951), "On the Green's functions of quantized fields". Proc Natl Acad Sci 37 (7) 452-455.

[24] G. Jona-Lasinio (1964), "Relativistic field theories with symmetry-breaking solutions", Il Nuovo Cimento, 34, 6, 1790.

[25] H. Lehmann, K. Symanzik, and W. Zimmerman (1955), "Zur Formulierung quantisierter Feldtheorien", Il Nuovo Cimento 1(1), 205.

[26] R F Streater and A S. Wightman (2000) "PCT, Spin and Statistics, and All That", Princeton University Press.

[27] K. Osterwalder; R. Schrader, (1973). "Axioms for Euclidean Green's functions". Comm. Math. Phys. 31 , no. 2, 83–112.

[28] Gell-Mann, M.; Low, F. E. (1954). "Quantum Electrodynamics at Small Distances". Phys. Rev.. 95 (5): 1300–1312.

[29] S. Tomonaga (1946). "On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields". Progress of Theoretical Physics. 1 (2): 27–42.

J. Schwinger (1948). "Quantum Electrodynamics. I. A Covariant Formulation". Phys. Rev.. 74 (10): 1439–61.

R. P. Feynman (1949). "Space–Time Approach to Quantum Electrodynamics". Phys. Rev.. 76 (6): 769–89.

[30] R. Percacci (2017), "An Introduction to Covariant Quantum Gravity and Asymptotic Safety", World Scientific Pub Co. Pte. Ltd., Singapore.

[31] Daniel F. Litim (2001) "Optimised renormalisation group flows". Phys. Rev. D 64, 105007.

[32] G. P. de Brito, Lecture Notes (unpublished).

[33] Maxwell, James Clerk (1865). "A dynamical theory of the electromagnetic field". Philosophical Transactions of the Royal Society of London. 155: 459–512.

[34] Yang, C. N.; Mills, R. (1954). "Conservation of Isotopic Spin and Isotopic Gauge Invariance". Phys. Rev.. 96 (1): 191–195.

[35] M. Gell-Mann (1964). "A Schematic Model of Baryons and Mesons". Phys. Lett. 8 (3): 214–215.

G. Zweig (1964). "An SU(3) Model for Strong Interaction Symmetry and its Breaking". CERN Report No.8182/TH.401.

[36] Goldstone, J; Salam, A and Weinberg, S (1962). "Broken symmetries". Phys. Rev. 127: 965.

[37] D. Litim, C. Wetterich, N. Tetradis, (1997). "Nonperturbative analysis of the Coleman-Weinberg phase transition". Mod. Phys. Lett. A, Vol. 12, No. 30 2287.

[38] L. F. Abbott (1981) “The Background Field Method Beyond One Loop” Nucl. Phys. B185, 189-203.

G. 't Hooft (1976), “The Background Field Method in Gauge Field Theories”, Acta Univ. Wratislav, No. 368.

B. De Witt, in “Quantum Gravity” 2, ed. C. Isham (Oxford Univ. Press, 1982).

B. Boulware (1981), “Gauge dependence of the effective action”, Phys. Rev. D23, 389.

[39] A. Codello, R. Percacci, C. Rahmede (2009). “Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation”, Annals Phys.324:414-469.

[40] N. Tetradis and C. Wetterich, “The high temperature phase transition for ϕ^4 theories”, Nucl. Phys. B398 (1993) 659—696.

[41] N. Tetradis and C. Wetterich, “Critical exponents from the effective average action”, Nucl. Phys. B 422 IFSI (1994) 541—592.

[42] S. Bornholdt, N. Tetradis, C. Wetterich, “Coleman-Weinberg phase transition in two-scalar models”, Phys. Lett. B 348 (1995) 89-99.

[43] B. Bergerhoff, F. Freire, D. F. Litim, S. Lola, and C. Wetterich, “Phase diagram of superconductors from nonperturbative flow equations”, Phys. Rev. B, V. 53, N. 9 (1996), 5734-5757.