Paola Carolina Moreira Delgado

Cosmological models with asymmetric quantum bounces

Rio de Janeiro, Brazil

June 29, 2020

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Dissertation presented to the Graduate Program at the Brazilian Center for Research in Physics as a partial requirement for obtaining the Master's degree in Physics.

Centro Brasileiro de Pesquisas Físicas

COSMO - Coordenação de Cosmologia, Astrofísica e Interações Fundamentais

Graduate Program

Supervisor: Nelson Pinto Neto

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“The natural desire of good men is knowledge.”
Leonardo da Vinci

Abstract

In quantum cosmology, one has to select a specific wave function solution of the quantum state equations under consideration in order to obtain concrete results. The simplest choices have been already explored, in different frameworks, yielding, in many cases, quantum bounces. As there is no consensually established boundary condition proposal in quantum cosmology, we investigate the consequences of enlarging known sets of initial wave functions of the universe, in the specific framework of the Wheeler-DeWitt equation interpreted along the lines of the de Broglie-Bohm Quantum Theory, on the possible quantum bounce solutions which emerge from them. In particular, we show that many asymmetric quantum bounces are obtained, which may incorporate non-trivial back-reaction mechanisms, as quantum particle production around the bounce, in the quantum background itself. In particular, the old hypothesis that our expanding universe might have arisen from quantum fluctuations of a fundamental quantum flat space-time is recovered, within a different and yet unexplored perspective.

Keywords: Quantum bounces. Quantum cosmology. Back-reaction.

Resumo

Em cosmologia quântica, é necessário selecionar uma função de onda específica que seja solução das equações de estado quântico consideradas a fim de obter resultados concretos. As escolhas mais simples já foram exploradas, em diferentes contextos, levando, em muitos casos, a bounces quânticos. Uma vez que não existe uma proposta de condição de contorno consensualmente estabelecida em cosmologia quântica, nós investigamos as consequências de generalizar conjuntos conhecidos de funções de onda iniciais do universo, espeficiamente no contexto da equação de Wheeler-DeWitt interpretada ao longo das linhas da Teoria Quântica de de Broglie-Bohm, nas possíveis soluções quânticas de bounce que emergem delas. Em particular, mostramos que muitos bounces quânticos assimétricos são obtidos, os quais podem incorporar mecanismos não triviais de contra-reação, como produção quântica de partículas em torno do bounce, no modelo de fundo quântico em si. Em particular, a antiga hipótese de que nosso universo em expansão pode ter surgido de flutuações quânticas de um espaço-tempo plano quântico fundamental é recuperada, dentro de uma perspectiva diferente e ainda inexplorada.

Palavras-chaves: Bounces quânticos. Cosmologia quântica. Contra-reação.

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Introduction

The Penrose-Hawking singularity theorems in General Relativity [1] predict a singularity in space-time as the beginning of our universe. Since it is outside of the scope of the theory, this singularity is understood as a pathology, which may be solved by incorporating quantum gravitational effects. The Quantum Theory is required due to the extremely high energy densities and curvature in this domain, leading to modifications of General Relativity that would constitute a Quantum Theory of Gravity.

In order to combine General Relativity and Quantum Mechanics, one should account to its practical and conceptual aspects, which represent a huge challenge. The fundamental nature of Quantum Mechanics implies in its validity for any system, including the universe itself. However, when considering the standard interpretation of Quantum Mechanics, namely the Copenhagen interpretation [2, 3, 4], issues related to the measurement problem and the postulate of the collapse of the wave function take place [5]. According to this approach, the collapse of the wave function depends on a measurement performed by a classical external observer. If the whole universe constitutes the system under consideration, this external domain does not exist and the collapse cannot occur. One important improvement in this direction was made by the concept of decoherence [6], which explains the emergence of the classical domain through an interaction between the system and the environment, thus independently of an observer. However, the collapse of the wave function remains an open question, since decoherence accounts only for the diagonalization of the reduced density matrix, but not for the selection of a single element in the diagonal. In other words, decoherence explains why a superposition of classical systems is not observed, but does not justify the unicity of facts [7]. Therefore, a further improvement is still required, which is achieved, for instance, by the spontaneous collapse approach [8], the Many-Worlds [9] and the de Broglie-Bohm [10, 11] interpretations of Quantum Mechanics. The later is the one adopted in this work.

The de Broglie-Bohm interpretation is a deterministic theory in which trajectories in the configuration space are completely determined by the evolution of dynamical variables. The probabilistic character of Quantum Mechanics arises as a practical aspect related to the initial field configurations in a statistical manner. It is not an inherent indeterminacy of properties as in Copenhagen interpretation. In this approach, the measurement problem is solved by an effective collapse, through which one unique branch in the wave function is occupied, depending on the initial configuration of the system, while the other branches remain empty and do not communicate to each other. Thus a quantization of the whole universe is possible independently of the existence of an external observer, which brings the conceptual coherence necessary to apply it to Cosmology [5].

The Quantum Cosmology that arises in this approach is able to avoid the initial singularity through the emergence of a bounce [5, 12, 13], i.e. a contracting phase of the scale factor followed by an expanding phase, or even multiple bounces [14, 15]. This aspect is a consequence of a de Broglie-Bohm quantum potential, which modifies the Friedmann equations.

The aim of this work is to generalize cosmological models previously obtained in [5, 13], which are characterized by symmetric bounces, i.e. solutions in which the contracting phase is identical to the expanding phase reversed in time. With this purpose, enlarged prescriptions for the wave functions of the universe are considered, which guide the trajectory of the scale factor in the configuration space. Our aim is to obtain asymmetric bounces, which could describe non-linear back-reactions coming from particle production around the bounce, modifying the background evolution in the expanding phase. We achieved a variety of asymmetric quantum bounces by enlarging the prescriptions for the initial wave function of the universe in different contexts, which are presented in chapter 6.

The content of this dissertation is presented in the following sequence: part I contains the Bohmian Quantum Gravity, where the Bohmian Quantum Theory and the Wheeler-DeWitt quantization are introduced and combined. Part II presents the minisuperspace models filled with a perfect fluid and the quantum bounces, including the standard symmetric trajectory and the original asymmetric solutions of this work. The standard symmetric bounces are obtained from initial Gaussian wave functions centered at the origin and without phase velocity. In section 5.1, generalized symmetric bounces are presented as the result of an unitary evolution of an initial Gaussian centered at the origin with phase velocity, which adds a new parameter to the system. In section 6.1, we gave up unitarity, since it is not a mandatory requirement for minisuperspace wave functions in the de Broglie-Bohm theory. As a result, asymmetric quantum bounces are obtained. In section 6.2, we consider a superposition of two Gaussians, each of them multiplied by an exponential factor containing an extra parameter, and obtain asymmetric quantum bounces with unitary evolution. Finally, the results and future perspectives are discussed in Conclusion.

Part I

BOHMIAN QUANTUM GRAVITY

1 DE BROGLIE-BOHM QUANTUM THEORY

In 1952, an alternative to the Copenhagen interpretation of Quantum Mechanics was presented by David Bohm [10] with the purpose of formulating a theory able to deterministically describe individual systems in the quantum domain. According to this theory, the precise behaviour of quantum systems is determined by hidden dynamical variables [1] and the probabilistic character of the theory arises as a practical necessity, not as an inherent inability to determine properties in the quantum level. The wave function guides the trajectory in the configuration space and can be represented in the polar form $\Psi = Re^{iS/\hbar}$, where the amplitude R and the phase S are real functions. The wave function Ψ is known as pilot wave, since the theory leads to a guidance equation that determines the trajectory on the configuration space. As long as the usual form of the Schrödinger equation is maintained, the results of measurements in this alternative interpretation are the same as the ones obtained by the usual interpretation, however with a broader conceptual framework. Louis de Broglie also made contributions in this direction [16], even before David Bohm, for the description of one particle by the theory of the Double Solution. In this theory, every solution of the Schrödinger equation corresponds to a physical wave $v = ae^{i\phi/\hbar}$, where a and ϕ are real functions. The idea was given up by de Broglie due to criticisms, especially from Wolfgang Pauli. Given these contributions and aspects, the deterministic interpretation of Quantum Mechanics presented hereafter is known as the de Broglie-Bohm Quantum Theory, the pilot wave Quantum Theory or Causal Interpretation.

According to this approach, an individual system is in a completely determined state, even when unobserved, which evolves in time according to the equations of motion. As just mentioned, the trajectory is guided by the wave function Ψ, which satisfies the Schrödinger equation. The probabilistic character of the system in the quantum domain arises as a practical consequence of the existence of hidden variables, i.e. the unknown initial conditions of the system lead to a practical indeterminacy, which is not intrinsic to the quantum nature. A good analogy can be made by considering thermodynamics and atomic structure: many phenomena can be understood in macrophysics through the thermodynamic description, which correctly relates averaged processes. However, an analysis in atomic level provides deterministic reasons to them, explaining their properties in a more detailed framework.

One could question the necessity of formulating an alternative interpretation of

[1] The hidden dynamical variables are hypothetical entities, yet unobserved, which provide the deterministic description of quantum phenomena.

Quantum Theory, given the efficiency of the description provided by the Copenhagen interpretation. A simple reason relies in the fact that other equally consistent interpretations are possible, leading to the question of which one truly corresponds to nature. In addition, in the case of the de Broglie-Bohm theory, the ontological interpretation spares us of giving up a detailed description of quantum systems, which in itself constitutes an advantage of the new approach. One could then impugn the falsifiability of these different interpretations, given that they predict the same outcomes from the experiments. However, the de Broglie-Bohm Quantum Theory can, in principle, be tested through systems in the so called quantum non-equilibrium [17, 18]. This possibility arises given that the Born rule is not an assumption in this interpretation, allowing probability densities to be different from $|\Psi|^2$. When $\rho = |\Psi|^2$, we say that quantum equilibrium is satisfied. Another reason to investigate other interpretations, extremely relevant in this work, is the solution of the measurement problem, which is present in the Copenhagen interpretation. This issue arises from the lack of a well-established description of the collapse of the wave function. According to the Copenhagen interpretation, the wave function is constituted by linear superpositions of states and satisfies the Schrödinger equation. Somehow, when a measurement is performed, the wave function collapeses into one unique state through a process that is not described by the equation of motion. As we are going to show, the de Broglie-Bohm interpretation solves this problem through an effective collapse, allowing an elegant correspondence between quantum and classical realities.

In order to present the formulation of the de Brolgie-Bohm Quantum Theory, we start by considering the one particle description and then generalize it to an arbitrary number of particles. Consider the Schrödinger equation given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi, \tag{1.1}$$

where m is the mass of the particle and Ψ is a complex quantity that can be represented as

$$\Psi = Re^{iS/\hbar}, \tag{1.2}$$

where R and S are real functions. Substituting (1.2) in (1.1), we obtain the following equations:

$$\frac{\partial R}{\partial t} = -\frac{1}{2m}(R\nabla^2 S + 2\nabla R\cdot\nabla S), \tag{1.3}$$

$$\frac{\partial S}{\partial t} = -\left[\frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right]. \tag{1.4}$$

Defining the probability density P as $P = R^2 = |\Psi|^2$, one can rewrite them as

$$\frac{\partial P}{\partial t} + \nabla\cdot\left(P\frac{\nabla S}{m}\right) = 0, \tag{1.5}$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{4m}\left[\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right] = 0. \tag{1.6}$$

In the classical limit, i.e. when $\hbar \to 0$, the physical interpretation is clear: S is solution of the Hamilton-Jacobi equation. From Classical Mechanics we know that, if all the trajectories of an ensemble of the system are normal to a given surface of constant S, then $\nabla S/m$ is the velocity vector $\mathbf{v}$. Taking this identification into account in equation (1.5), we conclude that P can be in fact understood as a probability density, since (1.5) assumes the form of a continuity equation.

In terms of R, the generalized Hamilton-Jacobi equation (1.6) can be rewritten as

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0, \tag{1.7}$$

allowing the identification of a quantum potential Q, which becomes relevant in the quantum domain, and an effective potential U, given respectively by

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{1.8}$$

$$U = V + Q = V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{1.9}$$

where V is the classical potential. Classical mechanics is obtained when the quantum potential Q becomes negligible. The solution of the generalized Hamilton-Jacobi equation (1.7) represents an ensemble of possible trajectories, which are under the action of both potentials V and Q. In its turn, R depends on S through the continuity equation (1.3).

Let us analyze the identity $\nabla S = m\mathbf{v} = \mathbf{p}$. It allows us to obtain the trajectory of the particle under consideration, since it is orthogonal to the surface $S =$ constant, resulting in the following differential equation:

$$m\frac{d\mathbf{x}}{dt} = \mathbf{p}(\mathbf{x}(t), t) = \nabla S. \tag{1.10}$$

In its turn, the total energy of the particle, including the kinetic term $\mathbf{p}^2/2m$ and the effective potential U, is given by

$$H = -\frac{\partial S}{\partial t}. \tag{1.11}$$

Thus equations (1.7) e (1.10) determine, together with the boundary conditions, the movement of the particle, which can be described as a Newton's equation of the form

$$\frac{d\mathbf{p}}{dt} = -\nabla U \equiv -\nabla(V + Q). \tag{1.12}$$

Note that the term $-\nabla Q$ can be understood as a quantum force acting on the particle in addition to the force related to the classical potential V. Thus the wave function Ψ represents a field, being the amplitude R, which appears in the quantum potential Q, directly related to the force on the particle. In order to determine the trajectory of the particle univocaly, it is necessary that the initial conditions are established. In this context, they are given by the initial wave function $\Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x})$ and by the initial position of the particle $\mathbf{x}(0) = \mathbf{x_0}$.

We now turn our attention to the continuity equation (1.5), where the probabilistic character of Quantum Mechanics arises. In the case of quantum equilibrium, this equation restricts the solutions of the Schrödinger equation to the ones that are compatible with the initial probability distribution $|\Psi_0|^2$. It is important to emphasize that in this interpretation the statistical approach is a practical necessity due to unpredictable disturbances caused by the measurement. [2] In [10] Bohm mentions that, in extremely small scales, i.e. distances of order of 10^{-13} cm (a characteristic nuclear size) or less [3], $|\Psi|^2$ could no longer satisfy the continuity equation and, therefore, could no longer represent the probability density. This was not confirmed experimentally, but, as already mentioned, a similar idea is still considered nowadays: in principle, a possible experimental verification of this interpretation is the detection of relaxation from quantum non-equilibrium to the quantum equilibrium configuration.

Finally, we generalize the theory to many bodies, starting with two particles of same mass m. The correspondent Schrödinger equation is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla_1^2\Psi + \nabla_2^2\Psi) + V(\mathbf{x_1}, \mathbf{x_2})\Psi. \tag{1.13}$$

The wave function is again represented in the polar form $\Psi = Re^{iS/\hbar}$, but now $R = R(\mathbf{x_1}, \mathbf{x_2})$ and $S = S(\mathbf{x_1}, \mathbf{x_2})$. As before, we define $P \equiv R^2$ and obtain the following equations:

$$\frac{\partial P}{\partial t} + \frac{1}{m}[\nabla_1 \cdot P\nabla_1 S + \nabla_2 \cdot P\nabla_2 S] = 0, \tag{1.14}$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla_1 S)^2 + (\nabla_2 S)^2}{2m} + V(\mathbf{x_1}, \mathbf{x_2}) - \frac{\hbar^2}{2mR}[\nabla_1^2 R + \nabla_2^2 R] = 0. \tag{1.15}$$

We thus have a six-dimensional wave function Ψ and velocities $\nabla_1 S/m$ and $\nabla_2 S/m$, each of them related to a surface associated to a particle. The quantum potential acting on each one of the particles is now given in terms of $\mathbf{x_1}$ and $\mathbf{x_2}$, evidencing an effective interaction:

$$Q(\mathbf{x_1}, \mathbf{x_2}) = -\left(\frac{\hbar^2}{2mR}\right)[\nabla_1^2 R + \nabla_2^2 R]. \tag{1.16}$$

For n particles the generalization is straight forward. The wave function is represented as $\Psi = R(\mathbf{x_1}...\mathbf{x_n})e^{iS(\mathbf{x_1}...\mathbf{x_n})/\hbar}$ and the velocity of each particle is given by $\mathbf{v_i} = \nabla_i S(\mathbf{x_1}...\mathbf{x_n})/m$. In its turn, the quantum potential reads

$$Q(\mathbf{x_1}...\mathbf{x_n}) = -\frac{\hbar^2}{2mR}\sum_{i=1}^{n}\nabla_i^2 R(\mathbf{x_1}...\mathbf{x_n}), \tag{1.17}$$

evidencing again that the force acting on a single particle depends on the positions of all the other particles in the system. This aspect of the theory is known as non-locality.

---

[2] In the Copenhagen interpretation, the probabilistic character is due to an inherent limitation of determining the state of the system.

[3] In [19] Bohm reduces this critical length scale to 10^{-16} cm.

After the introduction to the formalism, we are now able to detail the solution of the measurement problem. As already mentioned, in the Copenhagen interpretation there are two possible evolutions for the wave function: one given by the Schrödinger equation and other given by its collapse. In principle, the collapse takes place when a measurement is performed. However, it is not clear what kind of process is considered a measurement and what kind of system is a measurement device. What is necessary in order to collapse a wave function? An observation made by an human being? There is no objective reality independent of observations? These issues do not arise in the Bohmian interpretation, where the dynamics is always described by the dynamical equation. An effective collapse of the wave function arises as a result of the Bohmian mechanics, which describes both the system s and the measuring apparatus A.

In order to develop the mathematical formalism to describe a measurement process, let us consider a one-body system s associated with a wave function $\phi(\mathbf{x}_s, t)$ and the particle variable $O(\mathbf{x}_s, t)$ associated with a self-adjoint operator $\hat{O}(\hat{\mathbf{x}}_s, \hat{p}_s)$. This system interacts with a measuring apparatus A described by an initial wave function $\zeta_0(x_A)$, where the one-dimension coordinate $x_A(t, t_0)$ defines the location of the meter needle. Assuming an impulsive interaction between the system s and the apparatus A, one can write the interaction Hamiltonian as $H = g\hat{O}\hat{p}_A$, where g is a coupling constant and $\hat{p}_A$ is the momentum operator conjugate to $\hat{x}_A$. The initial wave function of s and A is given by $\Psi_0(\mathbf{x}_s, x_A) = \phi_0(\mathbf{x}_s)\zeta_0(x_A)$, since they are initially independent. During the interaction, the dynamics is governed by the Schrödinger equation

$$i\frac{\partial\Psi(\mathbf{x}_s, x_A, t)}{\partial t} = -ig\hat{O}\frac{\partial\Psi(\mathbf{x}_s, x_A, t)}{\partial x_A} \tag{1.18}$$

and the wave function $\Psi(\mathbf{x}_s, x_A, t)$ becomes entangled in the configuration space. Expanding it in a complete set of eigenfunctions $\phi_\alpha(\mathbf{x}_s)$ of the operator O with eigenvalue α, we have

$$\Psi(\mathbf{x}_s, x_A, t) = \Sigma_\alpha f_\alpha(x_A, t)\phi_\alpha(\mathbf{x}_s). \tag{1.19}$$

From the Schrödinger equation during the interaction (1.18), we obtain that the coefficients f_α satisfy

$$\frac{\partial f_\alpha}{\partial t} = -g\alpha\frac{\partial f_\alpha}{\partial x_A}, \tag{1.20}$$

which leads to the solution $f_\alpha(x_A, T) = f_{\alpha 0}(x_A - g\alpha T)$, where T is the period of the impulse and $f_{\alpha 0}(x_A)$ are the initial values. The initial wave function $\phi_0(\mathbf{x}_s)$ can be expanded in terms of eigenfunctions $\phi_\alpha(\mathbf{x}_s)$, where the coefficients of the expansion C_α are such that $f_{\alpha 0}(x_A) = C_\alpha\zeta_0(x_A)$. As a result, the propagated wave function reads

$$\Psi(\mathbf{x}_s, x_A, T) = \Sigma_\alpha C_\alpha\phi_\alpha(\mathbf{x}_s)\zeta_0(x_A - g\alpha T). \tag{1.21}$$

The eigenvalues α are now correlated with the coordinate x_A of the apparatus and the packets ζ_0 have macroscopically disjoint supports in configuration space. Thus the wave function Ψ splits up into branches that do not overlap and can be represented by only one of them, which is selected by the initial field configuration, i.e. in which the particle enters. In other words, Ψ can be effectively collapsed to a specific branch Ψ_α, without the need of a collapse postulate. The initial field configuration determines which branch of the wave function is occupied, while the others remain empty and do not communicate with each other. More details on the measurement process in Bohmian Quantum Theory can be found in [20].

For the sake of completeness, it is important to mention that an improvement to the Copenhagen interpretation, called decoherence, allows for the emergence of classical properties from the quantum formalism through interactions of the system with the environment. It explains why we do not see quantum superpositions of macroscopic objects [6]. However, it does not explain the unicity of facts, i.e. the single outcome of a measurement, requiring a further improvement such as the Many Worlds interpretation, where all branches of the wave function coexist, or the de Broglie-Bohm Quantum Theory, where a single branch is selected by the initial configuration.

Detailed descriptions of the de Broglie-Bohm Quantum Theory can be found in [10, 11, 20, 21].

2 CANONICAL QUANTUM GRAVITY

Quantum Mechanics and Special Relativity were successfully combined in Quantum Electrodynamics, the most accurate physical theory nowadays. Therefore, the first attempts to combine General Relativity with Quantum Mechanics naturally emerged, leading to the so called Canonical Quantum Gravity.

The canonical quantization procedure considers the Hamiltonian of the system in order to write the dynamical equation. Thus a first step to the canonical quantization of General Relativity is the construction of its Hamiltonian formalism. Since General Relativity is covariant under coordinate transformations, not all the components of the metric tensor represent degrees of freedom. It leads to constraints containing components of the metric tensor and its canonically conjugate momenta to be satisfied. In order to account for this gauge freedom, Paul Dirac proposed a quantization of constrained systems [22]. In this approach, terms containing the constraints and its Lagrangian multipliers are added to the Hamiltonian.

Later on, the Hamiltonian formalism of General Relativity was described by Richard Arnowitt, Stanley Deser and Charles Misner in [23] through geometrical quantities that charactrize a foliation of spatial hypersurfices in the timelike direction. Thus the topology of the manifold under consideration is restricted to be $\mathbb{M}^4 = \mathbb{R} \bigotimes \mathbb{M}^3$. The spatial hypersurfaces are defined through $f(x^\mu) =$ constant, while the one-forms $\eta = \eta_\mu dx^\mu = \partial_\mu f dx^\mu$ define their normals. Choosing the timelike coordinate to be $x^0 = t$, we have $\eta_\mu = -N\delta^0_\mu$, where N is called lapse function and is such that $g^{\mu\nu}\eta_\mu\eta_\nu = -1$. As a result, $g^{00} = -1/N^2$. The quantity $h^{\mu\nu} \equiv g^{\mu\nu} + \eta^\mu\eta^\nu$ defines a projector onto the hypersurfaces with components given by

$$
\begin{aligned}
h^{00} &= 0, \\
h^{0i} &= 0, \\
h^{ij} &= g^{ij} + N^2 g^{i0} g^{j0}. \qquad (2.1)
\end{aligned}
$$

The inverse matrix, i.e. h_{ij} is the metric tensor of the spatial hypersurfaces. Note that h_{ij} and h^{ij} are symmetric.

In its turn, the components of the contravariant metric tensor are

$$
\begin{aligned}
g^{00} &= -\frac{1}{N^2}, \\
g^{0i} &= \frac{N^i}{N^2}, \\
g^{ij} &= h^{ij} - \frac{N^i N^j}{N^2}, \qquad (2.2)
\end{aligned}
$$

where $N^i \equiv g^{i0}N^2$ is called shift vector.

With these definitions, the line element reads

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= N^2dt^2 + h_{ij}(N^idt + dx^i)(N^jdt + dx^j), \end{aligned} \tag{2.3}$$

where the covariant component of the shift vector is defined as $N_i = h_{ij}N^j$.

The physical interpretation of the lapse function N and the shift vector N^i can be inferred from the line element (2.3): N is the rate of change of the proper time of an observer with four-velocity η^μ with respect to the coordinate t, while N^i is the rate of change of the shift of x^i from one hyperfurface to another with respect to t.

Since we are considering three-dimensional hypersurfaces embedded in a four-dimensional manifold, an important quantity is the extrinsic curvature. From differential geometry, it is defined as

$$K_{\mu\nu} \equiv -h^\alpha_\mu h^\beta_\nu \nabla_{(\alpha}\eta_{\beta)}, \tag{2.4}$$

leading to the following components:

$$K_{ij} = -N\Gamma^0_{ij} = \frac{1}{2N}(2D_{(i}N_{j)} - \partial_t h_{ij}), \tag{2.5}$$

where D_i is the three-dimensional covariant derivative.

In order to find the Lagrangian density in terms of these quantities, we first compute the Ricci scalar. Splitting the components, it can be written as

$$R = 2R^{0i}{}_{0i} + R^{ij}{}_{ij}. \tag{2.6}$$

Using the Gauss-Codazzi equations, given by

$$R^m{}_{ijk} = {}^{(3)}R^m{}_{ijk} + \frac{1}{\eta^\alpha\eta_\alpha}(K_{ij}K_k{}^m) - K_{ik}K_j{}^m, \tag{2.7}$$

$$R^0{}_{ijk} = -\frac{1}{\eta^\alpha\eta_\alpha}(D_kK_{ij} - D_jK_{ik}), \tag{2.8}$$

$$R^{i0}{}_{i0} = \frac{1}{\eta^\alpha\eta_\alpha}(K^2 - K_{ij}K^{ij}) - (\eta^\alpha\eta^\beta{}_{;\beta})_{;\alpha} + (\eta^\alpha\eta^\beta{}_{;\alpha})_{;\beta}, \tag{2.9}$$

where $^{(3)}$ denotes the three-dimensional quantity, and the following expression obtained from (2.7)

$$R^{ij}{}_{ij} =^{(3)} R + \frac{1}{\eta^\alpha\eta_\alpha}(K_{ij}K^{ij} - K^2), \tag{2.10}$$

where $K = K^i{}_i$, we rewrite the Ricci scalar (2.6) as

$$R =^{(3)} R + \frac{1}{\eta^\alpha\eta_\alpha}(K^2 - K_{ij}K^{ij}) - 2(\eta^\alpha\eta^\beta{}_{;\beta})_{;\alpha} + 2(\eta^\alpha\eta^\beta{}_{;\alpha})_{;\beta}. \tag{2.11}$$

In its turn, the Lagrangian density of the Einstein-Hilbert action is given by

$$\mathscr{L} = (-g)^{\frac{1}{2}}R, \tag{2.12}$$

where we substitute (2.11) and obtain

$$\mathscr{L} = (-g)^{\frac{1}{2}} \left[{}^{(3)}R + \frac{1}{\eta^\alpha \eta_\alpha}(K^2 - K_{ij}K^{ij}) - 2(\eta^\alpha \eta^\beta{}_{;\beta})_{;\alpha} + 2(\eta^\alpha \eta^\beta{}_{;\alpha})_{;\beta} \right]. \tag{2.13}$$

Disregarding the boundary terms, since they will not contribute to the action after the integration and using $(-g)^{1/2}d^4x = Nh^{1/2}dtd^3x$ and $\eta^\alpha \eta_\alpha = -1$, we find

$$\mathscr{L} = Nh^{\frac{1}{2}}({}^{(3)}R + K^{ij}K_{ij} - K^2). \tag{2.14}$$

Note that (2.14) does not depend on $\partial_t N$ or $\partial_t N^i$. Thus the canonically conjugate momenta to N and N^i, given by $\delta\mathscr{L}/\delta(\partial_t N)$ and $\delta\mathscr{L}/\delta(\partial_t N^i)$, are zero. According to the formalism presented in [22], this defines these two quantities as primary constraints and therefore the Hamiltonian must include N and N^i as Lagrangian multipliers. On the other hand, the metric tensor of the hypersurfaces has a canonically conjugate momentum, which reads

$$\Pi_{ij} = \frac{\delta L}{\delta(\partial_t h^{ij})} = -h^{\frac{1}{2}}(K_{ij} - h_{ij}K). \tag{2.15}$$

The Hamiltonian density is given by

$$\mathcal{H} = \Pi^{ij}\dot{h}_{ij} - \mathscr{L}. \tag{2.16}$$

Substituting expressions (2.14, 2.15) we find

$$\mathcal{H} = N\left[h^{-\frac{1}{2}}\left(-\frac{\Pi^2}{2} + \Pi_{ij}\Pi^{ij}\right) - h^{\frac{1}{2}}{}^{(3)}R\right] + 2D_i(N_j\Pi^{ij}) - 2N_jD_i\Pi^{ij}, \tag{2.17}$$

where $\Pi = \Pi_a{}^a$. The term of the total derivative is a boundary term and, therefore, is neglected. Defining the DeWitt metric $G_{ijkl} \equiv h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})/2$ and noting that

$$G_{ijkl}\Pi^{ij}\Pi^{kl} = h^{-\frac{1}{2}}\left(\Pi_{kl}\Pi^{kl} - \frac{\Pi^2}{2}\right), \tag{2.18}$$

we rewrite the Hamiltonian density as

$$\mathcal{H} = N\left(G_{ijkl}\Pi^{ij}\Pi^{kl} - h^{\frac{1}{2}}{}^{(3)}R\right) - 2N_jD_i\Pi^{ij}. \tag{2.19}$$

Thus the gravitational action assumes the form

$$S = \frac{1}{16\pi}\int \left[\Pi^{ij}\dot{h}_{ij} - N\left(G_{ijkl}\Pi^{ij}\Pi^{kl} - h^{\frac{1}{2}}{}^{(3)}R\right) + N_i 2D_i\Pi^{ij}\right] dtd^3x. \tag{2.20}$$

Through the variational principle with respect to N and N_j, we obtain the components 00 and $0j$ of Einstein equations for $R_{\mu\nu}$=0, which correspond to

$$\mathscr{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - h^{\frac{1}{2}}{}^{(3)}R \approx 0, \tag{2.21}$$

$$\mathscr{H}^j = -2D_i\Pi^{ij} \approx 0, \tag{2.22}$$

where $\approx$ denotes the weakly equal sign. It means that the equation is satisfied when the constraints are taken into account, but not throughout the phase space. These are the secondary constraints and are called superhamiltonian and supermomentum, respectively.

By requiring that (2.21) and (2.22) do not vary in time, we do not obtain any other constraint. Thus we have concluded the analysis of the constraints and are able to write the Hamiltonian as

$$H = \int d^3x (N\mathscr{H} + N_j \mathscr{H}^j). \tag{2.23}$$

Note that the boundary terms disregarded when constructing the Lagrangian density would appear in the Hamiltonian if they were different from zero. It could be the case for spaces that are not closed. However, for closed spaces, we see that the Hamiltonian vanishes, since it is constituted by a linear combination of the constraints. Given that H represents the energy of the system, we conclude that the energy of a closed universe is zero.

In possession of the Hamiltonian (2.23), we are able to apply the quantization procedure to the theory. It starts by transforming the canonical variables into operators such that

$$i\hbar\{A, B\} \equiv [\hat{A}, \hat{B}], \tag{2.24}$$

where the braces represent the Poisson bracket and the brackets represent the commutator. The metric of hypersurfaces becomes an operator $\hat{h_{ij}}(\mathbf{x})$ that acts on wave functionals $\Psi(h_{ij}, \phi)$, where ϕ represents the matter degrees of freedom that can be present. Thus the Hamiltonian (2.23) leads to the following functional Schrödinger equation:

$$i\partial_t \Psi = \int d^3x (N\hat{\mathscr{H}} + N_i \hat{\mathscr{H}}^i)\Psi. \tag{2.25}$$

When it comes to a constraint $f(q,p) = 0$, the computation of $[f, A]$ vanishes for any operator A. In this case, this quantity in terms of operators is given by $f(\hat{q}, \hat{p})\Psi = 0$. Hence the constraints of General Relativity become

$$\hat{\mathscr{H}}\Psi = 0, \tag{2.26}$$

$$\hat{\mathscr{H}}^j\Psi = 0. \tag{2.27}$$

The wave functional $\Psi(h_{ij}, \phi)$ needs to satisfy both the Schrödinger equation (2.25) and the constraints (2.26,2.27). Equation (2.26) is known as the Wheeler-DeWitt equation. It is related to the constraint that corresponds to the invariance of the theory under general transformations of the time coordinate. Hence, it is expected that it establishes the dynamics of the wave function with respect to a time encompassed in the canonical variables. Taking into account that the time derivative in Schrödinger equation appears linearly, we usually search for time in linear terms of a conjugate momentum. However, $\hat{\mathscr{H}}$ does not have a term with this behaviour in general, such that identifying the time in this approach becomes a complicated task. Another way to identify the called problem of time is by remembering that the Hamiltonian (2.21) is constituted by the constraints, which are null. Thus, substituting it in the Schrödinger equation, one obtains $\partial_t \Psi = 0$, i.e. the wave function does not depend on time t. One possible solution is to understand time as a classical notion [24]. Another, which we apply in this work, is to identify the time in matter degrees of freedom, in a way that (2.26) assumes the Schrödinger form.

Another challenge is related to the probabilistic interpretation of the theory, because this presupposes a positive definite probability density, which results in 1 when integrated over all the possibilities. However, equation (2.26) is not always a Schrödinger equation. In the case of a Klein-Gordon equation, for instance, the probability density is not positive definite. One solution to this problem is to consider an interpretation of Quantum Mechanics that does not rely in probabilities, which is the case of the de Broglie-Bohm Quantum Theory.

In its turn, equation (2.27) is related to the invariance of the theory under changes of h_{ij} caused by spatial coordinates transformations and is called diffeomorphism constraint. Thus the wave function is a functional of the equivalence class of metrics h_{ij}, which describe the same geometry. The space of the three-dimensional spacelike geometries is called superspace.

Equation (2.26) is in general difficult to solve, which we overcome in this work by considering symmetries in order to construct the so called minisuperspace models.

For the sake of completeness, it is worth mentioning that an important improvement is achieved by simplifying (2.21) with the called Ashtekar variables [25], which allow us to rewrite this constraint in well known forms with well established quantizations. This procedure constitutes the Loop Quantum Gravity approach.

3 BOHMIAN QUANTUM GRAVITY

We now combine the formalism of the Bohmian Quantum Theory with the formalism of canonical quantization. We start by substituting (2.22) in the superspace constraint (2.27), where we also considered the matter field ϕ described by a canonical kinetic term and a potential V. We then obtain

$$-2D_i\Pi^i_j + \Pi_\phi \partial_j \phi = 0. \tag{3.1}$$

Substituting the canonically conjugate momenta by $\delta/\delta h_{ij}$, we find

$$-2h_{ik}D_j \frac{\delta\Psi}{\delta h_{jk}} + \frac{\delta\Psi}{\delta\phi}\partial_i\phi = 0. \tag{3.2}$$

Repeating this procedure with (2.21) and (2.26) and returning $\hbar$ and c in the expressions for a while, we obtain

$$\left[-\hbar^2\left(\kappa G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \frac{1}{2}h^{-\frac{1}{2}}\frac{\delta^2}{\delta\phi^2}\right) + V\right]\Psi = 0, \tag{3.3}$$

where $\kappa = 16\pi G/c^4$ and V is the classical potential. Equation (3.3) still needs to be regularized, since it has products of local operators at the same point.

Writing Ψ in the form $Re^{iS/\hbar}$ and substituting it in (3.2,3.3), we obtain the following real equations:

$$\begin{aligned}
-2h_{ik}D_j\frac{\delta S}{\delta h_{jk}} + \frac{\delta S}{\delta\phi}\partial_i\phi &= 0, &(3.4)\\
-2h_{ik}D_j\frac{\delta R}{\delta h_{jk}} + \frac{\delta R}{\delta\phi}\partial_i\phi &= 0, &(3.5)\\
\kappa G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} + \frac{1}{2}h^{-\frac{1}{2}}\left(\frac{\delta S}{\delta\phi}\right)^2 + V + Q &= 0, &(3.6)\\
\kappa G_{ijkl}\frac{\delta}{\delta h_{ij}}\left(R^2\frac{\delta S}{\delta h_{kl}}\right) + \frac{1}{2}h^{-\frac{1}{2}}\frac{\delta}{\delta\phi}\left(R^2\frac{\delta S}{\delta\phi}\right) &= 0. &(3.7)
\end{aligned}$$

In the non-regularized form, the quantum potential Q reads

$$Q = -\frac{\hbar^2}{R}\left(\kappa G_{ijkl}\frac{\delta^2 R}{\delta h_{ij}\delta h_{kl}} + \frac{1}{2}h^{-\frac{1}{2}}\frac{\delta^2 R}{\delta\phi^2}\right), \tag{3.8}$$

where one can see its dependence on $\hbar$.

Equations (3.4) and (3.6) correspond to generalized Hamilton-Jacobi equations in General Relativity, where the quantum potential Q brings new effects to the classical approach.

The guidance equations of the de Broglie-Bohm theory are given by

$$\Pi^{ij} = \frac{\delta S}{\delta h_{ij}}, \tag{3.9}$$

$$\Pi_{\phi} = \frac{\delta S}{\delta \phi}, \tag{3.10}$$

and result in the following trajectories:

$$\dot{h_{ij}} = 2NG_{ijkl}\frac{\delta S}{\delta h_{kl}} + D_i N_j + D_j N_i, \tag{3.11}$$

$$\dot{\phi} = Nh^{-\frac{1}{2}}\frac{\delta S}{\delta \phi} + N^i \partial_i \phi. \tag{3.12}$$

As we have seen, the dynamics described by the Bohmian theory does not depend on the choice of coordinates on the hypersurfaces, which is mathematically represented by the diffeomosphism constraint related to the shift vector N^i. Thus we choose $N^i = 0$ for simplicity. In its turn, the lapse function N describes the foliation. Two different choices of N that only differ by a factor that depends only on time t do not describe different foliations, but rather represent a time parameterization. Thus they do not lead to different dynamics. However, if they differ by something that does not depend only on time t, then the Bohmian dynamics is different. This constitutes a peculiar aspect of this theory, since General Relativity does not depend on the choice of foliation. A brief comparison between this situation and the case of non-locality and Lorentz invariance in Special Relativity is mentioned in [26] and more details on these calculations are given in [27].

Equation (3.11) plays an important role in the solution of the problem of time. Although $\partial_t \Psi = 0$, $\dot{h_{ij}}$ is not zero, allowing us to obtain information about the evolution of the scale factor of the universe. As mentioned before, in this work we also incorporate matter degrees of freedom that will account for the time dependence of the wave function Ψ.

Writing the components of the metric tensor in the ADM formalism, the Bohmian trajectory of h_{ij} given by (3.11) results in

$$G_{\mu\nu} = 8\pi G T_{Q\mu\nu}, \tag{3.13}$$

where $T_Q^{\mu\nu}$ reads

$$T_Q^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int d^4x' N(x')Q(x') \tag{3.14}$$

and

$$Q = -\frac{8\pi G G_{ijkl}}{|\Psi|}\frac{\delta^2|\Psi|}{\delta h_{ij}\delta h_{kl}} \tag{3.15}$$

is the quantum potential. Thus $T_Q^{\mu\nu}$ appears due to quantum effects and not due to the presence of matter. In the classical domain, $T_Q^{\mu\nu}$ vanishes and (3.13) returns to the classical

Einstein equations. The quantum potential (3.15) is responsible for the emergence of the bounce in cosmological scenarios, solving the singularity problem.

Now let us briefly mention a particularity about the probability interpretation in this context. As we have seen in chapter 1, $|\Psi|^2$ satisfies the continuity equation (1.5), hence its interpretation as a probability density is consistent from this point of view. However, when applied to Quantum Cosmology in order to describe the entire universe, Bohmian Quantum Gravity does not imply in an ensemble of universes (it is the case in the Many-Worlds interpretation). Thus a probabilistic procedure does not apply. The statistical approach plays an important role only when dealing with subsystems of the universe. It is an important aspect that allow us to consider non-unitary evolutions of the wave function of the universe, which is developed in section 6.1.

Part II

BOHMIAN QUANTUM COSMOLOGY

4 MINISUPERSPACE MODELS WITH PERFECT FLUID

As mentioned in chapter 2, the Wheeler-DeWitt equation (2.26) and the diffeomorphism constraint (2.27) are not easily solved in general. In order to reduce the degrees of freedom and make them simpler, we resort to symmetries, which is physically justified by the homogeneity and isotropy of the universe. Mathematically, this simplification is made by freezing degrees of freedom, reducing the superspace to a minisuperspace. We first perform expansions of the metric of the hypersurfaces h^{ij}, the matter degrees of freedom, which we denote by ϕ^A, and their canonically conjugate momenta, Π_{ij} and Π_A , in a complete set f_n, which represents space dependent modes. These expansions read

$$h^{ij}(x,t) = h^{ij}_{(0)}(t) + \Sigma_{n=1}^{\infty} h^{ij}_{(n)}(t) f_n(x), \tag{4.1}$$

$$\phi^{A}(x,t) = \phi^{A}_{(0)}(t) + \Sigma_{n=1}^{\infty} \phi^{A}_{(n)}(t) f_n(x), \tag{4.2}$$

$$\Pi_{ij}(x,t) = \Pi^{(0)}_{ij}(t) + \Sigma_{n=1}^{\infty} \Pi^{(n)}_{ij}(t) f_n(x), \tag{4.3}$$

$$\Pi_{A}(x,t) = \Pi^{(0)}_{A}(t) + \Sigma_{n=1}^{\infty} \Pi^{(n)}_{A}(t) f_n(x). \tag{4.4}$$

We then equal some of the $h^{ij}_{(n)}$, $\phi^{A}_{(n)}$, $\Pi^{(n)ij}$ and $\Pi^{(n)}_{A}$ to zero. The finite amount of them that remain constitutes the minisuperspace.

Let us see how the symmetries are incorporated in the classical action. Consider the line-element (2.3) written in the ADM formalism with $N^i = 0$:

$$ds^2 = -N^2(t)dt^2 + h_{ij}(x,t)dx^i dx^j. \tag{4.5}$$

The degrees of freedom that remain after the freezing of the other modes are represented by q^a, where $a = 1, ..., n$ enumerates them. The metric h_{ij} can be restricted in the following manner: we start by writing it in the form

$$h_{ij}(x,t)dx^i dx^j = a^2(t) d\Omega_3^2, \tag{4.6}$$

where Ω_3^2 represents the three-sphere and $q^1 = a$.

The Einstein-Hilbert action in the ADM formalism is given by the Lagrangian density (2.14). Considering the three-dimensional Ricci scalar from (2.10) and the expression for the extrinsic curvature (2.5) and substituting h_{ij} by (4.6), we obtain a Lagrangian of the form

$$L = N\left[\frac{1}{2N^2} f_{ab}(q)\dot{q}^a\dot{q}^b - V(q)\right], \tag{4.7}$$

where $f_{ab}(q)$ is the reduced DeWitt metric G_{ijkl} and $V(q)$ is a potential.

In order to obtain the Hamiltonian we first compute the conjugate momentum to a, which reads

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = f_{ab}(q)\frac{\dot{q}^b}{N}. \tag{4.8}$$

Substituting it in the Hamiltonian, we find

$$H = p_a\dot{q}^a - L = N\left[\frac{1}{2}f^{ab}(q)p_a p_b + V(q)\right]. \tag{4.9}$$

The classical equations of motion that result from the Hamiltonian (4.9) are

$$\dot{q}^a = Nf^{ab}(q)p_b, \tag{4.10}$$

$$\dot{p}_a = -N\left[\frac{1}{2}\frac{\partial f^{bc}(q)}{\partial q^a}p_b p_c + \frac{\partial V(q)}{\partial q^a}\right], \tag{4.11}$$

while the constraint $\mathscr{H} \approx 0$ yields

$$\frac{1}{2}f^{ab}(q)p_a p_b + V(q) \approx 0. \tag{4.12}$$

As we show in chapter 5, the canonical quantization of this model is made with a specific choice of operator ordering, leading to the following operator corresponding to the Hamiltonian (4.9):

$$\hat{\mathscr{H}} = -\frac{1}{2}\frac{1}{\sqrt{f}}\frac{\partial}{\partial q^a}\left[f^{ab}(q)\sqrt{f}\frac{\partial}{\partial q^b}\right] + V(q), \tag{4.13}$$

where $f \equiv \det(f^{ab^{-1}})$. The Wheeler-DeWitt equation becomes $\hat{\mathscr{H}} = 0$ and the guidance equations related to $\Psi = Re^{iS/\hbar}$ are given by

$$\dot{q}^a = Nf^{ab}(q)\frac{\partial S}{\partial q^b}. \tag{4.14}$$

The classical equations of motion (4.10,4.11) are recovered by identifying $\partial S/\partial q^a \equiv p_a$ and making $Q \to 0$, where the quantum potential Q is given by

$$Q = -\frac{1}{2\sqrt{f}|\Psi|}\frac{\partial}{\partial q^a}\left(f^{ab}(q)\sqrt{f}\frac{\partial}{\partial q^b}|\Psi|\right). \tag{4.15}$$

Note that the minisuperspace models do not encompass a full Quantum Theory of Gravity and neither represent a systematic approximation to it. However, it seems that the results obtained through this approach are consistent within the formalism and are reasonable enough in order to justify investigations of their physical content. We then assume the conjecture that these models bring some information from the full theory. More details and other possibilities of minisuperspace models can be found in [28].

The quantization procedure is detailed in the next chapter. For now we consider a classical cosmological model where the universe is filled with a perfect fluid, which is represented by the following Lagrangian:

$$L_M = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^n, \tag{4.16}$$

where ϕ is the potential related to the four-velocity of the fluid, which reads

$$U_\mu = \frac{\partial_\mu \phi}{\sqrt{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}}. \tag{4.17}$$

From the definition of the energy-momentum tensor, we obtain

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial L_M}{\partial g^{\mu\nu}} = 2n\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^n U_\mu U_\nu - g_{\mu\nu}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^n. \tag{4.18}$$

From the expression of the energy-momentum tensor for a perfect fluid with equation of state $P = \omega\rho$, given by

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu}, \tag{4.19}$$

where ρ is the energy density and P is the pressure, we identify

$$P = \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^n, \tag{4.20}$$

$$\omega = \frac{1}{2n-1}. \tag{4.21}$$

Defining the canonically conjugate momentum to ϕ as p_ϕ and using expressions (4.20, 4.21), we obtain the Hamiltonian

$$H_M = \frac{1}{\omega(\sqrt{2}n)^{1+\omega}} N \frac{p_\phi^{1+\omega}}{a^{3\omega}}. \tag{4.22}$$

In order to simplify expression (4.22), we perform a canonical transformation given by

$$T = \frac{\omega(\sqrt{2}n)^{1+\omega}}{1+\omega}\frac{\phi}{p_\phi^{1+\omega}}, \tag{4.23}$$

$$P_T = \frac{1}{\omega}\left(\frac{p_\phi}{\sqrt{2}n}\right)^{1+\omega}, \tag{4.24}$$

which leads to

$$H_M = N\frac{P_T}{a^{3\omega}}. \tag{4.25}$$

The same result can be achieved through the Schutz formalism, which is detailed in [29].

For the gravitational part of the action we consider the Friedmann-Lemaître-Robertson-Walker line-element

$$ds^2 = -N^2 dt^2 + a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{4.26}$$

where k is the spatial curvature.

The Lagrangian density (2.14) constitutes the gravitational part of the Einstein-Hilbert action, which, considering (4.26), reads

$$S_G = \int dt d^3x \left(\frac{\ddot{a}a^2}{N} - \frac{\dot{a}\dot{N}a^2}{N^2} + \frac{6\dot{a}^2 a}{N} + kNa\right). \tag{4.27}$$

Integrating by parts and disregarding the boundary terms, we obtain

$$S_G = \int dt d^3x \left(-\frac{a\ddot{a}^2}{N} + kaN \right). \quad (4.28)$$

The canonically conjugate momentum to a is

$$P_a = \frac{\partial \mathscr{L}_G}{\partial \dot{a}} = -\frac{2a\dot{a}}{N}, \quad (4.29)$$

yielding the following Hamiltonian:

$$H_G = N\left(-\frac{P_a^2}{4a} - 6ka \right). \quad (4.30)$$

Joining the gravitational part to the matter part in an universe with $k = 0$, which seems to be favored by observations, we obtain the following minisuperspace Hamiltonian:

$$H = N\left(-\frac{P_a^2}{4a} + \frac{P_T}{a^{3\omega}} \right), \quad (4.31)$$

which leads to the superhamiltonian constraint (2.21) of the form

$$N\left(-\frac{P_a^2}{4a} + \frac{P_T}{a^{3\omega}} \right) = 0. \quad (4.32)$$

This procedure yields a configuration space constituted by the dinamical variables a and T, together with their conjugate momenta P_a and P_T.

Note that the Hamiltonian (4.31) has a momentum appearing linearly, which corresponds to the expected behavior of a time variable in Schrödinger equation. Thus this variable related to the perfect fluid plays the role of time in this model. Beyond that, $\dot{T} = N/a^{3\omega}$, which means that T changes monotonically, being consistent with the interpretation of a time variable.

In this classical approach, we have $a \propto T^{2/3(1-\omega)}$ for $\omega \neq 1$, which represents a singularity at $T = 0$. In the next chapter we perform the quantization of this model and show how the bounce solutions are obtained. Chapters 5 and 6 correspond to the paper [30].

5 SYMMETRIC QUANTUM BOUNCES

As derived in chapter 4, leading to equation (4.31), for a flat, homogeneous and isotropic universe filled with a perfect fluid with equation of state $P = \omega\rho$, where P is the pressure, ρ is the energy density and ω is the equation of state parameter, the ADM [23] formalism leads to the following Hamiltonian

$$H = \frac{L_p^2}{V} N H_0, \tag{5.1}$$

where N is the lapse function, L_p is the Planck length and V is the volume of the co-moving homogeneous three-dimensional hypersurface, which we are supposing to be compact. The factor L_p^2/V was reinserted in order to make the Hamiltonian's dimension explicit. Since we are using natural units, $\hbar = c = 1$, all canonical variables contained in NH_0 are dimensionless, and the Hamiltonian has dimension of energy = 1/length, as it should be. As we have shown, H_0 is given by

$$H_0 \equiv \frac{P_T}{a^{3\omega}} - \frac{P_a^2}{4a}, \tag{5.2}$$

where a is the scale factor of the universe, T is the parameter related to the degree of freedom of the fluid, which plays the role of time, P_a and P_T are their respective canonically conjugate momenta. The constant L_p^2/V will be absorbed in the definition of time later on, yielding a dimensionless cosmic time. The Friedmann equations can be readily obtained from the Hamiltonian

$$H = NH_0. \tag{5.3}$$

Applying the Dirac quantization procedure for constrained systems, where the wave function is annihilated by the the constraint operator, $\hat{H}_0\Psi = 0$, and taking into account a particular choice of the factor ordering [31], which leads to a Schrödinger equation with a covariant Laplacian under redefinitions of a, we arrive at the following Wheeler-DeWitt equation:

$$i\frac{\partial}{\partial T}\Psi(a,T) = \frac{a^{(3\omega-1)/2}}{4}\frac{\partial}{\partial a}\left[a^{(3\omega-1)/2}\frac{\partial}{\partial a}\right]\Psi(a,T). \tag{5.4}$$

Performing the variable transformation given by

$$\chi = \frac{2}{3(1-\omega)} a^{3(1-\omega)/2}, \tag{5.5}$$

we obtain

$$i\frac{\partial\Psi(\chi,T)}{\partial T} = \frac{1}{4}\frac{\partial^2\Psi(\chi,T)}{\partial\chi^2}, \tag{5.6}$$

which can be identified as a Schödinger equation for a free particle of mass $m = 2$ and negative kinetic energy in one dimension. The solutions of equation (5.6) are the wave functions of the universe. With the choice $N = a^{3\omega}$ for the lapse function, the parameter T relates to the dimensionless cosmic time $t = (L_p^2/V)t_c$ through $dt = a^{3\omega}dT$, where t_c is the usual cosmic time, with dimension of length.

Since the scale factor a and, consequently, the variable χ must assume positive values, we are dealing with a Schrödinger equation for a particle with negative kinetic energy in the half axis [32]. In order to obtain unitary solutions and, as a consequence, a consistent probabilistic interpretation, it is necessary to perform a self-adjoint extension, that is, to consider the perfectly reflecting boundaries, which are given by the following condition:

$$\left(\Psi^*\frac{\partial\Psi}{\partial\chi} - \Psi\frac{\partial\Psi^*}{\partial\chi}\right)\Bigg|_{\chi=0} = 0. \tag{5.7}$$

Note, however, that the de Broglie-Bohm Quantum Theory is a dynamical fundamental theory, where probabilities arise in a secondary step, as in Classical Mechanics. And indeed, a probabilistic interpretation of the wave function of the Universe may not make sense, since there is only one universe in this approach. A probabilistic interpretation is required only for subsystems in the Universe, where we can perform measurements. In this situation, one can use the so called conditional wave functions for subsystems, in which the Wheeler-DeWitt equation reduces to an unitary Schrödinger form, and a probabilistic interpretation where the Born rule is valid can be recovered, which is called quantum equilibrium, see [33, 34] for details. Of course this opens the possibility that during this process violations of standard Quantum Mechanics might occur. Unfortunately, almost all systems in Nature have evolved to the quantum equilibrium phase, where the probability distribution is described by ρ, see [35, 36] for detailed investigations about this process, and possible exceptions. Concluding, in what follows, we will not require unitary evolution as necessary feature of the minisuperspace wave function.

Writing the wave function as $\Psi = Re^{iS}$, and substituting into equation (5.4), we obtain two real equations,

$$\frac{\partial\rho}{\partial T} - \frac{\partial}{\partial a}\left[\frac{a^{(3\omega-1)}}{2}\frac{\partial S}{\partial a}\rho\right] = 0, \tag{5.8}$$

$$\frac{\partial S}{\partial T} - \frac{a^{(3\omega-1)}}{4}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{a^{(3\omega-1)/2}}{4R}\frac{\partial}{\partial a}\left[a^{(3\omega-1)/2}\frac{\partial R}{\partial a}\right] = 0, \tag{5.9}$$

where $\rho(a, T) = a^{(1-3\omega)/2}|\Psi|^2$.

The key feature of the de Broglie-Bohm Quantum Theory is to assume that positions in configuration space (in our case a) have objective reality, independently of any observation, and satisfy the guidance equation

$$\dot{a} = -\frac{a^{(3\omega-1)}}{2}\frac{\partial S}{\partial a}, \tag{5.10}$$

or

$$\frac{d\chi}{dT} = -\frac{1}{2}\frac{\partial S}{\partial \chi}. \tag{5.11}$$

With equation (5.10), one can interpret (5.8) as a continuity equation for the distribution ρ and (5.9) as a generalized Hamilton-Jacobi equation supplemented by the quantum potential

$$Q \equiv -\frac{a^{(3\omega-1)/2}}{4R}\frac{\partial}{\partial a}\left[a^{(3\omega-1)/2}\frac{\partial R}{\partial a}\right]. \tag{5.12}$$

If one wants to recover the physical dimensions of equations (5.8, 5.9), one can easily verify that Planck constant $\hbar$ re-appears only multiplying the quantum potential, $Q \to \hbar^2 Q$. Hence Q brings the quantum effects to the dynamics. Since the total energy given by (5.9) includes also the quantum potential Q, the trajectory given by (5.10) will not be the same as the classical one, unless Q is negligible with respect to the other terms. This effect is responsible for the emergence of the quantum bounce, avoiding the standard classical initial singularity.

Let us consider an initial wave function of the universe given by

$$\Psi_0(\chi) = \left(\frac{8}{\pi\sigma^2}\right)^{\frac{1}{4}} \exp\left(-\frac{\chi^2}{\sigma^2}\right), \tag{5.13}$$

which satisfies the boundary condition (5.7). In order to obtain a unitary evolution, we must apply the correspondent propagator to the Wheeler-DeWitt equation (5.6) considering the boundary condition (5.7). It means that we must sum two propagators of a Schrödinger equation with negative kinetic energy and mass $m = 2$ (which corresponds to the obtained Wheeler-DeWitt equation), one to χ_0 and another to $-\chi_0$. The propagator of a Schrödinger equation for a free particle with mass $m = 2$ is given by

$$G(\chi, \chi_0, T) = \sqrt{-\frac{i}{\pi T}} \exp\left[\frac{i(\chi-\chi_0)^2}{T}\right]. \tag{5.14}$$

In order to account for the negative kinetic term in the Wheeler-DeWitt equation, we must take the complex conjugate of the argument in the exponential. The overall factor $\sqrt{-i/\pi T}$ does not need to be changed, since it simply multiplies the both sides of the Wheeler-DeWitt equation. Therefore, summing the propagator to χ_0 and the propagator to $-\chi_0$, we obtain

$$G(\chi, \chi_0, T) = \sqrt{-\frac{i}{\pi T}} \exp\left[-\frac{i(\chi-\chi_0)^2}{T}\right] + \sqrt{-\frac{i}{\pi T}} \exp\left[-\frac{i(\chi+\chi_0)^2}{T}\right]. \tag{5.15}$$

The propagator (5.15) is not the most general one that satisfies the boundary condition (5.7). One could, for instance, change the relative sign to minus in order to obtain $G(\chi = 0) = 0$. However, this propagator leads to a trivial solution for the propagated wave function of the universe. Thus, in practice, the propagator that results in a non-trivial solution satisfies a more restrictive boundary condition, which is given by the von Neumann condition $\partial_\chi G|_{\chi=0} = 0$. Superpositions of the propagators with relative signs plus and minus with a phase difference of $\pm\pi/2$ are also allowed. However, the only difference in the propagated wave function is a factor that does not modify the Bohmian trajectories.

Applying (5.15) to the initial wave function (5.13), i.e. performing the integration

$$\Psi(\chi, T) = \int_0^\infty G(\chi, \chi_0, T)\Psi_0(\chi_0, T)d\chi_0, \tag{5.16}$$

we arrive at the wave function for all times

$$\begin{aligned}\Psi(\chi, T) &= \left[\frac{8\sigma^2}{\pi(\sigma^4 + T^2)}\right]^{\frac{1}{4}} \exp\left[-\frac{\sigma^2\chi^2}{\sigma^4 + T^2}\right] \\ &\times \exp\left[-i\left(\frac{T\chi^2}{\sigma^4 + T^2} + \frac{1}{2}\arctan\left(\frac{\sigma^2}{T}\right) - \frac{\pi}{4}\right)\right],\end{aligned} \tag{5.17}$$

which also satisfies equation (5.7). Using the phase S of the above wave function, we are able to obtain the trajectory of the parameter χ through equation (5.11). It reads

$$\chi(T) = \chi_b\left[1 + \left(\frac{T}{\sigma^2}\right)^2\right]^{\frac{1}{2}}, \tag{5.18}$$

where χ_b is the value of χ at the bounce, which occurs at $T = 0$. One can recover the classical solution by taking a Gaussian infinitely peaked. In order to do that, one should consider the differential equation with initial condition $\chi_0 = \chi(T_0)$, which leads to the solution

$$\chi(T) = \chi_0\frac{\sqrt{T^2 + \sigma^4}}{\sqrt{T_0^2 + \sigma^4}}. \tag{5.19}$$

Then, by making $\sigma^2 \to 0$, the classical cosmology given by $\chi(T) = \chi_0 T/T_0$ is obtained.

In terms of the scale factor a one gets

$$a(T) = a_b\left[1 + \left(\frac{T}{\sigma^2}\right)^2\right]^{\frac{1}{3(1-\omega)}}, \tag{5.20}$$

where a_b and χ_b are related also through equation (5.5). Equation (5.20) describes a symmetric bounce, which is plotted in figure 1. It tends to the classical solution for large values of T: $a \propto t^{1/2}$ for radiation fluid ($\omega = 1/3$) and $a \propto t^{2/3}$ for dust fluid ($\omega = 0$).

A good model for the perfect hydrodynamical fluid in the early universe, where all particles are highly relativistic, is a radiation fluid with $w = 1/3$, which will be considered from now on. Note that, in this case, $T = \eta$, the conformal time (remember the relation of T with cosmic time t, $dt = a^{3w}dT$).

It is convenient to express the bounce solution in terms of cosmological quantities, which is achieved by relating the parameters of the wave function to observables. With this purpose, we will follow the same procedure developed in [37]. We first obtain the

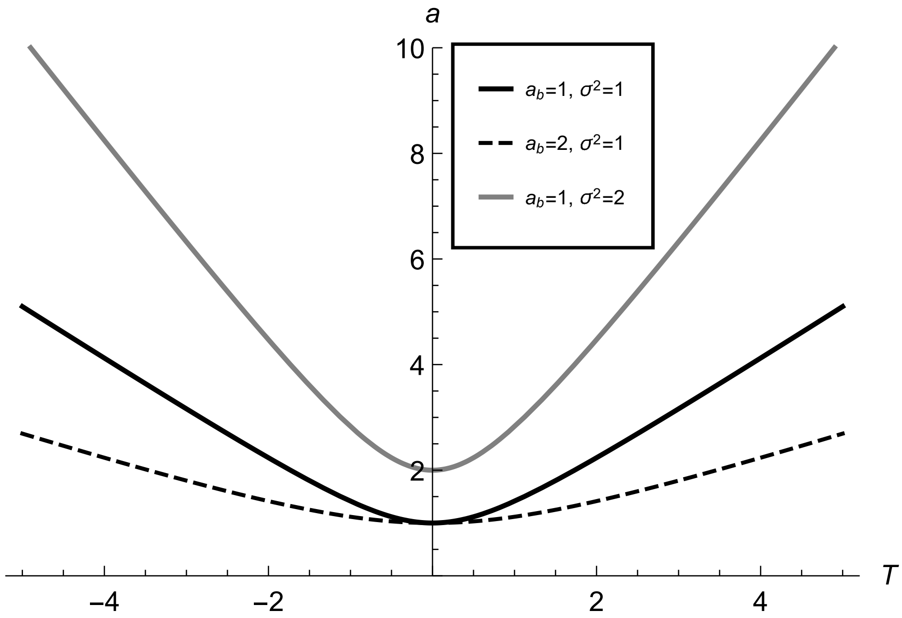

Figure 1 – a vs T for $\omega = \frac{1}{3}$.

Hubble function, given by $H = \frac{\dot{a}}{a}$, where dot denotes the derivative with respect to the physical cosmic time[1]. We then take an expansion of the Hubble function squared for large times T, which reads

$$H^2 = \frac{a_b^2}{a^4\sigma^4} = H_0^2\Omega_{r0}\frac{a_0^4}{a^4}, \tag{5.21}$$

where in the last equality we used the classical Friedmann equation, yielding

$$\Omega_{r0} = \frac{a_b^2}{a_0^4 H_0^2 \sigma^4}, \tag{5.22}$$

where $\Omega_{r0} = \rho_{r0}/\rho_{c0}$ is the dimensionless density parameter for radiation today. The subscript $_0$ in all quantities indicates their current values. The quantities ρ_{r0} and $\rho_{c0} = 3H_0^2/8\pi G$ are, respectively, the current energy density of radiation and the current critical density.

Performing the following transformation of variables

$$x_b = \frac{a_0}{a_b}, \tag{5.23}$$

$$\overline{\sigma} = \sigma\sqrt{a_0 H_0}, \tag{5.24}$$

we obtain

$$\overline{\sigma}^2 = \frac{1}{x_b\sqrt{\Omega_{r0}}}. \tag{5.25}$$

[1] When relating the parameters with cosmological observables, one must go back to the physical cosmic time, $t_c = (V/L_p^2)t$. The constant V/L_p^2 can be absorbed in the dimensionless variance σ, see equation (5.18), yielding a variance with dimensions of length$^{1/2}$. This turns the subsequent equations with the correct physical dimensions.

The Ricci scalar R in the case of a flat Friedmann-Lemaître-Robertson-Walker space-time reads

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{5.26}$$

In its turn, the curvature scale at the bounce is given by

$$L_b = \left.\frac{1}{\sqrt{R}}\right|_{T=0} = \frac{\overline{\sigma}^2}{\sqrt{6}x_b H_0} = \frac{1}{x_b^2 H_0\sqrt{6\Omega_{r0}}}. \tag{5.27}$$

To ensure that the Wheeler-DeWitt equation is a valid approximation for a more fundamental theory of Quantum Gravity [38], we must require that the bounce scale is larger than the Planck scale, i.e. $L_b > L_p$. Taking $H_0 \approx 70$ km $\times$ s^{-1} $\times$ Mpc^{-1}, $\Omega_{r0} \approx 10^{-4}$ and given that $L_p/R_{H0} \approx 1.25 \times 10^{-61}$, where $R_{H0} = 1/H_0$ is the Hubble radius today, we obtain the upper bound for x_b

$$x_b < 1.8 \times 10^{31}. \tag{5.28}$$

The lower limit can be obtained by requiring that the bounce occurs at energy scales much larger than the nucleosynthesis energy scale, i.e. $T_{BBN} = 10$ MeV. Using the CMB temperature equal to $T_{\gamma 0} = 2.7\ K$ in Mev and the inverse linear relation between the temperature and the scale factor, i.e. $T \propto a^{-1}$, which allows us to write

$$\frac{T_{\gamma 0}}{T_{\text{BBN}}} = \frac{a_{\text{BBN}}}{a_0} = x_{\text{BBN}}^{-1}, \tag{5.29}$$

we obtain

$$x_b \gg 10^{11}, \tag{5.30}$$

where BBN stands for Big Bang nucleosynthesis.

5.1 Generalized symmetric quantum bounces

Although the simplicity of the previous symmetric bounce, it represents a fine-tuning in the theory, since the contracting phase is restricted to be the same as the expansion reversed in time. For this reason, we aim to obtain cosmological models with asymmetric trajectories for the scale factor a.

Our initial proposal to obtain asymmetric solutions was to include a factor of the form $\exp(ip\chi)$ in the initial wave function, which represents a velocity for the Gaussian proposed in equation (5.13). Thus we have

$$\Psi_0(\chi) = \left(\frac{8}{\pi\sigma^2}\right)^{\frac{1}{4}} \exp\left(-\frac{\chi^2}{\sigma^2} + ip\chi\right). \tag{5.31}$$

Note that this initial wave function does not satisfy the boundary condition (5.7), which means that unitarity is not satisfied at $T = 0$. However, implementing a convolution between this initial wave function and a propagator that satisfies condition (5.7), we are, in practice, dealing with the projection of Ψ_0 onto the subspace of square-integrable functions on the χ half-line satisfying the von Neumann boundary condition. As a result, the propagated wave function that results from this convolution satisfies (5.7).

Propagating this initial wave function (5.31) with the propagator (5.15) from 0 to $+\infty$, that is, performing an unitary evolution, we obtain the following wave function for all times:

$$\Psi(\chi, T) \;=\; (2\pi\sigma^2)^{-\frac{1}{4}}\left(-1+\frac{iT}{\sigma^2}\right)^{-\frac{1}{2}}\Big[\phi(\chi,T)+\phi(-\chi,T)\Big], \tag{5.32}$$

where

$$\begin{aligned}\phi(\chi,T) \;\equiv\; & \exp\Bigg[-\frac{\sigma^2\chi^2}{T^2+\sigma^4}-\frac{T(p^2T\sigma^2-4p\sigma^2\chi)}{4(T^2+\sigma^4)} \\ &+\; i\left(-\frac{T\chi^2}{T^2+\sigma^4}+\frac{\sigma^2(p^2T\sigma^2-4p\sigma^2\chi)}{4(T^2+\sigma^4)}\right)\Bigg]\Big(1-\mathrm{Erf}\left[\epsilon(\chi,T)\right]\Big)\end{aligned} \tag{5.33}$$

and

$$\epsilon(\chi,T)\equiv\left(\frac{pT}{2}+\chi\right)\left[iT\left(-1+\frac{iT}{\sigma^2}\right)\right]^{-\frac{1}{2}}. \tag{5.34}$$

The wave function (5.32) satisfies the boundary condition (5.7). Thus, as mentioned before, the non-unitarity at the point $T=0$ for the initial wave function (5.31) does not spoil the unitarity after the convolution with the propagator (5.15).

We can see from equation (5.32) that the wave function was propagated equally to χ and to $-\chi$. Thus terms and arguments that are linear in χ are symmetrized with respect to $\chi=0$ by the unitary evolution with the propagator (5.15).

In order to exemplify a Bohmian trajectory for the scale factor a related to an unitary wave function with factors of the form $\exp(ip\chi)$, we are going to consider only the terms

$$\overline{\Psi}(\chi,T)=(2\pi\sigma^2)^{-\frac{1}{4}}\left(-1+\frac{iT}{\sigma^2}\right)^{-\frac{1}{2}}\Big[\overline{\phi}(\chi,T)+\overline{\phi}(-\chi,T)\Big],$$

where

$$\begin{aligned}\overline{\phi}(\chi,T) \;\equiv\; & \exp\Bigg[-\frac{\sigma^2\chi^2}{T^2+\sigma^4}-\frac{T(p^2T\sigma^2-4p\sigma^2\chi)}{4(T^2+\sigma^4)} \\ &+\; i\left(-\frac{T\chi^2}{T^2+\sigma^4}+\frac{\sigma^2(p^2T\sigma^2-4p\sigma^2\chi)}{4(T^2+\sigma^4)}\right)\Bigg],\end{aligned} \tag{5.35}$$

which also constitutes an unitary solution of the Wheeler-DeWitt equation (5.6). The choice to disregard the Gauss's error functions is for the sake of simplicity.

Inserting the global phase $\overline{S}$ of the wave function (5.35) into equation (5.11), it is possible to obtain a differential equation for the parameter χ. It reads

$$\frac{d\chi}{dT} = \frac{2T\chi\cos\left(\frac{2p\sigma^4\chi}{T^2+\sigma^4}\right) + 2T\chi\cosh\left(\frac{2pT\sigma^2\chi}{T^2+\sigma^4}\right) + pT\sigma^2\sin\left(\frac{2p\sigma^4\chi}{T^2+\sigma^4}\right) + p\sigma^4\sinh\left(\frac{2pT\sigma^2\chi}{T^2+\sigma^4}\right)}{2(T^2+\sigma^4)\left[\cos\left(\frac{2p\sigma^4\chi}{T^2+\sigma^4}\right) + \cosh\left(\frac{2pT\sigma^2\chi}{T^2+\sigma^4}\right)\right]}. \tag{5.36}$$

Using equation (5.5) in (5.36) and solving it numerically with initial condition $a_i = a(T_i)$, we obtain the trajectory of the scale factor $a(T)$, which is plotted in figure 2.

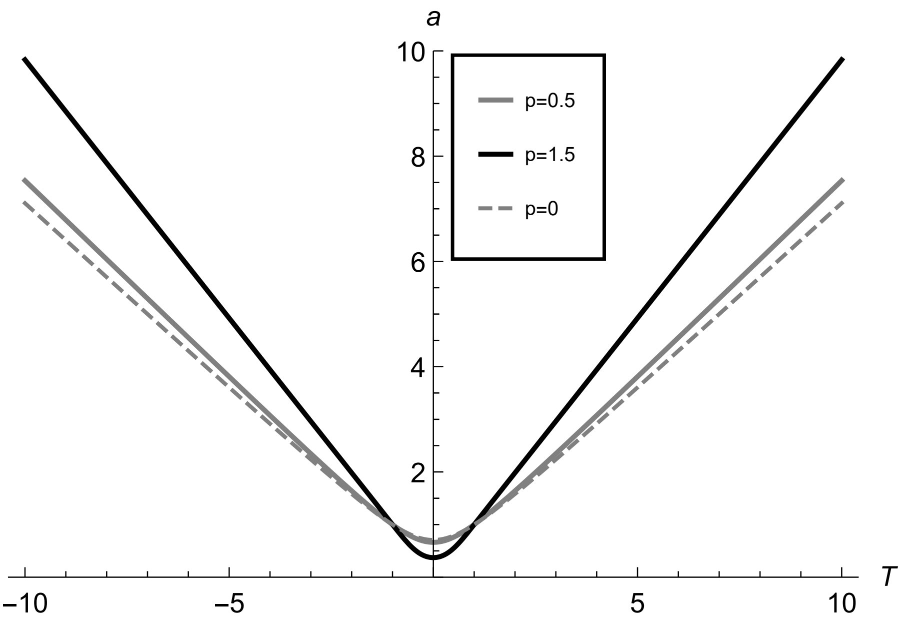

Figure 2 – a vs T for $\sigma = 1.0$, $a_i = 1.0$, $T_i = 1.0$, $\omega = \frac{1}{3}$.

The result is a symmetric bounce, regardless of the value of the parameter p related to the asymmetry. It happens when the unitary evolution for factors of the form $\exp(ip\chi)$ is maintained. As explained before, since these factors are linear in χ inside the exponential, they are going to be propagated equally to χ and to $-\chi$, resulting in a symmetrization of the propagated wave function and, as a consequence, of the trajectory of the scale factor a.

Note that different symmetric bounces can be obtained in other approaches to quantum cosmology. For instance, in references [39, 40], a relational quantization method was implemented, where unitarity is a necessary requirement in order to obtain a consistent probabilistic interpretation, and bouncing models were also found. On the other hand, our work relies on a deterministic interpretation of Quantum Mechanics, where probabilities are not fundamental, allowing to explore the consequences of wave functions of the Universe which are not restricted to evolve satisfying unitarity requirements.

6 ASYMMETRIC QUANTUM BOUNCES

6.1 Non-unitary asymmetric quantum bounces

An alternative to this hindrance is to give up unitarity, which is allowed according to the discussion previously made. In practise, it means to disconsider the boundary condition (5.7). The correspondent propagator is then only the first term of the propagator (5.15), given by

$$G^{NU}(\chi,\chi_0,T)=\sqrt{-\frac{i}{\pi T}}\exp\left[-\frac{i(\chi-\chi_0)^2}{T}\right], \tag{6.1}$$

where NU stands for non-unitary. Applying the propagator (6.1) to the initial wave function (5.31) without the normalization factor from $-\infty$ to $+\infty$, we obtain the following wave function for all times:

$$\Psi(\chi,T)=\left(-1+\frac{iT}{\sigma^2}\right)^{-\frac{1}{2}}\exp\left(\frac{\frac{ip^2T}{4}+ip\chi-\frac{\chi^2}{\sigma^2}}{1-\frac{iT}{\sigma^2}}\right). \tag{6.2}$$

We take the integration from $-\infty$ to ∞ in (6.1) in order to avoid terms containing Gauss error functions that arise if the integration is performed from 0 to ∞. In the end we must check that the restriction $\chi>0$ is still staisfied.

Writing equation (6.2) as $\Psi(\chi,T)=R(\chi,T)e^{iS(\chi,T)}$, we obtain

$$\Psi(\chi,T)=\left(-1+\frac{iT}{\sigma^2}\right)^{-\frac{1}{2}}\overline{\phi}(-\chi,T), \tag{6.3}$$

where $\overline{\phi}(\chi,T)$ is given by equation (5.35) (the first factor in the above equation does not depend on χ, hence it does not affect the calculation of the Bohmian trajectories). Then, by inserting S into (5.11), it is possible to obtain the trajectory in terms of χ. It reads

$$\chi(T)=\chi_b\left[1+\left(\frac{T}{\sigma^2}\right)^2+\left(\frac{p}{2\chi_b}\right)^2(T^2+\sigma^4)\right]^{\frac{1}{2}}-\frac{pT}{2}, \tag{6.4}$$

where $\chi_b=\chi(T_b)$ is the value of the variable χ at the moment of the bounce $T_b=\frac{p\sigma^4}{2\chi_b}$, which is not equal to zero as in the symmetric case. In terms of the scale factor, the trajectory reads

$$a(T)=\left\{-\frac{3p(1-\omega)}{4}T+a_b^{\frac{3(1-\omega)}{2}}\left[1+\left(\frac{T}{\sigma^2}\right)^2+\left(\frac{3p(1-\omega)}{4}\right)^2\frac{(T^2+\sigma^4)}{a_b^{3(1-\omega)}}\right]^{\frac{1}{2}}\right\}^{\frac{2}{3(1-\omega)}}, \tag{6.5}$$

where a_b relates to χ_b through equation (5.5). The trajectory (6.5) is shown in figure 3 for $w=1/3$, where it is evidenced that the value of the parameter p is directly related to the intensity of the asymmetry.

Note that equation (6.5) does not admit a singularity or negative values for $a(T)$, since we always have

$$\frac{3p(1-\omega)}{4}T < a_b^{\frac{3(1-\omega)}{2}}\left[1+\left(\frac{T}{\sigma^2}\right)^2+\left(\frac{3p(1-\omega)}{4}\right)^2\frac{(T^2+\sigma^4)}{a_b^{3(1-\omega)}}\right]^{\frac{1}{2}}. \tag{6.6}$$

This ensures that the restrictions $\chi > 0$ and $a > 0$ are satisfied, although we have disregarded the boundary condition (5.7) and propagated the wave function from $-\infty$ to ∞. A bounce solution is naturally obtained, without the need to impose restrictions to recover the positivity of the scale factor.

For $p = 0$ we re-obtain the symmetric bounce (5.20), which makes explicit the relation between the asymmetry and the factor $\exp(ip\chi)$.

As in the symmetric case, the classical solution arises for large values of T.

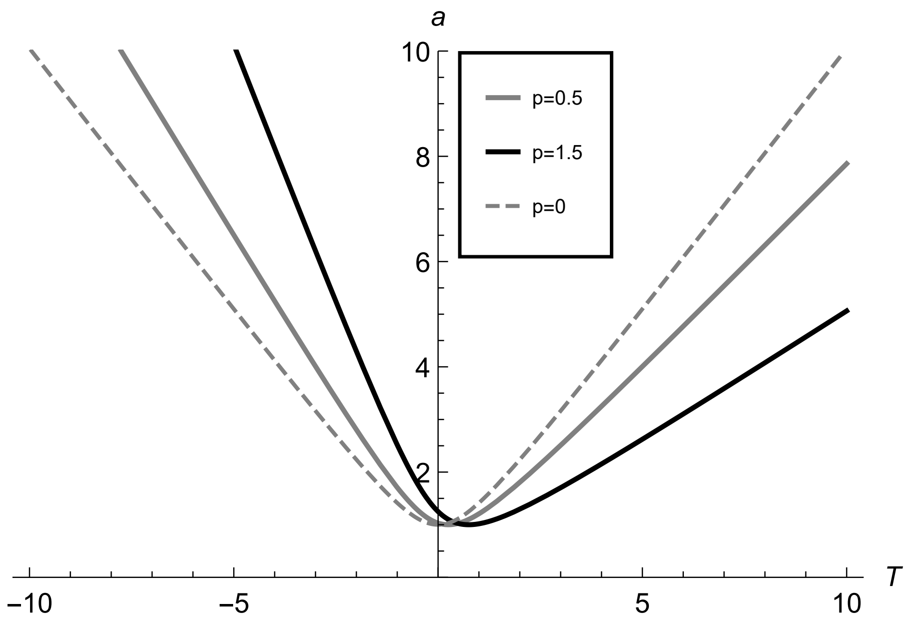

Figure 3 – a vs T for $\sigma = 1.0$, $a_b = 1.0$, $\omega = \frac{1}{3}$.

In order to obtain a slope in the contracting phase lower than the slope in the expanding phase, one has to take $p < 0$, or, equivalently, to change the factor from $\exp(ip\chi)$ to $\exp(-ip\chi)$ in the initial wave function (5.31) keeping $p > 0$. This case is particularly interesting, since the contracting phase may consist of an almost Minkowski universe. Applying the same procedure to obtain the Bohmian trajectory, we obtain a, which is plotted in figure 4.

Just as we did for the symmetric case, let us express the wave function parameters

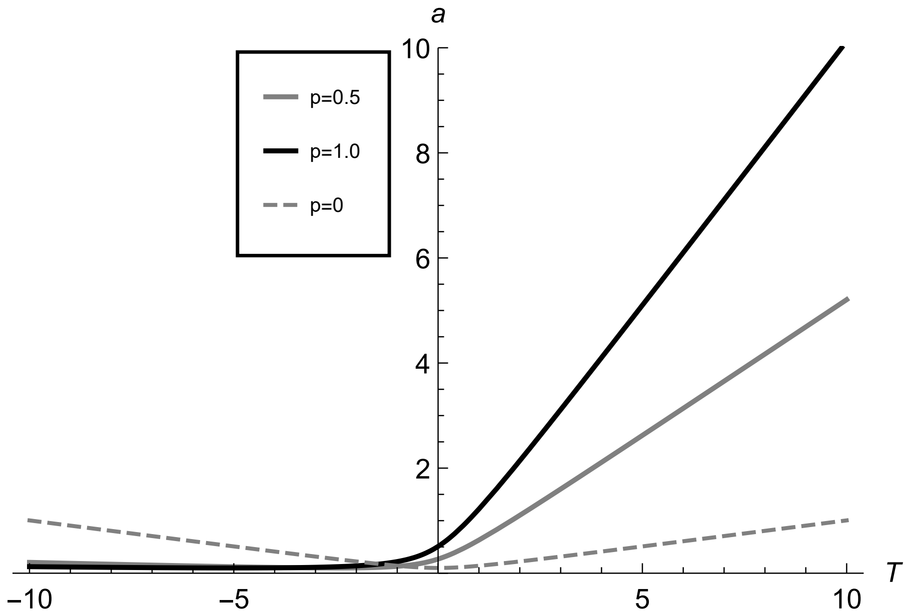

Figure 4 – a vs T for $\sigma = 1.0$, $a_b = 0.1$, $\omega = \frac{1}{3}$.

in terms of cosmological quantities for the case $w = 1/3$. Defining the parameters

$$x_b = \frac{a_0}{a_b}, \tag{6.7}$$

$$\overline{\sigma} = \sigma\sqrt{a_0 H_0}, \tag{6.8}$$

$$\overline{p} = \frac{p}{a_0^2 H_0}, \tag{6.9}$$

$$\overline{\eta} = \frac{\eta}{\sigma^2}, \tag{6.10}$$

$$y^2 = \frac{x_b \overline{p}\overline{\sigma}^2}{2}, \tag{6.11}$$

one can write

$$a = a_b\left(\pm y^2\overline{\eta} + \sqrt{1+y^4}\sqrt{1+\overline{\eta}^2}\right), \tag{6.12}$$

where the $\pm$ signs correspond to wave function phases $\exp(\mp ip\chi)$, with $p \geq 0$. In the limit $|\overline{\eta}| >> 1$, we get for the Hubble function,

$$H^2 = \frac{\left(\pm y^2 + \sqrt{1+y^4}\right)^2 a_b^2 H_0^2 a_0^2}{\overline{\sigma}^4 a^4} = H_0^2 \Omega_{r0} \frac{a_0^4}{a^4}, \tag{6.13}$$

in the expanding phase, and

$$H^2 = \frac{\left(\mp y^2 + \sqrt{1+y^4}\right)^2 a_b^2 H_0^2 a_0^2}{\overline{\sigma}^4 a^4} = H_0^2 \Omega_{rc} \frac{a_0^4}{a^4}, \tag{6.14}$$

in the contracting phase, where Ω_{rc} is the radiation energy density when the Universe has $H = H_0$ in the contracting phase divided by the critical density ρ_c. These equations imply that

$$\Omega_{r0} = \frac{\left(\pm y^2 + \sqrt{1+y^4}\right)^2}{\overline{\sigma}^4 x_b^2}, \tag{6.15}$$

$$\overline{\sigma}^2 = \left[x_b^2 \Omega_{r0} \left(1 \mp \frac{\overline{p}}{\sqrt{\Omega_{r0}}}\right)\right]^{-1/2}, \tag{6.16}$$

and

$$\Omega_{rc} = \Omega_{r0} \left(1 \mp \frac{\overline{p}}{\sqrt{\Omega_{r0}}}\right)^2. \tag{6.17}$$

Note that the $+$ sign in equation (6.12) implies, from equation (6.16), that $0 \leq \overline{p} < \sqrt{\Omega_{r0}}$. From equation (6.17), one can see that $\Omega_{rc} \leq \Omega_{r0}$, and in the limit $\overline{p} \to \sqrt{\Omega_{r0}}$ one has $\Omega_{rc} \to 0$. Hence, the contracting universe can be made arbitrarily flat, and the radiation fluid is created around the quantum phase, during the bounce.

In the $-$ sign case in equation (6.12), there is no constraint in $\overline{p}$, $0 \leq \overline{p} < \infty$, and $\Omega_{rc} \geq \Omega_{r0}$.

In this asymmetric case, the maximum curvature does not occur at the bounce, $\overline{\eta}_{\text{bounce}} = \mp y^2$, but at the conformal time $\overline{\eta}_{\max} = \mp\sqrt{\frac{\sqrt{1+y^4}-1}{2}}$. Hence, the minimum curvature scale reads

$$L_{\min} = \left.\frac{1}{\sqrt{R}}\right|_{\overline{\eta}_{\max}} = \frac{R_{H0}\left(1+\sqrt{1 \mp \frac{\overline{p}}{\sqrt{\Omega_{r0}}}}\right)^3}{8\sqrt{3\Omega_{r0}}\, x_b^2 \left(1 \mp \frac{\overline{p}}{\sqrt{\Omega_{r0}}}\right)^2 \sqrt{\left(2 \mp \frac{\overline{p}}{\sqrt{\Omega_{r0}}}\right)}}. \tag{6.18}$$

Note that equations (6.16, 6.18) reduce to their correspondents in the symmetric case given by (5.25, 5.27) for $\overline{p} = 0$.

As in the symmetric case, we require that the bounce scale is larger than the Planck scale, that is $L_{\min} > L_p$, and smaller then the curvature scale at nucleosynthesis. Hence, we demand

$$10^{-58} << \frac{L_{\min}}{R_{H0}} < 10^{-20}. \tag{6.19}$$

Note that, in the asymmetric case, there is no direct relation between x_b and $L_{\min}$ due to the presence of $\overline{p}$ in equation (6.18). Hence, neither x_b nor $\overline{p}$ have independent physical significance, just when combined to give $L_{\min}$. That is why, in this case, the condition must be put in terms of (6.19).

6.2 Unitary asymmetric quantum bounces

Another alternative to obtain asymmetric solutions is to perform superpositions of Gaussian wave functions multiplied by factors of the form $\exp[i(p\chi)^2]$. Since the term inside the exponential is not linear in χ, it is possible to generate asymmetry maintaining unitarity. Note that the asymmetry is achieved only when we perform superpositions. A single Gaussian in this format would lead to a symmetric bounce.

Considering the following superposition for the initial wave function

$$\Psi_0(\chi) = C\left[\exp\left(-\frac{\chi^2}{\sigma^2} + ip_1^2\chi^2\right) + \exp\left(-\frac{\chi^2}{\sigma^2} - ip_2^2\chi^2\right)\right], \tag{6.20}$$

where

$$C = \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left\{\left[-i(p_1^2+p_2^2) + \frac{2}{\sigma^2}\right]^{-\frac{1}{2}} + \left[i(p_1^2+p_2^2) + \frac{2}{\sigma^2}\right]^{-\frac{1}{2}} + \sqrt{2}\sigma\right\}^{-1/2}, \tag{6.21}$$

and applying the unitary propagator (5.15), we obtain a wave function for all times given by

$$\begin{aligned}\Psi(\chi,T) &= \frac{C\exp\left(-i\frac{\chi^2}{T}\right)}{\left[iT\left(-ip_1^2+\frac{i}{T}+\frac{1}{\sigma^2}\right)\left(ip_2^2+\frac{i}{T}+\frac{1}{\sigma^2}\right)\right]^{\frac{1}{2}}}\\ &\times \left\{\exp\left[\frac{i\chi^2}{T-iT^2(\frac{1}{\sigma^2}+ip_2^2)}\right]\left(-ip_1^2+\frac{i}{T}+\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\right.\\ &+ \left.\exp\left[\frac{i\chi^2}{T-iT^2(\frac{1}{\sigma^2}-ip_1^2)}\right]\left(ip_2^2+\frac{i}{T}+\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\right\}.\end{aligned} \tag{6.22}$$

Note that both (6.20) and (6.22) satisfy the boundary condition (5.7). Thus this case is unitary for all times.

Defining

$$\gamma_i = (-1)^i p_i^2 + \frac{1}{T}, \quad \beta_i = \gamma_i^2 + \frac{1}{\sigma^4}, \tag{6.23}$$

$$\alpha = \frac{\gamma_1}{\beta_1}\frac{\chi^2}{T^2} - \frac{\gamma_2}{\beta_2}\frac{\chi^2}{T^2} - \frac{1}{2}\arctan\left(\gamma_1\sigma^2\right) + \frac{1}{2}\arctan\left(\gamma_2\sigma^2\right) \tag{6.24}$$

and writing equation (6.22) as $\Psi(\chi,T) = R(\chi,T)e^{iS(\chi,T)}$, we can insert the phase S into (5.11) to obtain the differential equation for the parameter χ, given by

$$
\begin{aligned}
\frac{d\chi}{dT} &= -\Bigg\{ \exp\left(-\frac{2\chi^2}{\sigma^2\beta_1 T^2}\right)\left(-T+\frac{\gamma_1}{\beta_1}\right) 2\beta_2^{\frac{1}{2}}\frac{\chi}{T^2} + \exp\left[-\left(\frac{1}{\beta_1 T^2}+\frac{1}{\beta_2 T^2}\right)\frac{\chi^2}{\sigma^2}\right] \\
&\times \; (\beta_1\beta_2)^{\frac{1}{4}}\left[\frac{2\cos(\alpha)\chi}{T^2}\left(-2T+\frac{\gamma_1}{\beta_1}+\frac{\gamma_2}{\beta_2}\right)+\frac{2\sin(\alpha)\chi}{\sigma^2\beta_2 T^2}\right] + \exp\left[-\frac{2\chi^2}{\sigma^2\beta_2 T^2}\right] \\
&\times \; \left[-\frac{2\beta_1^{\frac{1}{2}}\left(\beta_2-\frac{\gamma_2}{T}\right)\chi}{\beta_2 T} - \exp\left[-\left(\frac{1}{\beta_1 T^2}-\frac{1}{\beta_2 T^2}\right)\frac{\chi^2}{\sigma^2}\right]\frac{2\beta_2^{\frac{1}{4}}\sin(\alpha)\chi}{\sigma^2\beta_1^{\frac{3}{4}}T^2}\right]\Bigg\} \\
&\times \; \Bigg\{ 2\exp\left(-\frac{2\chi^2}{\sigma^2\beta_2 T^2}\right)\beta_1^{\frac{1}{2}} + 2\exp\left(-\frac{2\chi^2}{\sigma^2\beta_1 T^2}\right)\beta_2^{\frac{1}{2}} + 4\exp\left[-\left(\frac{1}{\beta_1 T^2}+\frac{1}{\beta_2 T^2}\right)\frac{\chi^2}{\sigma^2}\right] \\
&\times \; (\beta_1\beta_2)^{\frac{1}{4}}\cos(\alpha)\Bigg\}^{-1}.
\end{aligned} \tag{6.25}
$$

For $p_1 = 0$ and $p_2 = 0$, i.e. $\gamma_1 = \gamma_2 = 1/T$ and $\beta_1 = \beta_2 = 1/T^2 + 1/\sigma^4$, we obtain

$$
\frac{d\chi}{dT} = \frac{T\chi}{T^2+\sigma^4}, \tag{6.26}
$$

which can be solved analytically and results in the trajectory (5.18) obtained before for the symmetric case.

Solving equation (6.25) numerically with initial condition $a_i = a(T_i)$, we obtain the trajectory for the parameter χ and then, using equation (5.5), for the scale factor a. The result is plotted in figure 5. Note that symmetric bounces are also obtained if $p_1 = p_2$.

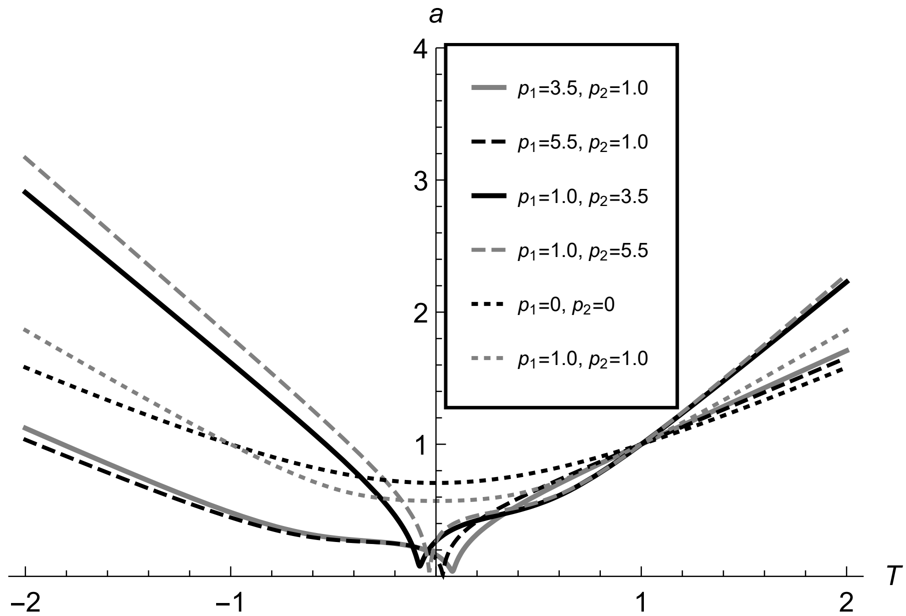

Figure 5 – a vs T for $\sigma = 1.0$, $a_i = 1.0$, $T_i = 1.0$ $\omega = \frac{1}{3}$.

The numerical solution of (6.25) also encompasses multiple bounces for certain values of the parameters σ, p_1 and p_2 and of the initial values a_i and T_i. See figure 6.

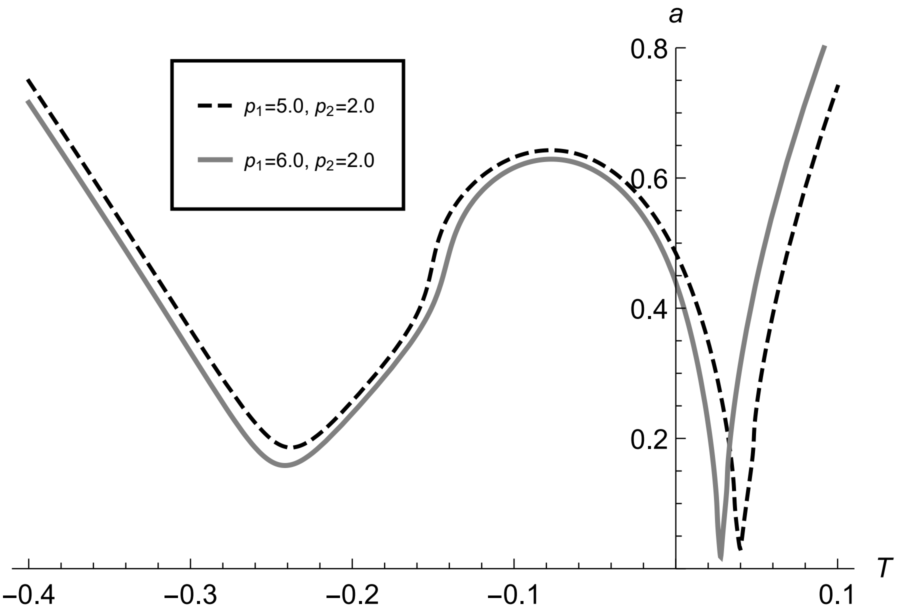

Figure 6 – a vs T for $\sigma = 1.5$, $a_i = 5.0$, $T_i = 1.0$, $\omega = \frac{1}{3}$.

As we did for the other bounce solutions, we express the wave function parameters in terms of cosmological quantities. Expanding the square of the correspondent Hubble function for large times T, we obtain

$$H^2 = \frac{a_i^2}{a^4(T_i^2 + \sigma^4)}. \tag{6.27}$$

Identifying the dimensionless density parameter for radiation today $\Omega_{r0} = \rho_{r0}/\rho_{c0}$ as the coefficient of $(a_0/a)^4$, we obtain

$$\Omega_{r0} = \frac{a_i^2}{a_0^4 H_0^2 (T_i^2 + \sigma^4)}. \tag{6.28}$$

In order to rewrite (6.28) in terms of a_b and T_b, we expand (6.25) for $p_1\sigma \ll 1$ and $p_2\sigma \ll 1$ up to the second order. Under these conditions, i.e. with small parameters related to asymmetry, we obtain a solution with a single bounce, given by

$$a(T) = a_b \left[1 - \frac{(p_1^2 - p_2^2)T}{2\left(1 + \frac{T^2}{\sigma^4}\right)}\right] \sqrt{1 + \frac{T^2}{\sigma^4}}, \tag{6.29}$$

where it is possible to relate T_b, p_1 and p_2 in the limit $T/\sigma^2 \ll 1$ up to first order by making $da/dT = 0$. As a result we obtain

$$T_b = \frac{(p_1^2 - p_2^2)\sigma^4}{2}. \tag{6.30}$$

Substituting (6.30) in (6.28), we note that the term $(p_1^2 - p_2^2)\sigma^4$ appears in Ω_{r0} squared. Thus $p_1\sigma$ and $p_2\sigma$ appear in fourth order and, according to our approach, should be disregarded. It means that the relations between the wave function parameters and the observables in this limit reduce to the relations of the symmetric case. We keep the term containing p_1 and p_2 in the expressions from now on in order to see where the corrections from the parameters related to asymmetry appear.

Performing the following transformation of variables

$$x_b = \frac{a_0}{a_b}, \tag{6.31}$$

$$\overline{\sigma} = \sigma\sqrt{a_0 H_0}, \tag{6.32}$$

$$\overline{p}_i^2 = \frac{p_i^2}{a_0 H_0}, \tag{6.33}$$

where $i = 1, 2$, we obtain

$$\overline{\sigma}^2 = \sqrt{\frac{2}{x_b^2\Omega_{r0} + \sqrt{x_b^2\Omega_{r0}[(\overline{p}_1^2 - \overline{p}_2^2)^2 + x_b^2\Omega_{r0}]}}}. \tag{6.34}$$

Note that equations (6.27, 6.28, 6.34) reduce to their correspondents in the symmetric case given by (5.21, 5.22, 5.25) if we disregard the terms $p_1\sigma$ and $p_2\sigma$ in fourth order. Obviously, the same applies for $\overline{p}_1 = \overline{p}_2 = 0$, which implies $T_i = T_b = 0$.

For this particular case, i.e. $T/\sigma^2 \ll 1$ up to first order and for $p_1\sigma \ll 1$ and $p_2\sigma \ll 1$ up to second order, the curvature scale at the bounce L_b assumes the same form of the symmetric case given by equation (5.27) with $\overline{\sigma}^2$ given by (6.34).

We now go back to the general case given by equation (6.25) and verify for which values of the parameters the bounce scale is larger than the Planck scale and smaller than the nucleosynthesis scale. We find L_b numerically for some non-multiple asymmetric bounces, and we obtain the correspondent bounce energy $E_b = L_b^{-1/2}$ for each case. The results are shown in table 1.

$p_1\sigma$	$p_2\sigma$	L_b (s)	E_b (MeV)
2.5	1.0	3.59934×10^{-3}	16.66820
3.5	1.0	5.95604×10^{-4}	40.97522
4.5	1.0	1.61263×10^{-4}	78.74681
5.5	1.0	5.75934×10^{-5}	131.76909
6.5	1.0	1.19055×10^{-5}	201.63933
7.5	1.0	4.78629×10^{-5}	289.81846
8.5	1.0	6.32385×10^{-6}	397.65741
9.5	1.0	3.60849×10^{-6}	526.42560
10.5	1.0	2.17979×10^{-6}	677.31783
1.0	2.5	5.64555×10^{-3}	13.30904
1.0	3.5	1.00531×10^{-3}	31.53917
1.0	4.5	2.75388×10^{-4}	60.25975
1.0	5.5	9.80995×10^{-5}	100.96402

Table 1 – L_b and E_b for $\sigma = 1.0, a_i = 1.0, T_i = 1.0, \omega = \frac{1}{3}$.

Since $L_p \approx 5 \times 10^{-44}\ s$, we see that $L_b >> L_p$ for all bounces considered. As mentioned before, this means that the validity of the Wheeler-DeWitt equation as an approximation to a more fundamental Theory of Gravity is well established. Beyond that, the bounce must occur at energy scales much larger than the nucleosynthesis scale, i.e. 10 MeV, which is not achieved by all cases considered. Indeed, as one can see from table I, the energy scale of such bounces are not much bigger than the nucleosynthesis energy scale, but they are many orders of magnitude smaller than the Planck energy scale. Hence, the physically relevant consistency check of such bouncing models is the upper limit of L_b, not its lower limit, which makes the distinction between L_b and L_{min} irrelevant.

The cases $p_1\sigma \geqslant 10.9$, $p_2\sigma = 1.0$ and $p_1\sigma = 1.0$, $p_2\sigma \geqslant 5.8$ represent multiple bounces. Consecutive bounces are also encountered in Quantum Reduced Loop Cosmology, in a scenario called emergent bounce [41]. It describes a series of bounces with successive increasing amplitudes. In our work, the multiple bounces do not necessarily present this behaviour. The solutions we found also allow for more than one bounce, but with similar amplitudes, before being launched to the expanding phase.

Conclusion

We have obtained generalizations of the quantum bounce solutions presented in references [5, 13], which are asymmetric with respect to the bounce and even possessing multiple bounces. These generalizations represent a key feature that allows for cosmological solutions where the contracting phase differs from the expansion.

In the case where the evolution of the wave function is non-unitary, the asymmetry is achieved through a phase velocity provided to the gaussian. We have shown that the unitary propagation in this case results in a symmetrization of the trajectory of the scale factor with respect to the bounce. This obstacle is overcome by disregarding the boundary condition (5.7), which is allowed by the de Broglie-Bohm Quantum Cosmology, since it does not lead to an ensemble of universes. Analytical asymmetric solutions are obtained and related to observables, which allows us to bound the minimum curvature scale. One particular class of interesting solutions for this case is the one exhibited in figure 4. It describes expanding cosmological solutions arising from an almost flat space-time. As discussed in section 6.1, the energy density at contraction can be made arbitrarily small, depending on the new quantum parameter p, related to the phase velocity of the initial wave function of the universe. The emerging picture is of an arbitrarily flat and almost empty space-time, which is launched through a bounce into the standard Friedmann expanding phase, containing the usual hot and dense radiation field. This fact open new windows to an old speculation, that our Universe arose from quantum fluctuations of a fundamental quantum vacuum. The de Broglie-Bohm theory allows a different regard to this hypothesis and the concrete possibility to extend this particular minisuperspace model by incorporating quantum cosmological perturbations to the system and quantitatively study their observational effects. This is subject for future work.

In the case where the asymmetry is a result of the superposition (6.20), unitary evolution is satisfied, although not required. The resulting differential equation for the trajectory of the scale factor can be solved numerically, opening possibilities for single or multiple asymmetric bounces, depending on the values of the parameters of the wave function and on the initial condition. A numerical computation of the curvature scale of these bounces is performed, pointing a proximity to the nucleosynthesis energy scale. An approximate solution for the limit $p_1\sigma \ll 1$ and $p_2\sigma \ll 1$ up to second order can be obtained, allowing us to relate the parameters of the wave function to observables. The resulting relations reduce to the ones of the symmetric case, which is expected given the smallness of the parameters related to asymmetry.

The asymmetric trajectories of the scale factor obtained in this work may be used to take into account significant back-reaction due to quantum particle production around the bounce, see references [37, 42]. As an example, we investigated, together with collaborators, a gravitational baryogenesis mechanism [44, 43] in those asymmetric solutions, which is encompassed in a broad work on baryogenesis in bouncing cosmologies, which we present soon.

In brief, our aim of finding asymmetric quantum bounces in the context of the Wheeler-DeWitt quantization with the de Broglie-Bohm interpretation of Quantum Mechanics was successfully achieved. Classes of asymmetric bounces were obtained both numerically and analytically, allowing for the emergence of our classical universe for large times. Interesting findings arose by investigating unitarity in these models, leading to different scenarios created by distinct formats of the initial wave function of the universe, including a bounce model with an almost Minkowski contracting phase.

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