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Testing Bekenstein Bounds and Inequalities Through Nonlinear Electrodynamics

Testando os Vínculos e Desigualdades de Bekenstein Através de Eletrodinâmicas Não Lineares

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To my parents, Miguel and Mirtha.

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"*...I might not be so sure about what interested me, I was absolutely sure about what did not interest me*"

A. Camus, The Stranger

Abstract

In this work we investigate the validity of Bekenstein bounds and inequalities within nonlinear electrodynamics (NLED). Bekenstein bounds provide relations between physical quantities such as energy, angular momentum and charge, and have been proved to hold within classical electrodynamics. We investigate what conditions must a NLED have in order to satisfy the inequalities. We provide proofs for the validity of these inequalities within some examples of NLED present in the literature and conjecture the violation condition for each of the inequalities. We provide examples of NLED violating each of the inequalities and explore the physical consequences of each violation. In particular, an inequality between energy and angular momentum that is considered to be related to the causal structure of physical theories is proven to not be defined univocally by causality. We show that within NLED the fulfillment of the partial inequalities does not assure the fulfillment of the complete inequality. We provide an additional requirement for the attainment of the complete inequaltiy.

Key-words: Nonlinear Electrodynamics; Bekenstein Bounds and Inequalities; Geometric Inequalities.

Resumo

Neste trabalho investigamos a validez dos vínculos e desigualdades de Bekenstein no contexto da eletrodinâmica não linear (NLED). As desigualdades de Bekenstein fornecem-nos relações entre distintas quantidades físicas tais como a energia, momento angular e carga, a validez delas foi provada na eletrodinâmica clássica. Investigamos quais condições deve satisfazer uma NLED de tal forma que ela cumpra as desigualdades. Fornecemos provas para a validez destas desigualdades em alguns exemplos de NLED presentes na literatura e conjecturamos as condições de violação para cada uma das desigualdades. Provamos exemplos de NLED que violam cada uma das desigualdades e exploramos as consequências físicas de cada uma das violações. Em particular, uma desigualdade entre a energia e o momento angular que se considera relacionada com a estrutura causal das teorias físicas é provada não ser definida univocamente pela causalidade. Mostramos que na NLED o cumprimento das desigualdades parciais não assegura a validade da desigualdade completa. Fornecemos uma condição adicional para a validade da desigualdade completa.

Palavras-chaves: Eletrodinâmica Não Linear; Vínculos e Desigualdades de Bekenstein; Desigualdades Geométricas.

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Notation and Conventions

The following notation and conventions will be adopted throughout the work.

- Bold characters, $\mathbf{A}$, denote vectors in 3 dimensions.
- Latin indices $i, j, k, ...$ run through $1, 2, 3$, while Greek indices $\mu, \nu, \lambda, ...$ take the values $0, 1, 2, 3$.
- The Minkowski metric for flat spacetime has the signature $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$.
- Gaussian-Heaviside-Lorentz units are adopted, this means $\epsilon_0 = 1$ and $\mu_0 = 1$. And $\hbar = c = 1$ unless otherwise stipulated.

Introduction

The research on Nonlinear electrodynamics (NLED) has its main cornerstone on two different assumptions: (1) Maxwell electrodynamics is only an approximate theory, hence it is necessary to look for a more general (nonlinear) theory of electrodynamics and (2), nonlinear phenomena of electrodynamics are due to the vacuum polarization in QED leading to nonlinear effective lagrangians [1]. Among the many motivations for postulating nonlinear extensions of electrodynamics it is worthy to highlight the quest for a formulation of electrodynamics with no divergences (Born-Infeld electrodynamics [2]) and the appearance of nonlinearities in the quantum domain (Euler-Heisenberg electrodynamics [3]), which led to the formulation of the first plausible theories.[1] Thus, the search for a nonlinear extension of electromagnetism in fact may be motivated by a wide range of philosophical reasons and theoretical evidence that Maxwell theory might not be complete. Then, the question that arises when formulating a NLED is: what should the *minimum* requirements for generalizing Maxwell electrodynamics be?

Due to the extreme success of Maxwell electrodynamics, an acceptable nonlinear extension of electrodynamics is supposed to agree with Maxwell's theory at the appropriate limit. We will accept this requirement as the *minimum* requirement for postulating a NLED. Thus a NLED is not *a priori* required to obey any other assumption rather than having the adequate limit. However, is the requirement of a NLED recovering Maxwell electrodynamics in an adequate limit a necessary and sufficient condition for it to be considered a physical theory? We should note that such a requirement, alone, opens the gate to potentially pathological theories, then there should be other criteria, either theoretical [4] or experimental [5], in order to discern between physical and unphysical theories. Nonetheless, there may also exist other criteria for testing NLED.

The appearance of geometric inequalities in General Relativity offers a powerful framework for characterizing different solutions and conjectures in the theory. One of the most remarkable geometric inequalities is the one given by Penrose [6, 7], which offers a criterion for characterizing gravitational collapse, specifically the cosmic censorship conjecture. Likewise, geometric inequalities appear equally when studying stationary black holes, and remarkably many of these inequalities also hold for more general cases under certain symmetry assumptions [8, 9]. When studying geometric inequalities for black holes generally the area is the best geometrical parameter to compare different physical quantities. However, it seems that when trying to expand such inequalities to bodies different than black holes, the area is no longer the best parameter for describing the size

[1] Since the earlier tentatives of generalizing Maxwell electrodynamics were not taken serious for suffering of several pathologies.

of a body, hence, other geometrical definitions for relating the size of a body with respect to other physical quantities must be explored. A special class of inequalities relating the radius of a circumscribing sphere to a body of arbitrary shape and different physical quantities known as Bekenstein inequalities [10, 11, 12, 13], which were initially proposed as an enforcement to the Generalized Second Law (GSL). Remarkably, despite its purely gravitational origin, these inequalities have proved to be valid *universally*, and "*provide a striking illustration of the unity of physics*"[2]. In fact, as showed by Schiffer and Bekenstein in [14], free scalar and electromagnetic fields satisfy the original bound. Also, when fixing the entropy to zero, it has been proved by Dain [9] that Maxwell electrodynamics satisfies the complete Bekenstein inequality, which relates the energy, angular momentum and charge. That being so, should any acceptable nonlinear extension of electrodynamics also satisfy Bekenstein inequalities?

The aim of this dissertation is to analyze the physical implications of Bekenstein inequalities through NLED. The fact that such inequalities are valid through a wide range of physical theories and that they are considered to have a universal character should provide us with an argument in favor of testing NLED through Bekenstein inequalities. Given this, an acceptable theory of NLED should satisfy identically all Bekenstein inequalities. However, the universality arguments of Bekenstein inequalities may also allow for pathological situations to arise, and since it is widely known that NLED may possess pathologies intrinsic to the theory the converse of the previous statement may also hold, i.e. a pathological theory satisfying Bekenstein inequalities may disprove the universality arguments behind the inequalities, or at least prove them too loose.

The dissertation is organized as follows: in Chapter 1 we give a review of geometric inequalities for black holes and Bekenstein bounds, making more emphasis in the latter and the arguments behind each of the bounds, together with its most important characteristics and its physical implications in the classical regime; in Chapter 2 we introduce nonlinear electrodynamics, first by reviewing the covariant formulation of electrodynamics and its lagrangian formulation, then by exposing some well known examples of NLED present in the literature together with its most important properties relevant for our work and finally by exposing the observer decomposition of the energy-momentum tensor and the energy conditions imposed to the energy momentum tensor; in Chapter 3 we analyze the behavior of each of the Bekenstein inequalities within the presented NLED, we look for counterexamples for each of the inequalities and analyze the physical implications of each of the counterexamples and finally derive a new inequality that should be required to a NLED for it to satisfy the complete inequality; finally, in Chapter 4 we present the conclusions together with some perspectives for future developments.

[2] As stated by Hod in [12].

1 Geometric Inequalities.

The mathematical considerations of geometry give place to the appearance of geometric inequalities. These inequalities very often relate properties of curves such as area, perimeter or radius between each other. In fact, the most famous geometric inequality in mathematics is the *isoperimetric inequality*, $L^2 \geq 4\pi A$, which relates the area of a closed curve with its perimeter. Very often the equal signs in such geometric inequalities hold for curves with a variational characterization.

The analysis of geometric inequalities is widely studied in mathematics, however, the study of such inequalities in physics is, as well, a very interesting research topic since physical theories can lead to the appearance of highly non-trivial geometrical inequalities from the mathematical point of view. With this in mind, physical geometric inequalities define an interplay between geometry and physics. In virtue that it is possible to geometrize many physical theories, the study of these type of inequalities seems to be a potentially powerful tool to define global inequalities that should be testable in many branches of physics.

The best known candidate to present geometric inequalities is, obviously, General Relativity (GR), since it is a geometric theory from the very beginning. Also, GR present a very special type of solutions (i.e. black holes) for the metric. Black holes can be characterized ideally by very few parameters and represent the perfect candidates for such studies. Over the years, many inequalities for black holes, relating different physical quantities, have been proposed. For example, a well known inequality for the characterization of gravitational collapse is the one given by Penrose [6]. Also, Dain [15] has widely studied a set of inequalities for black holes that remarkably hold even for dynamical black holes. Also, there have been proposals for relating a black hole's entropy with different quantities. This was initially presented by Bekenstein [10] and over the years there have appeared several generalizations to Bekenstein original inequality.

However, black holes represent only the most simple candidates for studying their physical characteristics in terms of geometric inequalities. It is always possible to try to generalize, or conjecture, such inequalities which would be valid for a wider range of objects. In the present section we will present a brief review of known black hole inequalities and a larger review of Bekenstein's inequality for ordinary objects. In the following section we will explicitly use the constants c, G, $\hbar$ and κ_B, unless otherwise stipulated.

1.1 Inequalities for Black Holes.

Black holes represent a very special kind of solutions for Einstein's equations. They are described by a few parameters, namely the area A, angular momentum J and charge Q, and, by virtue of black hole uniqueness theorem, stationary black holes in electro-vacuum are characterized by the Kerr solution of Einstein's equations. However, these solutions also describe naked singularities and it should be important to have a criterion for discriminating between Kerr solutions for black holes and naked singularities. Also it is important to note that black holes can be described by a few parameters only when they are stationary. However, it seems that restricting ourselves to stationary black holes is not the optimal way for modeling all of the phenomena that we can observe and predict, such as the black hole formation by gravitational collapse, which is not stationary at all.

As it will be seen in this section, it is possible to obtain several inequalities for stationary black holes, which are generalizable to dynamical black holes under certain assumptions. This fact is very important because it provides us with more evidence in order to prove the picture of gravitational collapse as right or wrong. Gravitational collapse relies on the weak cosmic censorship conjecture, that states that there cannot exist naked singularities due to gravitational collapse. The following section is mainly based in the review articles [9] and [8]. In this section we will use natural units $c = G = 1$ for simplicity.

1.1.1 Stationary Black Holes.

As stated before, in general, a stationary black hole can be characterized by a few parameters, namely the mass m, angular momentum, J and charge Q, which can be either electric, magnetic or both; however, we will not deal with charge throughout this section. And also, within General Relativity, all stationary black holes in vacuum are described by the Kerr exact solution. From this, it is interesting to note that some elementary geometric inequalities must be satisfied in order for the Kerr solution to be, indeed a black hole solution. We may start by describing the area of the horizon for a Kerr black hole, which is given by [16]

$$A = 8\pi\left(m^2 + \sqrt{m^4 - J^2}\right) \tag{1.1}$$

from which it is possible to derive three geometric inequalities; from the case where $J = 0$ in the latter we get,

$$\sqrt{\frac{A}{16\pi}} \leq m \tag{1.2}$$

which truly represent the total amount of rotational energy of the black hole, thus, if the difference between both quantities is zero, it means that the black hole is not rotating ($J = 0$). It is important to note that in the dynamical regime, this is the Penrose inequality. Also, from the fact that the square root in (1.1) should have real values it is possible to bound the mass and angular momentum as,

$$\sqrt{|J|} \leq m \tag{1.3}$$

Now, note that we asked for the square root to be real in order to have a well-defined area. In fact, inequality (1.3) gives us a criterion for discriminating between black holes and naked singularities. If the inequality is satisfied, then that Kerr solution represents a black hole; if the equality is satisfied, then that solution represents an extreme Kerr black hole, and thus an extreme Kerr black hole represents the 'optimal shape' for the corresponding inequality; and finally, if the inequality (1.3) is violated, then that Kerr solution is a naked singularity.

Finally by using (1.3) in (1.1), we arrive at

$$8\pi|J| \leq A \tag{1.4}$$

where the equality is attained when the equality in (1.3) holds. The inequality (1.4) is related to the temperature of the black hole, $\kappa/2\pi$, where κ is the surface gravity. The inequality holds when $\kappa > 0$, and the equality in the latter holds for a black hole with $\kappa = 0$. It is also possible to obtain other geometric inequalities relating the area, mass and angular momentum for Kerr black holes which are important in the spirit of the Penrose inequality, however we will not deal with such inequalities in this brief review. The three inequalities presented above were derived from the area of a stationary Kerr solution, however it is remarkable that these inequalities hold even for dynamical black holes. Now we will assess the inequalities for dynamical black holes.

1.1.2 Dynamical Black Holes.

Nature itself is highly dynamical, then it is important to obtain relations between physical quantities in dynamical cases. It is remarkable that the three geometric inequalities (1.2), (1.3) and (1.4) valid for the Kerr black holes are expected to hold also for axially symmetric, dynamical black holes. The main issue regarding the generalization of the inequalities to the dynamical case has to be with the fact that, in general, it is no longer possible to have well-defined quantities, so the way to deal with this is to define physically significant quantities that are either global or quasi-local[1], so it should be suitable to find inequalities relating only global or quasi-local quantities.

For instance, well-defined quasi-local quantities are the angular momentum and the area, and (1.4) only relates quasi-local quantities and should hold. Then, it will be possible to obtain a quasi-local mass from (1.4) and (1.1), which from the fact that it increases monotonically with the area and that the angular momentum is conserved in

[1] Quantities that are defined within an extended, but finite, region of spacetime. For a review on the subject see [17]

vacuum it is reasonable to think of this quasi-local mass as the non-stationary black hole's mass, then from (1.1) we define the black hole's mass as

$$m_{\rm bh} = \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}$$

where $m_{\rm bh}$ is a quasi-local mass, which is identified as the non-stationary black hole mass. The validity of $m_{\rm bh}$ describing adequately the black hole mass is related to the validity of the inequality (1.4) in axial symmetry. Nevertheless, the violation of (1.4) in the dynamical regime will only indicate us that $m_{\rm bh}$ is not an adequate parameter to describe the black hole's mass since it doesn't have the desired physical behavior. In that sense, the violation of (1.4) would not carry as much physical consequences as the violation of the other inequalities.

It is also possible to relate global and quasi-local quantities in the inequalities, a well-defined global quantity is the ADM mass, which is the mass of the whole spacetime. It is possible to relate the ADM mass with a quasi-local mass, and thus with other quasi-local quantities such as area and angular momentum. Namely, it is possible to bound the latter inequality with the ADM mass. From the fact that $m_{\rm bh} \leq m$ holds it is possible to obtain

$$m \geq \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}$$

which is the Penrose inequality with angular momentum. This inequality is intrinsically related to inequalities (1.2) and (1.3) and the way of deriving the latter is by the area theorem and the nontrivial fact that in axial symmetry the angular momentum is conserved. Penrose inequalities are intimately related to the standard picture of gravitational collapse, in fact, firstly Penrose [6] proposed the original inequality $m \geq \sqrt{A/16\pi}$ without any symmetry assumption in order to provide a criterion for proving that the standard picture of gravitational collapse is wrong, more specifically, the cosmic censorship conjecture which states that no naked singularities can be formed through gravitational collapse.

The standard picture of gravitational collapse is mainly characterized by the following two statements [9]: i) Gravitational collapse results in a black hole, i.e. naked singularities cannot be formed in gravitational collapse (cosmic censorship) and ii) The spacetime settles down to a stationary state, i.e. after a finite time all the matter surrounding the black hole falls into it, leaving a vacuum exterior region. So far, many Penrose-like inequalities have been proved, while the most general case remains open, for a review on the Penrose inequality see [7]. It is important to note that any counterexample of any of the inequalities (1.3), (1.2) in the dynamical regime above will prove that the actual picture of gravitational collapse is wrong.

1.2 Bekenstein Bounds and Inequalities

In the previous section we have seen inequalities relating the mass, area and angular momentum for black holes. However, it is also possible to relate other relevant quantities for black holes in such a way that the resulting inequalities will reach its 'optimal shape' for the black hole cases, while the inequalities will hold for every other body. Since it is possible to associate an entropy to a black hole, and it is known that such an entropy is in fact several times higher than the thermal entropy for a star of the same mass, the original idea presented by Bekenstein was to bound the black hole's entropy with its energy, such that the equality should be satisfied for the black hole, and the inequality satisfied for every other body.

Although the area represents an adequate size measure for black holes, it is not the case for every other body. For that purpose it will be necessary to change the geometrical parameter used to compare different physical quantities, such that, now, the resulting inequalities will be valid within an *effective radius*. Bekenstein based his original arguments in a *gedanken* experiment where a stationary black hole absorbs a body with energy $\mathcal{E}$ and effective radius $\mathcal{R}$ then the rate between entropy and energy must have a limit in order to respect the generalized second law (GSL). When a black hole absorbs a body of negligible self-gravity, its surface area increases by $8\pi\mathcal{E}\mathcal{R}$, then, in order for the GSL (the total entropy increase be nonnegative) to remain valid, the body's entropy must be bounded by $2\pi\mathcal{E}\mathcal{R}$ [18]. In the present section we will explicitly write all the constants in every inequality for purposes of clarity.

The original bound purposed by Bekenstein [10] relates entropy to energy as

$$\frac{\hbar c}{2\pi\kappa_B} S \leq \mathcal{E}\mathcal{R} \tag{1.5}$$

where the equality in the latter is attained for the case of a Schwarzschild black hole, thus, a black hole's entropy is the maximum reachable entropy. In (1.5) $\mathcal{R}$ represents the *effective radius* of an arbitrary system, $\mathcal{E}$ is interpreted as the energy above the ground state for such system. In fact it is interesting that from gravitational arguments of a black hole absorbing a body and independently attaining the bound in the case of the Schwarzschild black hole, the latter bound is *universal*, and it does not exhibit the constant G explicitly, hence it can always be probed in non-gravitational regimes.

Definition 1.2.1 (Effective radius $\mathcal{R}$). *The* effective radius $\mathcal{R}$ *is the radius of the smallest sphere* $\mathcal{B}_{\mathcal{R}}$ *that encloses the region* Σ *in flat space (see* Fig. 1*).*

However, we know that there are other relevant parameters in order to parametrize more general black holes, such as the charge and angular momentum. The generalization is non-trivial, and in fact, it took several attempts to try to generalize Bekenstein's original inequality to the more general, charged and rotating, case. It is important to note

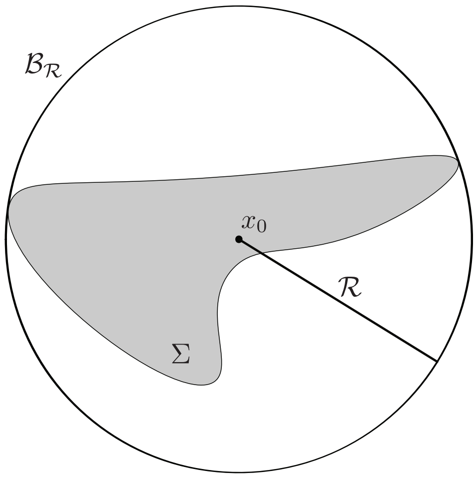

Figure 1 – The sphere $\mathcal{B}_{\mathcal{R}}$ is the smallest sphere of radius $\mathcal{R}$ centered in x_0 that encloses the region Σ.

that the bound (1.5) attains the equality for a Schwarzschild black holes, while a Kerr black hole respects the inequality. In the following subsections we will point out several generalizations that have been done in order for the bound (1.5) to be conformed by the other black hole solutions from General Relativity.

It is also important to note that (1.5) depends on the definition of $\mathcal{R}$, in fact, other definitions of $\mathcal{R}$ may be used in entropy bounds [19]. For the purposes of this work, $\mathcal{R}$ is defined as in 1.2.1. Bekenstein bound was originally proposed for systems to respect the GSL, however there are still many discrepancies about this argument being necessary for the validity of the GSL [18].

1.2.1 Zaslavskii's bound.

The first attempt to generalize (1.5) was done by Zaslavskii in [11], who derived an entropy bound considering a system composed of a charged black hole and thermal radiation. Zaslavskii noticed that the electromagnetic part of the system's energy is irrelevant for the statistical properties of the system, namely the entropy, finally arriving to the bound

$$\frac{\hbar c}{2\pi\kappa_B} S \leq \mathcal{R}\mathcal{E} - \frac{Q^2}{8\pi} \tag{1.6}$$

where $\mathcal{E}$ is the total energy of the system. It is possible to arrive to the bound by making the substitution $\mathcal{E}_0 = \mathcal{E} - \frac{Q^2}{8\pi\mathcal{R}}$ in (1.5), thus the remaining energy part in the bound $\mathcal{E}_0$ is the part that contributes to the statistical[2] properties of the system, where the term involving Q is a part of the electromagnetic energy *outside* the system. Originally, Zaslavskii made the derivation of the latter using a constant $\alpha \geq 1$ in the right hand side of (1.6), in order to simplify the proof. However, later on he argues that, indeed, that constant should be set to 1 from the facts that when treating uncharged bodies

[2] By making an analogy with Statistical Mechanics. Since the system can be a black hole, this analogy is not always true because a black hole's entropy does not have a statistical origin.

Bekenstein's original bound (1.5) must be recovered and that even in the case of charged bodies, the bound should not depend upon the charge density of the bodies, then that constant should be set to 1 in order to provide the *most general* character of the bound. Note that the bound (1.6) is saturated for the charged case of a black hole, i.e. the Reissner-Nordström solution. It is possible to arrive at an analogous reasoning for this inequality from merely classical arguments as it will be seen later on.

1.2.2 Hod's bound.

As it has been seen, Bekenstein's original bound is saturated for the case of a Schwarzschild black hole, and all the other black hole solutions respect the inequality. However it is still possible to perform *gedanken* experiments for different situations involving self-gravitating objects in order to probe the bound. Although, as argued by Hod in [12], the original bound may be only a necessary and not a sufficient condition for the GSL to hold, thus, it may be necessary to strengthen the original bound. Hod idealized an experiment where a rotating body is absorbed by a black hole, then, by virtue of the first law of black hole thermodynamics it is possible to find a minimal increase in the black hole's surface area $(\Delta A)_{\mathrm{min}}$, then, by applying the GSL, namely $(\Delta S)_{\mathrm{tot}} = \Delta S_{\mathrm{BH}} + \Delta S \geq 0$ and $S_{\mathrm{BH}} = A/4$ it is possible to find the bound

$$\frac{\hbar c}{2\pi\kappa_B} S \leq \sqrt{(\mathcal{E}\mathcal{R})^2 - c^2 J^2} \tag{1.7}$$

It is noteworthy that one of the most intriguing conclusions from Bekenstein's first bound (1.5), is that it imposes that the Schwarzschild black hole will have a maximum entropy since it attains the equality, while the other black hole solutions never attain the equality, leaving the Schwarzschild solution as a unique black hole among the general Kerr family of solutions. This led physicists to conjecture generalizations of the entropy bound for more general black holes, where the original bound is recovered for the appropriate limit, such that a generic black hole could always attain the bound. Thus, for the latter example it is possible to conclude that all Kerr black holes saturate (1.7). The bound is supposed to be essential for the validity of GSL, however there exist many discrepancies if the bound is truly necessary for the validity of GSL [18, p. 21]. It is important to note that *all* of the latter entropy bounds follow from the application of GSL, i.e. from gravitational considerations, however all of them represent *universal* bounds for the entropy for different systems. It is still possible to find a more general bound, which will be reviewed in the next subsection.

1.2.3 Bekenstein's full bound.

As we have seen, it is possible to tighten Bekenstein's original bound from diverse *gedanken* experiments. Then it is reasonable to conjecture tighter bounds for different quantities of a black hole. In accordance with the diverse generalizations presented before, Bekenstein and Mayo [13] conjectured a tighter bound for (1.5). First of all it is noteworthy that Zaslavskii's original arguments for obtaining a tighter bound were based not on a situation were the GSL may be violated, but rather on a feature of a static black hole in presence of thermal radiation. In their work, Bekenstein and Mayo prove that the bound derived by Zaslavskii is indeed recovered as a minimum area increase for the case of a charged black hole absorbing some charged object, where the area increase is given by the increase of the black hole's charge, and they have also obtained the same bound by the analysis of an electrically grounded black hole, where the potential at the event horizon is zero and the infall of a charged object produces an increase in the black hole's mass.

Thereafter, based on Hod's results of Sec. 1.2.2, they conjecture the following bound for entropy, energy, angular momentum and charge

$$\frac{\hbar c}{2\pi\kappa_B} S \leq \sqrt{(\mathcal{E}\mathcal{R})^2 - c^2 J^2} - \frac{Q^2}{8\pi} \tag{1.8}$$

it is important to note that a first order approximation (for $cJ \ll \mathcal{E}\mathcal{R}$) of the latter gives

$$\frac{\hbar c}{2\pi\kappa_B} S \leq \mathcal{R}\left(\mathcal{E} - \frac{J^2}{2\mu\mathcal{R}^2} - \frac{Q^2}{8\pi\mathcal{R}}\right)$$

where μ is the rest mass of the object and now the term involving the angular momentum resembles very much the classical rotational energy. The term in parenthesis in the latter inequality is in fact the maximal possible energy for a charged and rotating system. When deducting the rotational energy, the internal energy will be $\mathcal{E}_{\text{int}} = \mathcal{E} - \frac{J^2}{2I}$ and, for a fixed $\mathcal{E}$, in order for it to attain its maximum, the contribution of the rotational energy should attain its minimum, thus the moment of inertia I should be the maximum possible too, thus, it should be the one of a spherical shell. The same holds for a charged system where the maximum possible internal energy is achieved when the electrostatic energy is minimum, i.e. it is the energy of a charged shell with constant charge density, ρ, thus $\mathcal{E}_{\text{int}} = \mathcal{E} - \frac{Q^2}{8\pi\mathcal{R}}$. Now, the classical approximation showed above relates the entropy of a system with a system whose internal energy is not maximum since only 2/3 of the minimum possible rotational energy is deducted form the total energy. In fact, from purely classical arguments the bound should be stricter, and such restrictions could eventually lead to inconsistencies in the bound. However the expansion performed is not the only way of obtaining the bound above, so it does not represent a powerful argument for the validity of the conjectured inequality (1.8).

Nevertheless, the conjectured inequality saturates for any Kerr-Newman black hole, and this fact suggests that it may indeed be correct. Any variation of (1.8) will not saturate for a Kerr-Newman black hole. Besides, when conjecturing (1.8), Bekenstein and Mayo offered arguments in favor that this bound cannot be lowered in virtue of the no hair conjecture, thus, the bound presented above is the most general bound relating conserved quantities for black holes so far.

From (1.8) and the fact that the entropy is always nonnegative, three other inequalities can be obtained. Furthermore, the equality in those inequalities will only hold when the entropy is zero, which means that the remaining bound should be valid for a system with well-defined energy. These there inequalities relate energy, charge and angular momentum, namely; by making the angular momentum zero we get

$$\mathcal{E} \geq \frac{Q^2}{8\pi\mathcal{R}} \tag{1.9}$$

by making the charge zero

$$\mathcal{E}(\Sigma) \geq c\frac{|J(\Sigma)|}{\mathcal{R}} \tag{1.10}$$

and finally, when none of the quantities is zero

$$\mathcal{E}^2 \geq \frac{Q^4}{64\pi^2\mathcal{R}^2} + \frac{c^2J^2}{\mathcal{R}^2} \tag{1.11}$$

Note that in the more general inequality (1.11) the only constant that appears is c, then a suitable theory to prove such inequalities is electromagnetism. Each of the latter inequalities can, in fact, be proved for the case of Maxwell's electrodynamics. This work has been done by Dain in [20] and it is the main guide for the present dissertation. As it has been initially pointed out by Bekenstein, these bounds appear from gravitational considerations, however they have a universal character, then it should be possible (and a theoretical task) to prove such bounds in non-gravitational theories. The main questions that we want to investigate are related with the physical implication of the latter inequalities and whether or not these inequalities remain valid for nonlinear regimes of electrodynamics. By now we will start by exposing the main tools used to prove the inequalities (1.9) to (1.11) in electrodynamics, as well as the main physical consequences of these inequalities.

1.2.3.1 Bekenstein's full bound in electrodynamics.

Now we will review the theorems obtained by Dain in [20] for Bekenstein inequalities in Maxwell's electrodynamics.The first inequality (1.12) relates the total energy of the field with the electrostatic energy of a spherical shell of radius $\mathcal{R}$ and charge Q. In this case, we know that a charged shell has the minimum electrostatic energy, thus, the bound can be stated as follows:

Theorem 1.2.1. *Assume that the charge density ρ has compact support contained in the region Σ. In electrostatics, the following inequality holds:*

$$Q^2 \leq 8\pi\mathcal{E}\mathcal{R} \tag{1.12}$$

where Q is the charge contained in Σ, $\mathcal{R}$ is the radius of Σ defined in Def. 1.2.1, and $\mathcal{E}$ is the total electromagnetic energy. The equality in (1.12) holds if and only if the electric *field is equal to the electric field produced by a spherical thin shell of constant surface charge density and radius $\mathcal{R}$. In particular, this implies that the electric field vanishes inside Σ.*

The proof of (1.12) is simple. Since the above theorem involves the energy of a charged shell, it is possible to start from a generic potential which will be split into a potential (which coincides with the one of the charged spherical shell), and an auxiliary potential. Then, by virtue of Maxwell's equations, it is easy to show that the total energy of a generic field will, in fact, be greater than or equal to the energy of a spherical thin shell. This procedure will be detailed later on when treating the inequality (1.12) in nonlinear electrodynamics. However, by now it is important to point out that since the respective bound explicitly involves the electrostatic energy of a spherical shell within Maxwell electrodynamics, the violation of (1.12) seems very plausible for a nonlinear regime of electrodynamics.

The validity of the second inequality (1.10) is less obvious. It relates the energy of a field in a region, Σ, and the total angular momentum of the same region. A priori there is no physical reason to think that such a restriction may be imposed to *any* physical system. However it seems to be a fairly reachable condition for a wide variety of physical theories as it will be addressed later on. For the case of Maxwell electrodynamics, the inequality can be stated as follows [20]

Theorem 1.2.2. *Consider a solution of Maxwell's equations in the domain Σ. Let $\mathcal{R}$ be the radius of Σ defined in Def. 1.2.1 and let x_0 be the center of the corresponding sphere. Then the following inequality holds:*

$$c|J(\Sigma)| \leq \mathcal{R}\mathcal{E}(\Sigma) \tag{1.13}$$

where $J(\Sigma)$ is the angular momentum of the electromagnetic field with respect to the point x_0. Moreover, the equality holds if and only if the electromagnetic field vanishes in Σ.

Proof. The scheme of the proof is the following: via the definitions of energy for an electromagnetic field

$$\mathcal{E}(\Sigma) = \frac{1}{2}\int_\Sigma d^3x(|\mathbf{E}|^2 + |\mathbf{B}|^2)$$

and the angular momentum projected along the direction of the unit vector $\mathbf{k}$

$$J(\Sigma) = \int_\Sigma d^3x\,[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]\cdot\mathbf{k}$$

within Maxwell electrodynamics, it is possible to evaluate the difference $\mathcal{E}(\Sigma) - |J(\Sigma)|/\mathcal{R}$ of both quantities and maximize the difference as

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma d^3x\left\{\frac{|\mathbf{E}|^2+|\mathbf{B}|^2}{2} - \frac{1}{\mathcal{R}}\,|[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]\cdot\mathbf{k}|\right\}$$

where we have used the inequality $|\int_A dx f(x)| \leq \int_A dx|f(x)|$. It is possible to further reduce the integrand by applying the vector triangular inequalities $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$, and $|\mathbf{a}\times\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ together with the fact that $\mathbf{k}$ is a unitary vector, obtaining

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma d^3x \left\{ \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2} - \frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}\times\mathbf{B}| \right\}$$

furthermore it is possible to bound the last term in the integrand as

$$|\mathbf{E}\times\mathbf{B}|^2 = \left(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2}\right)^2 - \left\{ \left(\frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{2}\right)^2 + (\mathbf{E}\cdot\mathbf{B})^2 \right\} \leq \left(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2}\right)^2$$

where the equality is attained when $|\mathbf{E}| = |\mathbf{B}|$ and $\mathbf{E} \perp \mathbf{B}$. With this in hand it is possible to express the difference between energy and angular momentum is the region Σ as

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma d^3x \left(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2}\right)\left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right) \geq 0 \tag{1.14}$$

Since the evaluated difference in the latter is always positive, the corresponding integral will always be positive too. Thus, (1.13) is proved. Then, the equality will be achieved if and only if the integral is zero, since the integrand in (1.14) is a function of the fields, both electric and magnetic fields must be zero in order for the equality to be valid. Let us note that the inequality (1.13) can also be derived from the Dominant Energy Condition (DEC) for any field theory, the details and consequences of this fact will be explored later on. □

It is noteworthy that the inequality (1.13) can also be expressed for slow rotations as

$$\mathcal{E}(\Sigma) \geq \frac{1}{2}\frac{J^2}{\mu\mathcal{R}^2} \tag{1.15}$$

here it is possible to interpret the bound in classical terms as follows: a system of total energy $\mathcal{E}(\Sigma)$ is bounded from below by 2/3 of its minimum rotational energy. Thus, the bound can be recasted as follows

$$\mathcal{E}(\Sigma) \geq \frac{2}{3}\frac{J^2}{2I_s} \tag{1.16}$$

where I_s is the moment of inertia of a spherical shell. It is not clear when does the equality in the latter holds, since the right hand side represents a minimum rotational energy for a rigid body. Perhaps it is possible to tighten the classical bound, but such a bound will no longer be derivable from the relativistic inequality.

Finally, the complete inequality (1.11) relates the total energy of a system with its charge and angular momentum. As in the previous bound, there is *a priori* no physical reason to think that such a bound can be possible because of the presence of the angular momentum. The previous approximation (1.16) can be performed obtaining the same result for the rotational energy, however the rigidity statement is not clear at all. Moreover, for Maxwell electrodynamics the inequality can be stated in the following theorem [20]

Theorem 1.2.3. *Assume that $\rho(x, t_0)$, for some t_0, has compact support contained in Σ. Consider a solution of Maxwell's equations that decay at infinity. Then the following inequality holds at t_0:*

$$\frac{c|J(\Sigma)|}{\mathcal{R}} + \frac{Q^2}{8\pi R} \leq \mathcal{E}$$

that in particular implies

$$\frac{Q^4}{64\pi^2\mathcal{R}^2} + \frac{c^2|J(\Sigma)|^2}{\mathcal{R}^2} \leq \mathcal{E}^2 \tag{1.17}$$

If the equality holds, then the electromagnetic field is that produced by an electrostatic spherical thin shell of radius $\mathcal{R}$ and charge Q. For that case, the magnetic field vanishes everywhere and hence $J = 0$.

The proof of the theorem is performed in an analogous way as in the electrostatic case, i.e. splitting the potential in two and calculating the electromagnetic energy from Maxwell's equations for electrodynamics, then splitting the total energy in two different regions, namely Σ and $\mathbb{R}^3 \setminus \Sigma$, then it is possible to associate the angular momentum in Σ to the contribution of the energy in that region. Then, it is possible to estimate the difference as in the previous inequality and note that the remaining integrals will be always nonnegative in virtue of the fulfilment of the inequality between energy and angular momentum. With this in mind, from Maxwell electrodynamics it is possible to conclude that the complete inequality (1.17) *only* requires the fulfilment of inequality (1.13) and not (1.12). It will be seen later on that for the case of nonlinear electrodynamics there may be additional requirements for the fulfilment of the complete inequality.

2 Nonlinear Electrodynamics.

2.1 Covariant Formulation of Maxwell's Theory.

The genesis of Special Relativity is related to inconsistencies between Classical Dynamics and Electrodynamics. More precisely, the transformation between two inertial systems in Classical Dynamics, such that Newton's First Law remains invariant, requires the existence of Galilean transformations. Nevertheless, this invariance group does not leave the electromagnetic wave equation invariant, thus, Maxwell's electromagnetism is not invariant under Galilean transformations. Furthermore, it is possible to obtain the minimum set of transformations that leave the electromagnetic wave equation invariant for inertial observers, this set of transformations is known as Lorentz transformations and constitute the natural invariance group of electrodynamics.

Since electrodynamics is invariant under these transformations, it is suitable to obtain the well-known equations of the theory in a form invariant, i.e. manifestly covariant way. Maxwell's equations in vacuum read,

$$\nabla \cdot \mathbf{E} = \rho \tag{2.1a}$$

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{2.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.1d}$$

where (2.1a) and (2.1d) are known as the Gauss' law for the electric and magnetic field respectively, while (2.1c) is known as the *Faraday-Lenz* law and (2.1b) as the *Ampere-Maxwell* equation. From (2.1d), it is possible to associate the rotational of a vector field, $\nabla \times \mathbf{A}$, to the magnetic field, $\mathbf{B}$. Using this into (2.1c) it is possible to show that $\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$. Also, manipulating the equations with source terms we obtain,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = \partial_\mu J^\mu = 0 \tag{2.2}$$

which is the *continuity equation*, which implies the conservation of electric charge. Here, J^μ is the current 4-vector with components $(\rho, \mathbf{J})$, and the electric charge is given by

$$Q = \int_{\mathbb{R}^3} d^3x \rho(x) \tag{2.3}$$

Since we have shown that both the electric and magnetic field can be written in terms space and time derivatives of a scalar and a vector potential, Φ and $\mathbf{A}$ it should be possible to express both potentials as a 4-vector,

$$A^\mu = (\Phi, \mathbf{A})$$

such that Maxwell's equations should be written in a manifestly covariant way as

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{2.4a}$$

$$\partial_{[\alpha} F_{\beta\gamma]} = 0 \tag{2.4b}$$

where $F_{\mu\nu} = F_{[\mu\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric *Faraday tensor* that reproduces Maxwell's equations and J^μ is the current 4-vector with components $(\rho, \mathbf{J})$. Note that (2.4a) reproduces (2.1a) and (2.1b), while (2.4b) reproduces (2.1c) and (2.1d). The identity in (2.4b) is called *Bianchi identity* and can be rewritten as $\partial_\mu \widetilde{F}^{\mu\nu} = 0$, where $\widetilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$ given by

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{2.5}$$

and $\eta^{\mu\nu\alpha\beta} = -\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor, while $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor density. Writing explicitly both the Faraday tensor and its dual

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad ; \quad \widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix} \tag{2.6}$$

it may be seen that by performing the transformation $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$ it is possible to pass from $F^{\mu\nu}$ to $\widetilde{F}^{\mu\nu}$. In fact, Maxwell's equations (2.4a) and (2.4b) without sources are invariant under duality transformations

$$F'_{\mu\nu} = F_{\mu\nu}\cos\theta + \widetilde{F}_{\mu\nu}\sin\theta \tag{2.7a}$$

$$\widetilde{F}'_{\mu\nu} = -F_{\mu\nu}\sin\theta + \widetilde{F}_{\mu\nu}\cos\theta \tag{2.7b}$$

which represent rotations between the electric and magnetic fields. Note that this invariance argument is also valid for Maxwell's equations with sources such that there exists a magnetic monopole, i.e. (2.4b) should be of the form $\partial_\mu \widetilde{F}^{\mu\nu} = J^\nu_{\mathrm{m}}$ with J^μ_{m} the magnetic current 4-vector. If so, then it is possible to perform a duality transformation between J^μ and J^μ_{m} such that, for an appropriate rotation we obtain $J^\mu_{\mathrm{m}} = 0$ [21, p. 252]. Now, note that the transformation present in (2.6) implies $\theta = \pi/2$ in (2.7).

2.2 Lagrangian Formulation of Maxwell's Theory.

When treating a physical theory, specially in the context of classical mechanics, it is important to have a representation of it in a Lagrangian formalism since it will allow us to find the equations of motion and define conserved currents via Noether's theorem. The Lagrangian theory for fields resembles the particle treatment such that now, since the systems treated have infinite degrees of freedom, we will treat fields as the generalized coordinates for the particle analogous. Given this, the system is governed by an action of the form

$$S = \int_{\mathcal{U}} d^4x\sqrt{-g}\mathcal{L} \tag{2.8}$$

where $\mathcal{L}$ is the *Lagrangian density*, which will be called *Lagrangian* for shortage and $\mathcal{U}$ is a domain bounded by spacelike surfaces that vanish at infinity [22]. Here, $\mathcal{L}$ can depend on fields of any kind (scalar, tensorial, etc).

When treating a field theory in the Lagrangian formalism, it is important to construct invariant scalar quantities for the fields such that, after performing the usual variation we obtain the covariant Euler-Lagrange equations. For example, electrodynamics is invariant under Lorentz transformations, then the Lagrangian must be a Lorentz scalar. Now, since the aim is to reproduce Maxwell's equations through an action, then the invariant scalars in the action must be contractions of the two tensors defined in (2.6) and (2.5).

For such a purpose, we can construct the following invariants, $F^{\mu\nu}F_{\mu\nu}$ and $\widetilde{F}^{\mu\nu}F_{\mu\nu}$. For purposes of simplicity, we will define them in the following way,

$$F = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} = (|\mathbf{B}|^2 - |\mathbf{E}|^2) = E_\alpha E^\alpha - B_\alpha B^\alpha \tag{2.9}$$

$$G = \frac{1}{2}\widetilde{F}^{\mu\nu}F_{\mu\nu} = -2(\mathbf{B}\cdot\mathbf{E}) = 2B_\alpha E^\alpha \tag{2.10}$$

where E^α and B^α are spacelike vectors. It can be shown that any other invariant for electrodynamics can be expressed as a linear combination of the two invariants above [23, § 25]. Thus, those are the basic invariants for electrodynamics. The physical implication of these invariants is obvious, if the invariant F is zero in one reference frame, then it will be zero in every other reference frame, thus $|\mathbf{E}|^2 = |\mathbf{B}|^2$ in every inertial reference frame. Also, if the invariant G is zero in one reference frame, then it means that the electric and magnetic field are perpendicular in every other reference frame, and if this quantity is different from zero, $\mathbf{E}$ and $\mathbf{B}$ make an angle different than $\pi/2$ in every reference frame, we can find a frame where both fields are parallel at some point. And if $|\mathbf{E}| > |\mathbf{B}|$ (or $|\mathbf{E}| < |\mathbf{B}|$) in one reference frame, then the inequality will be valid in any other reference frame.

It is convenient to introduce the following algebraic identities for various contractions of $F^{\mu\nu}$ and $\widetilde{F}^{\mu\nu}$ which will be used later on,

$$\widetilde{F}^{\mu\alpha}\widetilde{F}_{\alpha\nu} - F^{\mu\alpha}F_{\alpha\nu} = F\delta^\mu_\nu \tag{2.11a}$$

$$\widetilde{F}^{\mu\alpha}F_{\alpha\nu} = -\frac{1}{2}G\delta^\mu_\nu \tag{2.11b}$$

$$F^\mu{}_\alpha F^\alpha{}_\beta F^\beta{}_\nu = -\frac{1}{2}G\widetilde{F}^\mu{}_\nu - FF^\mu{}_\nu \tag{2.11c}$$

$$F^\mu{}_\alpha F^\alpha{}_\beta F^\beta{}_\lambda F^\lambda{}_\nu = \frac{1}{4}G^2\delta^\mu_\nu - FF^\mu{}_\alpha F^\alpha{}_\nu \tag{2.11d}$$

Since the aim is to reproduce Maxwell's equations from the variation of $\mathcal{L}$, knowing that Maxwell's equations are linear on the fields it is reasonable to express a generic Lagrangian as linear on the invariants F and G (because both of them are quadratic on the fields, leading to linear field equations). Nevertheless, it can be shown that the variation of G is a total divergence of the action. Thus, linear (Maxwell's) electrodynamics has the following Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}F \tag{2.12}$$

which is known as the Larmor Lagrangian[1]. By varying the action for Maxwell's electrodynamics with respect to the potential A^μ and velocity $\partial_\alpha A^\mu$ we obtain the equations of motion as in (2.4a) and (2.4b).

It is possible to obtain the energy-momentum tensor due to a given Lagrangian by varying its action with respect to the metric $g^{\mu\nu}$. And, in addition we have the following useful identities: from the identity for a matrix M, $\ln\det M = \mathrm{Tr}\ln M$ we get $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta(g^{\mu\nu})$; from $\delta^\mu_\nu = g^{\mu\alpha}g_{\alpha\nu}$ and varying it we get $\delta(g^{\mu\nu}) = -g^{\mu\alpha}\delta(g_{\alpha\beta})g^{\beta\nu}$. The energy-momentum tensor by varying the action with respect to the metric is defined as,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L)}{\delta g^{\mu\nu}} \tag{2.13}$$

which is known as *metric energy-momentum tensor*. This strategy was initially proposed by Hilbert and it is remarkable to see that it always returns a symmetric tensor, so, with this technique there is no need of using the symmetrization procedures as in the Noether's theorem. Also this definition will allow us to define the energy-momentum tensor in curved spacetimes, something that can't be done by Noether's theorem.

However, it is important to emphasize the importance of Noether's theorem in the development of field theory. Noether's theorem states that for each continuous symmetry of $\mathcal{L}$ there exists a conservation law, which will allow us to find all the conserved quantities for a determined action. For example, if the action is invariant under the spacetime translations $x'^\mu = x^\mu - \epsilon^\mu$, then there exists a tensor $S^{\mu\nu}$ such that $\partial_\mu S^{\mu\nu} = 0$. However, in general $S^{\mu\nu}$ must be symmetrized in order to obtain the symmetric energy-momentum tensor. Throughout this work we will employ Hilbert's method for obtaining $T_{\mu\nu}$.

Now, applying the Hilbert operation to the Larmor Lagrangian, we obtain the energy-momentum tensor for Maxwell's electrodynamics,

$$T_{\mu\nu} = F_{\mu\lambda}F^{\lambda}_{\ \nu} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} \tag{2.14}$$

Note that in the following development we will consider only the flat space case ($g_{\mu\nu} = \eta_{\mu\nu}$). From the latter it is useful to define the *energy density*, u, and *momentum density*,

[1] It was discovered by Joseph Larmor in 1900.

s_i, as

$$u = T_{00} = \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) \tag{2.15}$$

$$s_i = T_{0i} = (\mathbf{E} \times \mathbf{B})_i \tag{2.16}$$

We may recognize (2.16) as the *ith* component of the Poynting vector. These quantities will allow us to define other relevant physical quantities.

Other interesting features of $T_{\mu\nu}$ for Maxwell's electrodynamics are that it is a traceless quantity, and in flat space it is easy to show that $\partial_\mu T^{\mu\nu} = 0$, which implies the conservation of $T_{\mu\nu}$.

Now, let η^μ be a Killing vector representing space rotations and t^μ a timelike vector normal to a spacelike surface U. Then, the angular momentum in the region Σ can be obtained from the energy-momentum tensor as

$$J(\Sigma) = \int_\Sigma d^3x T_{\mu\nu} t^\mu \eta^\nu \tag{2.17}$$

where, choosing $t^\mu = (1, 0, 0, 0)$ and x^i are spacelike Cartesian coordinates on Σ, the rotations are characterized byf

$$\eta_i = \epsilon_{ijk} k^j x^k \tag{2.18}$$

where k is a constant spacelike unit vector that represents the axis of rotation. Note that this definition coincides with the one given by Weinberg [24, p. 46]. Given this definition, we can derive the angular momentum on a region Σ for Maxwell's electrodynamics as

$$J(U) = \int_\Sigma d^3x \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\right] \cdot \mathbf{k}. \tag{2.19}$$

2.3 Theories of Electrodynamics of the form $\mathcal{L}(F)$ and $\mathcal{L}(F, G)$.

As it has been argued before, there exist only two independent scalar invariants that can be constructed from the Faraday tensor $F^{\mu\nu}$ and its dual $\widetilde{F}^{\mu\nu}$, which are given by the contractions $F^{\mu\nu}F_{\mu\nu}$ and $\tilde{F}^{\mu\nu}F_{\mu\nu}$. With this in mind, and the fact that the Larmor Lagrangian is linear and only depends on the invariant F, it is natural to search for more general Lagrangians that may describe situations that Maxwell's theory can't (e.g. self-interaction, singularity-free theories). Thus, it is possible to think of Maxwell's theory as an approximate theory such that every acceptable nonlinear extension of electrodynamics should recover Maxwell's electrodynamics. Among other alternative formulations of electrodynamics, it is important to note that the superposition principle is a consequence of the linearity of the field equations, thus, in nonlinear electrodynamics (NLED), there may not exist superposition of solutions and there could also be vacuum birefringence so that the field equations in vacuum can represent material phenomena in vacuum.

Historically, since its first formulation [25], the research on nonlinear electrodynamics has not been taken as a priority task for the community, mainly because the appearance of nonlinear differential equations for the fields results into complicated computations; as well as that, up to now, Maxwell electrodynamics has been a very successful theory for describing electromagnetic phenomena that we have been able to observe. Nonetheless, in the recent years, the research on NLED obtained much interest both from theoretical and experimental physicist due to the fact that NLED emerges naturally from String Theory, and effective actions for quantum electrodynamics can be obtained by perturbations of the Born-Infeld action. It is important to note that before the current interests behind NLED, some very important contributions have appeared [26, 27], as well as the analysis of causality for NLED and cosmological applications have been developed at CBPF [28, 29, 1].

In the following section we will introduce some known examples of NLED together with their most important consequences and finally expose the expressions for those important quantities for the work purposes for generic Lagrangians $\mathcal{L}(F)$ and $\mathcal{L}(F, G)$.

2.3.1 Born-Infeld Electrodynamics.

The most remarkable example of NLED is the one given by Born and Infeld [2, 30]. After the seminal works of Mie [25], Born, initially inspired by the Lagrangian for a relativistic free particle, proposed the following Lagrangian to generalize Maxwell's electrodynamics

$$\mathcal{L} = \beta^2 \left(1 - \sqrt{1 - \frac{F}{\beta^2}}\right) \tag{2.20}$$

based on the assumption that the problem of quantization of the electromagnetic field was due to the non-accounting of a radius for the electron [31]. Subsequently, based on Born's initial ideas, Born and Infeld derived a Lagrangian such that the action integral is an invariant, this is, the integrand should be a tensor density of weight -1, so that it may be decomposed in a symmetric part, $g_{\mu\nu}$, and an antisymmetric part, $F_{\mu\nu}$, namely

$$\mathcal{L} = \sqrt{-\det(a_{\mu\nu})} + A\sqrt{-\det(g_{\mu\nu})} + B\sqrt{-\det(F_{\mu\nu})} \tag{2.21}$$

with $a_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu}$. The constants A and B can be determined with the constraint that (2.21) must have the adequate Maxwellian limit arriving to $A = 1$ and $B = 0$. Thus, in cartesian coordinates, the Born-Infeld Lagrangian reads

$$\mathcal{L} = \beta^2 \left(1 - \sqrt{U}\right) \tag{2.22}$$

where $U = 1 + \frac{F}{\beta^2} - \frac{G^2}{4\beta^4}$, and β is a *maximum field parameter*. Noteworthy, the expression for the Lagrangian (2.21) has the particularity that can be generalizable to higher dimensions, a property that has given Born-Infeld electrodynamics a modern appeal. It is appropriate to point out that the initial formulation of Born's (2.20) action was proposed only in concordance with the principle of finiteness, and achieved its goal. The principle of finiteness states that in a satisfactory theory the physical quantities should not become infinite. However, in many theories the principle of finiteness is obtained as a consequence of deeper principles, e.g. a finite maximum speed is obtained from the principle of relativity in SR. Hence, Born and Infeld started looking for a theory that could recover Maxwell's theory under the appropriate symmetry assumptions. Their main guiding line was based that the passage from relativistic mechanics to classical mechanics implies a reduction on the symmetry group from the Lorentz group to the Galilean group. Similarly, they searched for a case where a Lagrangian invariant under general coordinate transformations reduces to the Larmor Lagrangian (which is Lorentz invariant) in an adequate field limit. Following this line of thought they postulated a simple invariant action and arrived to (2.22), where the invariant G arises naturally. This modification of electrodynamics was proposed in order to avoid the infinite self-energy of a point charge, such that a consistent theory of the electron could be developed. It was then believed that by avoiding these classical divergences it could be easier to find a complete quantum field theory of electrodynamics. It is noteworthy that the Larmor Lagrangian is recovered for the first order approximation of (2.22) (*weak-field approximation*). By varying the action with respect to the potential A^μ we obtain the corresponding field equations,

$$\partial_\mu \left[\frac{1}{\sqrt{U}} \left(-F^{\mu\nu} + \frac{G}{2\beta^2} \widetilde{F}^{\mu\nu} \right) \right] = -J^\nu \tag{2.23}$$

which can be recasted in vector notation, defining $\mathbf{D} \equiv \frac{1}{\sqrt{U}} \left(\mathbf{E} + \frac{(\mathbf{E}\cdot\mathbf{B})}{\beta^2} \mathbf{B} \right)$ and $\mathbf{H} \equiv \frac{1}{\sqrt{U}} \left(\mathbf{B} - \frac{(\mathbf{E}\cdot\mathbf{B})}{\beta^2} \mathbf{E} \right)$ in analogy with the electric displacement and magnetic field strength respectively, as

$$\nabla \cdot \mathbf{D} = \rho \qquad\qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j} \tag{2.24}$$

that resemble Maxwell's equations in matter. Note that in general the electric permittivity ε and the magnetic permeability μ in the constitutive relations $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ are nonlinear. Also, from the Bianchi identities the other pair of Maxwell's equations will remain valid. So, Born-Infeld electrodynamics can be interpreted as an electrodynamics presenting matter effects in vacuum due to the modification in the constitutive relations.

Given the Lagrangian we can derive the respective energy-momentum tensor for the theory,

$$T_{\mu\nu} = \frac{1}{\sqrt{U}} \left(F_\nu{}^\lambda F_{\lambda\mu} + \frac{1}{4\beta^2} G^2 g_{\mu\nu} \right) + g_{\mu\nu} \beta^2 (\sqrt{U} - 1) \tag{2.25}$$

where the energy-momentum tensor for Maxwell's electrodynamics is recovered in the limit when $\frac{F}{\beta^2} \to 0$, $\frac{G}{\beta^2} \to 0$. Note that the above expression is not traceless, this fact is related to Born-Infeld electrodynamics not being conformally invariant. For the electrostatic case, from the Gauss law in (2.24) it is possible to obtain the electric field for a point charge, where the charge density is given by $\rho = e\delta^3(\mathbf{x})$, such that $\mathbf{D} = e/4\pi r^2 \hat{\mathbf{r}}$, then the electrostatic constitutive relation will be.

$$\mathbf{D} = \frac{1}{\sqrt{1 - \frac{|\mathbf{E}|^2}{\beta^2}}}\mathbf{E}$$

inverting the latter equation and evaluating it at the point $r = 0$ leads

$$|\mathbf{E}|_{r=0} = \frac{e}{4\pi r^2 \sqrt{1 + \frac{e^2}{16\pi^2 r^4 \beta^2}}} = \beta \tag{2.26}$$

the behavior of the electric field for a point charge in (2.26) compared to the Maxwell case can be seen in Fig. 2, where it is clear that the electric field approaches a finite value at the origin.

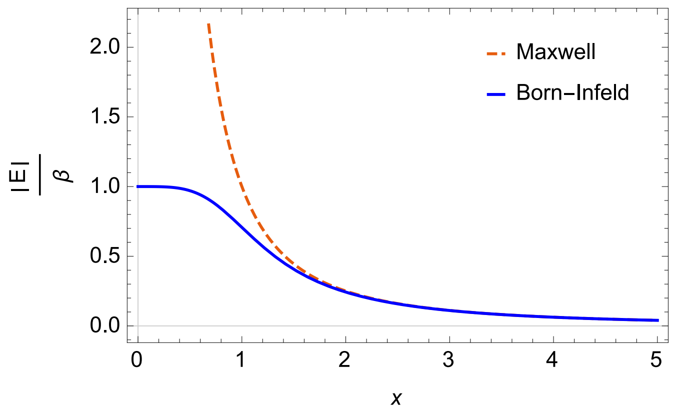

Figure 2 – The electrostatic field of a point charge as a function of $x = r\sqrt{\beta/e}$ for Born-Infeld (solid line) and Maxwell (dashed line).

Also, from the energy-momentum tensor (2.25) it is possible to obtain the energy for a point charge e by expressing the electrostatic part of the energy density u as

$$u = \beta^2 \left(\frac{1}{\sqrt{1 - \frac{|\mathbf{E}|^2}{\beta^2}}} - 1 \right) = \beta^2 \left(\sqrt{1 + \frac{|\mathbf{D}|^2}{\beta^2}} - 1 \right)$$

then, in virtue that we know the functional form of $|\mathbf{D}|$ it is possible to integrate the energy density to obtain the total energy as

$$\mathcal{E}_0 = \beta^2 \int_0^\infty dr\, r^2 \left(\sqrt{1 + \frac{e^2}{16\pi^2 \beta^2 r^4}} - 1 \right) = \sqrt{\frac{e^3 \beta}{4\pi}} \int_0^\infty dx\, x^2 \left(\sqrt{1 + \frac{1}{x^4}} - 1 \right) = 0.3486\sqrt{e^3\beta} \tag{2.27}$$

Now, from (2.25) it is possible to derive the angular momentum in a given region, Σ, as presented in the previous section, obtaining,

$$J(\Sigma) = \int_\Sigma d^3x \frac{1}{\sqrt{U}} \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\right] \cdot \mathbf{k} \tag{2.28}$$

Even though the motivations for proposing these electrodynamics turned out to be mistaken because of the fact that it is a non-renormalizable theory, Born-Infeld electrodynamics stands over the rest of nonlinear theories of electromagnetism due to some unique features that this theory has. For example, it is the only nonlinear theory that does not exhibit birefringence, which is associated with the fact that it has a single lightcone. This theory has gained attention in the past decades due to the fact that it emerges naturally from D-Brane theory, such that the way of coupling strings to electromagnetism is by Born-Infeld's Lagrangian, and effective actions in QED present perturbative corrections of the form of Born-Infeld's Lagrangian. It is also important to point out that at the moment when the theory was proposed there wasn't an adequate procedure for treating divergences in QFT, later on it was showed that it is possible to quantize the electromagnetic field despite of its divergences. Furthermore, Born-Infeld electrodynamics turns to be a non-renormalizable theory when treated in the quantum level. It turns out that, in this case, avoiding divergences in a Classical Field Theory makes the passage to a Quantum Field Theory impossible. It is important to note that at the moment of Born's proposal QFT was not developed at all, thus, it entirely made sense that avoiding classical divergences could help the passage from electromagnetism to QED.

Born-Infeld electrodynamics have been widely studied in many branches of physics since they were initially proposed. Among the contributions there are solutions for Born-Infeld electrostatics [32], the coupling of Born-Infeld electrodynamics to Einstein's equations [33], and the study of the magnetic sector [34]. Also, Born-Infeld construction of electrodynamics and the fact that it leads to a finite energy for a point charge has inspired alternative formulations of the gravitational action known as Born-Infeld inspired modifications of gravity[2] [35], where it is possible to obtain singularity-free solutions for black holes and cosmological scenarios.

2.3.2 Exponential Electrodynamics.

Another example of NLED is given by a Born-Infeld like Lagrangian presented originally by Hendi [36], the Lagrangian for this electrodynamics reads,

$$\mathcal{L} = \beta^2 \left(e^{-\frac{\mathcal{X}}{\beta^2}} - 1 \right) \tag{2.29}$$

with $\mathcal{X} = \frac{F}{2} - \frac{G^2}{8\beta^2}$. Note that (2.29) is constructed in such a way that Maxwell's electrodynamics is recovered when $F \ll \beta$ as in Born-Infeld electrodynamics. Hendi originally used this formulation of NLED in order to obtain Reissner-Nordström type of solutions for charged black holes. However, this formulation of NLED electrodynamics presents some other important aspects that will be exposed here.

[2] This formulation uses a metric affine, *à la* Palatini formalism.

Let us start by deriving the energy-momentum tensor for this electrodynamics,

$$T_{\mu\nu} = F_{\mu\alpha}F^{\alpha}{}_{\nu}e^{-\frac{\mathcal{X}}{\beta^2}} + g_{\mu\nu}\left(\frac{1}{4}\frac{G^2}{\beta^2}e^{-\frac{\mathcal{X}}{\beta^2}} - \beta^2 e^{-\frac{\mathcal{X}}{\beta^2}} + \beta^2\right) \tag{2.30}$$

while the angular momentum corresponding to exponential electrodynamics is given by

$$J(\Sigma) = \int_{\Sigma} d^3x e^{-\frac{\mathcal{X}}{\beta^2}}\left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\right] \cdot \mathbf{k} \tag{2.31}$$

The field equations for exponential electrodynamics will also be expressible in terms of the fields **D** and **H**, with

$$\mathbf{D} \equiv e^{-\frac{\mathcal{X}}{\beta^2}}\left(\mathbf{E} + \frac{(\mathbf{E}\cdot\mathbf{B})}{\beta^2}\mathbf{B}\right) \quad , \quad \mathbf{H} \equiv e^{-\frac{\mathcal{X}}{\beta^2}}\left(\mathbf{B} - \frac{(\mathbf{E}\cdot\mathbf{B})}{\beta^2}\mathbf{E}\right) \tag{2.32}$$

Hendi's initial arguments for proposing Born-Infeld type theories of electrodynamics were that these theories don't present shock waves, birefringence, and that they are duality invariant. However, despite being originally inspired by Born-Infeld electrodynamics, it possesses an infinite electric field at the origin of the point charge and birefringence, which are features not present in Born-Infeld electrodynamics. In fact, from the above equation it is easy to show that in the electrostatic case the dependence of **D** with respect to **E** is, $\mathbf{D} = \mathbf{E}\,e^{\frac{|\mathbf{E}|}{2\beta^2}}$, then $\frac{|\mathbf{D}|^2}{\beta^2} = \frac{|\mathbf{E}|^2}{\beta^2}e^{\frac{|\mathbf{E}|^2}{\beta^2}}$ which can be written in terms of the Lambert W function[3] as $\mathbf{E} = \beta\sqrt{W\left(\frac{Q^2}{\beta^2 r^4}\right)}$ with $Q = e/4\pi$, then performing the expansion for $\beta \to \infty$ leads to [37],

$$\mathbf{E} = \frac{Q}{r^2}\sqrt{1 + \frac{Q^2}{\beta^2 r^4}}\hat{\mathbf{r}} \tag{2.33}$$

the electric field in this case diverges slower than in the Maxwell case.

2.3.3 Logarithmic Electrodynamics.

The photon-photon scattering in QED gives rise to the nonlinear phenomena of vacuum birefringence, which is not present in Born-Infeld electrodynamics. Given this fact and that many effective actions in QED have lagrangians with logarithmic functions of F and G, Gaete and Helayël [38] proposed the following logarithmic Lagrangian,

$$\mathcal{L} = -\beta^2 \ln\left(1 + \frac{F}{2\beta^2} - \frac{G^2}{8\beta^4}\right) \tag{2.34}$$

among other features, this Lagrangian exhibits birefringence and a finite energy for the point particle. The field equations for this particular Lagrangian can also be recasted as in the Born-Infeld case such that the resulting equations resemble Maxwell's equation in material media, with the following expressions for **D** and **H**,

$$\mathbf{D} \equiv \frac{1}{V}\left(\mathbf{E} + \frac{(\mathbf{E}\cdot\mathbf{B})}{\beta^2}\mathbf{B}\right), \qquad \mathbf{H} \equiv \frac{1}{V}\left(\mathbf{B} - \frac{(\mathbf{E}\cdot\mathbf{B})}{\beta^2}\mathbf{E}\right) \tag{2.35}$$

[3] Or Product Log function, is the inverse function of $f(W) = We^W$.

with $V = 1 + \frac{F}{2\beta^2} - \frac{G^2}{8\beta^4}$. The energy-momentum tensor for such a Lagrangian reads

$$T_{\mu\nu} = \frac{1}{V}\left(F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{G}{2\beta^2}F_{\mu\alpha}\widetilde{F}^{\alpha}_{\ \nu}\right) + g_{\mu\nu}\beta^2 \ln V \tag{2.36}$$

and the respective angular momentum for a region Σ takes the form

$$J(\Sigma) = \int_{\Sigma} d^3x \frac{1}{V}\left(\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\right) \cdot \hat{\mathbf{k}} \tag{2.37}$$

This type of electrodynamics also possesses the feature that the field of a point charge is finite. It is possible to obtain the electric field for a point charge by setting $\rho = e\delta^3(\mathbf{r})$ and inverting the relation between $\mathbf{D}$ and $\mathbf{E}$ in (2.35) for the electrostatic case, doing this we finally obtain the expression for the electric field that reads,

$$\mathbf{E} = \frac{\beta^2}{Q}\left(-r^2 + \sqrt{r^4 + \frac{2Q^2}{\beta^2}}\right)\hat{\mathbf{r}} \tag{2.38}$$

Thus, for $r = 0$ the electric field has a maximum value of $\sqrt{2}\beta$, which is higher than the maximum allowed field in Born-Infeld electrodynamics. However it is important to note, for further considerations, that the parameter β can be associated to a *fundamental field* given by

$$|\mathbf{E}|_{\text{fund}} = \frac{m_e^2 c^3}{e\hbar}$$

or, adopting the natural units,

$$|\mathbf{E}|_{\text{fund}} = \frac{m_e^2}{e} \tag{2.39}$$

where m_e and e are the mass and charge of the electron respectively. Then, since the electric field is bounded by $|\mathbf{E}|_{\text{fund}}$, $\beta = \frac{|\mathbf{E}|_{\text{fund}}}{\sqrt{2}}$ such that the maximum possible value of the quotient $|\mathbf{E}|/|\mathbf{E}|_{\text{fund}} = 1$.

For this particular theory birefringence arises when studying the dispersion relations of a plane wave decomposition in the presence of a constant magnetic field. Nevertheless, this phenomenon disappears for the low field approximation, $\frac{B_0}{\beta} \ll 1$, and it is known that QED with one loop corrections has birefringence. This led Kruglov [39] to generalize (2.34), proposing the following Lagrangian ,

$$\mathcal{L} = -\beta^2 \ln\left(1 + \frac{F}{2\beta^2} - \frac{G^2}{8\beta^2\gamma^2}\right) \tag{2.40}$$

where the Lagrangian presented in (2.34) is recovered for $\gamma = \beta$. With this new Lagrangian the energy-momentum tensor reads,

$$T_{\mu\nu} = \frac{1}{W}\left(F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{G}{2\gamma^2}F_{\mu\alpha}\widetilde{F}^{\alpha}_{\ \nu}\right) + g_{\mu\nu}\beta^2 \ln W \tag{2.41}$$

with $W = 1 + \frac{F}{2\beta^2} - \frac{G^2}{8\beta^2\gamma^2}$. And the angular momentum for this electrodynamics has the same form as (2.37) with the substitution of W instead of V. All the other electrostatic characteristics presented before remain valid. But it is important to note that with the addition of the new parameter γ, the low field approximation also presents birefringence by virtue of $\gamma \neq \beta$. In fact, Kruglov estimates the difference between both parameters to be

$$\frac{1}{\gamma^2} - \frac{1}{\beta^2} \approx 10^{-20}\text{T}$$

Hence, Kruglov's proposal offers an alternative viewpoint where the birefringence phenomena is also present for the weak field approximation.

2.3.4 $\mathcal{L}(F, G)$ Electrodynamics.

The previously exposed lagrangians all come from a particular choice of functions, inspired by Born-Infeld theory. We have seen that each of those lagrangians has its own particularities. However, in principle a nonlinear electromagnetic Lagrangian can have any functional form with the only restriction that it must reduce to the Larmor Lagrangian in the appropriate limit. In virtue of this, it is useful for our analysis to derive the previously exposed quantities, as well as the invariance constraints, for an arbitrary Lagrangian of the type $\mathcal{L}(F, G)$.

In order to do so, let us first derive the energy-momentum tensor for an arbitrary Lagrangian by varying its action with respect to the metric. So, for this case it is suitable to write (2.13) as,

$$T_{\mu\nu} = 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}(F, G) \tag{2.42}$$

with $\delta\mathcal{L} = \mathcal{L}_F \delta F + \mathcal{L}_G \delta G$. Then, writing both invariants F and G as explicit functions of the metric and performing the respective variation we get

$$\delta F = F_\mu{}^\lambda F_{\nu\lambda}\delta g^{\mu\nu} = -F_{\mu\lambda}F^\lambda{}_\nu \delta g^{\mu\nu}$$

$$\delta G = \frac{1}{4}F_{\alpha\beta}\widetilde{F}^{\alpha\beta}g_{\mu\nu}\delta g^{\mu\nu} = -F_{\mu\lambda}\widetilde{F}^\lambda{}_\nu \delta g^{\mu\nu}$$

where the relation (2.11b) was used to arrive to the last equation. Given this, we finally obtain the energy-momentum tensor for an arbitrary Lagrangian,

$$T_{\mu\nu} = -2F_{\mu\alpha}\left(\mathcal{L}_F F^\alpha_\nu + \mathcal{L}_G \widetilde{F}^\alpha_\nu\right) - \mathcal{L}(F, G)g_{\mu\nu} = -F_{\mu\alpha}E^\alpha_\nu - g_{\mu\nu}\mathcal{L}(F, G) \tag{2.43}$$

where $E^{\mu\nu}$ is known as the *excitation tensor* defined by $E^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}$. It is also possible to rewrite $T_{\mu\nu}$ in terms of the Maxwell energy-momentum tensor $\tau_{\mu\nu}$ as

$$T_{\mu\nu} = -2\mathcal{L}_F \tau_{\mu\nu} + g_{\mu\nu}\left(\mathcal{L}_G G + \mathcal{L}_F F - \mathcal{L}(F, G)\right) \tag{2.44}$$

From the above equation (2.44), it is clear that not all the energy-momentum tensors for NLED will be traceless. The fact of having a traceless energy-momentum tensor is intimately related to conformal invariance. It is possible to obtain a condition for general energy-momentum tensors being traceless. By taking the trace of (2.44) and noting that $\tau^{\mu}{}_{\mu} = 0$ we get

$$T^{\mu}_{\mu} = 4\left[F\mathcal{L}_F + G\mathcal{L}_G - \mathcal{L}(F,G)\right] \tag{2.45}$$

For the trace to be zero, we obtain the following differential equation [22, p. 388],

$$F\mathcal{L}_F + G\mathcal{L}_G - \mathcal{L}(F,G) = 0$$

which can be conveniently written in logarithmic form and using the chain rule as

$$l_f + l_g = 1 \tag{2.46}$$

where $l = \ln \mathcal{L}$, $l_x = \frac{\partial \ln \mathcal{L}}{\partial \ln X}$. Solving this equation and expressing it in terms of $\mathcal{L}$, F, and G yields to the solution

$$\mathcal{L} = \sqrt{FG}\mathcal{H}\left(\frac{F}{G}\right) \tag{2.47}$$

where $\mathcal{H}$ is an arbitrary function. Note that the Larmor Lagrangian is recovered when $\mathcal{H} = -\frac{1}{2}\sqrt{\frac{F}{G}}$. Neither the Born-Infeld, exponential nor logarithmic lagrangians can be expressed as functions of F/G, hence, those electrodynamics are not conformally invariant.

It is also useful to derive the angular momentum for an arbitrary Lagrangian in a given region Σ,

$$J(\Sigma) = \int_{\Sigma} d^3x 2\mathcal{L}_F[\mathbf{x} \times (\mathbf{B} \times \mathbf{E})] \cdot \mathbf{k} \tag{2.48}$$

note that the above expression is independent of the invariant G.

Also, the field equations derived for an action given by an arbitrary Lagrangian read,

$$2\partial_\mu \left(\mathcal{L}_F F^{\mu\nu} + \mathcal{L}_G \widetilde{F}^{\mu\nu}\right) = \partial_\mu E^{\mu\nu} = -J^\nu \tag{2.49}$$

while the second pair of equations comes from the Bianchi identitiy. Since (2.49) can be written more simply in terms of the excitation tensor, resembling the original equation from Maxwell electrodynamics, the natural question that arises is whether it is possible or not to have any duality transformation between $\widetilde{F}^{\mu\nu}$ and $E^{\mu\nu}$. In fact, if $E^{\mu\nu}$ and $\widetilde{F}^{\mu\nu}$ were independent variables then such a transformation would take $E^{\mu\nu} \to \widetilde{F}^{\mu\nu}$. However, both tensors are not always independent, thus a reduced set of lagrangians will be duality invariant. In order to prove this let us set (2.49) to zero in order to perform the following rotation [40]

$$E'_{\mu\nu} = E_{\mu\nu}\cos\theta + \widetilde{F}_{\mu\nu}\sin\theta \tag{2.50a}$$

$$\widetilde{F}'_{\mu\nu} = -E_{\mu\nu}\sin\theta + \widetilde{F}_{\mu\nu}\cos\theta \tag{2.50b}$$

then, by considering infinitesimal transformations of the form $\delta F_{\mu\nu} = \widetilde{E}_{\mu\nu}$ and $\delta E_{\mu\nu} = \widetilde{F}_{\mu\nu}$ it is possible to arrive to the following condition of duality for an $\mathcal{L}(F,G)$ theory

$$\widetilde{E}^{\mu\nu}E_{\mu\nu} = \widetilde{F}^{\mu\nu}F_{\mu\nu} \tag{2.51}$$

it can be shown that Born-Infeld electrodynamics satisfy the latter, thus, it is duality invariant, while neither exponential nor logarithmic electrodynamics are duality invariant.

2.4 Observer Decomposition of $T_{\mu\nu}$ and Energy Conditions.

From considerations primarily concerning General Relativity and Cosmology, during the decade of the 1970s, Hawking, Ellis, Penrose, among others, developed a series of singularity theorems, most of them in accordance with some energy conditions. The energy conditions are requirements imposed to the energy-momentum tensor that imply specific relations between physical quantities. It is always possible to extend the energy conditions to other types of spacetime theories. In the following section we will start by reviewing a relativistic way of decomposing the energy-momentum tensor in an irreducible representation. This is done in order to introduce properly the most relevant energy conditions for our work and their respective consequences.

2.4.1 Observer decomposition of $T_{\mu\nu}$.

As we have seen before, it is possible to define the energy-momentum tensor corresponding to an arbitrary electromagnetic Lagrangian $\mathcal{L}(F,G)$, however, in this section we will introduce a relativistic way of decomposing $T_{\mu\nu}$ in terms of its irreducible components. Let λ^μ be a vector at a point P of a smooth, future directed timelike curve with unit tangent vector v^μ, which represents the normalized 4-velocity of an observer, i.e. $v^\mu v_\mu = 1$, then, it is always possible to decompose λ^μ into its components proportional to v^μ and orthogonal to v^μ as [41]

$$\lambda^\mu = \lambda^\nu v_\nu v^\mu + (\lambda^\mu - \lambda^\nu v_\nu v^\mu) \tag{2.52}$$

where the first term is obviously proportional to v^μ, while the second term is the difference between the original vector and it projection along v^μ, thus, it is perpendicular to v^μ. It is therefore suitable to introduce the following projectors:

$$k_{\mu\nu} = v_\mu v_\nu \tag{2.53}$$

$$h_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu \tag{2.54}$$

where (2.53) defines a projector along the direction of v^μ, while (2.54) is a projector in the orthogonal direction of v^μ, such that (2.52) can be now expressed as

$$\lambda^\mu = \lambda^\nu k_\nu{}^\mu + \lambda^\nu h_\nu{}^\mu$$

Note that they are true projectors, as $k_{\mu\alpha} k^\alpha{}_\nu = k_{\mu\nu}$, $h_{\mu\alpha} h^\alpha{}_\nu = h_{\mu\nu}$, and $k_\mu{}^\alpha h_{\alpha\nu} = 0$. Moreover, note that both of them represent *induced metrics* along the temporal direction (along v^μ) and the spatial direction respectively; thus, $h_{\mu\nu}$ is an induced metric in a hypersurface Σ which is orthogonal to v^μ.

With this in mind, it is possible to decompose any tensor in its irreducible representations. For instance, we can use this tool to decompose the Faraday tensor in its components, namely the electric and magnetic fields, E^μ and B^μ as

$$E^\mu = F^\mu{}_\nu v^\nu \ , \ \text{and} \quad B^\mu = \widetilde{F}^\mu{}_\nu v^\nu = \frac{1}{2}\eta^{\mu\nu\alpha\beta}v_\nu F_{\alpha\beta}$$

then, it is possible to invert both relations above in order to express $F^{\mu\nu}$ in terms of the fields E^μ and B^μ as

$$F_{\mu\nu} = 2E_{[\mu}v_{\nu]} + \eta_{\mu\nu\alpha\beta}v^\alpha B^\beta$$

Furthermore, it is also possible to decompose the energy-momentum 4-vector in a projection along the two directions as

$$T^\mu{}_\nu v^\nu = \left(T_{\alpha\nu}k^{\alpha\mu}\right)v^\nu + \left(T_{\alpha\nu}h^{\alpha\mu}\right)v^\nu \tag{2.55}$$

where the first term in parenthesis is the energy density, while the second is the momentum density. Then, it should also be possible to decompose the energy-momentum tensor, $T_{\mu\nu}$ in its irreducible representation. For that purpose, we will start by exposing the general decomposition of an energy-momentum tensor representing a fluid. Let us start by performing all the possible contractions of a tensor $T^{\mu\nu}$ with the respective projectors v^μ and $h^{\mu\nu}$,

$$T^{\mu\nu} = \rho v^\mu v^\nu + \left(q^\mu v^\nu + q^\nu v^\mu\right) + \pi^{\mu\nu} - ph^{\mu\nu} \tag{2.56}$$

it is possible to identify each of the components above with quantities representing the energy-momentum tensor of a fluid; hence, we identify ρ as the energy density, p as the isotropic pressure, $\pi^{\mu\nu}$ as the anisotropic pressure, which is a traceless quantity and q^μ as the heat flux. It is important to note that *any* energy-momentum tensor can be decomposed in a similar way as above, i.e. it is possible to relate each of the components of an arbitrary $T_{\mu\nu}$ with ρ, p, $\pi_{\mu\nu}$ and q_μ. For the following development of the energy conditions, it is useful to write the energy-momentum tensor for a perfect fluid, i.e. where the only non-vanishing components are the energy density and the isotropic pressure:

$$T^{\mu\nu} = (\rho + p)\, v^\mu v^\nu - pg^{\mu\nu} \tag{2.57}$$

With this tool in hand it is possible to obtain the irreducible decomposition for an energy-momentum tensor of an arbitrary nonlinear electromagnetic Lagrangian of the form $\mathcal{L}(F,G)$, which was previously presented in (2.43). By performing the corresponding contractions we arrive to the following identities for the hydrodynamical quantities: the energy density $\rho = T_{\mu\nu}v^\mu v^\nu$ reads,

$$\rho = 2\mathcal{L}_F E_\alpha E^\alpha + 2\mathcal{L}_G E_\alpha B^\alpha - \mathcal{L}(F,G)$$

recalling that both E^α and B^α are spacelike vectors ($E^\mu E_\mu < 0$). The heat flux, $q^\alpha = T_{\mu\nu}v^\mu h^{\nu\alpha}$, reads

$$q^\alpha = -2\mathcal{L}_F \eta^{\beta\alpha\rho\sigma}E_\beta v_\rho B_\sigma + 2\mathcal{L}_G \eta^{\beta\alpha\rho\sigma}E_\alpha v_\rho E_\sigma = -2\mathcal{L}_F \eta^{\beta\alpha\rho\sigma}E_\beta v_\rho B_\sigma$$

since the second term in the first equality is identically zero. Now, the isotropic pressure, $p = -\frac{1}{3}T_{\mu\nu}h^{\mu\nu}$ reads,

$$p = -\frac{4}{3}\mathcal{L}_F F - \frac{4}{3}\mathcal{L}_G G + \frac{2}{3}\mathcal{L}_F E_\alpha E^\alpha + \frac{2}{3}\mathcal{L}_G E_\alpha B^\alpha + \mathcal{L}(F,G)$$

and finally, the anisotropic pressure, $\pi_{\mu\nu} = T_{\alpha\beta}h^\alpha{}_\mu h^\beta{}_\nu + ph_{\mu\nu}$, is

$$\pi_{\alpha\beta} = -\frac{4}{3}\mathcal{L}_F F h_{\alpha\beta} + \frac{2}{3}\mathcal{L}_F E_\lambda E^\lambda h_{\alpha\beta} - 4\mathcal{L}_G E_\lambda B^\lambda h_{\alpha\beta} +$$

$$+2\mathcal{L}_G\left(-B_\alpha E_\beta + E_\alpha B_\beta\right) + 2\mathcal{L}_F\left(E_\alpha E_\beta + B_\alpha B_\beta - B_\lambda B^\lambda h_{\alpha\beta}\right)$$

Thus, we have derived all the irreducible parts of the energy-momentum tensor for an arbitrary Lagrangian of the form $\mathcal{L}(F,G)$, note that the corresponding Maxwellian limits are correctly recovered in the latter equations.

2.4.2 Energy Conditions.

Einstein's equations, in principle, have an infinite number of solutions, many of them representing *unphysical* situations. Hence, it is important to have some criteria for defining *physically reasonable* matter content, this is, imposing some constraints on the right hand side of Einstein's equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

by introducing these constraints, the range of possible solutions should be considerably reduced. In fact, initially such energy conditions were imposed in order to prove certain singularity theorems [42]. The most studied energy conditions are the Null Energy Condition (NEC), the Weak Energy Condition (WEC), the Dominant Energy Condition (DEC) and the Strong Energy Condition (SEC). Despite the energy conditions were initially formulated in the context of GR, they are in principle valid for any classical spacetime theory. Nevertheless, it is important to note that the ideal theoretical setup for imposing these energy conditions is clearly GR, since the dynamical equations depend explicitly on the energy-momentum tensor, while other alternative spacetime theories may also serve as an adequate framework for using the energy conditions. In this section we will introduce the most relevant energy conditions for the purposes of the present dissertation, presenting both the physical and geometrical interpretations for the energy conditions following [43].

2.4.2.1 Weak Energy Condition.

We begin by assessing the simplest energy condition for the sake of the present work. The Weak Energy Condition (WEC) reads:

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \tag{2.58}$$

for all timelike vectors ξ^{μ}. This means, the energy density must be nonnegative for any observer transversing a timelike curve. Note that by continuity ξ^{μ} may also be a null vector. In principle, such a condition seems rather consistent from the physical point of view, it may even seem to be a very basic assumption for the energy density of some classical matter field. However, the WEC in fact is not the *weakest* energy condition, in fact WEC reduces to the Null Energy Condition (NEC) when $\xi^{\mu} \to k^{\mu}$ in (2.58) with k^{μ} being a null vector. Hence, WEC includes NEC. It is important to note that many singularity theorems are formulated through the fulfillness of NEC, and so, they hold for all the other conditions that include NEC.

In order to see the physical implication of the WEC, let us invoke the energy-momentum tensor for a perfect fluid presented in (2.57), note that this $T_{\mu\nu}$ is diagonal, thus it is of type I according to the classification presented in [42, p. 89]. For a perfect fluid, performing the respective projections on (2.57), the requirement that (2.58) holds for all timelike vectors ξ^{μ} gives

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} = (\rho + p)\left(v_{\mu}\xi^{\mu}\right)^{2} - p \geq 0 \tag{2.59}$$

now by the wrong way Schwarz inequality [41] $\left(\eta^{\mu}\xi_{\mu}\right)^{2} \geq |\eta^{\mu}|^{2}|\xi^{\mu}|^{2} = 1$ the first condition is $\rho \geq 0$, then since WEC must be valid *for all* timelike vectors and clearly the wrong way Schwarz inequality is not bounded from above, the only form to ensure that the WEC (2.59) will hold for all timelike vectors is to impose that $\rho + p \geq 0$. It is important to note that if the inequality is satisfied for a unitary timelike vector, then it will hold for all timelike vectors. Here, the physical implication is clear, the WEC implies that the energy density must be nonnegative and the pressure is bounded from below by the energy density, this is, the pressure cannot exceed the energy density for any observer transversing a timelike curve. Note that the WEC can also be stated in a geometrical form as $G_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0$, however, there is not any straightforward interpretation of such a condition.

It is important to note that, despite being a vary basic condition, the WEC is violated for many situations in quantum regimes and even in some classical situations [43, p. 35]. In fact, an argument for imposing some stronger conditions on the energy-momentum tensor was the fact that there are some situations where an infinite number of particles could be created in a finite region of space, this apparent contradiction is saved with the Dominant Energy Condition [44].

2.4.2.2 Dominant Energy Condition.

As we have seen, the WEC only ensures us that the energy density will be nonnegative and the pressure is bounded form below by the energy density, nevertheless it does not restrain any other pathologies that a given energy-momentum tensor might have. For that sake, there exists a stricter energy condition, the Dominant Energy Condition (DEC) which is stated as

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0 \ , \ \text{and } T_{\mu\nu}\xi^{\mu} \text{ is a causal vector} \tag{2.60}$$

which can also be restated as $T_{\mu\nu}\xi^{\mu}k^{\nu} \geq 0$ for all ξ^{μ} and k^{μ} timelike or null vectors. The second condition in (2.60) implies directly the prohibition of superluminal propagation of energy-momentum. There is also a stronger version of DEC, called the Strengthened Dominant Energy Condition (SDEC), the only difference between DEC and SDEC is the fact that SDEC imposes $T_{\mu\nu}\xi^{\mu}$ to be a timelike vector, while DEC only asks the energy-momentum density to be a causal vector. Now, for a perfect fluid, DEC imposes the conditions $\rho \geq 0$, which comes from WEC and from the causality imposition on $T_{\mu\nu}\xi^{\nu}$ we find

$$\left(T^{\alpha}_{\ \beta}\xi^{\beta}\right)\left(T_{\alpha\gamma}\xi^{\gamma}\right) = \left(\rho^2 - p^2\right)\left(v_{\beta}\xi^{\beta}\right)^2 + p^2 \geq 0$$

which again by applying the wrong way Schwarz inequality we find that $\rho \geq p$, which together from the lower bound that WEC imposes on the pressure turns to be $\rho \geq |p|$, such that now, the pressure has an upper and a lower bound imposed by the energy density.

As the fact that DEC imposes the restriction of superluminal propagation of the energy-momentum density is very important for the development of this work, we will review the conservation theorem as presented in [42, p. 94]

Theorem 2.4.1 (Conservation theorem)**.** *If the energy-momentum tensor obeys the dominant energy condition and is zero on $(\partial\mathcal{U})_3$, a timelike boundary of $\mathcal{U}$ and on the initial surface $(\partial\mathcal{U})_1$, a past non-timelike boundary of $\mathcal{U}$, then it is zero everywhere on $\mathcal{U}$.*

Proof. Let

$$x(t) = \int_{\mathcal{U}(t)} T^{\alpha\beta}\xi_{\alpha}\xi_{\beta}dv = \int^{t}\left(\int_{\mathcal{H}(t')\cap\mathcal{U}} T^{\alpha\beta}\xi_{\alpha}d\sigma_{\beta}\right)dt' \geq 0$$

Then $dx/dt \leq Px$ (by Lemma 4.3.1 in [42, p. 92]). But for sufficiently early values of t, $\mathcal{H}(t)$, a surface in $\mathcal{U}$ at $t =$constant, will not intersect $\mathcal{U}$ and so x will vanish. Thus x will vanish for all t which implies that $T^{\alpha\beta}$ is zero on $\mathcal{U}$. □

Then, if the energy-momentum tensor vanishes on the initial surface then it will be zero everywhere, since if no matter is present at the beginning there is no possibility of finding matter in the future domain of dependence.

The latter theorem Thm. 2.4.1 can be rephrased as follows [43]: *if a covariantly divergence-free $T_{\mu\nu}$ is required to satisfy the DEC and it vanishes on a closed, achronal set, then it vanishes in the domain of dependence of that set.* An achronal surface S is a subset $S \in M$ where no two points are connected by timelike curves, it allows us to determine the domain of dependence $D(S)$. An achronal surface becomes a Cauchy surface when its domain of dependence is the whole manifold M.

The energy-momentum cannot propagate locally outside the lightcone. This consequence of DEC constitutes an important building block for spacetime theories since a theory satisfying DEC will not present the possibility of causality violations. So far, all known classical matter fields satisfy DEC, and as it will be seen later on, it seems that such a condition obeys the very nontrivial inequality $\mathcal{RE}(\Sigma) \geq |J(\Sigma)|$ (1.13). By now, we will begin the study of DEC for arbitrary nonlinear electrodynamics.

2.4.2.3 The Dominant Energy Condition for Nonlinear Electrodynamics.

It is useful to derive a general expression for DEC in the context of general, $\mathcal{L}(F, G)$, nonlinear electrodynamics. From the general energy-momentum tensor (2.43) with the use of the identities (2.11) we can arrive to the following result for $T_{\mu\nu}\xi^{\nu}$ being a causal vector, i.e. $T_{\mu\nu}T^{\nu\lambda}\xi^{\mu}\xi^{\lambda} \geq 0$,

$$\mathcal{L}^2 + G^2\left(\mathcal{L}_G^2 + \mathcal{L}_F^2\right) - 2\mathcal{L}_G\mathcal{L}G - 4\left(\mathcal{L}_F^2 F + \mathcal{L}_F\mathcal{L}_G G - \mathcal{L}_F\mathcal{L}\right)E^2 \geq 0 \tag{2.61}$$

this expression will allow us to determine whether or not the nonlinear lagrangians treated throughout this work satisfy DEC. Note that the latter simplifies drastically for the case of a Lagrangian of the form $\mathcal{L}(F)$,

$$\mathcal{L}^2 + G^2\mathcal{L}_F^2 - 4\mathcal{L}_F^2 F E^2 + 4\mathcal{L}_F\mathcal{L}E^2 \geq 0$$

it is easy to check that the Larmor Lagrangian satisfies the latter, as well as the Lagrangian for Born $\mathcal{L}(F)$ electrodynamics.

3 Bekenstein Bounds and Inequalities for Nonlinear Electrodynamics.

In the previous chapters we have introduced the main theoretical information and tools for the analysis that will be done in this chapter. Bekenstein inequalities have the particularity that are formulated in order to have a *universal* character, i.e. they should remain valid in all the branches of physics. Particularly in electrodynamics there are still many open questions on whether Maxwell's theory can be considered *complete*[1] or just a very good approximation of a wider theory. With this in mind, many reformulations and generalizations of electrodynamics have been proposed over the years, including theories with massive photons, formulated in higher and lower dimensions and nonlinear theories of electrodynamics, which themselves have a huge range of motivations for being formulated (the main motivation being the finiteness of the point charge energy and the photon's self interaction).

Hence, following Bekenstein's universal argument, *all* the bounds presented in Sec. 1.2 should remain valid in every acceptable generalization of Maxwell's electrodynamics. This may sound a very strong statement to be taken for granted. As it has been argued before, there does not exist any trivial conjecture about the validity of neither of the inequalities. The main goal of this chapter is to test these inequalities in particular situations of nonlinear electrodynamics and to see what conditions must be imposed to an arbitrary $\mathcal{L}(F, G)$ lagrangian in order to satisfy these inequalities. As long as one of the reasons of a nonlinear theory of electrodynamics is to *generalize* Maxwell's theory it is expected that any reasonable extension must have the corresponding Maxwell limit.

In the present chapter we will start by analyzing the validity of Bekenstein bounds and inequalities for Born-Infeld electrodynamics.

3.1 Born-Infeld Electrodynamics.

3.1.1 Inequality between charge and energy.

We will begin by exploring the inequality between charge and energy (1.12) in Born-Infeld electrodynamics. It is important to note that this inequality is valid for Maxwell's electrostatics, then the natural way to test it in Born-Infeld theory is to use its static limit, $F \to -|\mathbf{E}|^2$ and $G = 0$ in (2.22) which represents the particular case of Born's original lagrangian (2.20). Therefore, we are interested in the electrostatic energy

[1] In the sense that it suffice to describe all electromagnetic phenomena in nature.

density for this particular theory, recasting the general energy-momentum tensor (2.25) and applying the respective limit we get that the electrostatic energy is given by

$$\mathcal{E}_{\mathrm{BI}} = \int_{\mathbb{R}^3} \left(\frac{\beta^2}{\sqrt{U}} - \beta^2 \right) d^3x \tag{3.1}$$

where the expression in parenthesis is the energy density. Now, since we are treating the electrostatic case, the quantity $\sqrt{U}$ is just $\sqrt{1 - \frac{|\mathbf{E}|^2}{\beta^2}}$. So it is possible to expand the integrand as,

$$\mathcal{E}_{\mathrm{BI}} = \int_{\mathbb{R}^3} d^3x \left[\beta^2 \left(1 + \frac{|\mathbf{E}|^2}{2\beta^2} + \frac{3}{8}\frac{|\mathbf{E}|^4}{\beta^4} + ... \right) - \beta^2 \right] \tag{3.2}$$

Note that this expansion is performed because of the fact that the electric field is bounded form above by β, then for the series above to be convergent $\frac{|\mathbf{E}|}{\beta} < 1$, thus, we will exclude the case where $|\mathbf{E}| = \beta$. All the terms in the series expansion of the latter are positive, and we can recognize the Maxwell part in the integrand above so

$$\mathcal{E}_{\mathrm{BI}} = \int_{\mathbb{R}^3} d^3x \left[\frac{|\mathbf{E}|^2}{2} + \frac{3}{8}\frac{|\mathbf{E}|^4}{\beta^2} + ... \right] = \mathcal{E}_{\mathrm{M}} + \int_{\mathbb{R}^3} d^3x \left[\frac{3}{8}\frac{|\mathbf{E}|^4}{\beta^2} + ... \right] \tag{3.3}$$

where $\mathcal{E}_{\mathrm{M}}$ is the Maxwell energy. Consequently, we have that, for the electrostatic case,

$$\mathcal{E}_{\mathrm{BI}} \geq \mathcal{E}_{\mathrm{M}} \tag{3.4}$$

and the equality holds *only* when the fields are very small compared to β, i.e. $\frac{|\mathbf{E}|}{\beta} \to 0$. For every other case, where $\frac{|\mathbf{E}|}{\beta} \in (0, 1)$ we have $\mathcal{E}_{BI} > \mathcal{E}_M$.

Since for the Maxwell case it is proven that the inequality (1.12) holds [20], then for the electrostatic Born energy the inequality will also hold as well as the rigidity statement for $\frac{|\mathbf{E}|}{\beta} \to 0$. Moreover, for every other case, the inequality will also hold, but it is no longer possible to assume the rigidity statement because of (3.4). Thus, in general we have,

$$\mathcal{E}_{\mathrm{BI}} \geq \frac{Q^2}{8\pi\mathcal{R}} \tag{3.5}$$

where the equality will hold if and only if the electric field is equal to the electric field produced by a spherical shell of radius $\mathcal{R}$ and total charge Q, and $\frac{|\mathbf{E}|}{\beta} << 1$.

As for now, the static bound seems to work for Born-Infeld electrodynamics because this nonlinear theory has an adequate Maxwell limit since the equality is reached for the weak field approximation. Furthermore, such a scenario should not be plausible when the nonlinear energy density is lower that the Maxwell one. We will explore this scenario later on.

3.1.2 Inequality between energy and angular momentum.

In order to analyze the inequality between energy and angular momentum (1.13) in the regime of Born-Infeld electrodynamics we will proceed to take the difference between the energy and angular momentum in the region Σ. Using the energy-momentum tensor (2.25), the energy density after some algebraic manipulations can be expressed as

$$u = \frac{1}{\sqrt{U}}\left(\beta^2 + |\mathbf{B}|^2 - \beta^2\sqrt{U}\right)$$

Then, taking the difference between this energy density and the angular momentum defined in (2.28) we obtain

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| = \int_\Sigma d^3x \left(\frac{1}{\sqrt{U}}\left(\beta^2 + |\mathbf{B}|^2 - \beta^2\sqrt{U}\right)\right) - \frac{1}{\mathcal{R}}\left|\int_\Sigma d^3x \left(\frac{1}{\sqrt{U}}\left[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})\right]\cdot\mathbf{k}\right)\right| \tag{3.6}$$

We will estimate the difference above, showing that the integrand after all the minimizations remains positive hence the inequality will be proved; this strategy was first employed by Dain in [20]. Then, using the inequality $|\int f(x)| \leq \int |f(x)|$, the vector triangular inequalities and the fact that $\mathbf{k}$ is a unitary vector we arrive at

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \frac{1}{\sqrt{U}}\left(\beta^2 + |\mathbf{B}|^2 - \beta^2\sqrt{U} - \frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}||\mathbf{B}|\right) \tag{3.7}$$

Now in order to see whether the integrand is positive for every field configuration we must note that in Born-Infeld electrodynamics both the electric and magnetic fields are bounded from above by the parameter β. Then, defining the numbers $\alpha := \frac{|\mathbf{E}|}{\beta}$ and $\gamma := \frac{|\mathbf{B}|}{\beta}$, we obtain,

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \frac{\beta^2}{\sqrt{U}}\left(\gamma^2 + 1 - \sqrt{U} - \frac{|\mathbf{x}|}{\mathcal{R}}\alpha\gamma\right) \tag{3.8}$$

Now, we can write U in (2.22) in terms of α and γ as,

$$U = 1 + \gamma^2 - \alpha^2 - \alpha^2\gamma^2\cos^2\theta$$

where $\theta = \arccos(\mathbf{E}\cdot\mathbf{B}/|\mathbf{E}||\mathbf{B}|)$. Using this definition, it is possible to write (3.8) as

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \frac{\beta^2}{2\sqrt{U}}\left[\left(\gamma - \frac{|\mathbf{x}|}{\mathcal{R}}\alpha\right)^2 + \left(1 - \sqrt{U}\right)^2 + \alpha^2\gamma^2\cos^2\theta + \alpha^2\left(1 - \frac{|\mathbf{x}|^2}{\mathcal{R}^2}\right)\right] \tag{3.9}$$

It is obvious that the right-hand side of the latter is nonnegative. Thus, the inequality (1.13) between energy and angular momentum is proven directly. Also we may infer that the equality will be reached when the integrand in (3.9) is zero, which means that the electric and magnetic fields must vanish in Σ, which is the same condition as in Maxwell's electrodynamics. Moreover it is remarkable that in the case where $\alpha = \gamma$ and $\theta = (2n - 1)\pi/2$, i.e. $U = 1$ the integrand is still positive and has the value

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \frac{\beta^2}{2}\left(\gamma^2 + \alpha^2\right)\left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right) = \int_\Sigma d^3x \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)\left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right) \tag{3.10}$$

The last equality in the latter equation is, remarkably, the same expression as the one obtained for the case of Maxwell electrodynamics! Therefore, the same conclusions for the rigidity statement in the classical case hold for the Born-Infeld case. Moreover, this result is particularisable for the Born $\mathcal{L}(F)$ electrodynamics.

In conclusion, since we have seen that the equality in the bound between charge and energy is attained in the classical approximation for the Born-Infeld case, it is reasonable to expect that it will also happen with the inequality between energy and angular momentum. However, as we have proved above, that assumption is not necessary at all in order to prove the inequality and the rigidity statement. Furthermore, the exact same expression as in the Maxwell case is recovered for the particular configuration of $\mathbf{E} \perp \mathbf{B}$ and $|\mathbf{E}| = |\mathbf{B}|$. This result is non-trivial since it is expected to recover the Maxwell expression for the weak field limit, in fact, by making the assumption of $U = 1$ and $G = 0$ in the energy-momentum tensor it is impossible to obtain the classical expression, as well as by doing the same in the lagrangian.

3.1.3 Inequality between charge, energy and angular momentum.

For the analysis of the full inequality (1.17) we begin with the total energy for Born-Infeld electrodynamics expressed in terms of α and γ from the previous section

$$\mathcal{E} = \int_{\mathbb{R}^3} d^3x \frac{\beta^2}{\sqrt{U}} \left(1 + \gamma^2 - \sqrt{U} + \sqrt{U}\left(\frac{\alpha^2}{2} - \frac{\alpha^2}{2}\right)\right) \tag{3.11}$$

we have explicitly added and subtracted the term $|\mathbf{E}|^2/2$, which might be identified as the Maxwell energy. This is done in pursuance of isolating the electrostatic term in order to obtain the corresponding bound between energy and charge (1.12) as it will be seen later on. Now, as it has been discussed, it is possible to express both the electric and magnetic fields in terms of the vector and scalar potentials as $\mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Now, we can express the scalar potential as

$$\Phi = \Phi_0 + \Phi_1$$

such that Φ_1 is an auxiliary potential and Φ_0 the potential of an spherical shell of radius $\mathcal{R}$,

$$\Phi_0 = \begin{cases} \frac{Q}{r}, & \text{if } r \geq \mathcal{R}, \\ \frac{Q}{\mathcal{R}}, & \text{if } r \leq \mathcal{R}, \end{cases} \tag{3.12}$$

note that $\nabla\Phi_0 = 0$ inside $\mathcal{B}_\mathcal{R}$, then, by construction $\nabla\Phi = \nabla\Phi_1$ inside $\mathcal{B}_\mathcal{R}$. Then, it is possible to show that the integral over the space of the squared modulus of the electric field is

$$\int_{\mathbb{R}^3} d^3x |\mathbf{E}|^2 = \int_{\mathbb{R}^3} d^3x \left\{ |\nabla\Phi|^2 + |\partial_t \mathbf{A}|^2 + 2\nabla\Phi \cdot \partial_t \mathbf{A} \right\}$$

but the last term in the above integral is indeed zero using the Coulomb gauge because it can be split as $\int_{\mathbb{R}^3} d^3x \{\nabla\Phi \cdot \partial_t \mathbf{A}\} = \int_{\mathbb{R}^3} d^3x \{\nabla \cdot (\Phi \partial_t \mathbf{A}) - \Phi \partial_t (\nabla \cdot \mathbf{A})\} = 0$, where the first term is a surface term and the second vanishes because of the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$). Meanwhile, the term with the scalar potential can be expressed in terms of both Φ_0 and Φ_1 as $|\nabla\Phi|^2 = |\nabla\Phi_0|^2 + |\nabla\Phi_1|^2 + 2\nabla\Phi_0 \cdot \nabla\Phi_1$ where the last term also vanishes, it can be written as

$$\int_{\mathbb{R}^3} d^3x \nabla\Phi_o \cdot \nabla\Phi_1 = \int_{\mathbb{R}^3\setminus\mathcal{B}_\mathcal{R}} d^3x \left\{\nabla \cdot (\Phi_0\nabla\Phi_1) - \Phi_0\nabla^2\Phi_1\right\} = 0$$

where the integral over the domain $\mathcal{B}_\mathcal{R}$ vanishes from the fact that Φ_0 is constant, hence the remaining domain of integration is $\mathbb{R}^3 \setminus \mathcal{B}_\mathcal{R}$; and the the first term in the right side is a surface term, and the second term vanishes because by construction $\nabla^2\Phi_1 = 0$; this strategy of splitting the scalar potential and using the Coulomb gauge was first applied by Dain in [20] in the proof of the inequality in Maxwell electrodynamics. Therefore, (3.11) can be rewritten as

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \mathcal{E}(\Sigma) + \int_{\mathbb{R}^3\setminus\Sigma} d^3x \left[\frac{1}{2}|\nabla\Phi_1 + \partial_t\mathbf{A}|^2 + \frac{\beta^2}{\sqrt{U}}\left(1 + \gamma^2 - \sqrt{U}\left(1 + \frac{\alpha^2}{2}\right)\right)\right] \tag{3.13}$$

where we have split the integral for the energy in two domains Σ and $\mathbb{R}^3 \setminus \Sigma$ and used the properties described above for the scalar potentials, i.e. the fact that in Σ, $|\nabla\Phi_1|^2 = |\nabla\Phi|^2 = |\mathbf{E}|^2$, so the total integral evaluated in Σ is, effectively, the energy in that region, $\mathcal{E}(\Sigma)$. Since we have been able to isolate the energy in the region Σ we can invoke the result from Sec. 3.1.2 for the relation between the energy and angular momentum in that region (3.9), then we get

$$\begin{aligned}\mathcal{E} - \frac{Q^2}{2\mathcal{R}} - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_{\mathbb{R}^3\setminus\Sigma} d^3x \left[\frac{1}{2}|\nabla\Phi_1 + \partial_t\mathbf{A}|^2 + \frac{\beta^2}{\sqrt{U}}\left(1 + \gamma^2 - \sqrt{U}\left(1 + \frac{\alpha^2}{2}\right)\right)\right] + \\ + \int_\Sigma d^3x \frac{\beta^2}{2\sqrt{U}}\left[\left(\gamma - \frac{|\mathbf{x}|}{\mathcal{R}}\alpha\right)^2 + \left(1 - \sqrt{U}\right)^2 + \alpha^2\gamma^2\cos^2\theta + \alpha^2\left(1 - \frac{|\mathbf{x}|^2}{\mathcal{R}^2}\right)\right]\end{aligned} \tag{3.14}$$

The only term that might be negative in the right hand side of the latter expression is the last term in the integral in $\mathbb{R}^3 \setminus \Sigma$, notwithstanding we can write it as

$$1 + \gamma^2 - \sqrt{U}\left(1 + \frac{\alpha^2}{2}\right) = \frac{1}{2}\left[\left(1 - \sqrt{U}\right)^2 + \alpha^2\gamma^2\cos^2\theta + \alpha^2\left(1 - \sqrt{U} + \frac{\gamma^2}{\alpha^2}\right)\right]$$

where it is not clear that the last term is nonnegative since $U \in [0, 2]$, however, $U > 1$ implies $\gamma > \alpha$, thus, the last term in the latter expression is nonnegative for every field configuration. Finally, since all the terms in the right hand side of (3.14) are nonnegative, we can state that the inequality

$$\mathcal{E} \geq \frac{Q^2}{8\pi\mathcal{R}} + \frac{|J(\Sigma)|}{\mathcal{R}} \tag{3.15}$$

holds for Born-Infeld electrodynamics, and thus, inequality (1.17) is satisfied. Now, for the equality to hold in (3.15) it is necessary for both the integrals in the right side to vanish, as in the Maxwell case, this will be achieved only when the electric field is the field of a spherical shell, i.e. it vanishes in Σ, then $\nabla\Phi_1 = 0$ and for $\mathbf{A} = 0$, then $\mathbf{B} = 0$ everywhere, hence $U = 1$ and both integrands will vanish. Thus, the equality is achieved only by an static spherical shell, which means that there is no dynamics present in the fields when the equality is achieved.

Born-Infeld electrodynamics constitute a very important example of NLED. This theory, despite respecting the Maxwell limit, offers a wide variety of peculiar consequences, in the sense that it possesses many unique features among NLED. The proofs performed in this section can be interpreted as follows: since Bekenstein's inequalities are valid *universally* in physics, they should offer us a criteria for discerning between physically realizable and unrealizable theories, then, following this argument, Born-Infeld electrodynamics should constitute a physically plausible theory because it satisfies all of Bekenstein's inequalities. We shall test other theories of NLED in order to probe the validity of these partial conclusions.

3.2 Exponential Electrodynamics.

3.2.1 Inequality between charge and energy.

Now we will begin the analysis of Bekenstein bounds in the regime of exponential electrodynamics as defined in Sec. 2.3.2. This formulation of NLED has the particularity that despite being a Born-Infeld inspired formulation of NLED, the electric field diverges near the origin, then it will not be possible anymore to normalize the electric field with the parameter β, however there is still an analogous procedure of working the inequality. We will begin with the electrostatic energy for exponential electrodynamics,

$$\mathcal{E} = \int_{\mathbb{R}^3} d^3x \left[-Fe^{-\frac{F}{2\beta^2}} - \beta^2 \left(e^{-\frac{F}{2\beta^2}} - 1 \right) \right] \tag{3.16}$$

as we are dealing with the electrostatic case we have assumed $G = 0$ in (2.29), as well as for now, since $|\mathbf{B}| = 0$, then the invariant $F = -|\mathbf{E}|^2$, so we can write the latter only in terms of the electric field. Additionally we will subtract explicitly the Maxwell energy density $|\mathbf{E}|^2/2$, leaving the expression

$$\mathcal{E} - \mathcal{E}_{\mathrm{M}} = \int_{\mathbb{R}^3} d^3x \left[|\mathbf{E}|^2 e^{\frac{|\mathbf{E}|^2}{2\beta^2}} - \beta^2 \left(e^{\frac{|\mathbf{E}|^2}{2\beta^2}} - 1 \right) - \frac{|\mathbf{E}|^2}{2} \right] \tag{3.17}$$

now we proceed in an analogous way as it has been done in the previous section, i.e. we define $\omega := |\mathbf{E}|/\beta$. Note that we do this in order to be able to compare the electric field with the parameter β which has field dimensions. Since in the case given by this electrodynamics the electric field is not bounded by β as in the standard Born-Infeld electrodynamics, we explicitly introduce the number ω which can be greater than unity[2].

[2] For purposes of clarity in the calculations we change ω which in fact is different form α in the previous section which was a bounded number.

Consequently, we can write the energy of the field as

$$\mathcal{E} - \mathcal{E}_{\mathrm{M}} = \int_{\mathbb{R}^3} d^3x \beta^2 \left[e^{\frac{\omega^2}{2}} \left(\omega^2 - 1 \right) + 1 - \frac{\omega^2}{2} \right] \tag{3.18}$$

so far it is not possible to see directly if the difference above is nonnegative always. Since we know that $\omega \in [0, \infty]$, we can start by taking the respective limits both when $\omega \to 0$ and $\omega \to \infty$, and we can see that, effectively both limits are nonnegative, in fact

$$\lim_{\omega \to 0} \left[e^{\frac{\omega^2}{2}} \left(\omega^2 - 1 \right) + 1 - \frac{\omega^2}{2} \right] = 0$$

$$\lim_{\omega \to \infty} \left[e^{\frac{\omega^2}{2}} \left(\omega^2 - 1 \right) + 1 - \frac{\omega^2}{2} \right] = \infty$$

Moreover, since we are not able anymore to fix a maximum possible value for the electric field, in fact the difference between the energy density in this exponential electrodynamics compared to standard Maxwell electrodynamics diverges rapidly as $\omega > 1$, this behavior can be seen in Fig. 3. It is remarkable that the energy density in this formulation of exponential electrodynamics diverges more rapidly than in the standard case. This shows us that even in Born-Infeld inspired theories of electrodynamics it is not possible to obtain regular field distributions on the origin, and many 'pathologies' of the classical theory are exaggerated by nonlinear generalizations of the theory. In fact, the divergence in this case is exponential, while the divergence in Maxwell's case is only quadratic. Note that even though the field diverges slower than in the Maxwell case, we have the converse situation for the energy density.

As for the case of the inequality (1.12), since we have proved that $\mathcal{E} \geq \mathcal{E}_{\mathrm{M}}$ we can conclude that exponential electrodynamics satisfy the inequality between charge and energy. As in the Born-Infeld case, we can state that the equality will hold in the case where the field is small compared to β, i.e. it is possible to obtain the Maxwell limit for the energy density.

3.2.2 Inequality between energy and angular momentum.

Regarding the inequality (1.13) that relates the energy and angular momentum in the region Σ, we will proceed in the same way as before, estimating the difference between the energy and the angular momentum in that region. The energy density from (2.30) is

$$u = \beta^2 e^{\frac{-\mathcal{X}}{\beta^2}} \left(\omega^2 + \frac{G^2}{4\beta^4} - 1 \right) + \beta^2$$

where we have explicitly isolated the factor β^2. Furthermore, the angular momentum is defined in (2.31). Then we can estimate the difference between the two quantities as

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}} |J(\Sigma)| = \int_{\Sigma} d^3x \left\{ \beta^2 e^{\frac{-\mathcal{X}}{\beta^2}} \left(\omega^2 + \frac{G^2}{4\beta^4} - 1 \right) + \beta^2 \right\} - \frac{1}{\mathcal{R}} \left| \int_{\Sigma} d^3x e^{\frac{-\mathcal{X}}{\beta^2}} \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \right] \cdot \mathbf{k} \right| \tag{3.19}$$

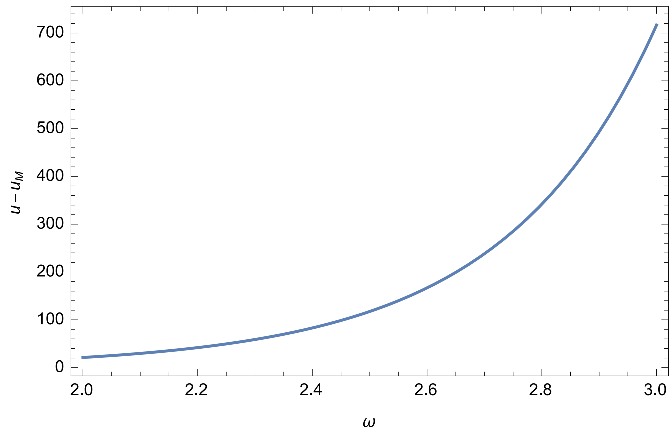

Figure 3 – The difference between the energy density for exponential electrodynamics in the electrostatic regime compared to the Maxwell electrostatic energy density.

Now, using the inequalities for the absolute value of the integral and the vector triangular inequalities as in the previous section, we estimate the difference as

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \left\{ \beta^2 e^{\frac{-\mathcal{X}}{\beta^2}} \left(\omega^2 + \frac{G^2}{4\beta^4} - 1 \right) + \beta^2 - e^{\frac{-\mathcal{X}}{\beta^2}} \frac{|\mathbf{x}|}{\mathcal{R}} |\mathbf{E} \times \mathbf{B}| \right\} \tag{3.20}$$

in order to reduce the latter integral, we will use the vector inequality $|\mathbf{E} \times \mathbf{B}| \leq \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)/2$, then we will extremize the inequality by maximizing the negative contribution, which is given when $|\mathbf{x}| = \mathcal{R}$. After rearranging the terms, and defining $\delta := |\mathbf{B}|/\beta$, it is possible to express the remaining integral as

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \beta^2 \int_\Sigma d^3x \left[e^{-\mathcal{X}/\beta^2} \left(-\frac{\mathcal{X}}{\beta^2} - 1 + \frac{1}{2}\omega^2\delta^2 \cos^2\theta \right) + 1 \right] \geq 0 \tag{3.21}$$

Again, since the fields are no longer upper bounded it is no longer possible to assume that $\mathcal{X}$ will be bounded too, nevertheless it is still possible to see how this function behaves at its critical points, both when $\mathcal{X}$ goes to $\pm\infty$ the expression in parenthesis goes to 0, leaving only a constant, β^2 in the integral, when $\mathcal{X}$ goes to 0,which is the case when both the electric and magnetic field have equal magnitude and are mutually perpendicular, the expression in the parenthesis goes to -1 (because $\mathbf{E} \perp \mathbf{B}$), which will cancel the other positive term in the integral in (3.21), leaving a null integrand, which is the case when the equality is identically satisfied. By this, we can argue that the rigidity statement, as in the cases treated before, is attained when both of the fields vanish inside $\mathcal{B}_\mathcal{R}$ and, by continuity, they are also zero in $|\mathbf{x}| = \mathcal{R}$. Hence, we can conclude that in the regime of exponential electrodynamics the inequality between energy and angular momentum (1.13) is satisfied.

3.2.3 Inequality between charge, energy and angular momentum.

Continuing the analysis for the complete inequality (1.17), we begin by expressing the total energy adding and subtracting the term $|\mathbf{E}|^2/2$ which we will later associate with the term $Q^2/2\mathcal{R}$ as in the previous section. The total energy reads

$$\mathcal{E} = \int_{\mathbb{R}^3} d^3x\beta^2 \left[e^{\frac{-\mathcal{X}}{\beta^2}} \left(\omega^2 + \frac{G^2}{4\beta^4} - 1 \right) + 1 - \frac{\omega^2}{2} \right] + \int_{\mathbb{R}^3} d^3x \frac{|\mathbf{E}|^2}{2} \tag{3.22}$$

As in the previous section, it is possible to express the scalar potential as a sum of two potentials, where one is the potential for an spherical thin shell. Then, it is possible to split both integrals in two domains, Σ and $\mathbb{R}^3 \setminus \Sigma$ respectively, consequently, the last term in the latter equation leads to our desired term of $Q^2/2\mathcal{R}$ and a positive term that depends on the potential Φ_1 when the integral is evaluated in the domain $\mathbb{R}^2 \setminus \Sigma$, while inside Σ, the terms added an subtracted in (3.22) will identically vanish because $|\nabla\Phi_1| = |\mathbf{E}|$ in that region, hence, as in the previous section, we obtain

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \mathcal{E}(\Sigma) + \int_{\mathbb{R}^3\setminus\Sigma} d^3x \left[\frac{1}{2} \left(|\nabla\Phi_1|^2 + |\partial_t\mathbf{A}|^2 \right) + \beta^2 \left(e^{\frac{-\mathcal{X}}{\beta^2}} \left(\omega^2 + \frac{G^2}{4\beta^2} - 1 \right) + 1 - \frac{\omega^2}{2} \right) \right] \tag{3.23}$$

and now that we have isolated the term involving the energy in the region Σ, we can use the bound obtained in Sec. 3.2.2 to bound our result with the angular momentum, leaving the expression

$$\begin{aligned} \mathcal{E} - \frac{Q^2}{2\mathcal{R}} - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_{\mathbb{R}^3\setminus\Sigma} d^3x \left[\beta^2 \left(e^{\frac{-\mathcal{X}}{\beta^2}} \left(\omega^2 + \frac{G^2}{4\beta^2} - 1 \right) + 1 - \frac{\omega^2}{2} \right) + \right. \\ \left. \frac{1}{2} \left(|\nabla\Phi_1|^2 + |\partial_t\mathbf{A}|^2 \right) \right] + \beta^2 \int_{\Sigma} d^3x \left[e^{-\mathcal{X}/\beta^2} \left(-\frac{\mathcal{X}}{\beta^2} - 1 + \frac{1}{2}\omega^2\delta^2\cos^2\theta \right) + 1 \right] \end{aligned} \tag{3.24}$$

From the discussion in Sec. 3.2.2 we know that the inequality between energy and angular momentum (1.13) is satisfied because the integral evaluated in the region Σ above is positive, then we are only concerned with the integral in the first line of (3.24), which we do not know if it is positive or negative. Note that when assessing the inequality between energy and charge we have used an analogous expression when comparing the *electrostatic* energy density of this formulation of exponential electrodynamics and Maxwell electrodynamics, we have shown that indeed the difference is always nonnegative. However, this might not hold when treating the dynamical case, for specific field configurations, indeed, when $|\mathbf{B}| \gg |\mathbf{E}|$ the respective integrand is negative, since it is no longer possible to assume that $|\mathbf{E}|$ nor $|\mathbf{B}|$ are bounded by β. Then, the fulfillness of the partial inequalities (1.12) and (1.13) does not guarantee the fulfillness of the total inequality (1.17).

3.3 Logarithmic Electrodynamics.

So far the two electrodynamics studied satisfy the inequalities (1.12) and (1.13). For purposes of clarity in the development of this study, it would be useful to find examples of physically plausible NLED which violate one of the inequalities, or both at the same time. Thus, it is plausible to study one more case of NLED in order to better see how is the behavior of these inequalities under certain NLED regimes. Now we will study such inequalities in logarithmic electrodynamics presented in Sec. 2.3.3.

3.3.1 Inequality between charge and energy.

For logarithmic electrodynamics, the electrostatic energy density is the same for both formulations of [38] and [39] and reads

$$u = \frac{2|\mathbf{E}|^2\beta^2}{2\beta^2 - |\mathbf{E}|^2} + \beta^2 \ln\left(1 - \frac{|\mathbf{E}|^2}{2\beta^2}\right)$$

We shall express the latter energy density in terms of a 'normalized' field as in the last section. Noteworthy, logarithmic electrodynamics possess a regular field in the origin but it is not fixated to the constant β, then we will define the numbers $\alpha := |\mathbf{E}|/\sqrt{2}\beta$ and $\gamma := |\mathbf{B}|/\sqrt{2}\beta$, with, $\alpha, \gamma \in [0, 1]$. Then, we can write the difference between logarithmic energy density and the Maxwell energy density as

$$\mathcal{E} - \mathcal{E}_M = \beta^2 \int_{\mathbb{R}^3} d^3x \left\{ \frac{2\alpha^2}{1-\alpha^2} + \ln\left(1-\alpha^2\right) - \alpha^2 \right\} \tag{3.25}$$

Analogously to the case of exponential electrodynamics, it is possible to see that, in fact, the above expression is always positive, i.e. the energy corresponding to logarithmic electrodynamics is greater than the Maxwell energy. We can see this behavior in Fig. 4. In fact, it is possible to see that the logarithmic energy density goes to infinity faster than the Maxwell part. Then, since the logarithmic static energy is greater that the Maxwell energy, the energy of a spherical shell within logarithmic electrodynamics will be greater that $Q^2/2\mathcal{R}$. Thus, (1.12) will be satisfied within this regime of electrodynamics.

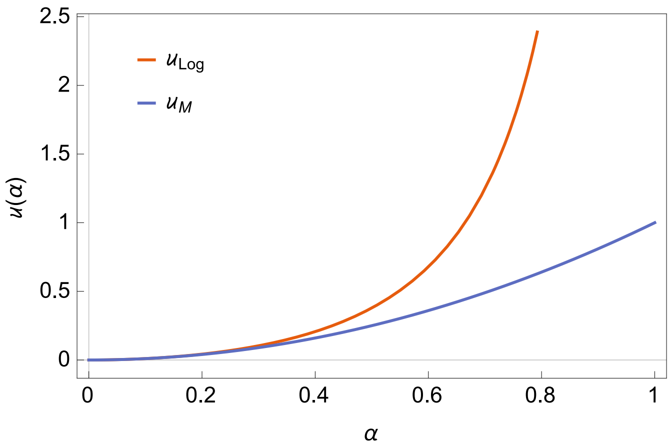

Figure 4 – The electrostatic energy density for logarithmic electrodynamics and the electrostatic energy density for Maxwell electrodynamics.

3.3.2 Inequality between energy and angular momentum.

With respect to the inequality between energy and angular momentum in logarithmic electrodynamics, since in this formulation of electrodynamics the field is finite and bounded by $\sqrt{2}\beta$, we will express our quantities using the definitions $\alpha := |\mathbf{E}|/\sqrt{2}\beta$ and $\gamma := |\mathbf{B}|/\sqrt{2}\beta$ as in the Born-Infeld case, where α and γ are bounded from above by 1. With these definitions, it is possible to write the energy density for logarithmic electrodynamics as

$$u = \beta^2 \left[\frac{1}{V} \left(2\alpha^2 + \alpha^2\gamma^2 \cos^2\theta \right) + \ln(V) \right] \tag{3.26}$$

where $V = 1 + \gamma^2 - \alpha^2 - \alpha^2\gamma^2\cos^2\theta$, and the angular momentum in the region Σ can be written as

$$J(\Sigma) = \int_\Sigma d^3x \frac{1}{V} \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \right] \cdot \mathbf{k} \tag{3.27}$$

In order to analyze the relationship between these two quantities, we will follow the same procedure as in the previous sections. Consequently, estimating the difference between energy and angular momentum in the region Σ leads to the following inequality for the difference

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \left\{ \frac{1}{V} \left(2\alpha^2\beta^2 + \beta^2\alpha^2\gamma^2\cos^2\theta \right) + \beta^2 \ln(V) - \frac{|\mathbf{x}|}{\mathcal{R}} \frac{1}{V} |\mathbf{E} \times \mathbf{B}| \right\} \tag{3.28}$$

where we have performed the same steps for reducing the difference as in the previous sections. Now, we use the inequality $|\mathbf{E}\times\mathbf{B}| \leq (|\mathbf{E}|^2 + |\mathbf{B}|^2)/2$, then, we can further reduce the difference as,

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x\beta^2 \left[\frac{1}{V} \left\{ \alpha^2 \left(2 - \frac{|\mathbf{x}|}{\mathcal{R}} \right) - \frac{|\mathbf{x}|}{\mathcal{R}}\gamma^2 + \alpha^2\gamma^2\cos^2\theta \right\} + \ln(V) \right] \tag{3.29}$$

Since α and γ are bounded by 1, then $0 \leq V \leq 2$, it may be possible for the inequality to not be valid when $V = 0$ because $\ln(0) = -\infty$, however, the above integrand is positive for every allowed field configuration. For instance, when the contribution of the integrand is most negative ($|\mathbf{x}|/\mathcal{R} = 1$), the above expression (3.29) reduces to

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma d^3x\beta^2 \left\{ \frac{1}{V}(1 - V) + \ln(V) \right\} \tag{3.30}$$

then, taking the limit where $V \to 0$ leads

$$\lim_{V\to 0^+} \left(\frac{1}{V}(1 - V) + \ln(V) \right) = \infty$$

Consequently, we can state that the inequality (1.13) between energy and angular momentum is satisfied within logarithmic electrodynamics. Note that in the limit when $V = 1$, the latter integrand for the case when $\mathbf{E} \perp \mathbf{B}$ can be reduced to the form of the standard Maxwell case, as well as it happened in the Born-Infeld case, thus, the same arguments hold for the rigidity statement in this formulation of electrodynamics. It is possible to see the behavior of the integrand present in (3.30) where $|\mathbf{x}| = \mathcal{R}$ in Fig. 5. The only point where the difference between enegy and angular momentum is null is when $V = 1$ which means that all the rigidity conditions from Maxwell electrodynamics hold.

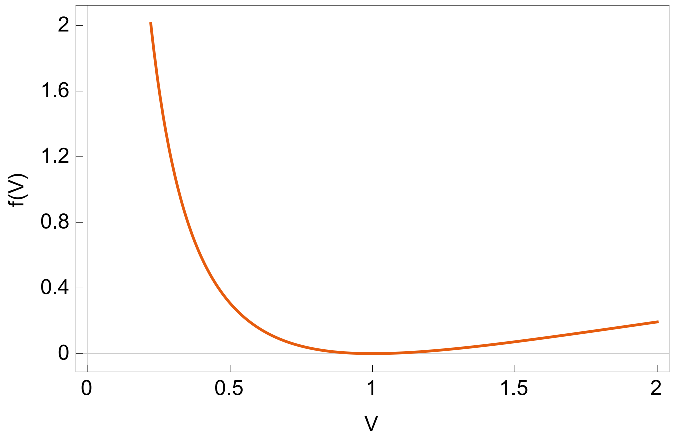

Figure 5 – The integrand in the difference, $f(V)$, between energy and angular momentum in (3.30) as a function of V.

3.3.3 Inequality between charge, energy and angular momentum.

With respect to the complete inequality (1.17) between the charge, energy and angular momentum, as well as in the other sections, we will use the results for the inequality between the angular momentum and energy presented in Sec. 3.3.2 and an analogous development for the inequality as for the other examples of NLED. We start form the total energy, given by (3.26), with an added an subtracted term of $|\mathbf{E}|^2/2$

$$\mathcal{E} = \int_{\mathbb{R}^3} d^3x\beta^2 \left[2\frac{\alpha^2}{V} + \ln(V) + \frac{1}{V}\alpha^2\gamma^2\cos^2\theta - \alpha^2 \right] + \int_{\mathbb{R}^3} d^3x \frac{|\mathbf{E}|^2}{2} \tag{3.31}$$

We know that it is possible to split the scalar potential associated to the electric field in two, and the last integral in the above expression can also be split in two domains, Σ and $\mathbb{R}^3 \setminus \Sigma$. Then, by the construction of the scalar potentials exposed above, it is possible to express the above expression as

$$\begin{aligned} \mathcal{E} = \mathcal{E}(\Sigma) + \frac{Q^2}{2\mathcal{R}} + \int_{\mathbb{R}^3\setminus\Sigma} d^3x\beta^2 \left[\alpha^2 \left(\frac{2}{V} - 1 \right) + \frac{1}{V}\alpha^2\gamma^2\cos^2\theta + \ln(V) \right] + \\ + \int_{\mathbb{R}^3\setminus\Sigma} d^3x \frac{1}{2} |\nabla\Phi_1 + \partial_t \mathbf{A}|^2 \end{aligned} \tag{3.32}$$

then by using the result obtained in the previous section, we can bound the energy with the angular momentum as

$$\begin{aligned} \mathcal{E} - \frac{Q^2}{2\mathcal{R}} - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_{\Sigma} d^3x\beta^2 \left[\frac{1}{V} \left\{ \alpha^2 \left(2 - 1\frac{|\mathbf{x}|}{\mathcal{R}} \right) - \frac{|\mathbf{x}|}{\mathcal{R}}\gamma^2 + \alpha^2\gamma^2\cos^2\theta \right\} + \ln(V) \right] + \\ + \int_{\mathbb{R}^3\setminus\Sigma} d^3x\beta^2 \left[\alpha^2 \left(\frac{2}{V} - 1 \right) + \frac{1}{V}\alpha^2\gamma^2\cos^2\theta + \ln(V) \right] + \int_{\mathbb{R}^3\setminus\Sigma} d^3x \frac{1}{2} |\nabla\Phi_1 + \partial_t \mathbf{A}|^2 \end{aligned} \tag{3.33}$$

Since the values of V are bounded it is possible to see that the only term that could be negative above (the integral in $\mathbb{R}^3 \setminus \Sigma$) is positive for every field configuration, in fact, when $V = 0$, the limit of the integrand is ∞. Consequently, logarithmic electrodynamics satisfy the complete inequality (1.17).

3.4 Counterexamples.

As we have discussed in the previous sections, a *reasonable* formulation of NLED should include Maxwell electrodynamics as a limit in the Lagrangian. The NLEDs that we have discussed in this chapter all possess the Maxwell limit for the low-field approximation; the Born-Infeld lagrangian reduces to the Maxwell lagrangian when $F/\beta^2 \to 0$, and also, the energy densities for each of the cases were higher than the Maxwell one. Since the inequality involving the charge and the energy is derivable from the electrostatic energy density, it should be natural to think that a reasonable formulation of NLED might have, indeed, a lower energy density than that of Maxwell, resulting on a violation of the inequality (1.12).

A similar reasoning may also be valid for the inequality involving energy and angular momentum. It is important to note that this inequality, a priori, is not derivable from any physical principle, and in fact, a classical limit for this inequality would result in a fulfillness of it for every system. All the NLEDs studied above satisfy the inequality and there is no physical reason whatsoever to conjecture a violation of this inequality by the arguments exposed above. However, it should be highlighted that this inequality is also derivable from DEC. Given that DEC states that there cannot be superluminal propagation of energy-momentum, in flat space this inequality is intimately related to the causal structure of the theory. Then, the natural conjecture from the fact that the inequality is a consequence of DEC is to argue that a violation of such inequality will be given by a *noncausal* NLED.

This section is aimed to look for counterexamples of both inequalities based on the arguments exposed here.

3.4.1 Counterexample for the inequality between charge and energy.

Given the behavior of the NLED studied above and the fact that all of them satisfy the respective Maxwell limit, it is suitable to look for a counterexample resembling one of the above electrodynamics, with the same Maxwell limit, but with a different series expansion. Since in this section we are only interested in the electrostatic case, we will only focus our attention to Lagrangians of the form $\mathcal{L}(F)$, then, a suitable electromagnetic Lagrangian should always recover the term $-F/2$ in a series expansion, even though we will only focus in the electric part of the Lagrangian. Then, we can write the logarithmic lagrangian (2.34) as a $\mathcal{L}(F)$ as

$$\mathcal{L}(F) = -\beta^2 \ln\left(1 + \frac{F}{2\beta^2}\right) \sim -\frac{F}{2} + \frac{F^2}{8\beta^2} - \frac{F^3}{24\beta^4} + ...$$

and we note that, indeed, the Larmor Lagrangian is recovered as a first order approximation. However, in principle, there is no reason why this functional form of a logarithmic

Lagrangian must be unique. For example, we can try by inverting the order of signs in the latter Lagrangian and expressing it as a series expansion

$$\mathcal{L}(F) = \beta^2 \ln\left(1 - \frac{F}{2\beta^2}\right) \sim -\frac{F}{2} - \frac{F^2}{8\beta^2} - \frac{F^3}{24\beta^4} - \ldots \tag{3.34}$$

remarkably, this last expression also possesses the adequate limit as a first order approximation, so, in principle there is no reason why this should be taken as a physically reasonable Lagrangian. Therefore, we will begin by looking for the properties that a Lagrangian of this form possess. First of all, the energy density for this Lagrangian is

$$u = \frac{|\mathbf{E}|^2}{1 - \frac{F}{2\beta^2}} - \beta^2 \ln\left(1 - \frac{F}{2\beta^2}\right) \tag{3.35}$$

Now arises the question on whether this Lagrangian provides a bound for the electric field, in order to analyze this we must see how is the constitutive relation between $\mathbf{D}$ and $\mathbf{E}$ for the electrostatic regime. Then, for the electrostatic regime we have

$$\mathbf{D} = \frac{\mathbf{E}}{1 + \frac{|\mathbf{E}|^2}{2\beta^2}} \tag{3.36}$$

for a point charge, e, by virtue of Gauss theorem we have $\mathbf{D} = e/4\pi r^2 \hat{\mathbf{r}}$, then we can obtain the electric field at the point $r = 0$ by inverting (3.36), nevertheless, the field at that point becomes complex. Since we are not interested particularly in a theory with a regular electric field at $r = 0$ but a modification of Maxwell electrodynamics we can impose a *minimum* radius for the electron within this theory. Then, by inverting the constitutive relation (3.36) for the point charge we get

$$|\mathbf{E}| = \frac{1 \pm \sqrt{1 - 2q^2/r^4\beta^2}}{q/r^2\beta^2} = \sqrt{2}\beta r^2 \left(\frac{1 \pm \frac{1}{r^2}\sqrt{r^4 - r_0^4}}{r_0^2}\right) \tag{3.37}$$

where $q = e/4\pi$ and $r_0^2 := \sqrt{2}Q/\beta$. Consequently, evaluating the field at the point $r = r_0$ leads to $|\mathbf{E}|_{r_0} = \sqrt{2}\beta$, which coincides with the value obtained for the standard logarithmic electrodynamics in Sec. 2.3.3. Note that with this definition it is not necessary to choose a sign for the solution of the resulting quadratic equation. Now that we know that logarithmic electrodynamics return a finite field for a given critical radius, let us see how the energy density of this alternative formulation behaves with respect to the standard (Maxwell) energy density. It can be seen (Fig. 6) the electrostatic energy density for this modification of logarithmic electrodynamics is smaller than the Maxwell counterpart for strong fields. Therefore, since the equality in the inequality between charge and energy in Maxwell electrodynamics is characterized by the charge distribution of a spherical thin shell, which has the variational characterization of possessing the minimal electrostatic energy, and the modified Lagrangian exposed here leads to a lower energy density than in the Maxwell case, the inequality between charge and energy (1.12) will no longer be satisfied for every field configuration in this formulation.

In order to clarify the previous statement let us calculate the binding energy for a spherical thin shell of radius r_0 and charge Q within Maxwell electrodynamics and this modified version of logarithmic electrodynamics. For Maxwell electrodynamics we know that the binding energy for a spherical shell of radius r_0 is

$$\mathcal{E}_{\mathrm{M}} = \frac{Q^2}{8\pi r_0} \sim 0.1186\sqrt{Q^3\beta} \tag{3.38}$$

For the case of modified logarithmic electrodynamics we get that the binding energy for a thin shell of charge Q and radius r_0 is

$$\begin{aligned}\mathcal{E}_{\log} &= \int_{r_0}^{\infty} r^2 dr \left\{ \frac{|\mathbf{E}|^2}{1+\frac{|\mathbf{E}|^2}{2\beta^2}} - \beta^2 \ln\left(1+\frac{|\mathbf{E}|^2}{2\beta^2}\right)\right\} = \int_{r_0}^{\infty} dr \left\{ \frac{Q^2/4\pi r^2}{1+\frac{Q^2}{8\pi\beta^2 r^4}} - \beta^2 r^2 \ln\left(1+\frac{Q^2}{8\pi\beta^2 r^4}\right)\right\} \\ &= \sqrt{\frac{Q^3\beta}{8\pi\sqrt{2}}} \int_{\sqrt{2}}^{\infty} dy \left\{ \frac{2}{y^2+\frac{1}{y^2}} - y^2 \ln\left(\frac{1}{y^4}+1\right)\right\} \sim 0.1108\sqrt{Q^3\beta}\end{aligned} \tag{3.39}$$

where we have made the substitution $y := r\sqrt{4\pi\sqrt{2}\beta/Q}$. Finally, we can compare both results obtained in (3.38) and (3.39),

$$\mathcal{E}_{\mathrm{M}} > \mathcal{E}_{\log} \tag{3.40}$$

and proving that in fact, for this case the Maxwellian energy of a spherical thin shell is greater than the logarithmic one. Since the equality for the bound between energy and charge (1.12) is attained when the field is the one given by a spherical thin shell, the inequality will be violated within this modified logarithmic electrodynamics.

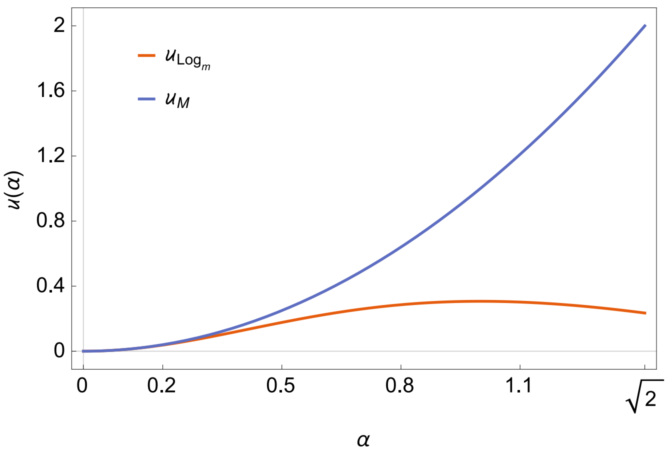

Figure 6 – The electrostatic energy density for modified logarithmic electrodynamics and the electrostatic energy density for Maxwell electrodynamics.

Now, does the violation of one of the inequalities implies the violation of the other inequalities? The worked inequalities only have in common the energy, thus, it is natural to expect that the violation of one partial inequality will have no consequences on the behavior of the other partial inequality. In order to see this, we will evaluate the inequality between energy and angular momentum for this modified logarithmic electrodynamics. The energy density expressed in terms of the normalized numbers α and γ for this electrodynamics is

$$u = \frac{2\alpha^2\beta^2}{1-\gamma^2+\alpha^2} - \beta^2 \ln\left(1-\gamma^2+\alpha^2\right) \tag{3.41}$$

while the angular momentum for this electrodynamics reads,

$$J(\Sigma) = \int_\Sigma d^3x \frac{1}{1-\gamma^2+\alpha^2} \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\right] \cdot \mathbf{k} \tag{3.42}$$

With this quantities in hand, we proceed in the same way as in the other examples, then, the estimate for the difference between energy and angular momentum in the region Σ is

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \left\{ \frac{|\mathbf{E}|^2}{1-\gamma^2+\alpha^2} - \beta^2 \ln\left(1-\gamma^2+\alpha^2\right) - \frac{|\mathbf{x}|}{\mathcal{R}} \frac{1}{1-\gamma^2+\alpha^2} |\mathbf{E} \times \mathbf{B}| \right\} \tag{3.43}$$

which can be further minimized as

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| \geq \int_\Sigma d^3x \beta^2 \left[\frac{1}{1-\gamma^2+\alpha^2} \left\{ \alpha^2 \left(2 - \frac{|\mathbf{x}|}{\mathcal{R}}\right) - \frac{|\mathbf{x}|}{\mathcal{R}}\gamma^2 \right\} - \ln\left(1-\gamma^2+\alpha^2\right) \right] \tag{3.44}$$

Which is possible to see that is always positive. In fact, the same conclusions as in all the cases can be obtained in this case: when $\alpha = \gamma$ then the integrand reduces exactly to the one of Maxwell electrodynamics, which ensures us that the rigidity statement will hold too. Thus, this modified logarithmic Lagrangian satisfies the inequality between energy and angular momentum.

3.4.2 Counterexample for the inequality between energy and angular momentum.

The quest for a counterexample of the inequality between energy and angular momentum based on the ground of NLED is obscure, since all the worked examples preserve such inequality, and *a priori* there is no physical reason to conjecture such an inequality nor to conjecture the violation of it. However, it is very important to note that the inequality can be obtained as a consequence of DEC [20], due to the importance of this result, the explicit calculation can be found in Appendix A.

Since the DEC is associated with the energy-momentum vector being causal, in flat space it is directly associated with causal structure of the theory. Thus, *any* causal theory will, indeed, satisfy the inequality between energy and angular momentum. As nonlinear photons propagate along effective null geodesics then it is expected that every causal theory, i.e. with effective null geodesics either null or timelike in the Minkowski background will satisfy the inequality. This leads to the following question: is the inequality between energy and angular momentum governed univocally by DEC? If that were the case, then *any* noncausal theory of NLED will violate the given inequality. Thus, if that were the case, such an inequality will provide us of a strong criterion for discerning between physically viable (causal) theories, and pathological, unphysical (noncausal) ones.

In order to prove whether the assumption of the inequality $\mathcal{E}(\Sigma)\mathcal{R} \geq |J(\Sigma)|$ is related to the DEC, and consequently to a well behaved causal structure, we will begin by assessing the *causality conditions* for NLED as given by Shabad in [4], as well as by Goulart and Perez Bergliaffa in [28]. For the case where $\mathcal{L} = \mathcal{L}(F)$ the causality principle requires

$$\mathcal{L}_F \leq 0 \tag{3.45}$$

together with the *unitarity* principle

$$\mathcal{L}_F + \mathcal{L}_{FF} \leq 0 \tag{3.46}$$

which requires that the residue of the propagator be positive. Shabad has showed that these requirements are analogous to requiring DEC to hold.

Hence, it should be possible to probe lagrangians where the causality is violated for all field configurations, as well as lagrangians where the causality is violated only for certain field configurations. The aim of this section is to analyze the behavior of the inequality (1.13) in the case where noncausal NLED is present.

3.4.2.1 Noncausal lagrangian.

A strong noncausal lagrangian violates (3.45) for every field configuration. Then, it should be suitable to probe a lagrangian with a similar structure as the Born lagrangian, for example

$$\mathcal{L} = \beta^2 \left(\sqrt{1 + \frac{F}{\beta^2}} - 1 \right) = \beta^2 \left(\sqrt{U} - 1 \right) \tag{3.47}$$

with $U := 1 + \frac{|\mathbf{B}|^2}{\beta^2} - \frac{|\mathbf{E}|^2}{\beta^2}$. Note that despite being very alike the Born-Infeld lagrangian, this lagrangian does not reduce correctly to the Larmor lagrangian for the adequate field limit. Moreover, the causality condition (3.45) is broken since $\mathcal{L}_F = \frac{1}{2\sqrt{U}} > 0$. Furthermore, the energy density and angular momentum for this lagrangian read

$$u = -\frac{|\mathbf{E}|^2}{\sqrt{U}} + \beta^2 - \beta^2\sqrt{U} \tag{3.48}$$

$$J(\Sigma) = \int_\Sigma d^3x \frac{1}{\sqrt{U}} \left(\mathbf{x} \times (\mathbf{B} \times \mathbf{E}) \right) \cdot \mathbf{k} \tag{3.49}$$

Then, estimating the difference between energy and angular momentum leads to

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} = \int_\Sigma d^3x \frac{\beta^2}{\sqrt{U}} \left(-\alpha^2 + \sqrt{U} - U \right) - \frac{1}{\mathcal{R}} \left| \int_\Sigma d^3x \frac{1}{\sqrt{U}} \left(\mathbf{x} \times (\mathbf{B} \times \mathbf{E}) \right) \cdot \mathbf{k} \right| \tag{3.50}$$

which we can further estimate by the same ways as it has been done before: first we will use the inequality $|\int f(x)| \leq \int |f(x)|$ where the equality holds when $f(x)$ is positive on the integration domain, and later use the inequality $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$, the fact that $\mathbf{k}$ is a unitary vector, and the inequality $|\mathbf{a}\times\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ for the remaining product, leaving the estimate as

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma d^3x \frac{\beta^2}{\sqrt{U}}\left(\sqrt{U} - 1 - \gamma^2 - \frac{|\mathbf{x}|}{\mathcal{R}}\frac{|\mathbf{B}\times\mathbf{E}|}{\beta^2}\right) \tag{3.51}$$

now we can make a further estimation, as before, $|\mathbf{a}\times\mathbf{b}| \leq (|\mathbf{a}|^2 + |\mathbf{b}|^2)/2$ and, with the conventions for α and γ we obtain

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma d^3x \frac{\beta^2}{\sqrt{U}}\left(\sqrt{U} - 1 - \gamma^2 - \frac{|\mathbf{x}|}{\mathcal{R}}\frac{\alpha^2+\gamma^2}{2}\right) \tag{3.52}$$

Note that the integrand in the last expression may be negative for certain field configurations, however since we are only estimating the difference there is, up to now, no straightforward conclusion regarding the overall sign of the integral. Nevertheless, it is still possible to assume that all the inequalities in the above lines will be indeed equalities, namely: the integrand in the absolute value in (3.50) is always positive, $\mathbf{x}$ is perpendicular to the Poynting vector, and, furthermore, the fields $\mathbf{E}$ and $\mathbf{B}$ are perpendicular and have the same magnitude. With the fulfillment of these assumptions, then the total estimate between the energy and angular momentum is

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} = -\int_\Sigma d^3x \beta^2\left(\gamma^2\left(1 + \frac{|\mathbf{x}|}{\mathcal{R}}\right)\right) < 0 \tag{3.53}$$

where the overall sign of the integral is *negative* for every field configuration such that $|\mathbf{E}| = |\mathbf{B}|$. However, this result implies the particularity that $F = 0$, then it should also be suitable to perform a similar estimative for the case of $F \neq 0$, i.e. without the estimative done in the last step. Such a procedure leads to the result

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} = \int_\Sigma d^3x \frac{\beta^2}{\sqrt{U}}\left(\sqrt{U} - 1 - \gamma^2 - \frac{|\mathbf{x}|}{\mathcal{R}}\alpha\gamma|\sin\theta|\right) \tag{3.54}$$

which is possible to see that is, in fact, negative for every field configuration, since $U > 1$ implies $\gamma^2 > \alpha^2$ and $\sqrt{U} < U + \gamma^2$ and when $U < 1$, $\sqrt{U} < 1$.

This case constitutes an example of how the inequality (1.13) can be violated. Nevertheless, the present example represents a physically unrealizable example and an extremely pathological one, because besides it violates the causality for every field configuration, it also violates the WEC, which means the energy density for such a lagrangian is mostly negative. Then, it is suitable to see how is the behavior of the inequality for a lagrangian that is noncausal only for certain field configurations.

3.4.2.2 Weakly noncausal lagrangian.

In order to find a partially noncausal lagrangian, due to condition (3.45), it is suitable that $\mathcal{L}_F$ be a linear combination of constants and the invariant F. Following this, one possible lagrangian may be the one proposed by Kruglov in [45], which reads

$$\mathcal{L} = -\frac{F}{2}e^{-\frac{F}{2\beta^2}} \tag{3.55}$$

this particular lagrangian constitutes a modification of exponential electrodynamics presented before, with the particularity that Maxwell electrodynamics is recovered only in the zeroth order of the expansion for the exponential. The derivative of this lagrangian is given by

$$\mathcal{L}_F = \frac{1}{2}e^{-\frac{F}{2\beta^2}}\left(-1+\frac{F}{2\beta^2}\right) \tag{3.56}$$

Since the requirement of causality is $\mathcal{L}_F \leq 0$, this particular lagrangian has a causality bound given by

$$|\mathbf{B}|^2 \leq 2\beta^2 + |\mathbf{E}|^2 \tag{3.57}$$

note that this is a bound only on the magnetic field, the electric field is unbounded with respect to causality conditions. Because a causal theory satisfies directly the inequality in question, we will only focus our following study in the sector where $2\beta^2 < F$, i.e. $\mathcal{L}_F > 0$. Therefore, we begin by calculating the energy density for this type of electrodynamics, which is given by

$$u = e^{-\frac{F}{2\beta^2}}\left(\frac{|\mathbf{E}|^2+|\mathbf{B}|^2}{2} - \frac{F}{2\beta^2}|\mathbf{E}|^2\right)$$

Note that the causality condition does not always imply that the energy density will be positive, however, for this particular example the causality condition does imply the non-negativity of the energy density, since for the case where $F = 2\beta^2$, which is in accordance to $\mathcal{L}_F = 0$, the energy density will be $u = F/2 = \beta^2$, which is positive. The angular momentum is

$$J(\Sigma) = \int_\Sigma d^3x e^{-\frac{F}{2\beta^2}}\left(\frac{F}{2\beta^2}-1\right)[\mathbf{x}\times(\mathbf{B}\times\mathbf{E})]\cdot\mathbf{k}$$

Then, the difference between energy and angular momentum in the region Σ is given by

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} = \int_\Sigma d^3x e^{-\frac{F}{2\beta^2}}\left(\frac{|\mathbf{E}|^2+|\mathbf{B}|^2}{2} - \frac{F}{2\beta^2}|\mathbf{E}|^2\right) - \left|\int_\Sigma d^3x e^{-\frac{F}{2\beta^2}}\left(\frac{F}{2\beta^2}-1\right)[\mathbf{x}\times(\mathbf{B}\times\mathbf{E})]\cdot\mathbf{k}\right| \tag{3.58}$$

which, by using the same inequalities for the absolute value of the integral $|\int f(x)| \leq \int |f(x)|$, and the vector triangular inequalities and noting that since we are treating the case where $\mathcal{L}_F > 0$, then

$$\left|\int_\Sigma d^3x e^{-\frac{F}{2\beta^2}}\left(\frac{F}{2\beta^2}-1\right)[\mathbf{x}\times(\mathbf{B}\times\mathbf{E})]\cdot\mathbf{k}\right| \leq \int_\Sigma d^3x e^{-\frac{F}{2\beta^2}}\left(\frac{F}{2\beta^2}-1\right)|\mathbf{x}\times(\mathbf{B}\times\mathbf{E})\cdot\mathbf{k}|$$

where the equal will hold when $|\mathbf{x} \times (\mathbf{B} \times \mathbf{E}) \cdot \mathbf{k}|$ is positive in Σ. Consequently, we can write the estimative as

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} = \int_\Sigma d^3x e^{-\frac{F}{2\beta^2}} \left(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2} - \frac{F}{2\beta^2}|\mathbf{E}|^2 - \frac{1}{\mathcal{R}} \left(\frac{F}{2\beta^2} - 1 \right) |\mathbf{x} \times (\mathbf{B} \times \mathbf{E}) \cdot \mathbf{k}| \right) \tag{3.59}$$

$$\geq \int_\Sigma d^3x e^{-\frac{F}{2\beta^2}} \left(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2} - \frac{F}{2\beta^2}|\mathbf{E}|^2 - \frac{|\mathbf{x}|}{\mathcal{R}} \left(\frac{F}{2\beta^2} - 1 \right) |\mathbf{B} \times \mathbf{E}| \right) \tag{3.60}$$

We know that $F > 2\beta^2$, then we will assign the define $a := F/2\beta^2$, such that $a > 1$, leaving the integral

$$\geq \int_\Sigma d^3x e^{-\frac{F}{2\beta^2}} \left\{ \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2} + \frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}||\mathbf{B}||\sin\theta| - a \left(|\mathbf{E}|^2 + \frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}||\mathbf{B}||\sin\theta| \right) \right\} \tag{3.61}$$

It is possible to see that the certain noncausal field configurations can, indeed, preserve the inequality between energy and angular momentum. This particular behavior as compared with the causality condition for different values of θ can be seen in Fig. 7.

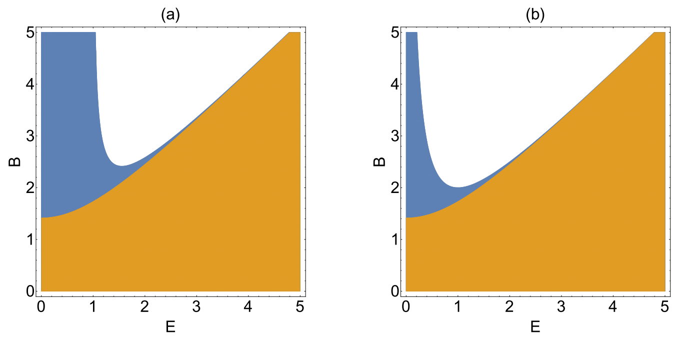

Figure 7 – The values where the integrand in (3.61) is positive (blue region) compared to the values where the causality condition (3.57) holds (red region) for (a) $\theta = 0$ and (b) $\theta = \pi/2$.

A very interesting and peculiar consequence of the inequality between energy and angular momentum (1.13) is the fact that it can be derived as a consequence of DEC. It is particularly encouraging to think that such a condition could allow us to have a criterion whether a physical theory may be valid or not[3]. However, as we have seen in this section such an inequality can also be attained even when noncausal propagation is present. As for the inequality in question allows pathological cases to occur it should be necessary to search for a *stricter* inequality such that DEC can univocally determine the validity of it.

3.5 Arbitrary $\mathcal{L}(F)$ Electrodynamics.

We have seen examples of NLED that satisfy both of the inequalities (1.12) and (1.13), and counterexamples of NLED that violate one of those inequalities. It is expected

[3] In the sense that all *reasonable* matter fields obey DEC.

that the fulfillment of both the inequalities implies the fulfillment of the full inequality (1.17). Nevertheless, we have seen that such an assumption might not hold for every case since there are configurations in exponential electrodynamics for which the electromagnetic energy density is smaller than the electrostatic energy density, which allows for the complete inequality (1.17) to be violated despite the partial inequalities being satisfied. Then, it should be necessary to obtain which extra conditions must be imposed in order for the complete inequality to hold.

In the present section we will begin deriving general requirements for the inequality between energy and angular momentum to hold within a general NLED, then, using the result obtained we will obtain the general requirements for the inequality between energy, charge and angular momentum to hold.

3.5.1 Inequality between energy and angular momentum.

Since the inequality between energy and angular momentum is purely quasi-local, it follows that all quantities involved in the inequality must be evaluated within a region in space, Σ. The direct way of evaluating such inequalities is by comparing them, as we have done in the previous examples in this chapter. For arbitrary NLED the difference between energy and angular momentum is

$$\mathcal{E}(\Sigma) - \frac{1}{\mathcal{R}}|J(\Sigma)| = \int_\Sigma -2\mathcal{L}_F|\mathbf{E}|^2 - \mathcal{L}(F) - \frac{1}{\mathcal{R}}\left|\int_\Sigma 2\mathcal{L}_F[\mathbf{x}\times(\mathbf{B}\times\mathbf{E})]\cdot\mathbf{k}\right| \tag{3.62}$$

Then, we can perform the same estimations for the absolute value of the integral and the vector triangular inequalities, obtaining

$$\geq \int_\Sigma -2\mathcal{L}_F|\mathbf{E}|^2 - \mathcal{L}(F) - \frac{1}{\mathcal{R}}2\left|\mathcal{L}_F\right||\mathbf{x}||\mathbf{B}||\mathbf{E}| \tag{3.63}$$

Since the causality condition imposes a sign on $\mathcal{L}_F$ but does not guarantees that the energy density will be nonnegative, the reasonable situation to evaluate the integral in (3.63) is where the causality is assured. However, as we have seen in the previous section, there exist noncausal theories that can satisfy the inequality between energy and angular momentum, then it is also important to study the case where $\mathcal{L}_F > 0$. Then, $|\mathcal{L}_F|$ must be evaluated as

$$|\mathcal{L}_F| = \begin{cases} \mathcal{L}_F & \text{if } \mathcal{L}_F \geq 0, \\ -\mathcal{L}_F & \text{if } \mathcal{L}_F < 0, \end{cases}$$

We get a set of two inequalities depending on the overall sign of $\mathcal{L}_F$, which can be stated as follows,

$$\int_\Sigma d^3x\left\{-2\mathcal{L}_F\left(|\mathbf{E}|^2 \pm \frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{B}||\mathbf{E}|\right) - \mathcal{L}(F)\right\} \geq 0 \tag{3.64}$$

Where the sign inside the parenthesis is positive for $\mathcal{L}_F \geq 0$ and negative for $\mathcal{L}_F < 0$. If the integrand in (3.64) is positive then the inequality between energy and angular momentum (1.13) will be immediately satisfied.

3.5.2 Inequality between energy, charge and angular momentum.

For deriving the conditions that an arbitrary NLED lagrangian of the form $\mathcal{L}(F)$ must accomplish in order to satisfy the complete inequality between energy, charge and angular momentum (1.17) we will start by obtaining the total energy in $\mathbb{R}^3$ for an arbitrary NLED from the energy-momentum tensor presented in (2.43) and add and subtract the term $|\mathbf{E}|^2/2$ corresponding to the Maxwell electrostatic energy density. Then, the energy is

$$\mathcal{E} = \int_{\mathbb{R}^3} d^3x \left\{ \frac{|\mathbf{E}|^2}{2} - 2\mathcal{L}_F|\mathbf{E}|^2 - \mathcal{L}(F) - \frac{|\mathbf{E}|^2}{2} \right\} \tag{3.65}$$

Since the Bianchi identities remain unaltered for NLED of the form $\mathcal{L}(F, G)$, it is always possible to perform the same decomposition of the electric and magnetic fields in terms of the scalar potential Φ and the vector potential $\mathbf{A}$. Furthermore, the construction of the potentials performed in the other sections will also be valid within general NLED, then it is possible to express the electric field in terms of two potentials Φ_0 and Φ_1 and then separate the integral corresponding to the positive contribution of the Maxwell part in the domains Σ and $\mathbb{R}^3 \setminus \Sigma$ obtaining

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \int_{\Sigma} d^3x \frac{|\mathbf{E}|^2}{2} + \int_{\mathbb{R}^3\setminus\Sigma} d^3x \frac{|\nabla\Phi_1 + \partial_t\mathbf{A}|^2}{2} + \int_{\mathbb{R}^3} d^3x \left\{ -|\mathbf{E}|^2 \left(2\mathcal{L}_F + \frac{1}{2} \right) - \mathcal{L}(F) \right\} \tag{3.66}$$

It is possible to split the last integral in the same two domains where the other integrals are evaluated, then, the integral evaluated in the region Σ will be exactly the energy in that region $\mathcal{E}(\Sigma)$ leaving

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \mathcal{E}(\Sigma) + \int_{\mathbb{R}^3\setminus\Sigma} d^3x \left\{ \frac{|\nabla\Phi_1 + \partial_t\mathbf{A}|^2}{2} - |\mathbf{E}|^2 \left(2\mathcal{L}_F + \frac{1}{2} \right) - \mathcal{L}(F) \right\} \tag{3.67}$$

Now we use the result obtained in (3.64) to bound $\mathcal{E}(\Sigma)$ with $|J(\Sigma)|/\mathcal{R}$, obtaining,

$$\begin{aligned} \mathcal{E} - \frac{Q^2}{2\mathcal{R}} - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_{\mathbb{R}^3\setminus\Sigma} d^3x \left\{ \frac{|\nabla\Phi_1 + \partial_t\mathbf{A}|^2}{2} - |\mathbf{E}|^2 \left(2\mathcal{L}_F + \frac{1}{2} \right) - \mathcal{L}(F) \right\} + \\ + \int_{\Sigma} d^3x \left\{ -2\mathcal{L}_F \left(|\mathbf{E}|^2 \pm \frac{|\mathbf{x}|}{\mathcal{R}} |\mathbf{B}||\mathbf{E}| \right) - \mathcal{L}(F) \right\} \end{aligned} \tag{3.68}$$

From the condition obtained in the previous section we know that the inequality between energy and angular momentum will only be valid when the integral evaluated in Σ is positive, then the other integral must also be positive in order for the inequality to be satisfied. Then, since the first term in the remaining integral evaluated in $\mathbb{R}^3 \setminus \Sigma$ is also nonnegative, we may impose the following condition,

$$\int_{\mathbb{R}^3\setminus\Sigma} d^3x \left\{ -2|\mathbf{E}|^2\mathcal{L}_F - \mathcal{L}(F) \right\} \geq \int_{\mathbb{R}^3\setminus\Sigma} d^3x \frac{|\mathbf{E}|^2}{2} \tag{3.69}$$

Which can finally be rewritten as,

$$\mathcal{E}_{\mathcal{L}(F)}(\mathbb{R}^3 \setminus \Sigma) \geq \mathcal{E}_{\mathrm{Ms}}(\mathbb{R}^3 \setminus \Sigma) \tag{3.70}$$

where $\mathcal{E}_{\mathrm{Ms}}$ is the electrostatic energy form Maxwell electrodynamics and $\mathcal{E}_{\mathcal{L}(F)}$ is the electromagnetic energy for NLED. Then, if the inequality (3.70) holds as well as the inequality between energy and angular momentum (1.13) then the complete inequality (1.17) will be valid within NLED, hence, the fulfillment of the partial inequality between energy and charge (1.12) is not needed to prove the complete inequality. It is important to note that the inequality (3.70) obtained in this section relates the *total electromagnetic energy* for a NLED with the Maxwell electrostatic energy in the region Σ. As it has been discussed when treating the inequality between energy and charge, it is natural to assume that a NLED with bigger electrostatic energy density than the Maxwellian one will immediately satisfy the inequality $2\mathcal{E}\mathcal{R} \geq Q^2$. However, NLED may reproduce self-interaction phenomena, and, in general, it is possible that the total electromagnetic energy of a NLED is smaller than the electrostatic part. This fact will influence directly in the validity of the static inequality between $\mathcal{E}$ and Q. Consequently, if the inequality between energy and charge holds within a NLED in the electrostatic regime, then for the inequality to be valid in the electrodynamic regime, the total electromagnetic energy must be necessarily greater or equal than the electrostatic part

$$\mathcal{E}_{\mathcal{L}(F)} \geq \mathcal{E}_{\mathcal{L}(F)\mathrm{s}} \geq \mathcal{E}_{\mathrm{Ms}}$$

This inequality in fact offers a stronger restriction for NLED than the previous inequality $2\mathcal{E}\mathcal{R} \geq Q^2$ as the fulfillment of the inequality between Q and $\mathcal{E}$ does not always imply the fulfillment of the total inequality, while the inequality (3.70) is a necessary condition for the fulfillment of the complete inequality.

4 Conclusions and Perspectives.

Geometric inequalities represent a powerful framework for obtaining bounds for different classes of theories. Spacetime theories constitute a particular example of field theories for which geometry is embedded in spacetime, hence, it is natural to expect that geometric inequalities arise within these theories. Then, it is natural to conjecture the validity of such inequalities under certain conditions both in spacetime theories and in other field theories. As for Bekenstein's inequality, despite it being formulated for sustaining the GLS, it is important to note that throughout the years it has proved to be more important providing universal arguments, because such inequality is respected both in the classical domain as well as in the quantum domain.

When fixing the entropy in Bekenstein's bounds to zero we get non-trivial relations between dynamical variables, which are not expected to hold from classical arguments. It is necessary to make assumptions about the relation between such dynamical variables in order to get analogies in classical mechanics, for example, the case of slow rotation and the deduction of the minimum rotational energy to the total system's energy. By performing these analogies we infer that in fact, such inequalities may be considered too loose, since the 'minimum' rotational energy is a fraction of the actual minimum rotational energy for a rigid body.

We have provided proofs for each of the inequalities within particular examples of NLED in the literature, where each of the partial inequalities holds. In particular, these proofs have allowed us to conjecture the regime of validity of each of the inequalities when nonlinear regimes of electrodynamics are present. Then we have provided a counterexample for each of the partial inequalities. It is important to note that each of the counterexamples were due to Lagrangians with the adequate limit for weak fields. In particular, the violation of the inequality between energy and charge is given by a non-linear logarithmic Lagrangian that represents a modification of the original logarithmic electrodynamics presented by Gaete and Helayël [38] which recovers the same limit for weak fields, such a Lagrangian provides a smaller energy density than the one given by the Larmor Lagrangian. Hence, the physical assumption behind the conjecture of violation of the inequality is clear: the inequality will be violated when a theory possesses a lower energy than in the Maxwell case, where the inequality holds.

The quest for a counterexample for the inequality between energy and angular momentum was far from obvious from the physical point of view. There does not exist any physical assumption derived from the nature of nonlinear Lagrangians in order to conjecture the violation of such an inequality. However, we have used the fact that the inequality can indeed be derived as a consequence of the Dominant Energy Condition. Therefore, it is reasonable to check if the inequality is violated when a physical theory does not satisfy DEC. We have seen that the violation of DEC is, indeed, necessary for the violation of the inequality. Furthermore, we have showed that DEC is not a necessary condition for the fulfillment of the inequality but only a sufficient one. This fact is very important since it has allowed us to provide an example of noncausal propagation of light present in the literature [45] where the inequality is satisfied for particular cases of noncausal propagation. The inequality between energy and angular momentum seemed to offer a strong criterion for discerning between physically reasonable theories and unphysical ones, nevertheless, the fact that this inequality is not univocally determined by DEC has the consequence that this inequality is not strict enough because pathological theories satisfy the inequality. We conclude that the inequality between energy and angular momentum is not directly connected with causality.

Finally, we have shown that the fulfillment of both partial inequalities does not assure the fulfillment of the complete inequality between energy, charge and angular momentum in NLED. This is due to the fact that, in general, there may appear situations arising from NLED where the total electromagnetic energy density is smaller than the electrostatic energy density. Thus, we have derived an additional inequality that must be satisfied for NLED to satisfy the complete inequality. Furthermore, the general requirements for the fulfillment of the complete inequality are that the inequality between energy and angular momentum and the new inequality must both hold.

There is a broad range of future developments that can be addressed from this work. First of all, it is particularly interesting that the original arguments behind Bekenstein's inequalities are of gravitational origin, despite it is extremely difficult to have a notion of radial distance in curved spacetimes and the only proofs of gravitating objects satisfying the inequalities are for black holes. Since we have proved that there exist NLED that satisfy such inequalities, it could be interesting to analyze the behavior of such inequalities when an *effective geometry* is taken into account. Basically, it is possible to 'covariantize' the field equations for NLED, such that the covariant equations are analogous to the linear case but the covariant derivative is taken with respect to an *effective metric*. The effective metric arises when analyzing photon propagation in NLED, nonlinear photons indeed propagate along *effective* null geodesics, rather than null geodesics in the Minkowski background as linear photons do, this gives rise to the phenomenon of birefringence and the possibility of having noncausal theories of NLED (where the effective light cones are defined outside the Minkowski light cone). With an effective geometry approach it could be possible to provide proofs of the inequalities due to analog gravitational scenarios, which might provide analogies for purely gravitational systems.

Second, it has been proved by Schiffer and Bekenstein [14] that the free scalar and electromagnetic fields satisfy the original bound (1.5), nevertheless up to now there is no proof for these fields' entropy satisfying the complete inequality. Then, the possible developments regarding this are twofold: (i) provide a proof of the complete inequality for free fields and (ii) provide a proof of the original bound (1.5) for fields generated by NLED.

Third, the universality assumptions behind Bekenstein's inequalities provide a wide range of work for proving such inequalities in scenarios different than electromagnetism or general relativity.

Last, since the inequality between energy and angular momentum is somehow connected to DEC but it allows pathological theories to satisfy the inequality, it is particularly encouraging the try to strengthen the inequality in order for it to only be univocally determined for causal theories, i.e. only to theories that satisfy DEC.

APPENDIX A – The Inequality $\mathcal{R}\mathcal{E}(\Sigma) \geq |J(\Sigma)|$ from the DEC.

The inequality (1.13), $\mathcal{E}(\Sigma)\mathcal{R} \geq |J(\Sigma)|$, can be derived as a consequence of the Dominant Energy Condition (DEC)

$$T_{\mu\nu}\xi^\mu l^\nu \geq 0 \tag{A.1}$$

for every ξ^μ and l^μ causal vectors. Then, choosing an adequate coordinate system where $\xi^\mu = \delta^\mu_0$ is a timelike vector, and constructing the null vector $l^\mu = \delta^\mu_0 - \hat{\eta}^\mu$ where $\eta^\mu = \eta^i$ is a spacelike vector that represents spatial rotations,

$$\eta^i = \epsilon^{ijk} k_j x_k$$

and, consequently, $\hat{\eta}^\mu = \eta^\mu/\sqrt{\eta}$, is a unitary vector and $\eta = -\eta_\alpha \eta^\alpha$ is the square norm of this vector. Then, it is easy to show that $l^\mu l_\mu = 0$ since $\delta^\mu_0 \eta_\mu = 0$. By applying this construction into the DEC (A.1) we get

$$T_{\mu\nu}\xi^\mu \xi^\nu \geq T_{\mu\nu}\xi^\mu \hat{\eta}^\nu \tag{A.2}$$

Now, since η^μ depends linearly on x, the distance to the axis, then the square norm, η, will be bounded by

$$\eta \leq \mathcal{R}^2$$

being $\mathcal{R}$ the minimum sphere that encloses Σ. Then, integrating both sides of (A.2) in the domain Σ we get

$$\int_\Sigma T_{\mu\nu}\xi^\mu \xi^\nu \geq \int_\Sigma T_{\mu\nu}\xi^\mu \hat{\eta}^\nu \tag{A.3}$$

$$\geq \frac{1}{\mathcal{R}} \int_\Sigma T_{\mu\nu}\xi^\mu \eta^\nu \tag{A.4}$$

Finally, by using the definitions for ξ^μ and η^μ, we obtain

$$\mathcal{E}(\Sigma) \geq \frac{1}{\mathcal{R}} J(\Sigma) \tag{A.5}$$

which is the desired inequality.

Nevertheless, it is interesting to point out that, from (A.2), such an inequality is indeed valid at *every point*. This is a striking consequence of the DEC, since it directly relates the energy and angular momentum densities at each and every point. However, as far as we know, there does not exist a good notion of angular momentum as a global quantity, and that may be the reason behind defining quasi-local quantities from the DEC. Moreover, the equality will hold when $T_{\mu\nu}\xi^\mu l^\nu = 0$, which means that $T^{\mu\nu}\xi_\mu$ is a null vector and is zero everywhere but at the point where $\eta = \mathcal{R}^2$.

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