

Invariant partial differential equations of conformal Galilei algebra as deformations: cryptohermiticity and contractions

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We investigate the general class of second-order partial differential equations (PDEs), invariant under the $d = 1$ $\ell = \frac{1}{2} + \mathbb{N}_0$ centrally extended conformal Galilei algebras, pointing out that they are deformations of decoupled systems. For $\ell = \frac{3}{2}$ the unique deformation parameter γ belongs to the fundamental domain $\gamma \in]0, +\infty[$. We show that, for any $\gamma \neq 0$, invariant PDEs with discrete spectra (either bounded or unbounded) induce crypto-Hermitian operators possessing the same spectrum as two decoupled oscillators, provided that their frequencies are in the special ratio $r = \frac{\omega_2}{\omega_1} = \pm \frac{1}{3}, \pm 3$ (the negative energy solutions correspond to a special case of the Pais–Uhlenbeck oscillator), where ω_1, ω_2 are two different parameters of the invariant PDEs. We also consider the $\gamma = 0$ decoupled system for any value r of the ratio. It possesses enhanced symmetry at the critical values $r = \pm \frac{1}{3}, \pm 1, \pm 3$. Two inequivalent 12-generator symmetry algebras are found at $r = \pm \frac{1}{3}, \pm 3$ and $r = \pm 1$, respectively. The $\ell = \frac{3}{2}$ conformal Galilei algebra is not a subalgebra of the decoupled symmetry algebra. Its $\gamma \rightarrow 0$ contraction corresponds to an 8-generator subalgebra of the decoupled $r = \pm \frac{1}{3}, \pm 3$ symmetry algebra. The features of the $\ell \geq \frac{5}{2}$ invariant PDEs are briefly discussed.
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1. Introduction

In Refs. [1,2], second-order partial differential equations (PDEs), invariant under the $\widehat{\text{cga}}_\ell$ ($\ell = \frac{1}{2} + \mathbb{N}_0$) centrally extended conformal Galilei algebra [3], were constructed. They were shown to possess a spectrum that is either continuous [1] or discrete (positive and bounded) [2]. In Ref. [1] the invariant PDEs were obtained via Verma module representation, while in Ref. [2] the so-called on-shell condition was used (for the cases at hand the two approaches are proven to be equivalent).

In this paper we address several important issues that were not touched on in these two previous works. We name a few: the identification of the general class of invariant PDEs (which turns out to depend on real parameters belonging to a fundamental domain), the existence of a contraction algebra, the reason for the cryptohermiticity (here we use the word adopted in Ref. [4]) of the discrete spectrum, the construction of the Hilbert space connected with Pais–Uhlenbeck oscillators with unbounded spectra, etc.

Specifically, the following list of results is derived in the present paper (we limit ourselves here in the introduction to discussing the first nontrivial case obtained for $\ell = \frac{3}{2}$; the $\ell > \frac{3}{2}$ cases are commented on in Sect. 8): two special differential realizations of $\widehat{\text{cga}}_\ell$ produce, as invariant PDEs, Schrödinger equations with continuous and respectively discrete spectra. Both realizations depend on a parameter $\gamma \neq 0$. Unitarily inequivalent theories are recovered for γ belonging to the fundamental domain $\gamma \in]0, +\infty[$.

The $\gamma = 0$ PDEs are decoupled equations. The continuum spectrum case corresponds to the free Schrödinger equation in $1 + 1$ dimensions, while the discrete spectrum case corresponds to a system of two decoupled *crypto-Hermitian oscillators* (namely, despite being non-Hermitian, possessing the same spectrum as two decoupled oscillators with the given frequencies). The parameter γ can therefore be regarded as a deformation parameter and as a coupling constant.

Without loss of generality we can fix $\omega_1 = 1$ to be the energy mode of the first oscillator in the coupled PDE. Then, the $\widehat{\text{cga}}_{\frac{3}{2}}$ invariance of the PDE is recovered if the energy mode of the second oscillator possesses the critical values $\omega_2 = \pm\frac{1}{3}, \pm 3$ ($\omega_2 = 3$ is the solution given in Ref. [2]). The negative values correspond to an unbounded spectrum and, as explained later, are connected with special cases of the Pais–Uhlenbeck oscillators. At fixed $\omega_{1,2}$, the spectrum of the operators does not depend on the value of γ .

The $\gamma \rightarrow 0$ limit of the $\widehat{\text{cga}}_{\frac{3}{2}}$ algebra produces a contraction algebra that is a symmetry subalgebra of the decoupled systems. For the decoupled oscillators (the analysis can be limited to the $\omega_1 = 1, \omega_2 \geq 1$ domain without loss of generality), the PDE possesses a 9-generator symmetry algebra at generic values, with enhanced symmetry at the critical values $\omega_2 = 1$ and $\omega_2 = 3$ (two different 12-generator symmetry algebras are obtained at these special points). The $\gamma \rightarrow 0$ contraction algebra is an 8-generator subalgebra of the $\omega_2 = 3$, 12-generator decoupled symmetry.

In this paper we discuss the subtle connection with Pais–Uhlenbeck oscillators. The Pais–Uhlenbeck model is a higher-derivative system [4–6] that admits, via the Ostrogradskii construction [7], a Hamiltonian formulation. The Ostrogradskii Hamiltonian is canonically equivalent to a set of decoupled harmonic oscillators with alternating (positive and negative) energy modes. In a series of papers [8–12] the Pais–Uhlenbeck oscillators with energy modes given (up to a normalization factor) by the arithmetic progression $\omega_i = 2i - 1$ were linked to the conformal Galilei algebras $\widehat{\text{cga}}_\ell$ (with $\ell = n - \frac{1}{2}$). As discussed in Sect. 8, the connection is rather subtle. To avoid any confusion, we should stress that the symmetry operators considered here are required to be realized as *first-order* differential operators.

The scheme of the paper is as follows: in Sect. 2 we present the ($\gamma \neq 0$ -dependent) differential realization for the deformation of the free Schrödinger equation at $\ell = \frac{3}{2}$ and the differential realization for the coupled oscillator. The connection of the two differential realizations obtained by similarity transformations and change of the time coordinate is shown in Sect. 3. In Sect. 4 the most general solution of the $\widehat{\text{cga}}_{\frac{3}{2}}$ -invariant oscillator is given. The symmetry of the decoupled oscillator (with enhanced critical points at $\omega_2 = 1$ and $\omega_2 = 3$) is presented in Sect. 5. The $\ell = \frac{3}{2}$ contraction algebra in the $\gamma \rightarrow 0$ limit is given in Sect. 6. The Hilbert space for the oscillators and the relation with PT-symmetry is discussed in Sect. 7. In Sect. 8 the extension to the $\ell > \frac{3}{2}$ cases and the relation to Pais–Uhlenbeck oscillators are commented on. Generalizations of the present construction are discussed in the conclusion, Sect. 9.

2. Differential realizations of the conformal Galilei algebra $\widehat{\text{cga}}_{\frac{3}{2}}$

The $d = 1$, $\ell = \frac{3}{2}$ centrally extended conformal Galilei algebra $\widehat{\text{cga}}_{\frac{3}{2}}$ consists of eight generators $(z_0, z_{\pm}, w_{\pm 1}, w_{\pm 3}, c)$, obeying the following nonvanishing commutation relations:

$$\begin{aligned} [z_0, z_{\pm}] &= \pm 2iz_{\pm}, \\ [z_+, z_-] &= -4iz_0, \\ [z_0, w_k] &= ikw_k, \\ [z_{\pm}, w_k] &= i(k \mp 3)w_{k \pm 2}, \\ [w_{|k|}, w_{-|k|}] &= (3 - 2|k|)16c. \end{aligned} \quad (1)$$

In this section we introduce two differential realizations of the above algebra. As will be explained in the following, the first differential realization corresponds to a symmetry of a deformed free Schrödinger equation in $1 + 1$ dimensions, while the second differential realization corresponds to a symmetry of another system (possessing the same spectrum as two decoupled oscillators).

The first differential realization is a one-parameter ($\gamma \neq 0$) extension of the differential realization obtained in Ref. [1] via standard left action on the space of functions defined on the coset (see also Ref. [13]). The presence of γ can be understood by the fact that the combined rescalings of the space and time coordinates that preserve the conformal Galilei algebra structure lead to just one free parameter. In this case we obtain first-order differential operators acting on functions of τ, x, y . They are given by

$$\begin{aligned} \bar{z}_+ &= \partial_{\tau}, \\ \bar{z}_0 &= -2i\tau\partial_{\tau} - ix\partial_x - 3iy\partial_y - 2i, \\ \bar{z}_- &= -4\tau^2\partial_{\tau} - 4\left(\tau x - \frac{3}{\gamma}y\right)\partial_x - 12\tau y\partial_y - 8(\tau - ix^2), \\ \bar{w}_{+3} &= \partial_y, \\ \bar{w}_{+1} &= -2i\tau\partial_y + \frac{2i}{\gamma}\partial_x, \\ \bar{w}_{-1} &= -4\tau^2\partial_y + \frac{8}{\gamma}\tau\partial_x - \frac{8i}{\gamma}x, \\ \bar{w}_{-3} &= 8i\tau^3\partial_y - \frac{24i}{\gamma}\tau^2\partial_x - 48\left(\frac{1}{\gamma}\tau x + \frac{1}{\gamma^2}y\right), \\ \bar{c} &= \frac{1}{\gamma^2}. \end{aligned} \quad (2)$$

The second differential realization is the γ extension of the differential realization presented in Ref. [2]. It is given by the first-order differential operators acting on functions of t, x, y :

$$\begin{aligned} \widehat{z}_0 &= \partial_t, \\ \widehat{z}_+ &= e^{2it}(\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\ \widehat{z}_- &= e^{-2it}\left(\partial_t - ix\partial_x - 3iy\partial_y + \frac{12}{\gamma}y\partial_x + 7ix^2 + \frac{12}{\gamma}xy - 2i\right), \\ \widehat{w}_{+3} &= e^{3it}\partial_y, \end{aligned}$$

$$\begin{aligned}
\widehat{w}_{+1} &= e^{it} \left(\partial_y + \frac{2i}{\gamma} \partial_x + \frac{2i}{\gamma} x \right), \\
\widehat{w}_{-1} &= e^{-it} \left(\partial_y + \frac{4i}{\gamma} \partial_x - \frac{4i}{\gamma} x \right), \\
\widehat{w}_{-3} &= e^{-3it} \left(\partial_y + \frac{6i}{\gamma} \partial_x - \frac{18i}{\gamma} x - \frac{48}{\gamma^2} y \right), \\
\widehat{c} &= \frac{1}{\gamma^2}.
\end{aligned} \tag{3}$$

We now consider the three elements $\Omega_0, \Omega_{\pm 1}$ of the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ enveloping algebra of degree $0, \pm 1$ (measured by the degree generator $-\frac{i}{2}z_0$), respectively:

$$\begin{aligned}
\Omega_{+1} &= iz_+ + \frac{\gamma^2}{16} (\{w_{+3}, w_{-1}\} - \{w_{+1}, w_{+1}\}), \\
\Omega_0 &= i\bar{z}_0 + \frac{\gamma^2}{32} (\{w_{+3}, w_{-3}\} - \{w_{+1}, w_{-1}\}), \\
\Omega_{-1} &= i\bar{z}_- + \frac{\gamma^2}{16} (\{w_{+1}, w_{-3}\} - \{w_{-1}, w_{-1}\}),
\end{aligned} \tag{4}$$

where the curly brackets on the r.h.s. denote anticommutators.

The three operators $\Omega_{\pm 1, 0}$ close the $\mathfrak{sl}(2)$ algebra, with Ω_0 the Cartan element:

$$\begin{aligned}
[\Omega_0, \Omega_{\pm 1}] &= \mp 2\Omega_{\pm 1}, \\
[\Omega_{+1}, \Omega_{-1}] &= 4\Omega_0.
\end{aligned} \tag{5}$$

It is important to note that, for both differential realizations (2) and (3), $\Omega_0, \Omega_{\pm 1}$ are presented as second-order differential operators.

For the differential realization (2) we have

$$\begin{aligned}
\overline{\Omega}_{+1} &= i\partial_\tau - i\gamma x \partial_y + \frac{1}{2} \partial_x^2 = i\bar{z}_+ - \overline{H}_+, \\
\overline{\Omega}_0 &= -2i\tau \overline{\Omega}_{+1} = i\bar{z}_0 - \overline{H}_0, \\
\overline{\Omega}_{-1} &= -4\tau^2 \overline{\Omega}_{+1} = i\bar{z}_- - \overline{H}_-.
\end{aligned} \tag{6}$$

The $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ on-shell invariant condition for $\overline{\Omega}_{\pm 1, 0}$ (see Refs. [2, 14, 15] for a definition) is guaranteed by the fact that their only nonvanishing commutators with the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ generators are expressed as

$$\begin{aligned}
[\bar{z}_0, \overline{\Omega}_{+1}] &= 2i\overline{\Omega}_{+1}, \\
[\bar{z}_-, \overline{\Omega}_{+1}] &= 4i\overline{\Omega}_0 = 8\tau \overline{\Omega}_{+1}, \\
[\bar{z}_+, \overline{\Omega}_0] &= -2i\overline{\Omega}_{+1} = \tau^{-1} \overline{\Omega}_0, \\
[\bar{z}_-, \overline{\Omega}_0] &= 2i\overline{\Omega}_{-1} = 4\tau \overline{\Omega}_0, \\
[\bar{z}_+, \overline{\Omega}_{-1}] &= -4i\overline{\Omega}_0 = 2\tau^{-1} \overline{\Omega}_{-1}, \\
[\bar{z}_0, \overline{\Omega}_{-1}] &= -2i\overline{\Omega}_{-1}.
\end{aligned} \tag{7}$$

The degree 1 invariant equation

$$\widehat{\Omega}_1 \Psi(\tau, x, y) = 0 \Rightarrow i\partial_\tau \Psi = -\frac{1}{2}\partial_x^2 \Psi + i\gamma x \partial_y \Psi \quad (8)$$

is a Schrödinger equation with τ playing the role of a time coordinate. The parameter γ is a coupling constant. This equation can be regarded as a γ -deformation of the free Schrödinger equation in 1 + 1 dimensions.

For the differential realization (3), $\Omega_0, \Omega_{\pm 1}$ are given by

$$\begin{aligned} \widehat{\Omega}_{+1} &= e^{2it} \widehat{\Omega}_0 = i\widehat{z}_+ - \widehat{H}_+, \\ \widehat{\Omega}_0 &= i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 - 3y\partial_y - i\gamma x \partial_y - \frac{3}{2} = i\widehat{z}_0 - \widehat{H}_0, \\ \widehat{\Omega}_{-1} &= e^{-2it} \widehat{\Omega}_0 = i\widehat{z}_- - \widehat{H}_-. \end{aligned} \quad (9)$$

The on-shell invariant condition is guaranteed by the fact that their only nonvanishing commutators with the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ generators are expressed as

$$\begin{aligned} [\widehat{z}_0, \widehat{\Omega}_{+1}] &= 2i\widehat{\Omega}_{+1}, \\ [\widehat{z}_-, \widehat{\Omega}_{+1}] &= 4i\widehat{\Omega}_0 = 4ie^{-2it}\widehat{\Omega}_{+1}, \\ [\widehat{z}_+, \widehat{\Omega}_0] &= -2i\widehat{\Omega}_{+1} = -2ie^{2it}\widehat{\Omega}_0, \\ [\widehat{z}_-, \widehat{\Omega}_0] &= 2i\widehat{\Omega}_{-1} = 2ie^{-2it}\widehat{\Omega}_0, \\ [\widehat{z}_+, \widehat{\Omega}_{-1}] &= -4i\widehat{\Omega}_0 = -4ie^{2it}\widehat{\Omega}_{-1}, \\ [\widehat{z}_0, \widehat{\Omega}_{-1}] &= -2i\widehat{\Omega}_{-1}. \end{aligned} \quad (10)$$

The degree 0 invariant equation

$$\widehat{\Omega}_0 \Psi(t, x, y) = 0 \Rightarrow i\partial_t \Psi = \left(-\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + 3y\partial_y + i\gamma x \partial_y + \frac{3}{2} \right) \Psi \quad (11)$$

is a Schrödinger equation with t playing the role of the time coordinate. The non-Hermitian operator on the r.h.s. possesses a discrete and positive spectrum. The parameter γ is a coupling constant. Equation (11) can be regarded as a γ -deformation of a decoupled “crypto-Hermitian oscillator” discussed in the following.

3. Connection of the two differential realizations

The two differential realizations (2) and (3) of the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ algebra introduced in Sect. 2 induce Schrödinger equations from, respectively, degree 1 and degree 0 invariant operators.

The two differential realizations are connected via a similarity transformation coupled with a redefinition of the time coordinate.

Let us denote as \widehat{g} an operator entering (3) or (9) and as \overline{g} its corresponding operator entering (2) or (6). For convenience we introduce the operator \widehat{X}_\pm by setting, for \widehat{z}_\pm in (3),

$$\widehat{z}_\pm = e^{\pm 2it}(\partial_t + \widehat{X}_\pm), \quad \widehat{X}_\pm = ix\partial_x + 3iy\partial_y + ix^2 + 2i. \quad (12)$$

The connection is explicitly realized by the similarity transformation

$$\begin{aligned}\widehat{g} &\mapsto \overline{g} = e^S \widehat{g} e^{-S}, & (e^S = e^{S_2} e^{S_1}), \\ S_1 &= t\widehat{X}_+, \\ S_2 &= \frac{1}{2}x^2,\end{aligned}\tag{13}$$

supplemented by the redefinition of the time coordinate

$$t \mapsto \tau = \frac{i}{2}e^{-2it}.\tag{14}$$

The first similarity transformation (induced by S_1) allows us to map

$$\widehat{z}_+ \mapsto \widetilde{z}_+ = e^{S_1} \widehat{z}_+ e^{-S_1} = e^{2it} \partial_t = \partial_\tau,\tag{15}$$

so that

$$\widehat{\Omega}_{+1} \mapsto \widetilde{\Omega}_{+1} = e^{S_1} \widehat{\Omega}_{+1} e^{-S_1} = ie^{2it} \partial_t - \widetilde{H}_{+1},\tag{16}$$

with

$$\widetilde{H}_{+1} = e^{2it} \left(i\widehat{X}_+ + e^{i\widehat{X}_+} \widehat{H}_0 e^{-i\widehat{X}_+} \right).\tag{17}$$

Due to the commutators

$$[\widehat{X}_+, \widehat{H}_0] = 2i\widehat{K}_+, \quad [\widehat{X}_+, \widehat{K}_+] = -2i\widehat{K}_+,\tag{18}$$

where

$$\widehat{K}_+ = \frac{1}{2}(\partial_x + x)^2 - i\gamma x \partial_y,\tag{19}$$

we obtain

$$\widetilde{H}_{+1} = e^{2it} (i\widehat{X}_+ + \widehat{H}_0 + \widehat{K}_+) - \widehat{K}_+.\tag{20}$$

The remarkable identity

$$i\widehat{X}_+ + \widehat{H}_0 + \widehat{K}_+ = 0\tag{21}$$

implies that \widetilde{H}_{+1} does not depend on the time coordinate (either t or τ).

The second similarity transformation (induced by S_2) allows us to express

$$\widetilde{\Omega}_{+1} \mapsto \overline{\Omega}_{+1} = e^{S_2} \widetilde{\Omega}_{+1} e^{-S_2} = i\partial_\tau + \frac{1}{2}\partial_x^2 - i\gamma x \partial_y,\tag{22}$$

in the form that reduces, in the $\gamma \rightarrow 0$ limit, to the standard free Schrödinger equation in $1+1$ dimensions.

One should observe that the similarity transformation preserves the symmetry properties of the equations, mapping first-order invariant operators into first-order invariant operators.

The following commutative diagram is obtained:

$$\begin{array}{ccccc}\text{coupled } (\gamma \neq 0) : & \mathbf{Free}_\gamma^{0,\pm 1}(\tau) & \xleftrightarrow{S} & \mathbf{Osc}_\gamma^{0,\pm 1}(t) & \\ & \downarrow r & & \downarrow r & \\ \text{decoupled } (\gamma = 0) : & \mathbf{Free}^{0,\pm 1}(\tau) & \xleftrightarrow{S} & \mathbf{Osc}^{0,\pm 1}(t) & \end{array}\tag{23}$$

The left- (right-)hand part of the diagram denotes the equations obtained from the differential realizations (2) and (3), respectively. The horizontal arrows indicate the similarity transformation together with the change of the time coordinate, τ and t , respectively.

The three invariant PDEs (at degree 0, ± 1) are mapped into each other.

In the left-hand part, the deformed Schrödinger invariant PDEs correspond to deg 1 and possess a continuous spectrum.

In the right-hand part the deformed Schrödinger invariant PDEs correspond to deg 0. They possess a real, discrete spectrum that, as shown in Sect. 7, coincides with the spectrum of two decoupled harmonic oscillators. We will call this system the $\ell = \frac{3}{2}$ oscillator.

The vertical arrows denote the mapping to the decoupled systems. This mapping can be reached in two ways:

i) the singular similarity transformation

$$g \mapsto R_1 g R_1^{-1}, \quad \text{with} \quad R_1 = e^{\alpha y \partial_y} \quad (24)$$

(such that $\gamma \rightarrow e^{-\alpha} \gamma$ in the $\alpha \rightarrow \infty$ limit¹).

Despite the singularity of the limit, the invariant equations of the upper part of the diagram admits as a nonsingular limit the decoupled equations of the lower part of the diagram. This similarity transformation preserves the symmetry of the equations, mapping first-order invariant operators into first-order invariant operators;

ii) the nonsingular similarity transformation

$$g \mapsto R_2 g R_2^{-1}, \quad R_2 = e^{\left(\frac{3i}{8}\gamma x + \frac{i}{8}\gamma \partial_x - \frac{1}{96}\gamma^2 \partial_y\right) \partial_y}. \quad (25)$$

This nonsingular transformation does not preserve the symmetry of the equation because some of the transformed generators are no longer first-order differential operators.

The four Schrödinger equations associated with the commutative diagram (23), starting from the upper right-hand corner and proceeding clockwise, are: (I) the $\ell = \frac{3}{2}$ oscillator (11), (II) the decoupled (i.e., $\gamma = 0$) $\ell = \frac{3}{2}$ oscillator, (III) the free Schrödinger equation in 1 + 1 dimensions, and, finally, (IV) the deformed free Schrödinger equation (8).

The three inequivalent (with constant, linear, and quadratic potentials; see Refs. [14–17]) Schrödinger equations in 1 + 1 dimensions invariant under the Schrödinger algebra are recovered as restrictions of the $\ell = \frac{3}{2}$ invariant PDEs. Indeed, if we introduce the x, y separation of variables, the equation of the harmonic oscillator and the free Schrödinger equation are recovered by setting $\partial_y \equiv 0$ from, respectively, Eqs. (I) and (IV). The linear Schrödinger equation is recovered from Eq. (IV) after setting $\Psi(\tau, x, y) = \psi(\tau, x)\phi(y)$, with the restriction $\partial_y \phi(y) = k\phi(y)$.

4. The general $\ell = \frac{3}{2}$ oscillator

The $\widehat{\text{cga}}_{\frac{3}{2}}$ conformal Galilei invariance requires the coupling parameter $\gamma \neq 0$. Since a unitary transformation changes its phase, we can assume without loss of generality that γ belongs to the fundamental domain $\gamma \in]0, +\infty[$.

¹ For α real the transformation (24) is not unitary. The choice $\alpha = \ln \gamma$ allows us to set $\gamma = 1$. The free parameter $\gamma \in]0, +\infty[$ labels different unitarily inequivalent theories with testable consequences. See, e.g., the probability decay to the vacuum state presented in (62).

For γ real, the invariant PDEs in the left-hand part of the commutative diagram (23) are Hermitian. This is not the case for the invariant PDEs in the right-hand part of the diagram. Under Hermitian conjugation, the $\ell = \frac{3}{2}$ oscillator equation is transformed into its conjugate

$$\widehat{\Omega}_0^\dagger(\gamma)\Psi(t, x, y) = 0 \equiv \left(i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 + 3y\partial_y - i\gamma x\partial_y + \frac{3}{2}\right)\Psi(t, x, y). \quad (26)$$

All operators

$$\Theta = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + \omega y\partial_y - i\gamma x\partial_y + C, \quad (27)$$

for any arbitrary constant C and any $\gamma \neq 0$, induce a Schrödinger equation with $\ell = \frac{3}{2}$ conformal Galilei symmetry, if ω is restricted to the values

$$\omega = \pm\frac{1}{3}, \pm 3. \quad (28)$$

The $\omega \leftrightarrow -\omega$ change of sign is explained by the Hermitian conjugation. Understanding the $\omega \leftrightarrow \frac{1}{\omega}$ transformation is subtler. One should note at first that in the $\gamma = 0$ decoupled case the role of the space coordinates x, y can be exchanged by performing the canonical transformation

$$y \leftrightarrow \frac{1}{\sqrt{2}}(x - \partial_x), \quad \partial_y \leftrightarrow \frac{1}{\sqrt{2}}(x + \partial_x).$$

Next, the coupling term is introduced in terms of the nonsingular similarity transformation given by the inverse of Eq. (25). As it turns out, this procedure guarantees the conformal Galilei invariance of the resulting PDE.

An explicit check of the symmetries of this class of PDEs proves that, in order to have the on-shell invariant equations $[\widehat{z}_\pm, \widehat{\Omega}_0] = f_\pm \cdot \widehat{\Omega}_0$, with f_\pm arbitrary functions of the coordinates and symmetry generators of the form $\widehat{z}_\pm = e^{\pm i\lambda t}(\partial_t + X_\pm)$ (X_\pm time-independent operators and $\lambda \neq 0$), the following necessary and sufficient condition has to be satisfied: the two equations

$$\begin{aligned} \lambda \left(\omega^2 + 1 - \frac{5}{2}\lambda^2 \right) &= 0, \\ -3\lambda^2 + 3\lambda^4 + 2\lambda\omega + 4\lambda^3\omega - \lambda^2\omega^2 - 2\lambda\omega^3 &= 0, \end{aligned} \quad (29)$$

must be simultaneously solved. The only nonvanishing solutions for λ are encountered at $\omega = \pm 3$ and $\omega = \pm\frac{1}{3}$. Therefore, the $\omega = \pm\frac{1}{3}, \pm 3$ critical values are special points of enhanced symmetry.

5. Symmetry of the decoupled $\ell = \frac{3}{2}$ oscillator

By applying the same considerations as in Sect. 4, it is sufficient to analyze the symmetry of the decoupled ($\gamma = 0$) $\ell = \frac{3}{2}$ operator

$$\Omega = i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 + \omega y\partial_y \quad (30)$$

in the range $\omega \in [1, \infty[$.

For a generic ω the following invariant operators can be encountered at degree $0, \pm\frac{1}{2}, \pm\frac{\omega}{2}, \pm 1$:

$$\begin{aligned}
 z_{\pm} &= e^{\pm 2it}(\partial_t \pm ix\partial_x + i\omega y\partial_y + ix^2 \pm \frac{i}{2}), \\
 z_0 &= \partial_t + i\omega y\partial_y, \\
 d &= -\frac{i}{2}\partial_t, \\
 c &= 1, \\
 w_{\omega} &= e^{i\omega t}\partial_y, \\
 w_1 &= e^{it}(\partial_x + x), \\
 w_{-1} &= e^{-it}(\partial_x - x), \\
 w_{-\omega} &= e^{-i\omega t}y.
 \end{aligned} \tag{31}$$

d is the degree operator. Explicitly, the degree is

$$\pm 1 : z_{\pm}; \quad 0 : z_0, d, c; \quad \frac{\pm\omega}{2} : w_{\pm\omega}; \quad \frac{\pm 1}{2} : w_{\pm 1}. \tag{32}$$

This 9-generator symmetry algebra closes the $\mathfrak{u}(1) \oplus (\mathfrak{sch}(1) \oplus \mathfrak{h}(1))$ algebra, with nonvanishing commutation relations given by

$$\begin{aligned}
 [d, z_{\pm}] &= \pm z_{\pm}, \\
 [d, w_k] &= \frac{k}{2}w_k, \\
 [z_0, z_{\pm}] &= \pm 2iz_{\pm}, \\
 [z_+, z_-] &= -4iz_0, \\
 [z_0, w_{\pm 1}] &= \pm iw_{\pm 1}, \\
 [z_{\pm}, w_{\mp 1}] &= \mp 2iw_{\pm 1}, \\
 [w_1, w_{-1}] &= -2c, \\
 [w_{\omega}, w_{-\omega}] &= c.
 \end{aligned} \tag{33}$$

d is the generator of the $\mathfrak{u}(1)$ subalgebra, while $z_0, z_{\pm}, w_{\pm 1}, c$ generate the Schrödinger algebra $\mathfrak{sch}(1)$ and $w_{\pm\omega}, c$ generate the Heisenberg algebra $\mathfrak{h}(1)$.

The critical values $\omega = 1$ and $\omega = 3$ are points of enhanced symmetry for the decoupled system.

5.1. The enhanced symmetry for the decoupled $\omega = 1$ system

At the critical value $\omega = 1$ three extra generators are found at degrees 0 and -1 :

$$\begin{aligned}
 q_1 &= y(\partial_x + x), \\
 q_2 &= e^{-2it}y^2, \\
 q_3 &= e^{-2it}y(\partial_x - x).
 \end{aligned} \tag{34}$$

They have to be added to the previous set of (generic) symmetry generators

$$\begin{aligned}
 z_{\pm} &= e^{\pm 2it} (\partial_t \pm ix\partial_x + iy\partial_y + ix^2 \pm \frac{i}{2}), \\
 z_0 &= \partial_t + iy\partial_y, \\
 d &= -\frac{i}{2}\partial_t, \\
 c &= 1, \\
 w_{1b} &= e^{it}\partial_y, \\
 w_{1a} &= e^{it}(\partial_x + x), \\
 w_{-1a} &= e^{-it}(\partial_x - x), \\
 w_{-1b} &= e^{-it}y,
 \end{aligned} \tag{35}$$

where we have denoted, for $\omega = 1$, $w_{\pm\omega}$ entering (31) as “ $w_{\pm 1b}$ ”.

The extra nonvanishing commutation relations involving the q_i generators are

$$\begin{aligned}
 [z_0, q_1] &= iq_1, \\
 [d, q_2] &= -q_2, \\
 [d, q_3] &= -q_3, \\
 [z_+, q_3] &= -2iq_3, \\
 [z_-, q_1] &= 2iq_3, \\
 [w_{1b}, q_1] &= w_{1a}, \\
 [w_{-1a}, q_1] &= 2w_{-1b}, \\
 [w_{1b}, q_2] &= 2w_{-1b}, \\
 [w_{1b}, q_3] &= w_{-1a}, \\
 [w_{1a}, q_3] &= -2w_{-1b}, \\
 [q_1, q_3] &= -2q_2.
 \end{aligned} \tag{36}$$

The symmetry algebra closes as a nonsemisimple, 12-generator Lie algebra.

5.2. The enhanced symmetry for the decoupled $\omega = 3$ system

At the $\omega = 3$ critical value the three extra generators $r_{-j}, j = 1, 2, 3$, of degree $-j$ are encountered. We have, explicitly,

$$\begin{aligned}
 r_{-1} &= e^{-2it}y(\partial_x + x), \\
 r_{-2} &= e^{-4it}y(\partial_x - x), \\
 r_{-3} &= e^{-6it}y^2.
 \end{aligned} \tag{37}$$

At $\omega = 3$ the symmetry algebra is a 12-generator algebra that differs from the 12-generator symmetry algebra of the $\omega = 1$ decoupled system.

The extra nonvanishing commutation relations involving the r_{-j} generators are given by

$$\begin{aligned}
 [d, r_{-j}] &= -jr_{-j}, \\
 [z_0, r_{-1}] &= ir_{-1}, \\
 [z_0, r_{-2}] &= -ir_{-2}, \\
 [z_-, r_{-1}] &= 2ir_{-2}, \\
 [z_+, r_{-2}] &= -2ir_{-1}, \\
 [w_{+3}, r_{-1}] &= w_{+1}, \\
 [w_{-1}, r_{-1}] &= 2w_{-3}, \\
 [w_{+3}, r_{-2}] &= w_{-1}, \\
 [w_{+1}, r_{-2}] &= -2w_{-3}, \\
 [w_{+3}, r_{-3}] &= 2w_{-3}, \\
 [r_{-1}, r_{-2}] &= -2r_{-3}.
 \end{aligned} \tag{38}$$

6. The contraction algebra

A contraction algebra is recovered from (3) by taking the $\gamma \rightarrow 0$ limit and by suitably rescaling the generators. The contraction requires the rescaling $\hat{g} \mapsto \tilde{g} = \gamma^s \hat{g}$ (\hat{g} is any generator entering (3)), with the power s given by

$$\begin{aligned}
 s = 0 &: \hat{z}_0, \hat{z}_+, \hat{w}_3, \hat{c}, \\
 s = 1 &: \hat{z}_-, \hat{w}_1, \hat{w}_{-1}, \\
 s = 2 &: \hat{w}_{-3}.
 \end{aligned} \tag{39}$$

The contracted 8-generator algebra expressed by $\tilde{z}_\pm, \tilde{z}_0, \tilde{c}, \tilde{w}_k$ ($k = \pm 1, \pm 3$) is a subalgebra of the full 12-generator symmetry algebra of the $\omega = 3$ decoupled system with generators given by (31) and (37). The identification goes as follows:

$$\begin{aligned}
 \tilde{z}_+ &= e^{\tilde{S}} z_+ e^{-\tilde{S}} = e^{2it} (\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\
 \tilde{z}_0 &= e^{\tilde{S}} (2id - \frac{3i}{2}c) e^{-\tilde{S}} = \partial_t, \\
 \tilde{z}_- &= e^{\tilde{S}} (12ir_{-1}) e^{-\tilde{S}} = 12ie^{-2it} y (\partial_x + x), \\
 \tilde{w}_{+3} &= e^{\tilde{S}} w_{+3} e^{-\tilde{S}} = e^{3it} \partial_y, \\
 \tilde{w}_{+1} &= e^{\tilde{S}} (-2iw_{+1}) e^{-\tilde{S}} = 2ie^{it} (\partial_x + x), \\
 \tilde{w}_{-1} &= e^{\tilde{S}} (-4iw_{-1}) e^{-\tilde{S}} = 4ie^{-it} (\partial_x - x), \\
 \tilde{w}_{-3} &= e^{\tilde{S}} (48w_{-3}) e^{-\tilde{S}} = -48e^{-3it} y, \\
 \tilde{c} &= e^{\tilde{S}} c e^{-\tilde{S}} = 1,
 \end{aligned} \tag{40}$$

with the similarity transformation given by $\tilde{S} = -\frac{3}{2}it$.

The contraction algebra corresponds to the 2D Euclidean algebra acting on two sets of creation/annihilation operators. We have the $\mathfrak{e}(2) \oplus \mathfrak{h}(2)$ algebra, with nonvanishing commutators given by

$$\begin{aligned} [\tilde{z}_0, \tilde{z}_{\pm}] &= \pm 2i\tilde{z}_{\pm}, \\ [\tilde{z}_0, \tilde{w}_k] &= k\tilde{w}_k, \\ [\tilde{z}_+, \tilde{w}_{-1}] &= -4i\tilde{w}_{+1}, \\ [\tilde{z}_-, \tilde{w}_{+3}] &= 6i\tilde{w}_{+1}, \\ [\tilde{z}_-, \tilde{w}_{-1}] &= 2i\tilde{w}_{-3}, \\ [\tilde{w}_{|k|}, \tilde{w}_{-|k|}] &= (3 - 2|k|)16\tilde{c}. \end{aligned} \quad (41)$$

7. Non-Hermitian deformed oscillators with real eigenvalues

The non-Hermitian operator derived from the degree 0 invariant equation (11) is

$$H_0(\gamma) = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + 3y\partial_y + i\gamma x\partial_y + \frac{3}{2}. \quad (42)$$

It may be written (see (3)) in the form

$$H_0(\gamma) = \frac{\gamma^2}{16}(\hat{w}_{-1}\hat{w}_{+1} - \hat{w}_{-3}\hat{w}_{+3}) + 2. \quad (43)$$

A discrete spectrum can be calculated algebraically from a lowest-weight representation. The ground state $\psi_{0,0}$ is defined by

$$\hat{w}_{+1}\psi_{0,0} = \hat{w}_{+3}\psi_{0,0} = 0. \quad (44)$$

The solution of the above equations gives the explicit expression of the ground state (up to normalization):

$$\psi_{0,0} = e^{-x^2/2}. \quad (45)$$

The excited states are given by

$$\psi_{n,m} = \hat{w}_{-1}^n \hat{w}_{-3}^m \psi_{0,0}. \quad (46)$$

The corresponding eigenvalue $E_{n,m}$ is computed by the commutation relations

$$[H_0(\gamma), \hat{w}_{-1}] = \hat{w}_{-1}, \quad [H_0(\gamma), \hat{w}_{-3}] = 3\hat{w}_{-3}. \quad (47)$$

One finds that the operator $H_0(\gamma)$ has real discrete eigenvalues that are identical to the decoupled harmonic oscillator eigenvalues:

$$E_{n,m} = n + 3m + 2. \quad (48)$$

The explicit forms of the eigenfunctions are readily obtained from (3). We list some of them:

$$\begin{aligned}\psi_{1,0} &= \widehat{w}_{-1} \psi_{0,0} = -\frac{8i}{\gamma} x e^{-x^2/2-it}, \\ \psi_{2,0} &= \widehat{w}_{-1}^2 \psi_{0,0} = 2 \left(\frac{4i}{\gamma} \right)^2 (2x^2 - 1) e^{-x^2/2-2it}, \\ \psi_{0,1} &= \widehat{w}_{-3} \psi_{0,0} = -\frac{24}{\gamma^2} (i\gamma x + 2\gamma) e^{-x^2/2-3it}, \\ \psi_{1,1} &= \widehat{w}_{-1} \widehat{w}_{-3} \psi_{0,0} = \frac{48}{\gamma^2} \left(1 - 4x^2 + \frac{4i}{\gamma} xy \right) e^{-x^2/2-4it}.\end{aligned}\quad (49)$$

In general, the excited-state wavefunctions have the form of

$$\psi_{n,m} = \mathcal{P}_{n,m}(x,y) e^{-x^2/2} e^{-(n+3m)it}, \quad (50)$$

where $\mathcal{P}_{n,m}(x,y)$ is a degree $n+m$ polynomial in x,y . The phase factor $e^{-(n+3m)it}$ is not essential to the eigenvalue problem.

Following Ref. [4], we can call any non-Hermitian operator (such as (42)) with a real spectrum “crypto-Hermitian”. The operator (42), being invariant under the transformations $x \mapsto -x, y \mapsto y, i \mapsto -i$, is PT-symmetric. Therefore, it belongs to the class of non-Hermitian PT-symmetric operators (see Refs. [18–20]) with real spectra. It was pointed out in Refs. [21–23] that PT-symmetric operators are often pseudo-Hermitian. It turns out that this is not the case for the operator (42), since the similarity transformation (25) is not realized by a Hermitian operator. For this reason the stronger notion of quasi-hermiticity, as well as the constructions based upon it (see Refs. [24–26]), is not applicable in our context.

In Sect. 8 we discuss the connection of the operator (42) with a special class of Pais–Uhlenbeck oscillators, recovered at given algebraic frequencies. It is worth mentioning that PT-symmetry in the context of Pais–Uhlenbeck oscillators (at arbitrary, unequal frequencies) was investigated in Refs. [27–29].

In order to pave the way to discuss the connection between conformal Galilei algebras and Pais–Uhlenbeck oscillators at special frequencies, we need to introduce the operator $K(\overline{\gamma})$, obtained from (42) via a noncanonical transformation that preserves the canonical commutation relations, but does not preserve the Hermitian conjugation property. Up to a vacuum energy constant, we can set

$$K(\overline{\gamma}) = a^\dagger a + 3b^\dagger b + \frac{1}{2} + \overline{\gamma}(a + a^\dagger)b, \quad (51)$$

through the positions

$$a = \frac{1}{\sqrt{2}}(x + \partial_x), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x), \quad b = \frac{1}{\sqrt{2}}(z + \partial_z), \quad b^\dagger = \frac{1}{\sqrt{2}}(z - \partial_z), \quad \overline{\gamma} = \frac{i}{\sqrt{2}}\gamma. \quad (52)$$

The two independent creation/annihilation operators $(a, a^\dagger$ and $b, b^\dagger)$ have nonvanishing commutators: $[a, a^\dagger] = [b, b^\dagger] = 1$.

The similarity transformation (25) maps $K(\overline{\gamma})$ into the decoupled Hermitian operator $K(0)$ through

$$K(\overline{\gamma}) = RK(0)R^{-1}, \quad R = e^{-(\frac{\overline{\gamma}}{2}a^\dagger b + \frac{\overline{\gamma}}{4}ab + \frac{\overline{\gamma}^2}{48}b^2)}. \quad (53)$$

Since, as mentioned before, R is not Hermitian, $K(\bar{\gamma})$, for $\bar{\gamma} \neq 0$, is not a pseudo-Hermitian operator.

The operator $K(\bar{\gamma})$ acts on the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$, the space of square-integrable functions defined on the coordinates x, z .

One can read from the commutators

$$[K(\bar{\gamma}), A_\lambda] = \lambda A_\lambda \quad (54)$$

which excited modes are created.

For any $\bar{\gamma} \neq 0$, the solutions of Eq. (54) are obtained for $\lambda = \pm 3, \pm \frac{1}{3}$. The corresponding modes are

$$\begin{aligned} A_{-3} &= b, \\ A_{-1} &= a + \frac{1}{2}\bar{\gamma}b, \\ A_{+1} &= a^\dagger - \frac{1}{4}\bar{\gamma}b, \\ A_{+3} &= b^\dagger - \frac{1}{2}\bar{\gamma}a^\dagger - \frac{1}{4}\bar{\gamma}a + \frac{1}{24}\bar{\gamma}^2b. \end{aligned} \quad (55)$$

In this basis the nonvanishing commutators are

$$[A_{-i}, A_j] = \delta_{ij}, \quad (i, j = 1, 3). \quad (56)$$

The non-Hermitian operator $K(\bar{\gamma})$ commutes with the “non-Hermitian analog of the number operator”, $N(\bar{\gamma})$. In terms of the A_k modes, the operators are given by

$$\begin{aligned} K(\bar{\gamma}) &= 3A_3A_{-3} + A_1A_{-1} + \frac{1}{2}, \\ N(\bar{\gamma}) &= A_3A_{-3} + A_1A_{-1}, \\ [K(\bar{\gamma}), N(\bar{\gamma})] &= 0. \end{aligned} \quad (57)$$

The Fock vacuum $|\text{vac}\rangle$ satisfies

$$a|\text{vac}\rangle = b|\text{vac}\rangle = 0, A_{-1}|\text{vac}\rangle = A_{-3}|\text{vac}\rangle = 0. \quad (58)$$

The Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$ can be spanned by both sets of (unnormalized) states:

$$\begin{aligned} |n, m\rangle &= (a^\dagger)^n (b^\dagger)^m |\text{vac}\rangle, \\ |\bar{n}, \bar{m}\rangle &= A_1^n A_3^m |\text{vac}\rangle. \end{aligned} \quad (59)$$

The invertibility of the Bogoliubov-type transformation (55) implies that the $|\bar{n}, \bar{m}\rangle$ eigenstates form a complete (albeit nonorthogonal) set that can be expressed in terms of the decoupled eigenvectors $|n, m\rangle$ (the converse is also true).

We can therefore write

$$|\text{vac}\rangle = |0, 0\rangle = |\bar{0}, \bar{0}\rangle. \quad (60)$$

The spectrum of $K(\bar{\gamma}), N(\bar{\gamma})$ coincides with the spectrum of the Hamiltonian and number operator of a decoupled harmonic oscillator. $|\bar{n}, \bar{m}\rangle$ is an eigenvector for $K(\bar{\gamma}), N(\bar{\gamma})$ with respective eigenvalues

$n + 3m + \frac{1}{2}$ and $n + m$. In increasing order of $K(\bar{\gamma})$ eigenvalues, the first (unnormalized) common eigenvectors of $K(\bar{\gamma})$, $N(\bar{\gamma})$ are

$$\begin{aligned}
 \left(\frac{1}{2}, 0\right): \quad & |\bar{0}, \bar{0}\rangle = |0, 0\rangle = |\text{vac}\rangle, \\
 \left(\frac{3}{2}, 1\right): \quad & |\bar{1}, \bar{0}\rangle = |1, 0\rangle, \\
 \left(\frac{5}{2}, 2\right): \quad & |\bar{2}, \bar{0}\rangle = |2, 0\rangle, \\
 \left(\frac{7}{2}, 1\right): \quad & |\bar{0}, \bar{1}\rangle = |0, 1\rangle - \frac{1}{2}\bar{\gamma}|1, 0\rangle, \\
 \left(\frac{7}{2}, 3\right): \quad & |\bar{3}, \bar{0}\rangle = |3, 0\rangle, \\
 \left(\frac{9}{2}, 2\right): \quad & |\bar{1}, \bar{1}\rangle = |1, 1\rangle - \frac{1}{2}\bar{\gamma}|2, 0\rangle - \frac{1}{4}\bar{\gamma}|0, 0\rangle, \\
 \left(\frac{9}{2}, 4\right): \quad & |\bar{4}, \bar{0}\rangle = |4, 0\rangle.
 \end{aligned} \tag{61}$$

Since the operators are non-Hermitian, their eigenvectors are nonorthogonal. This implies measurable physical consequences. Let us suppose that we are able to prepare the system in a given common eigenvector of $K(\bar{\gamma})$, $N(\bar{\gamma})$, let us say the state $|\bar{1}, \bar{1}\rangle$. Following the standard rule of quantum mechanics, we can compute the probability of this state collapsing, after a measurement operation, to the vacuum state. A simple computation shows that the probability is $p = |{}_N\langle \bar{1}, \bar{1} | \bar{0}, \bar{0} \rangle_N|^2$ ($|\bar{0}, \bar{0}\rangle_N$, $|\bar{1}, \bar{1}\rangle_N$ are the normalized states). In the case at hand we have

$$p = \frac{|\bar{\gamma}|^2}{16 + 9|\bar{\gamma}|^2}. \tag{62}$$

This probability is restricted in the range $0 \leq p < \frac{1}{9} < 1$. The deformation coupling parameter $\bar{\gamma}$, via its squared modulus, has testable consequences.

8. Comment on Pais–Uhlenbeck oscillators and the $\ell \geq \frac{5}{2}$ cases

The same spectrum of eigenvalues is obtained for

- i) the coupled ($\gamma \neq 0$) crypto-Hermitian operator (42),
- ii) the decoupled ($\gamma = 0$) crypto-Hermitian operator, and (up to a vacuum energy shift)
- iii) the Hermitian Hamiltonian (given by (51) for $\bar{\gamma} = 0$) of two decoupled oscillators.

The construction of Sect. 7 can be repeated by starting with the Hermitian conjugate of the operator (42). In this case the spectrum of the three resulting operators is unbounded. It is given, up to the vacuum energy shift, by $E_{n,m} = n - 3m$. The Hilbert space of the decoupled harmonic oscillators with energy modes $1, -3$ continues to be $\mathcal{L}^2(\mathbb{R}^2)$, obtained by applying the creation operators a^\dagger, b^\dagger to the Fock state $|0, 0\rangle$ ($a|0, 0\rangle = b|0, 0\rangle = 0$). Due to the unboundedness of the spectrum, $|0, 0\rangle$ can no longer be interpreted as the vacuum state.

The system with an unbounded spectrum is related to the Pais–Uhlenbeck oscillators. We recall [4,5] that the Pais–Uhlenbeck model is a higher-derivative system. It admits, via the Ostrogradskiĭ

construction [7] (see Ref. [30] for a review), a Hamiltonian formulation. The resulting Ostrogradskiĭ Hamiltonian is canonically equivalent to a set of decoupled harmonic oscillators with alternating (positive and negative) energy modes. The n -oscillator Pais–Uhlenbeck system is canonically expressed as

$$H_n = \sum_{i=1}^n (-1)^{i+1} \omega_i a_i^\dagger a_i, \quad (63)$$

where $\omega_i \in \mathbb{R}$ and the constraint $\omega_i < \omega_{i+1}$ is satisfied.

The harmonic oscillator with energy modes $1, -3$ is a special case of the 2-oscillator Pais–Uhlenbeck model. In a series of papers [8–12], the Pais–Uhlenbeck oscillators with energy modes given (up to a normalization factor) by the arithmetic progression $\omega_i = 2i - 1$ were linked to the conformal Galilei algebras $\widehat{\mathfrak{ga}}_\ell$ (with $\ell = n - \frac{1}{2}$).

The present analysis proves that this association is rather subtle. We would like to stress that in this paper we consider (as it is standard in PDE theory) the symmetry generators to be at most *first-order* differential operators. In this respect the Pais–Uhlenbeck PDE given by the decoupled harmonic oscillators does not possess any enhanced symmetry (not even at the special $1, -3$ energy modes). The PDE, invariant under the conformal Galilei algebra, is obtained for the coupled operator only for $\gamma \neq 0$. The decoupled operator ($\gamma = 0$) possesses the 12-generator symmetry algebra (introduced in Sect. 6) that does not contain the conformal Galilei algebra as a subalgebra.

Even so, these results are not in contradiction with the findings in previous works on Pais–Uhlenbeck oscillators. For example, in Ref. [9] it was shown that on the Hamiltonian level the $\omega_1 = 1, \omega_2 = -3$ Pais–Uhlenbeck oscillators possess the (centrally extended) $\frac{3}{2}$ -conformal Galilei symmetry in terms of Poisson brackets among conserved charges. It can be shown, on the other hand, that at the quantum level, some of the Noether charges entering Eq. (39) in Ref. [9] are *second-order* differential operators.

For general half-integer ℓ , the invariant PDEs that possess the conformal Galilei algebra $\widehat{\mathfrak{ga}}_\ell$ (for a definition, see Ref. [3]) as a symmetry algebra depend on $\ell + \frac{3}{2}$ coordinates. The invariant PDEs are deformations of decoupled equations, depending on $\ell - \frac{1}{2}$ deformation parameters $\gamma_j \neq 0$ ($j = 1, \dots, \ell - \frac{1}{2}$). The decoupled systems are recovered in the limit, for any $j, \gamma_j \rightarrow 0$.

The invariant PDEs with a continuous spectrum are

$$i\partial_\tau \Psi(\tau, \vec{x}) = \left(-\frac{1}{2} \partial_{x_1}^2 + i \sum_{j=1}^{\ell-\frac{1}{2}} \gamma_j x_j \partial_{x_{j+1}} \right) \Psi(\tau, \vec{x}). \quad (64)$$

The invariant PDEs with a discrete spectrum are

$$i\partial_t \Psi(t, \vec{x}) = \left(-\frac{1}{2} \partial_{x_1}^2 + \frac{1}{2} x_1^2 + \sum_{i=2}^{\ell+\frac{1}{2}} \omega_i x_i \partial_{x_i} + i \sum_{j=1}^{\ell-\frac{1}{2}} \gamma_j x_j \partial_{x_{j+1}} \right) \Psi(t, \vec{x}). \quad (65)$$

The energy modes ω_i are normalized so that $|\omega_i| = 2i - 1$. The solution $\omega_i = \epsilon_i |\omega_i|$ with all positive signs ($\forall i, \epsilon_i = +1$) corresponds to the bounded discrete spectrum discussed in Ref. [2]. By taking the Hermitian conjugation, the solution with flipped signs, $\epsilon_i = -1$ for all i , also leads to a $\widehat{\mathfrak{ga}}_\ell$ -invariant PDE.

An explicit computation of the on-shell condition for $\ell = \frac{5}{2}$ (similar to the one presented in Sect. 4) proves that the $\widehat{\mathfrak{ga}}_{\frac{5}{2}}$ invariance is guaranteed by both choices of signs, $\epsilon_2 = \pm 1$ and $\epsilon_3 = \pm 1$. As

explained above, the alternating choice ($\epsilon_2 = -1, \epsilon_3 = +1$) is related to a special case of three Pais–Uhlenbeck oscillators (with 1, -3 , 5 energy modes).

An open problem is finding a general proof, valid for all half-integer ℓ , that every choice of $\epsilon_i = \pm 1$ signs leads to the $\widehat{\mathfrak{cga}}_\ell$ symmetry algebra of the PDE equation (65).

9. Conclusions

In this section we summarize the new results obtained in this paper.

We constructed the most general class of second-order PDEs, invariant under the $d = 1$ centrally extended conformal Galilei algebras $\widehat{\mathfrak{cga}}_\ell$ with half-integer ℓ , proving that they are Schrödinger equations that are deformations of decoupled equations. For $\ell = \frac{3}{2}$ the unique deformation parameter is $\gamma \neq 0$ (the decoupled systems being recovered in the $\gamma \rightarrow 0$ limit).

At $\ell = \frac{3}{2}$ the invariant PDEs with discrete spectra, besides γ , depend on two frequencies ω_1, ω_2 (entering the equation in nonsymmetric form). The invariance under $\widehat{\mathfrak{cga}}_{\ell=\frac{3}{2}}$ is only recovered at special critical values of the ratio $r = \frac{\omega_2}{\omega_1}$ given by $r = \pm\frac{1}{3}, \pm 3$. The $r = 3$ value reproduces the bounded spectrum presented in Ref. [2], while the negative values $r < 0$ produce an unbounded spectrum that coincides with the spectrum of two Pais–Uhlenbeck oscillators at the given ratio r .

We further investigated the symmetry algebra of the $\gamma = 0$ decoupled systems for a generic value of $r = \frac{\omega_2}{\omega_1}$, obtaining the following results. Enhanced symmetries are encountered at the critical values $r = \pm\frac{1}{3}, \pm 1, \pm 3$. Two inequivalent 12-generator symmetry algebras are recovered at $r = \pm\frac{1}{3}, \pm 3$ and $r = \pm 1$, respectively.

The $\widehat{\mathfrak{cga}}_{\ell=\frac{3}{2}}$ conformal Galilei algebra is not a subalgebra of the 12-generator, $\gamma = 0$ and $r = \pm\frac{1}{3}, \pm 3$, decoupled symmetry. From $\widehat{\mathfrak{cga}}_{\ell=\frac{3}{2}}$, in the $\gamma \rightarrow 0$ limit, a contraction algebra is recovered. The contraction algebra (see Sect. 6) is an 8-generator subalgebra of the full $r = \pm\frac{1}{3}, \pm 3$ symmetry algebra of the $\gamma = 0$ decoupled system.

As a corollary of this analysis we showed that the contraction algebra, rather than the conformal Galilei algebra itself, is an enhanced symmetry of the $r = -3$ decoupled Pais–Uhlenbeck oscillators.

Concerning the $\ell = \frac{3}{2}$ PDE with a discrete spectrum, we showed that, besides the $r = 3$ real bounded spectrum, a real unbounded spectrum is obtained at $r = -\frac{1}{3}, -3$. For all admissible values r the induced operators on the r.h.s. are not Hermitian. They are, nevertheless, PT-symmetric [18]. They are not, on the other hand, pseudo-Hermitian [21], since the nonsingular similarity transformations (see Eqs. (25) and (53)), mapping the coupled into the decoupled system, are not realized by a Hermitian operator. The non-Hermitian operator (51) acts on the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$. Its real discrete spectrum coincides with the spectrum of two decoupled harmonic oscillators. The Bogoliubov transformations relating the coupled and decoupled systems are explicitly given.

For generic half-integer ℓ , the number of nonvanishing deformation parameters γ_j is $\ell - \frac{1}{2}$. For $\ell = \frac{5}{2}$ the arithmetic progression of ω_i frequencies entering Eq. (65) and producing the $\widehat{\mathfrak{cga}}_{\ell=\frac{5}{2}}$ symmetry algebra is given by $\pm 1, \pm 3, \pm 5$. The 1, 3, 5 sequence corresponds to the bounded spectrum, while the 1, -3 , 5 sequence corresponds to a special case of three Pais–Uhlenbeck oscillators.

The extension of this construction to the $\ell = \frac{1}{2} + \mathbb{N}_0$ centrally extended conformal Galilei algebras with $d > 1$ (see Ref. [3]) is immediate. The invariant PDEs with discrete spectrum correspond to non-Hermitian operators whose spectrum is real and given by d copies of the energy modes created in the $d = 1$ case.

In the class of systems investigated here, the oscillators all turn out to have different frequencies. Equal frequency oscillators are obtained from a different class of second-order PDEs invariant under the $d = 2$ centrally extended conformal Galilei algebras with integer ℓ ; see Ref. [31].

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