$$\mathscr{Z} = |[1,1] + [5,5] + [7,7] + [10,10] + [1,22] + [22,1] + [2,11] + [11,2] + [5,14] + [14,5] + [7,10] + [10,7]|^2$$
$$+ |[1,7] + [7,1] + [1,16] + [16,1] + [5,8] + [8,5] + [5,11] + [11,5] + [7,16] + [16,7] + [8,11] + [11,8]|^2$$

with [a,b] = (a-1,b-1)

$$\mathscr{F} = \lambda_{(0,0)} \oplus \lambda_{(4,4)} \oplus \lambda_{(6,6)} \oplus \lambda_{(9,9)} \oplus \lambda_{(0,21)} \oplus \lambda_{(21,0)}$$
$$\oplus \; \lambda_{(1,10)} \oplus \lambda_{(10,1)} \oplus \lambda_{(4,13)} \oplus \lambda_{(13,4)} \oplus \lambda_{(6,9)} \oplus \lambda_{(9,6)}$$

$\mathscr{E}_{21} \hookleftarrow \mathscr{A}_{21}$
$0 \; \hookleftarrow \; (0,0),(4,4),(10,1),(1,10),(6,6),(9,6),$ $(6,9),(13,4),(4,13),(10,10),(21,0),(0,21)$
$21 \; \hookleftarrow \; (6,0),(0,6),(7,4),(4,7),(10,4),(4,10),$ $(15,0),(0,15),(10,7),(7,10),(15,6),(6,15)$

Figure 5: The $\mathscr{E}_{21}$ graph and $\mathscr{E}_{21} \hookleftarrow \mathscr{A}_{21}$ induction rules (for vertices $\in J$).
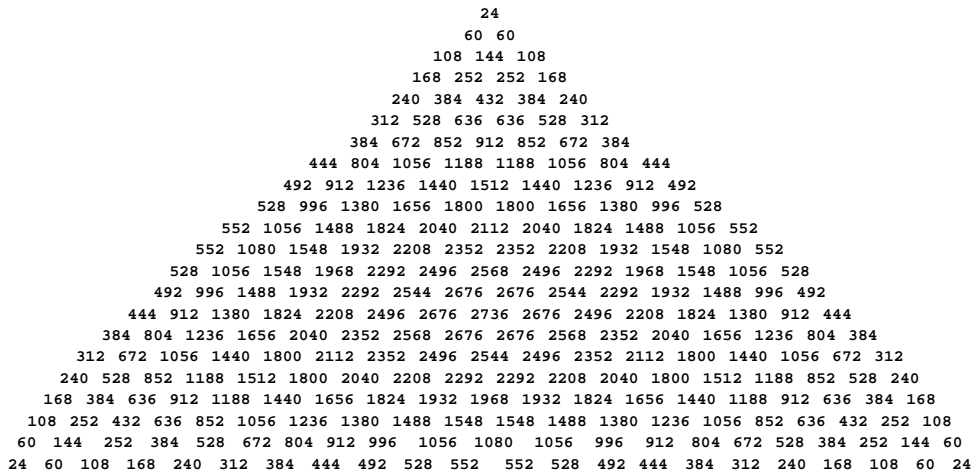
```
                                    24
                                  60  60
                               108  144  108
                            168  252  252  168
                         240  384  432  384  240
                      312  528  636  636  528  312
                   384  672  852  912  852  672  384
                444  804  1056  1188  1188  1056  804  444
             492  912  1236  1440  1512  1440  1236  912  492
          528  996  1380  1656  1800  1800  1656  1380  996  528
       552  1056  1488  1824  2040  2112  2040  1824  1488  1056  552
     552  1080  1548  1932  2208  2352  2352  2208  1932  1548  1080  552
   528  1056  1548  1968  2292  2496  2568  2496  2292  1968  1548  1056  528
 492  996  1488  1932  2292  2544  2676  2676  2544  2292  1932  1488  996  492
444  912  1380  1824  2208  2496  2676  2736  2676  2496  2208  1824  1380  912  444
 384  804  1236  1656  2040  2352  2568  2676  2676  2568  2352  2040  1656  1236  804  384
 312  672  1056  1440  1800  2112  2352  2496  2544  2496  2352  2112  1800  1440  1056  672  312
240  528  852  1188  1512  1800  2040  2208  2292  2292  2208  2040  1800  1512  1188  852  528  240
168  384  636  912  1188  1440  1656  1824  1932  1968  1932  1824  1656  1440  1188  912  636  384  168
108  252  432  636  852  1056  1236  1380  1488  1548  1548  1488  1380  1236  1056  852  636  432  252  108
60  144  252  384  528  672  804  912  996  1056  1080  1056  996  912  804  672  528  384  252  144  60
24  60  108  168  240  312  384  444  492  528  552  552  528  492  444  384  312  240  168  108  60  24
```

Figure 6: Dimension of the blocks labelled by vertices of the $\mathscr{A}_{21}$ graph for $\mathscr{E}_{21}$.

q-dim	$\mathscr{E}_9$	$\hookleftarrow$	$\mathscr{A}_9$
$[1]=1$	0_0	$\hookleftarrow$	$(0,0),(4,1),(1,4),(4,4),(9,0),(0,9)$
$[1]$	1_0	$\hookleftarrow$	$(2,2),(5,2),(2,5)$
$[1]$	2_0	$\hookleftarrow$	$(2,2),(5,2),(2,5)$
$3+2\sqrt{3}$	3_0	$\hookleftarrow$	$(1,1),(3,0),(0,3),(6,0),(0,6),(7,1),(1,7),(6,3),(3,6),$
			$(2,2)_2,(4,1)_2,(1,4)_2,(5,2)_2,(2,5)_2,(4,4)_2,(3,3)_3$
$[3]=1+\sqrt{3}$	0_1	$\hookleftarrow$	$(1,0),(4,0),(1,3),(3,2),(0,5),(5,1),(2,4),(4,3),(3,5),$
			$(0,8),(8,1),(5,4)$
$[3]$	1_1	$\hookleftarrow$	$(2,1),(1,3),(3,2),(5,1),(2,4),(4,3),(1,6),(6,2),(3,5)$
$[3]$	2_1	$\hookleftarrow$	$(2,1),(1,3),(3,2),(5,1),(2,4),(4,3),(1,6),(6,2),(3,5)$
$3+\sqrt{3}$	3_1	$\hookleftarrow$	$(0,2),(2,1),(4,0),(1,3),(0,5),(5,1),(7,0),(1,6),(6,2),$
			$(3,5),(5,4),(2,7),(3,2)_2,(2,4)_2,(4,3)_2$
$[3]$	0_2	$\hookleftarrow$	$(0,1),(3,1),(0,4),(5,0),(2,3),(4,2),(1,5),(3,4),(8,0),$
			$(5,3),(4,5),(1,8)$
$[3]$	1_2	$\hookleftarrow$	$(1,2),(3,1),(2,3),(4,2),(1,5),(6,1),(3,4),(5,3),(2,6)$
$[3]$	2_2	$\hookleftarrow$	$(1,2),(3,1),(2,3),(4,2),(1,5),(6,1),(3,4),(5,3),(2,6)$
$3+\sqrt{3}$	3_2	$\hookleftarrow$	$(2,0),(1,2),(3,1),(0,4),(5,0),(1,5),(6,1),(0,7),(5,3),$
			$(2,6),(7,2),(4,5),(2,3)_2,(4,2)_2,(3,4)_2$
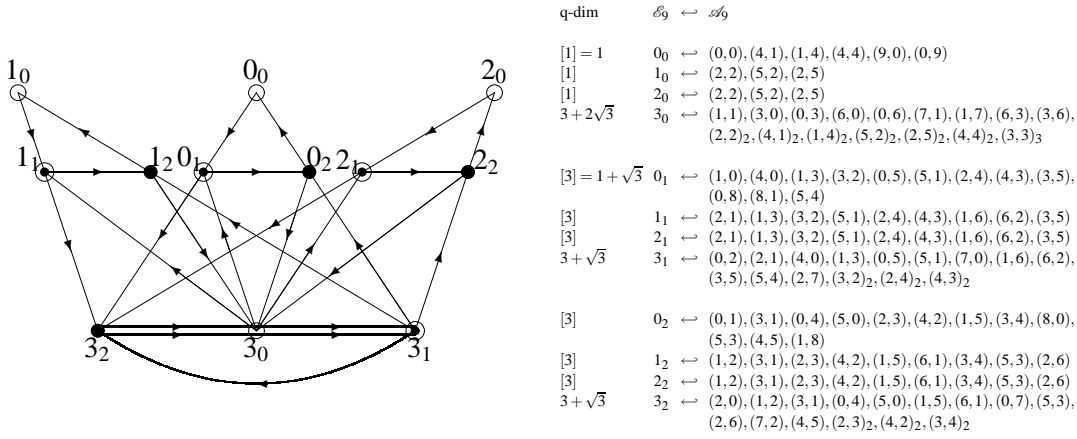
Figure 3: The $\mathscr{E}_9$ graph, quantum dimensions and $\mathscr{E}_9 \hookleftarrow \mathscr{A}_9$ induction rules.
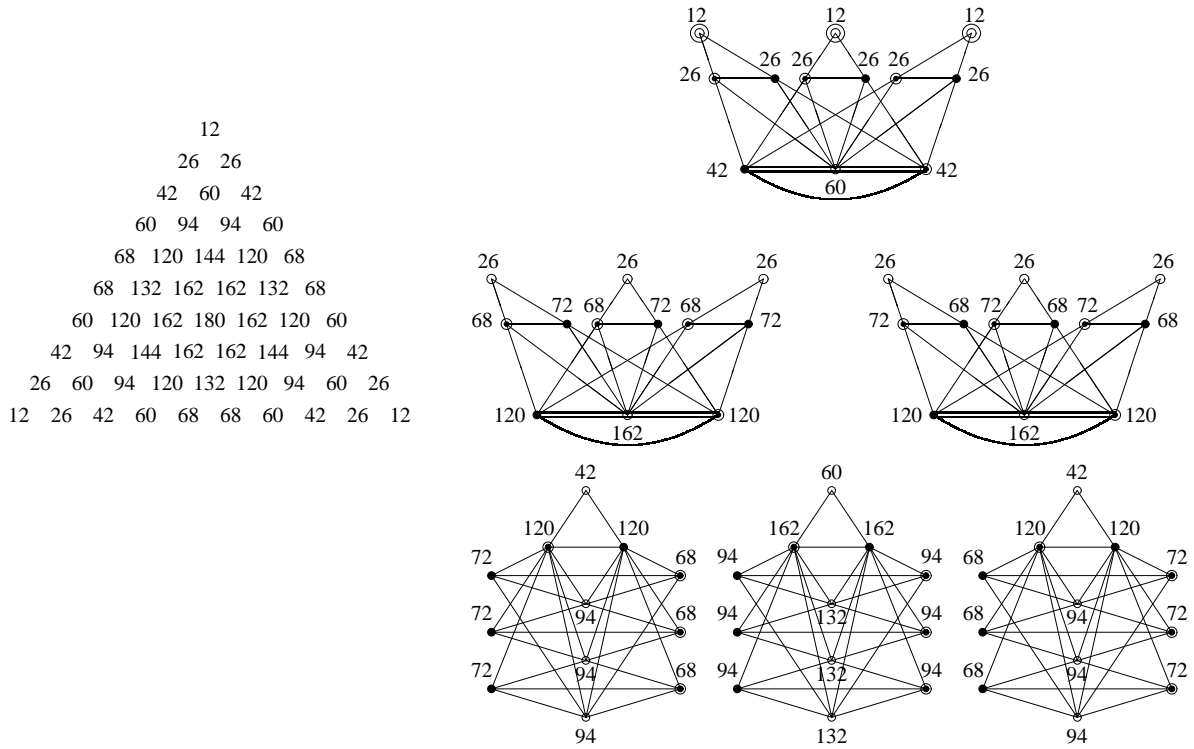
Figure 4: Dimensions d_n and d_x for the $\mathscr{E}_9$ graph.

$$\mathscr{Z} = |[1,1]+[3,3]|^2 + |[1,3]+[4,3]|^2 + |[2,3]+[6,1]|^2$$
$$+ |[4,1]+[1,4]|^2 + |[3,2]+[1,6]|^2 + |[3,1]+[3,4]|^2$$

with $[a,b] = (a-1,b-1)$.

$$\mathscr{F} = \lambda_{(0,0)} \oplus \lambda_{2,2}$$
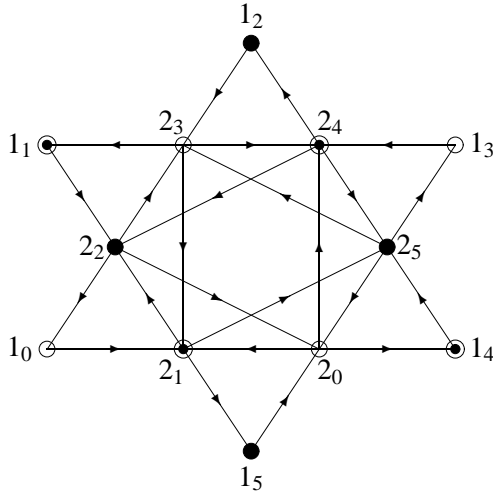
q-dim	$\mathscr{E}_5 \hookleftarrow \mathscr{A}_5$
$[1] = 1$	$1_0 \hookleftarrow (0,0),(2,2)$
$[1]$	$1_1 \hookleftarrow (0,2),(3,2)$
$[1]$	$1_2 \hookleftarrow (1,2),(5,0)$
$[1]$	$1_3 \hookleftarrow (3,0),(0,3)$
$[1]$	$1_4 \hookleftarrow (2,1),(0,5)$
$[1]$	$1_5 \hookleftarrow (2,0),(2,3)$
$[3] = 1+\sqrt{2}$	$2_0 \hookleftarrow (1,1),(3,0),(2,2),(1,4)$
$[3]$	$2_1 \hookleftarrow (1,0),(2,1),(1,3),(3,2)$
$[3]$	$2_2 \hookleftarrow (0,1),(1,2),(3,1),(2,3)$
$[3]$	$2_3 \hookleftarrow (1,1),(0,3),(2,2),(4,1)$
$[3]$	$2_4 \hookleftarrow (0,2),(2,1),(4,0),(1,3)$
$[3]$	$2_5 \hookleftarrow (2,0),(1,2),(3,1),(0,4)$

Figure 1: The $\mathscr{E}_5$ graph, quantum dimensions and $\mathscr{E}_5 \hookleftarrow \mathscr{A}_5$ induction rules.
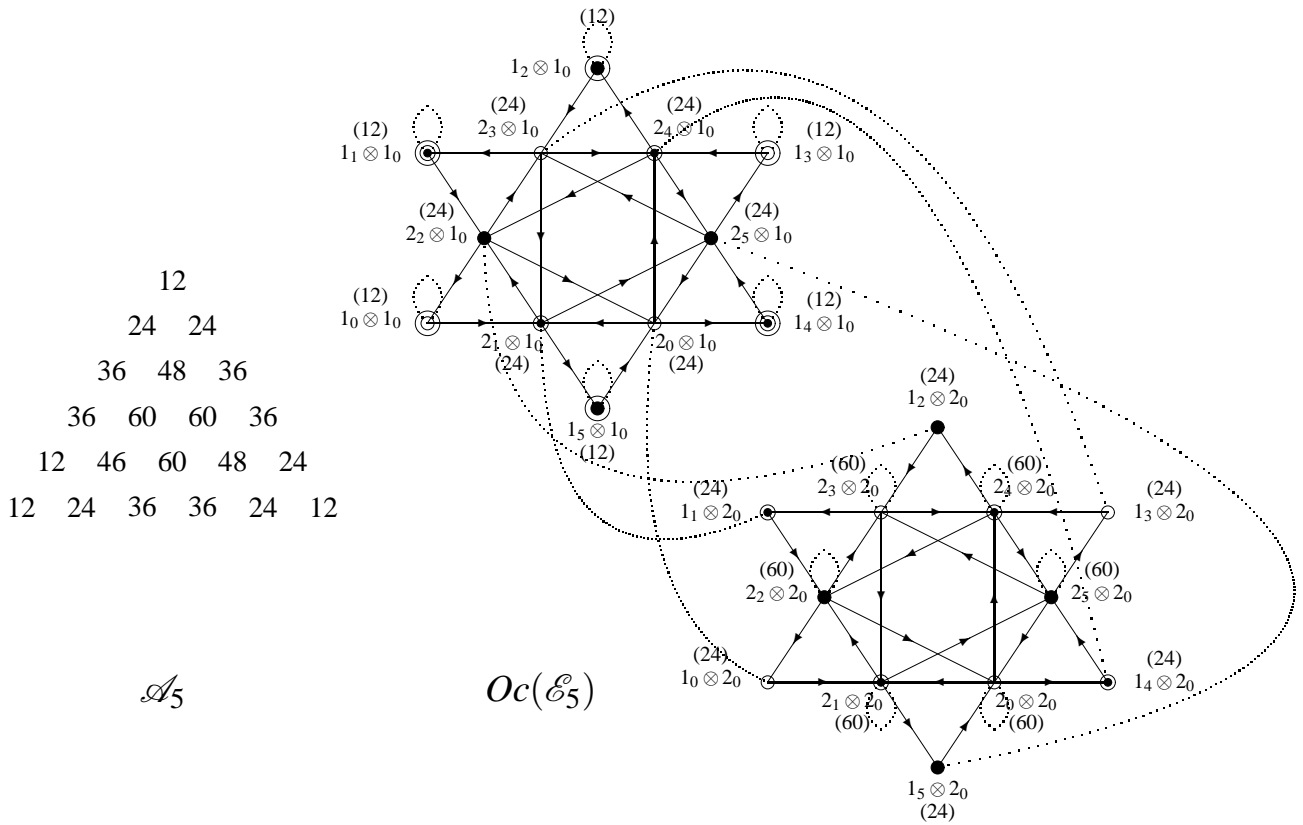
Figure 2: Dimension d_n and d_x of the blocks for $\mathscr{E}_5$.

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Graph	κ	r_E, r_A, r_O	d_H	$d_V - d_H$	$d_{\mathscr{B}} = d_{\mathscr{\hat{B}}}$	$|\mathscr{E}|$	$|\mathscr{A}/\mathscr{E}|$	$|J|$
$\underline{\mathscr{A}_1}$	4	$3,3,3$	9	0	27	3	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_2}$	5	$6,6,6$	45	0	351	$\frac{3}{2}(5+\sqrt{5})$	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_3}$	6	$10,10,10$	164	0	2920	36	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_4}$	7	$15,15,15$	486	0	17766	106.027	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_5}$	8	$21,21,21$	1242	0	85644	$48(3+2\sqrt{2})$	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_9}$	12	$55,55,55$	21307	0	10517299	$432(7+4\sqrt{3})$	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_{21}}$	24	$253,253,253$	2729870	0	41644127980	$288\left(18+10\sqrt{3}+\sqrt{6\left(97+56\sqrt{3}\right)^2}\right)$	1	$=|\mathscr{E}|$
$\underline{\mathscr{A}_k}$	$k+3$	$r_E=r_A=r_O=\frac{(k+1)(k+2)}{2}$	$\frac{(\kappa-2)(\kappa-1)\kappa(\kappa+1)(\kappa+2)(\kappa^2+5)}{1680}$	0	$d_{\mathscr{B}}(\mathscr{A}_k)$	$3\frac{\kappa^2 csc^6(\pi/\kappa)sec^2(\pi/\kappa)}{256}$	1	$=|\mathscr{E}|$
$\mathscr{A}_3^*$	6	$2,10,10$	36	0	144	2	4	.
$\mathscr{A}_4^*$	7	$3,15,15$	102	0	798	2.86294	3	.
$\mathscr{A}_5^*$	8	$3,21,21$	.	0	.	.	.	.
$\mathscr{A}_{k\geq1}^*$	$k+3$	$.,r_A,r_A$	.	0	.	.	.	.
$\underline{\mathscr{D}_3}$	6	$6,10,18$	96	30	1032	$12=\frac{1}{3}|\mathscr{A}_3|$	3	4
$\underline{\mathscr{D}_6}$	9	$12,28,36$	1218	.	64698	$671.56=\frac{1}{3}|\mathscr{A}_6|$	3	223.853
$\underline{\mathscr{D}_9}$	12	$21,55,63$	8193	622	1573275	$144(7+4\sqrt{3})=\frac{1}{3}|\mathscr{A}_9|$	3	$48(7+4\sqrt{3})$
$\underline{\mathscr{D}_{k=0mod3}}$	$k+3$	$\frac{r_A-1}{3}+3,r_A,3r_E$	.	.	.	$=\frac{1}{3}|\mathscr{A}_k|$	3	$\frac{1}{3}|\mathscr{E}|$
$\mathscr{D}_4$	7	$5,15,15$	$\frac{1}{3}d_H(\mathscr{A}_4)=162$	0	$\frac{1}{9}d_{\mathscr{B}}(\mathscr{A}_4)=1974$	$\frac{1}{3}|\mathscr{A}_4|=35.3424$	3	
$\mathscr{D}_5$	8	$7,21,21$	$\frac{1}{3}d_H(\mathscr{A}_5)=414$	0	$\frac{1}{9}d_{\mathscr{B}}(\mathscr{A}_5)=9516$	$\frac{1}{3}|\mathscr{A}_5|=16(3+2\sqrt{2})$	3	
$\mathscr{D}_{k=1,2mod3}$	$k+3$	$\frac{1}{3}r_A,r_A,r_A$	$\frac{1}{3}d_H(\mathscr{A}_k)$	0	$\frac{1}{9}d_{\mathscr{B}}(\mathscr{A}_k)$	$\frac{1}{3}|\mathscr{A}_k|$	3	
$\mathscr{D}_3^*$	6	$6,10,18$	$3d_H(\mathscr{A}_3^*)=108$	.	$9d_{\mathscr{B}}(\mathscr{A}_3^*)=1296$	.	.	.
$\mathscr{D}_9^*$	12	$18,55,54$	$3d_H(\mathscr{A}_9^*)$	.	$9d_{\mathscr{B}}(\mathscr{A}_9^*)$	.	.	.
$\mathscr{D}_{k=0mod3}^*$	$k+3$	$3r_{A^*},r_A,r_O(\mathscr{D}_k)$	$3d_H(\mathscr{A}_k^*)$	.	$9d_{\mathscr{B}}(\mathscr{A}_k^*)$	.	.	.
$\mathscr{D}_4^*$	7	$9,10,10$	$3d_H(\mathscr{A}_4^*)=306$	0	$9d_{\mathscr{B}}(\mathscr{A}_4^*)=7182$	.	.	.
$\mathscr{D}_5^*$	8	$9,21,21$	$3d_H(\mathscr{A}_5^*)$	0	$9d_{\mathscr{B}}(\mathscr{A}_5^*)$	.	.	.
$\mathscr{D}_{k=1,2mod3}^*$	$k+3$	$3r_{A^*},r_A$	$3d_H(\mathscr{A}_k^*)$	0	$9d_{\mathscr{B}}(\mathscr{A}_k^*)$	.	.	.
$\mathscr{D}_9^t$	12	$17,55,63$	7001	.	1167355	$72(2+\sqrt{3})$	$6(2+\sqrt{3})$	.
$\mathscr{D}_9^{t*}$	12	$11,55,63$	.	.	.	.	.	.
$\underline{\mathscr{E}_5}$	8	$12,21,24$	720	0	29376	$12(2+\sqrt{2})$	$2(2+\sqrt{2})$	6
$\underline{\mathscr{E}_5/3}$	8	$4,21,24$	$\frac{1}{3}d_H(\mathscr{E}_5)=240$	0	$\frac{1}{9}d_{\mathscr{B}}(\mathscr{E}_5)=3264$	$\frac{1}{3}|\mathscr{E}_5|=4(2+\sqrt{2})$	$6(2+\sqrt{2})$	.
$\underline{\mathscr{E}_9}$	12	$12,55,72$	4656	792	518976	$36(2+\sqrt{3})$	$12(2+\sqrt{3})$	3
$\underline{\mathscr{E}_9/3}$	12	$12,55,72$	5616	936	754272	$\frac{1}{3}|\mathscr{E}_9|=12(2+\sqrt{3})$	$36(2+\sqrt{3})$	.
$\underline{\mathscr{E}_{21}}$	24	$24,253,288$	288576	0	480701952	$24\left(18+10\sqrt{3}+\sqrt{6\left(97+56\sqrt{3}\right)}\right)$	$12\left(18+10\sqrt{3}+\sqrt{6\left(97+56\sqrt{3}\right)}\right)$	2

Table 3: Dimensions and quantum masses for $sl(3)$ cases.
$$d_{\mathscr{B}}(\mathscr{A}_k)=\frac{(\kappa-2)(\kappa-1)\kappa^2(\kappa+1)(\kappa+2)(1052+325\kappa^2+58\kappa^4+5\kappa^6)}{4435200}$$

Graph	induction	$q-dim$	$|\mathscr{E}|$	$|\mathscr{A}/\mathscr{E}|$	$|\mathscr{J}|$
$\mathscr{A}_2 = A_3$	.	$1, \sqrt{2}, 1$	4	1	$=|\mathscr{A}|$
$\mathscr{A}_3 = A_4$	.	$1, \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1+\sqrt{5}), 1$	$5+\sqrt{5}$	1	$=|\mathscr{A}|$
$\mathscr{A}_4 = A_5$	.	$1, \sqrt{3}, 2, \sqrt{3}, 1$	12	1	$=|\mathscr{A}|$
$\mathscr{A}_5 = A_6$	.	$1, 2\cos(\frac{\pi}{7}), 1+2\cos(\frac{\pi}{7}), 1+2\cos(\frac{\pi}{7}), 1$	18.59	1	$=|\mathscr{A}|$
$\mathscr{A}_k = A_{r=k+1}$	.	$[1], [2], \ldots, [2], [1]$	$(\kappa/2)\csc^2(\pi/\kappa)$	1	$=|\mathscr{A}|$
$\mathscr{A}_{10} = A_{11}$	.	$1, \sqrt{2+\sqrt{3}}, 1+\sqrt{3}, \sqrt{3(2+\sqrt{3})}, 2+\sqrt{3}; 2\sqrt{2+\sqrt{3}}; +\text{sym.}$	$24(2+\sqrt{3})$	1	$=|\mathscr{A}|$
$\mathscr{A}_{16} = A_{17}$	.	$\ldots$	$9\csc^2(\pi/18) = 298.471$	1	$=|\mathscr{A}|$
$\mathscr{A}_{28} = A_{29}$	.	$\ldots$	$30\left(12+5\sqrt{5}+\sqrt{3\left(85+38\sqrt{5}\right)}\right)$	1	$=|\mathscr{A}|$
$\mathscr{D}_4 = D_4$			$\frac{|\mathscr{A}_5|}{2} = 6$	2	$\frac{|\mathscr{A}_5|}{4}$
$\mathscr{D}_8 = D_6$		$\sqrt{\frac{1}{2}(5+\sqrt{5})}, \frac{1}{2}(3+\sqrt{5}), \sqrt{5+2\sqrt{5}}; \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1+\sqrt{5})$	$\frac{|\mathscr{A}_9|}{2} = 5(3+\sqrt{5})$	2	$\frac{|\mathscr{A}_9|}{4}$
$\mathscr{D}_{16} = D_{10}$	$\ldots$	$[1], [2], [3], [4], [5], [6], [7], [8]; [9]/2, [9]/2$	$\frac{|\mathscr{A}_{17}|}{2} = 149.235$	2	$\frac{|\mathscr{A}_{17}|}{4}$
$\mathscr{D}_{28} = D_{16}$	$\ldots$	$\ldots \ldots$	$\frac{|\mathscr{A}_{29}|}{2}$	2	$\frac{|\mathscr{A}_{29}|}{4}$
$\mathscr{D}_6 = D_5$		$1, \sqrt{2+\sqrt{2}}, 1+\sqrt{2}; \sqrt{1+\frac{1}{\sqrt{2}}}, \sqrt{1+\frac{1}{\sqrt{2}}}$	$|A_7|/2 = 2csc^2(\pi/8) = 13.65$	$?$	.
$\mathscr{D}_{10} = D_7$		$1, \sqrt{2+\sqrt{3}}, 1+\sqrt{3}, \sqrt{3(2+\sqrt{3})}, 2+\sqrt{3}; \sqrt{2+\sqrt{3}}, \sqrt{2+\sqrt{3}}$	$|A_{11}|/2 = 3csc^2(\pi/12)$	.	.
$\mathscr{E}_{10} = E_6$			$4(3+\sqrt{3})$	$3+\sqrt{3}$	4
$\mathscr{E}_{16} = E_7$		$q = \exp(\frac{i\pi}{18})$	$\begin{aligned}|\mathscr{E}| &= 2([2]^2+[4]^2+[4]^2/[3]^2) \\ &= 38.468 \\ |A_{17}| &= |D_{10}||D_{10}|/|J| \\ |D_{10}| &= |A_{17}|/2 \end{aligned}$	$[1]+[9]+[17] = 7.758$	$\begin{aligned}|J| &= |D_{10}|/2 = |A_{17}|/4 \\ |J| &= [1]^2+[3]^2+[5]^2+ \\ &[7]^2+[9]^2/4+[9]^2/4 \end{aligned}$
$\mathscr{E}_{28} = E_8$		$q = \exp(\frac{i\pi}{30})$	$\frac{1}{2}\left(15\left(3+\sqrt{5}\right)+\sqrt{30\left(65+29\sqrt{5}\right)}\right)$	$\frac{1}{2}\left(3\left(5+\sqrt{5}\right)+\sqrt{150+66\sqrt{5}}\right)$	$\frac{1}{2}\left(5+\sqrt{5}\right)$

Table 2: Quantum dimensions for $sl(2)$ cases.
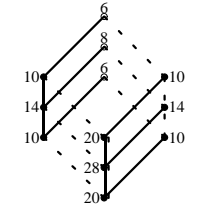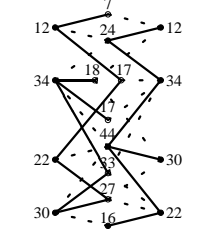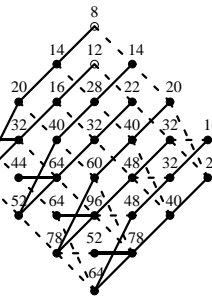
Graph	$\kappa = k+2$	d_n, Z and exponents, $\mathscr{F}$	d_x	$d_H = \kappa(\kappa+1)r/6$	$d_V - d_H$	$d_{\mathscr{B}} = d_{\tilde{\mathscr{B}}}$
$\mathscr{A}_1 = A_2$	3	$(2,2)$	$(2,2)$	4	0	$8 = 2^3$
$\mathscr{A}_2 = A_3$	4	$(3,4,3)$	$(3,4,3)$	10	0	$34 = 2^1 17^1$
$\mathscr{A}_3 = A_4$	5	$(4,6,6,4)$	$(4,6,6,4)$	20	0	$104 = 2^3 13^1$
$\mathscr{A}_4 = A_5$	6	$(5,8,9,8,5)$	$(5,8,9,8,5)$	35	0	$259 = 7^1 37^1$
$\mathscr{A}_5 = A_6$	7	$(6,10,12,12,10,6)$	$(6,10,12,12,10,6)$	56	0	$560 = 2^4 5^1 7^1$
$\mathscr{A}_{10} = A_{11}$	12	$(11,20,27,32,35,36,35,32,27,20,11)$	$\ldots$	$286 = 2^1 11^1 13^1$	0	$8294 = 2^1 11^1 13^1 29^1$
$\mathscr{A}_{16} = A_{17}$	18	$(17,32,45,56,65,72,77,80,81,80,77,72,65,56,45,32,17)$	$\ldots$	$3^1 17^1 19^1$	0	$3^1 5^1 13^1 17^1 19^1$
$\mathscr{A}_{28} = A_{29}$	30	$(29,56,81,104,125,144,161,176,189,200,209,216,221,224;225;\mathrm{sym.})$	$\ldots$	$5^1 29^1 31^1$	0	$17^1 29^1 31^1 53^1$
$\mathscr{A}_k = A_{r=k+1}$	$k+2$	$d_n = (n+1)(k+1-n),\ n=0,\ldots,k$	$d_x = d_n$	$(\kappa-1)\kappa(\kappa+1)/6$	0	$\kappa(\kappa^4-1)/30$
$\mathscr{D}_4 = D_4$	6	$(4,6;8;6,4)$	$(4,6;4,4)\,(4,6;4,4)$	$28 = 4^1 7^1$	$(4+4)$	$168 = 2^3 3^1 7^1$
$\mathscr{D}_8 = D_6$	10	$(6,10,14,16;18;16,14,10,6)$	$(6,10,14,16;9,9)\,(6,10,14,16;9,9)$	$110 = 2^1 5^1 11^1$	$(9+9)$	$1500 = 2^2 3^1 5^3$
$\mathscr{D}_{16} = D_{10}$	18	$(10,18,26,32,38,42,46,48;50;\mathrm{sym.})$	$(10,18,26,32,38,42,46,48;25,25)_2$	$570 = 2^1 3^1 5^1 19^1$	$(25+25)$	$2^2 5 5 0 1^1$
$\mathscr{D}_{28} = D_{16}$	30	$(16,30,44,56,68,78,88,96,104,110,116,120,124,126\,;128\,;\mathrm{sym.})$	$\ldots\ldots$	$2480 = 2^4 5^1 31^1$	$(64+64)$	$2^5 7 7 5 7^1$
$\mathscr{D}_k = D_{r_e = \frac{k}{2}+2}$	$2r-2$	$(r,\kappa,\ldots;\frac{1}{2}(1+\frac{\kappa}{2})^2;\ldots,\kappa,r),\quad \mathscr{F} = \lambda_0 \oplus \lambda_k$	$(r,\kappa,\ldots;\frac{1}{4}(1+\frac{\kappa}{2})^2,\frac{1}{4}(1+\frac{\kappa}{2})^2)_2$	$\kappa(\kappa+1)(\kappa+2)/12$	$\frac{1}{2}(1+\frac{\kappa}{2})^2$	$\frac{(2+\kappa)(120+\kappa(28+\kappa(26+\kappa(17+4\kappa))))}{480}$
$\mathscr{D}_6 = D_5$	8	$(5,8,11,12,11,8,5)$	$(5,8,11,12,11,8,5)$	$60 = 2^2 3^1 5^1$	0	$564 = 2^2 3^1 47^1$
$\mathscr{D}_{10} = D_7$	12	$(7,12,17,20,23,24,23,20,17,12,7)$	$(7,12,17,20,23,24,23,20,17,12,7)$	$182 = 2^1 7^1 13^1$	0	$3398 = 2^1 1699^1$
$\mathscr{D}_k = D_{r_o = \frac{k}{2}+2}$	$2r-2$	$(r,\kappa,\ldots,\kappa,r),\quad \mathscr{F} = \lambda_0 \oplus \lambda_k$	$(r,k+2,\ldots,k+2,r)$	$\kappa(\kappa+1)(\kappa+2)/12$	0	$\frac{\kappa(176+\kappa(80+\kappa(60+\kappa(25+4\kappa))))}{480}$
$\mathscr{E}_{10} = E_6$	12	$(6,10,14,18,20,20,20,18,14,10,6)$ $Z = \|1+7\|^2 + \|4+8\|^2 + \|5+11\|^2$ $\mathscr{F} = \lambda_0 \oplus \lambda_6$		$156 = 2^2 3^1 13^1$	0	$2512 = 2^4 157^1$
$\mathscr{E}_{16} = E_7$	18	$(7,12,17,22,27,30,33,34,35,34,33,30,27,22,17,12,7)$ $Z = \|1+17\|^2 + \|5+13\|^2 + \|7+11\|^2 + \|9\|^2 + ((3+15)\ \overline{9} + h.c.)$ $\mathscr{F} = \lambda_0 \oplus \lambda_8 \oplus \lambda_{16}$		$399 = 3^1 7^1 19^1$	0	$10905 = 3^1 5^1 727^1$
$\mathscr{E}_{28} = E_8$	30	$(8,14,20,26,32,38,44,48,52,56,60,62,64,64,64;\mathrm{sym.})$ $Z = \|1+11+19+29\|^2 + \|7+13+17+23\|^2$ $\mathscr{F} = \lambda_0 \oplus \lambda_{10} \oplus \lambda_{18} \oplus \lambda_{28}$		$1240 = 2^3 5^1 31$	0	$63136 = 2^5 1973^1$

Table 1: Horizontal dimensions, vertical dimensions, and bialgebras dimensions for $sl(2)$ cases.

3.5.2 The $sl(3)$ **family**

Many results about the $sl(3)$ graphs can be found in the original article [16] and in the book [3], chap.17. Other aspects of members of this system, in particular their quantum symmetries, are described in [33], [23], [14], [24] and [22]. Here we are only interested in giving tables describing the block structure of the quantum groupoid $\mathscr{B}$, i.e., dimensions d_n, d_x, the quantum dimensions and the quantum masses. Many such numbers were certainly obtained by [34], several examples are studied in [12] and [39]. Generic formulae are ours. Results presented here are complete only for the cases with self - fusion, for which we also give tables describing induction rules (actually the induction tables for $\mathscr{E}_{21}$ could not fit in this publication and will appear in a unabridged version of our work). A description of the cases without self-fusion, along the same lines, should be made available in [17] and [22]. Some of the results presented here already appeared in [12].

Acknowledgments

After simplification by a common denominator the *ADE* trigonometric identities read[9] explicitly

$$\sum_{m,n=1}^{r} (Z)_{m,n} \sin(m\pi/\kappa) \sin(n\pi/\kappa) = \kappa/2$$

For instance in the E_8 case, $\kappa = 30$, we have

$$\left\{ sin(7\tfrac{\pi}{30}) + sin(13\tfrac{\pi}{30}) + sin(17\tfrac{\pi}{30}) + sin(23\tfrac{\pi}{30}) \right\}^2 \; + \; \left\{ sin(1\tfrac{\pi}{30}) + sin(11\tfrac{\pi}{30}) + sin(19\tfrac{\pi}{30}) + sin(29\tfrac{\pi}{30}) \right\}^2 = 15$$

After simplification, the higher *ADE* trigonometric identities of $sl(3)$ type read[9]:

$$\sum_{\substack{n=(n_1,n_2) \\ m=(m_1,m_2)}} (Z)_{m,n}\, sin(m_1\tfrac{\pi}{\kappa})\, sin(m_2\tfrac{\pi}{\kappa})\, sin((m_1+m_2)\tfrac{\pi}{\kappa})\, sin(n_1\tfrac{\pi}{\kappa})\, sin(n_2\tfrac{\pi}{\kappa})\, sin((n_1+n_2)\tfrac{\pi}{\kappa}) = 3\kappa^2/64$$

For instance in the $\mathscr{E}_{21}$ case, $\kappa = 24$, setting $s_n = sin(n\pi/24)$, we have

$$(2s_1s_7s_8 + 2s_5s_8s_{13} + 2s_5s_{11}s_{16} + 2s_1s_{16}s_{17} + 2s_8s_{11}s_{19} + 2s_7s_{16}s_{23})^2 + $$
$$+ (s_1^2 s_2 + s_5^2 s_{10} + 2s_2s_{11}s_{13} + s_7^2 s_{14} + 2s_7s_{10}s_{17} + 2s_5s_{14}s_{19} + s_{11}^2 s_{22} + 2s_1s_{22}s_{23})^2 = 27$$

Product of quantum dimensions and the discriminant formula:

Define $D = (\sum_{\lambda_n \in \mathscr{A}} \mu_n^2)^r / \Pi_{\lambda_n \in \mathscr{A}} \mu_n^2$, where r is the total number of irreps of the fusion algebra, $r = k+1$ for $sl(2)$, $r = (k+1)(k+2)/2!$ for $sl(3)$, $r = (k+1)(k+2)(k+3)/3!$ for $sl(4)$. In the case of $sl(2)$, using previous results for $\sum_{\lambda_n \in \mathscr{A}} \mu_n^2$ and a classical trigonometric identity giving $\Pi_{\lambda_n \in \mathscr{A}} \mu_n^2 = [2^{-(\kappa-1)} \kappa (Sin(\pi/\kappa))^{-(\kappa-1)}]^2$ where results are written in terms of the Coxeter number κ, one finds $D = 2^{\kappa-1} \kappa^{\kappa-3}$. More generally, it was shown recently in [25], that D is the square of the determinant of the matrix S_{mn}/S_{m0} (this is the matrix of "quantum conjugacy classes") which, in the case of $sl(2)$ could also be defined as the discriminant of the characteristic polynomial of the adjacency matrix of the graph A_r. Example: Take A_{11} of the $sl(2)$ system, the characteristic polynomial of the adjacency matrix is $-s^{11} + 10s^9 - 36s^7 + 56s^5 - 35s^3 + 6s$ and its discriminant is 10567230160896, indeed equal to $2^{11} 12^9$. D is an integer and general expressions for this quantity are obtained in the quoted reference. In the case of $sl(2)$, one recovers the expression calculated previously, whereas [25] gives $D = 3^{(\kappa-2)(\kappa-1)/2} \kappa^{(\kappa-4)(\kappa-2)}$ for $sl(3)$, where $\kappa = k+3$. Since $|\mathscr{A}_k|$ is already known, one can use the Gepner formula for D to provide an efficient way for obtaining the square product of quantum dimensions. For $sl(3)$ we find:

$$\Pi_{\lambda_n \in \mathscr{A}}\, \mu_n^2 = 16^{-(\kappa-2)(\kappa-1)} \kappa^{3(\kappa-2)} \left(\cos\left(\frac{\pi}{\kappa}\right) \sin^3\left(\frac{\pi}{\kappa}\right) \right)^{-(\kappa-2)(\kappa-1)}$$

3.5 Tables

3.5.1 The $sl(2)$ family

There are two tables. Notations should be clear. Most results are hardly new. In particular, the dimensions d_n and d_v given in table 1 were obtained already in [34], [8], [38] and [11]. The compact expression for d_H can be seen in [30], and several quantum dimensions or masses given in table 2 can also be found in [11] and [39]. Nevertheless, both tables contain new entries and explicit formulae.

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[9]Warning: In this expression, we shifted by $+1$ the indices labelling irreps.

when the space of sections Γ_a contains λ_n in its reduction, i.e., when the integer $(\mathfrak{e})_{na} \geq 1$. Possible multiplicities being understood, we shall write

$$dim(\Gamma_a) = \sum_{\lambda_n \uparrow \Gamma_a} \mu_n$$

In particular, for the space of functions $\mathscr{F}$ over the "discrete non commutative space": $\mathscr{A}/\mathscr{E}$, we have $dim(\Gamma_0) = \sum_n (\mathfrak{e})_{n0} \mu_n$ and we shall write (see also [9]):

$$|\mathscr{A}/\mathscr{E}| = dim(\Gamma_0) = dim(\mathscr{F}) = \sum_{\lambda_n \uparrow \Gamma_0} \mu_n$$

Then, we have the following result :

$$dim_{\mathscr{E}}(a) = \frac{dim\Gamma_a}{dim\Gamma_0} = \frac{dim\Gamma_a}{|\mathscr{A}/\mathscr{E}|}$$

Moreover, $|\mathscr{A}/\mathscr{E}| = |\mathscr{A}|/|\mathscr{E}|$. To obtain the quantum dimensions μ_a of vertices of $\mathscr{E}$, and therefore the order $|\mathscr{E}|$, it is therefore enough to know the μ_n for $\mathscr{A}$ diagrams and the induction rules.

Example: take $\mathscr{E} = E_6$, and consider the central vertex σ_2 (the triple point). Its quantum dimension can be calculated using Perron Frobenius, or directly, using $\sigma_2 = \sigma_1^2 - \sigma_0$:

$$dim_E(\sigma_2) = dim_E(\sigma_1)^2 - dim_E(\sigma_0) = (2cos(\pi/12))^2 - 1 = 1 + \sqrt{3}$$

but one can also obtain from A_{11}: induction from σ_0 gives $\Gamma_0 = \lambda_0 \oplus \lambda_6$, so that $|A_{11}/E_6| = [1] + [7] = 3 + \sqrt{3}$, and induction from σ_2 gives $\Gamma_2 = \lambda_2 \oplus \lambda_4 \oplus \lambda_6 \oplus \lambda_8$, so

$$dim_E(\sigma_2) = \frac{[3] + [5] + [7] + [9]}{[1] + [7]} = \frac{6 + 4\sqrt{3}}{3 + \sqrt{3}} = 1 + \sqrt{3},.$$

Call *J* the modular subalgebra of $\mathscr{E}$. For graphs with self - fusion, we have both $|\mathscr{O}| = |\mathscr{E}| \times |\mathscr{E}|/|J|$ and $|\mathscr{O}| = |\mathscr{A}|$, so that we have also $|\mathscr{A}|/|\mathscr{E}| = |\mathscr{E}|/|J|$.

Using quantum dimensions of $\mathscr{E}$, one can compute $|J| = \sum_{\sigma_c \in J \subset \mathscr{E}} \mu_c^2 = \sum_{\sigma_c \in J} |\Gamma_c|^2/|\Gamma_0|^2$, but the relation for $|\mathscr{O}|$ can be written $|J| = |\mathscr{A}|/|\Gamma_0|^2$, so that comparing the two expressions implies $|A| = \sum_{c \in J} |\Gamma_c|^2$. Actually the following *ADE* or generalized *ADE* trigonometric identities, which seem to belong to the folklore of CFT (we shall write them later in a non standard but suggestive way), imply the previous result. Let *Z* be a modular invariant matrix of an *sl(N)* system at level *k*, with or without self-fusion. Its matrix elements $(Z)_{m,n}$ are indexed by a pair of irreps λ_m and λ_n. For *sl(N)*, these are of course multi-indices. As before, call μ_m the quantum dimension of λ_m. Then we have the following identity :

$$\sum_{m,n} \mu_m (Z)_{m,n} \mu_n = \sum_m \mu_m^2$$

The right hand side of this identity is $|\mathscr{A}_k|$ and can be calculated from the modular matrix *S*.

$$|\mathscr{A}| = \frac{\kappa}{2} \frac{1}{\sin^2\left(\frac{\pi}{\kappa}\right)} \quad \text{for } sl(2), \text{ with } \kappa = k+2$$

$$|\mathscr{A}| = \frac{3}{256} \kappa^2 \frac{1}{\sin^6\left(\frac{\pi}{\kappa}\right)} \frac{1}{\cos^2\left(\frac{\pi}{\kappa}\right)} \quad \text{for } sl(3), \text{ with } \kappa = k+3$$

3.3.1 $sl(2)$

We shall give the values of d_n, d_x, d_H, d_V and $d_{\mathscr{B}}$ in tables, but thanks to the above identification with the vector space underlying the preprojective algebra of quiver theories, we have a closed formula for d_H that work for all $sl(2)$ cases, namely $d_H = \frac{\kappa(\kappa+1)r}{6}$. Recall that k is the level, $r = k+1$ is the rank (the number of vertices), and $\kappa = k+2$ is the Coxeter number. In terms of the dimension $dim(E)$ of the Lie group corresponding to the chosen *ADE* diagram, and using the Kostant formula $dim(E) = (\kappa+1)r$, we can also write $d_H = \frac{\kappa \, dim(E)}{6}$. For instance, $E_6 \rightarrow 156 = 78\frac{12}{6}$ $E_7 \rightarrow 399 = 133\frac{18}{6}$ $E_8 \rightarrow 1240 = 248\frac{30}{6}$. For A_r graphs, the rank $r = k+1 = \kappa-1$ so that $d_H(A_r) = (\kappa-1)\kappa(\kappa+1)/6$, this can be obtained directly from the fact that $d_n = (n+1)(k+1-n)$, the trivial representation being labelled $n = 0$. Notice that 2κ is the period (in n) of matrices F_n. It is interesting to summarize how the general formula for d_H is obtained in quiver theories [30]: one constructs a generating function for the d_n (matrix Hilbert series) and obtains d_H as twice the sum of matrix elements of the inverse of the Cartan matrix of the chosen graph $(2\,\mathbb{1} - F_1)$; this, in turn, is given by the Freudenthal - de Vries strange formula, which, using Kostant relation, can be written $\kappa(\kappa+1)r/12$, hence the result.

3.3.2 $sl(3)$

Dimensions are later given in tables. In the case of d_H, there is no known formula that works in all cases. We however obtained the following closed result for $\mathscr{A}_k$ diagrams: $d_H = (k+1)(k+2)(k+3)(k+4)(k+5)(k^2+6k+14)/1680$ or, using $\kappa = k+3$, $d_H = (\kappa-2)(\kappa-1)\kappa(\kappa+1)(\kappa+2)(\kappa^2+5)/1680$. There is no Lie group theory associated with the Di Francesco Zuber diagrams, nevertheless it is natural to call rank the quantity $r = (k+1)(k+2)/2$, since it is the number of vertices of the $\mathscr{A}_k$ Weyl alcove. Our previous result for d_H uses the fact that the sum of matrix elements of the inverse of $(3\,\mathbb{1} - (F_1 + F_1^t)/2)$ can be shown to be equal to $r(\kappa+1)(\kappa+2)/60$.

3.4 Quantum dimensions

As usual, q-integers are denoted $[n] = (q^n - q^{-n})/(q - q^{-1})$ with $q = exp[i\pi/\kappa]$. One can define quantum dimensions μ of the vertices of $\mathscr{E}$ from the Perron Frobenius eigenvector of its adjacency matrix. If the graph $\mathscr{E}$ has self - fusion, they can be calculated from the quantum dimensions of the unit (which is 1) and of the generator (which is $[2] = 2cos(\pi/\kappa)$ for $sl(2)$ and $[3] = 1 + 2cos2\pi/\kappa$ for $sl(3)$), by using the character property of μ. Let $\mu_a = dim_{\mathscr{E}}(\sigma_a)$ the quantum dimension of the vertex σ_a. Call also $|\mathscr{E}| = \sum_a \mu_a^2$. Graphs $\mathscr{A}$ and $\mathscr{O}$ describing fusion and quantum symmetries have self - multiplication, and quantum dimensions $\mu_n = dim_{\mathscr{A}}(\lambda_n)$ and $\mu_x = dim_{\mathscr{O}}(o_x)$ for their vertices can be obtained in a similar way. In the particular case $\mathscr{E} = \mathscr{A}$, the μ_n can be obtained from the modular generator S (a unitarizing matrix for the adjacency matrix of the graph): $\mu_n = S_{0n}/S_{00}$. For $sl(2)$, $\mu_n = [n+1]$. For $sl(3)$, $\mu_{p,q} = [p+1][q+1][p+q+2]/[2]$. Unitarity of S implies $|\mathscr{A}| = \sum_n \mu_n^2 = 1/S_{00}^2$ and several explicit expressions given at the end of this section.

Call $\mathfrak{e}$ the intertwiner describing induction - restriction between $\mathscr{A}$ and $\mathscr{E}$ (also called essential matrix relative to the unit vertex). It is a rectangular matrix with r_A lines and r_E columns. Essential matrices $\mathfrak{e}_a$ are obtained from annular matrices F^n as follows: $(\mathfrak{e}_a)_{nb} = (F^n)_{ab}$. The intertwiner is $\mathfrak{e} = \mathfrak{e}_0$. From induction, one obtains $dim(\Gamma_a) = \sum_n (\mathfrak{e})_{na} \mu_n$. It is convenient to write $\lambda_n \uparrow \Gamma_a$

of Z (or exponents of $\mathscr{E}$). Forgetting multiplicities, the matrix Z defines a partition on this set: two exponents m, n are in the same modular block iff $Z_{mn} \neq 0$. For instance, at level 10, in the case of E_6, we have $Z = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$ and therefore a partition of $K = \{0, 6, 3, 7, 4, 10\}$ into three subsets $\{0, 6\}$, $\{3, 7\}$, $\{4, 10\}$. At level 17, in the case of E_7, we have $Z = |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 + (\overline{\chi}_8)(\chi_2 + \chi_{14}) + (\overline{\chi}_2 + \overline{\chi}_{14})\chi_8$, so we have a partition of $K = \{0, 16, 4, 12, 6, 10, 8\}$ into four subsets $\{0, 16\}$, $\{4, 12\}$, $\{6, 10\}$, $\{8\}$. The modular block of the origin (containing 0) is denoted K_0. We observe that when $\mathscr{E}$ is monoidal (self-fusion), Z is a sum of squares and there is a one to one correspondence between modular blocks K_a and the modules Γ_a sitting above the irreducible objects a belonging to the modular subalgebra J of the Grothendieck ring of $\mathscr{E}$. For instance the first modular block $K_0 = \{0, 6\}$ of E_6 corresponds to the algebra $\mathscr{F} = \Gamma_0 = \lambda_0 \oplus \lambda_6$. This is not so when there is no self-fusion: the first modular block K_0 of E_7 is $\{0, 16\}$ although $\mathscr{F} = \Gamma_0 = \lambda_0 \oplus \lambda_8 \oplus \lambda_{16}$.

3.3 Dimensions of horizontal and vertical spaces H and V, dimension of the weak Hopf algebra $\mathscr{B}$

The horizontal space $H = \bigoplus_n H_n$, coming in the construction of the first algebra structure on the weak Hopf algebra $\mathscr{B}$ was defined before, in terms of categorical data. In the $sl(2)$ case H can be realized as the (Ocneanu) vector space of essential paths on ADE graphs but also as the vector space underlying the Gelfand-Ponomarev preprojective algebra associated with the corresponding unoriented quiver. In the first realization H_n is defined as a particular subspace of the space *Paths* of all paths on the graph $\mathscr{E}$, whereas in the second construction it is defined as a quotient. Identification stems, for instance, from the fact that dimensions of these finite dimensional vector spaces, calculated according to the two definitions, are equal. In the case of $sl(3)$, the grading label of the horizontal space H_n refers to a pair of integers (n_1, n_2), specifying an irreducible representation (it can also be seen as a Young tableau), this suggests a generalization of the notion of preprojective algebras associated with quivers.

In the case of $sl(2)$, dimensions $d_n = dimH_n$, $d_H = \sum_n d_n$ and $d_{\mathscr{B}} = \sum_n d_n^2$, where n runs in the set of irreducible objects of $\mathscr{A}_k$, and $d_x = dimV_x$, $d_V = \sum_x d_x$ and $d_{\widehat{\mathscr{B}}} = \sum_x d_x^2$, where x runs in the set of irreducible objects of $\mathscr{O}$ have been calculated first by [34], then by [8], [38], [11]. In the case of $sl(3)$, they have been calculated by [34], [12], [39] and [22]. One check, of course, that $d_{\mathscr{B}} = d_{\widehat{\mathscr{B}}}$ in all cases, since the underlying vector space is the same. Surprisingly, one also observes[7] that $d_H = d_V$ in most cases. The collection of known results giving d_H for all $sl(2)$ cases can be condensed into a closed formula by using a recent result obtained by [30] for the dimensions of preprojective algebras associated with *ADE* quivers[8]. To compute d_n, the pedestrian approach, that works in all cases, is to calculate the annular matrices F_n, describing the module action of $\mathscr{A}_k$ on $\mathscr{E}$, using recursion formulae giving irreps of $sl(2)$ or $sl(3)$, then to sum over all matrix elements, since $H_{ab}^m = Hom(\lambda_n \otimes \lambda_a, \lambda_b)$. To compute d_x, one has first to determine $\mathscr{O}$, for instance by solving the modular splitting equation, then the dual annular matrices S_x describing the module action of $\mathscr{O}$ on $\mathscr{E}$, and finally to sum over all matrix elements, since $V_{ab}^x = Hom(\lambda_a \otimes \lambda_x, \lambda_b)$.

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[7]When it is not so, in particular when the graph $\mathscr{E}$ is a $\mathbb{Z}_2$ or $\mathbb{Z}_3$ orbifold, one knows how to "correct" this curious linear sum rule, which was first observed in the case of $sl(2)$ by [38].

[8]Warning: the preprojective algebra is a multiplicative structure on H, at least for $sl(2)$ cases, it cannot be identified with either $\mathscr{B}$ or $\widehat{\mathscr{B}}$, see also [10].

sion and toric matrices; a particular case of this equation reads $\sum_x (W_{0x})_{\lambda\mu} W_{x0} = N_\lambda Z N_\mu^{tr}$, called "equation of modular splitting", was presented by A.Ocneanu in Bariloche (2000). Given fusion matrices N_p (known in general) and a modular invariant matrix $Z = W_{00}$, solving this equation, i.e., finding the W_{x0}, allows one to reconstruct the character ring of $\mathcal{O}$. A practical method to solve this equation is given in [24], with several $sl(2)$ and $sl(3)$ examples. Left and right chiral categories $\mathcal{O}_{L,R}$ are defined, using alpha-induction functors, as additive and monoidal subcategories of $\mathcal{O}$ whose objects are direct summands of $\alpha_{LR}(\lambda)$, for all λ in $\mathscr{A}_k$. They are not braided but their intersection, the ambichiral subcategory $\mathscr{J}$ is. When $\mathscr{E}$ is monoidal, the Grothendieck ring of $\mathscr{J}$, called ambichiral, is isomorphic with the modular subalgebra J already defined.

3. Notations and miscellaneous results

3.1 Notations (summary)

From now on we shall work at the level of Grothendieck groups, or rings, but use for them the same notation as for the categories themselves. So, we have a commutative and associative algebra $\mathscr{A}$ with[5] a base λ_n, structure constants $(N_n)_{pq}$, an associative algebra $\mathcal{O}$, with a base o_x, structure constants $(O_x)_{yz}$, a vector space $\mathscr{E}$ with a base σ_a which is a module over $\mathscr{A}$ and $\mathcal{O}$, with structure constants $(F_n)_{ab}$ and $(S_x)_{ab}$. When $\mathscr{E}$ has self fusion, its structure constants are $(E_a)_{bc}$. The ring $\mathcal{O}$ is a $\mathscr{A}$ bimodule with structure coefficients $(W_{xy})_{m,n}$. The modular invariant partition function is $Z = W_{00}$. Like before, the notation $\mathscr{E}$ refers to a generic example, unless it denotes an exceptional case (the context should be clear). $\mathscr{E}$ being chosen, the numbers of irreducible objects in categories $\mathscr{A}$, $\mathscr{E}$ and $\mathcal{O}$ are respectively denoted r_A, $r = r_E$ and r_O. They are the number of vertices of the associated graphs. In the case of $\widehat{sl}(2)_k$, $r_A = k+1$. In the case of $\widehat{sl}(3)_k$, $r_A = (k+1)(k+2)/2$. The script notation using the level k as an index can be generalized to all $sl(N)$ theories but it is incompatible with the traditional notation for $sl(2)$ (the Dynkin diagrams), where the index refers to the rank. We have the identifications: $E_6 = \mathscr{E}_{10}$, $E_7 = \mathscr{E}_{16}$, $E_8 = \mathscr{E}_{28}$, $A_r = \mathscr{A}_{k=r-1}$, $D_{s+2} = \mathscr{D}_{k=2s}$. There are no $\mathscr{D}$ cases with odd level in the $sl(2)$ family. Notations for higher ADE diagrams of type $sl(3)$ were given in the previous section. Generalized Coxeter numbers are $\kappa = k+N$ for members of $sl(N)$ families.

3.2 Modular blocks

Call $Z = W_{00}$ the modular invariant. It is a matrix Z_{mn} indexed by (classes of) irreducible objects $m,n \in Irr(\mathscr{A})$. It can also be written as a sesquilinear quadratic form $\sum_{mn} \chi_{\overline{m}} Z_{\overline{m}n} \chi_n$ (the partition function). The following results are attributed to [34] and [4, 5, 6]: $r_E = Tr(Z)$, and $r_O = Tr(ZZ^t)$; the Grothendieck ring of $\mathscr{A}$ is commutative but the one of $\mathcal{O}$, which is not necessarily commutative, is isomorphic to the direct sum of matrix algebras of sizes Z_{mn}. For example, take $sl(2)$ at level 8; if we are given $Z = |\chi_0 + \chi_8|^2 + |\chi_2 + \chi_6|^2 + 2|\chi_4|^2$, we know a priori that $r_A = 9$, $r_E = 6$, $r_O = 12$, and that the algebra of quantum symmetries is isomorphic with $\bigoplus_{x=1}^{x=8} \mathbb{C}_x \oplus M(2, \mathbb{C})$. Actually this is the D_6 module of A_9. The collection K of those irreducible objects λ_n of $\mathscr{A}$ that appear on the diagonal of Z, with multiplicity Z_{nn}, is called the multiset[6] of exponents

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[5]In the previous section, we used the notation $\lambda_n, \lambda_a, \lambda_x$ for λ_n, σ_a, o_x.

[6]Exponents are conventionally defined as indices n shifted by $+1$ for $sl(2)$, or by $+(1,1)$ for $sl(3)$.

ture (compatibility) rests on the properties of the pairing, or, equivalently, on the properties of the coefficients[2] (Ocneanu cells) obtained by pairing two bases of matrix units[3] for the two products. Being obtained by pairing double triangles, Ocneanu cells (generalized $6J$ symbols) are naturally associated with tetrahedra with two types (black "b", or white "w") of vertices, so that edges bb, bw or ww refer to labels n, a, x of $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{O}$. Using the two product and coproduct structures[4] one can obtain representation categories $Rep(\mathscr{B})$ and $Rep(\widehat{\mathscr{B}})$ respectively equivalent to $\mathscr{A}_k$ and $\mathscr{O}$. As already discussed, the character ring of the first (the fusion algebra) is always described by a (generalized) Dynkin diagram of type $\mathscr{A}$. The character ring of the next (the algebra of quantum symmetries) is not always commutative, it has two generators, called chiral, in the case of $sl(2)$, together with their complex conjugated in the case of $sl(3)$. The Cayley graph of multiplication by the two chiral generators (two types of lines), called the Ocneanu graph of $\mathscr{E}$, encodes the structure. As already mentioned, $\mathscr{B}$ is weak. This should not be a surprise since not any monoidal category arises as representation category of a bialgebra, however (at least when the category has finitely many simple objects, and this is our case), it can be realized as the category of representations of a weak bialgebra. In the next paragraph, we shall see that the knowledge of a modular invariant is sufficient to reconstruct the character rings of $\mathscr{A}_k$ (that we already know), of $\mathscr{O}$, and the semisimple and co semisimple structure of $\mathscr{B}$. The $\mathscr{A}_k \times \mathscr{O}$ module corresponding to $\mathscr{E}$ itself can be recovered from the study of the source and target subalgebras of $\mathscr{B}$. Results obtained in operator algebra by [34] and [4, 5, 6] have been translated to a categorical language by [36]: choice of a braiding or of the opposite braiding in the category $\mathscr{A}_k$ can be used to construct two tensor functors from $(\mathscr{A}_k)$ to $\mathscr{O}$, called $\alpha^{L,R}$ ("alpha induction" in the language of [5]). Here our presentation differs from [36], because we find easier to think that there exists a functor $\mathscr{A}_k \times \mathscr{O} \times \mathscr{A}_k \mapsto \mathscr{O}$, so that the previous $\alpha^{L,R}$ are obtained as particular cases ($\mathbb{1}_A \times \mathbb{1}_O \times \mathscr{A}_k$ or to $\mathscr{A}_k \times \mathbb{1}_O \times \mathbb{1}_A$). At the level of Grothendieck rings, we have a bimodule property, that reads (we only use labels to denote the corresponding irreducible objects): $m x n = \sum_y (W_{xy})_{mn} y$, where m, n refer to irreducible objects of $\mathscr{A}_k$, x, y to irreducible objects of $\mathscr{O}$, and where W_{xy} constitute a family of so - called toric matrices, with matrix elements $(W_{xy})_{mn}$, again non negative integers. When both x and y refer to the unit object (that we label 0), one recovers the modular invariant partition function $Z = W_{00}$ of conformal field theory. As explained in [38], when one or two indices x and y are non trivial, toric matrices are interpreted as partition functions on a torus, in a conformal theory of type $\mathscr{A}_k$, with boundary conditions specified by $\mathscr{E}$, but with defects (one or two) specified by x and y. Only Z is modular invariant (it commutes with the generator S and T of $SL(2,\mathbb{Z})$ in the Hurwitz - Verlinde representation). Toric matrices were first introduced and calculated by Ocneanu (unpublished). Various methods to compute or define them can be found in [8], [38]. Ref. [11] gives explicit expressions for all W_{x0}, for all members of the $sl(2)$ family. The modular invariant partition functions themselves have been known for many years: we have the *ADE* classification of [7] for $sl(2)$, and the classification [21] for $sl(3)$, encoded by Di Francesco - Zuber diagrams. Left and right associativity constraints $(m(nxp)q) = (mn)x(pq)$ for the $\mathscr{A} \times \mathscr{A}$ bimodule structure of $\mathscr{O}$ can be written in terms of fu-

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[2]Constructions of $\mathscr{B}$, inspired from [35], and using these properties, were given in [38] and [13].

[3]Definition of cells involve normalization choices: the spaces H_{ab}^m are not always one-dimensional, moreover one may decide to use bases made of vector proportional to matrix units rather than matrix units themselves.

[4]In the operator algebra community, one would usually define a star operation and a scalar product on $\mathscr{B}$, so that both products could be defined on the same underlying vector space[35].

in [27] that $\mathscr{E}$ is monoidal (self-fusion) if and only if $\mathscr{F}$ is commutative. In the next section we shall describe $\mathscr{F}$ as an object in $\mathscr{A}$ for all $sl(2)$ and $sl(3)$ actegories, in particular for those that are monoidal, but we shall not use the above characterization. Its simple summands play the role of quantum Klein invariants. The same algebra $\mathscr{F}$, called Frobenius algebra, plays a prominent role in the approach of [20]. Notice that module categories associated with *ADE* Dynkin diagrams for $sl(2)$, or with Di Francesco - Zuber diagrams for $sl(3)$ are never modular (unless $\mathscr{E} = \mathscr{A}$ itself) but they contained a subcategory J, also denoted $Mod^0_{\mathscr{A}_k}(\mathscr{F})$ which is modular. When $\mathscr{E}$ is monoidal, the subring of its Grothendieck ring associated with this modular subcategory is called the modular subring (or subalgebra) and is also denoted J.

The third and final needed ingredient is the centralizer category of $\mathscr{E}$ with respect to the action of $\mathscr{A}_k$. It is sometimes called the "dual category" (not a very good name) and is defined as the category of module functors from $\mathscr{E}$ to itself: these endofunctors should be functors F "commuting" with the action of $\mathscr{A}_k$, i.e., such that $F(\lambda_n \otimes \lambda_a)$ is isomorphic with $\lambda_n \otimes F(\lambda_a)$, for $\lambda_n \in Ob(\mathscr{A}_k)$ and $\lambda_a \in Ob(\mathscr{E})$, via a family of morphisms c_{λ_n,λ_m} obeying triangular and pentagonal constraints. We simply call $\mathscr{O} = Fun_{\mathscr{A}_k}(\mathscr{E},\mathscr{E})$ this centralizer category[1], but one should remember that its definition involves both $\mathscr{A}_k$ and $\mathscr{E}$. Because of the previous compatibility property, if $\mathscr{E}$ is a left actegory over $\mathscr{A}_k$, it is probably better to consider it as a right actegory over $\mathscr{O}$ (actually over its opposite, since we are permuting the two factors). The category $\mathscr{O}$ is additive, semi-simple in our case, and monoidal (use composition of functors as tensor product). $\mathscr{E}$ is therefore both a module category over $\mathscr{A}_k$ and over $\mathscr{O}$. The Grothendieck group of $\mathscr{E}$ is therefore not only a $\mathbb{Z}_+$ module over the fusion ring, but also a $\mathbb{Z}_+$ module over the Grothendieck ring of $\mathscr{O}$, called the Ocneanu ring (or algebra) of quantum symmetries and denoted by the same symbol. Structure constants of the ring of quantum symmetries are encoded by matrices O_x, called "matrices of quantum symmetries"; structure constants of the module, with respect to the action of quantum symmetries, are encoded by the so called "dual annular matrices" S_x. The next problem is to find a way to describe explicitly this centralizer category. The solution lies in the construction of a finite dimensional weak bialgebra $\mathscr{B}$, which is going to be such that the monoidal category $\mathscr{A}_k$ can be realized as $Rep(\mathscr{B})$, and also such that the monoidal category $\mathscr{O}$ can be realized as $Rep(\widehat{\mathscr{B}})$ where $\widehat{\mathscr{B}}$ is the dual of $\mathscr{B}$. These two algebras are finite dimensional (actually semisimple in our case) and one algebra structure (say $\widehat{\mathscr{B}}$) can be traded against a coalgebra structure on its dual. $\mathscr{B}$ is a weak bialgebra, not a bialgebra, because $\Delta \mathbb{1} \neq \mathbb{1} \otimes \mathbb{1}$, where Δ is the coproduct in $\mathscr{B}$, and $\mathbb{1}$ is its unit. Actually, in our cases, it is not only a weak bialgebra but a weak Hopf algebra (we can define an antipode, with the expected properties [2], [31], [32]). One categorical construction of $\mathscr{B}$ is given in [36]. We propose another one that should lead to the same bialgebra, and may be simpler. Label irreducible objects λ_- of categories $\mathscr{A}_k$ by $m, n, \ldots$, of $\mathscr{E}$ by $a, b, \ldots$, and of $\mathscr{O}$ by $x, y, \ldots$. Call $H^m_{ab} = Hom(\lambda_n \otimes \lambda_a, \lambda_b)$, the "horizontal space of type n from a to b" (also called space of essential paths of type n from a to b, space of admissible triples, or triangles...) Call $V^x_{ab} = Hom(\lambda_a \otimes \lambda_x, \lambda_b)$ the "vertical space of type x from a to b". We just take these horizontal and vertical spaces as vector spaces and consider the graded sums $H^m = \sum_{ab} H^m_{ab}$ and $V^x = \sum_{ab} V^x_{ab}$. To construct the weak bialgebra, we take the (graded) endomorphism algebras $\mathscr{B} = \sum_m End(H^m)$ and $\widehat{\mathscr{B}} = \sum_x End(V^x)$. For obvious reasons, $\mathscr{B}$ and $\widehat{\mathscr{B}}$ are sometimes called "algebra of double triangles". Existence of the bialgebra struc-

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[1] For $sl(2)$, the structure of $Fun_{\mathscr{A}_k}(\mathscr{E}_1, \mathscr{E}_2)$, where $\mathscr{E}_{1,2}$ can be distinct module categories was obtained by [34].

correspond to the graphs A_r (with $k = r - 1$), D_{even}, with $k = 0 \mod 4$, E_6, with $k = 10$, and E_8, with $k = 28$. This was already known, at the level of rings, in [37], [16] and was proved, at the categorical level, by [27]. The cases D_{odd}, with level $k = 2 \mod 4$, and E_7, at level 16, are non monoidal actegories. $\mathscr{A}_k$ is always modular, but the corresponding actegories are not, even when they happen to be monoidal; however, they always contain a subcategory which is modular (we shall come back to it). At the level of graphs, the D diagrams (even of odd) are $\mathbb{Z}_2$ orbifolds of the A diagrams at the same level. In the case of $sl(3)$, the classification of $\mathbb{Z}_+$ modules over the corresponding fusion rings at level k is not tractable (or not useful), however there is another route stemming from the classification of $sl(3)$ modular invariants [21]. The graphs encoding all $sl(3)$ module categories are called the di Francesco - Zuber diagrams [16]. Existence of the corresponding categories was shown by A. Ocneanu [33], actually one of the candidates had to be discarded, very much like the tadpole graphs of $sl(2)$. Several $sl(3)$ actegories have monoidal structure (graphs with self - fusion), namely: $\mathscr{A}_k$ itself, the $\mathscr{D}_k$, whose McKay diagrams are $\mathbb{Z}_3$ orbifolds of those of $\mathscr{A}_k$, when k is divisible by 3, and three exceptional cases called $\mathscr{E}_5$, $\mathscr{E}_9$ and $\mathscr{E}_{21}$, at levels 5, 9 and 21. The other actegories (not monoidal) are: the series $\mathscr{A}_k^*$, for which the number of simple objects is equal to the number of self dual simple objects in $\mathscr{A}_k$, the $\mathscr{D}_k$ series, when $k = 1$ or $2 \mod 3$, the series $\mathscr{D}_k^*$, for all k, and several modules of exceptionals called $\mathscr{E}_5^*$, $\mathscr{E}_9^*$, $\mathscr{D}_9^t$ (a generalization of E_7) and $\mathscr{D}_9^{t*}$. Some of the graphs of that system have double lines, like $\mathscr{E}_9$, so that it is not appropriate to say that Di Francesco - Zuber diagrams are the "simply laced" diagrams of type $sl(3)$: better to call them "higher ADE". In all cases however, with self-fusion or not, the rigidity property implied by $\mathscr{A}_k$ holds (the condition $(F_{\bar{n}})_{ab} = (F_n)_{ba}$ does not forbid double lines). Taking quotients of the above diagrams by discrete groups gives higher analogues of the non *ADE* Dynkin diagrams which define modules over the Grothendieck ring of $\mathscr{A}_k$, but the rigidity condition is not satisfied and the corresponding category should not exist (see however the previous footnote). Classification of $sl(4)$ module categories is also claimed to be completed [33].

Let us pause to develop a tentative pedagogical analogy that should make the next result look natural. Consider a finite group, a subgroup, and the corresponding homogenous space (the space of right cosets, for example). The group can be fibered as a principal bundle over the coset space, the structure group being the chosen subgroup. This subgroup has representations, in particular irreducible ones. For any such, let us say a, one can build an associated vector bundle, and consider the space Γ_a of its sections. It carries a representation of the big group, although not irreducible (theory of induced representations). For the particular choice of the trivial representation of the small group, call it 0, the space of sections Γ_0 is an algebra, namely the space of functions $\mathscr{F}$ on the coset. Moreover every space of sections, say Γ_a, is a module over the algebra $\mathscr{F}$. In our case we have a non commutative geometry which is still, in a sense, finite, but the situation is similar. Simple objects, labelled by a, of the module category $\mathscr{E}$ can be thought as points of a graph, as irreducible representations of a would-be quantum subgroup of $su(2)$ or $su(3)$ at some root of unity, or as spaces of sections Γ_a above a quantum space determined by the pair $(\mathscr{A}_k, \mathscr{E})$, i.e., as modules over some particular algebra $\mathscr{F}$ which is an algebra in a monoidal category ($\mathscr{A}_k$ in our case), and right modules over $\mathscr{F}$ form an additive category $Mod_{\mathscr{A}_k}(\mathscr{F})$. Reciprocally, we have the following theorem proved in [36] under actually weaker assumptions than those listed previously for the action of ($\mathscr{A}_k$ on $\mathscr{E}$: there exists a semisimple indecomposable algebra $\mathscr{F}$, belonging to the set of objects of $\mathscr{A}_k$ such that the module categories $\mathscr{E}$ and $Mod_{\mathscr{A}_k}(\mathscr{F})$ are equivalent. It is shown

both orientations) and are oriented in the case of $sl(3)$. Irreducible representations are denoted by λ_p, with $p \geq 0$, in the first case, and λ_{pq}, with $p, q \geq 0$ in the next. Notice the shift of indices: the unit $\mathbb{1}$ of the category corresponds to λ_0 or to λ_{00}. One should certainly keep in mind the distinction between the monoidal category, with its objects and morphisms, and its Grothendieck ring, but they will often be denoted by the same symbol. Actually, the Mckay graph itself is also denoted $\mathscr{A}_k$. In the $sl(2)$ case, it can be identified with the Coxeter - Dynkin diagram A_r, with $r = k + 1$. In both $sl(2)$ and $sl(3)$ cases, it is a truncated Weyl chamber at level k (a Weyl alcove). It is often useful to think of $\mathscr{A}_k$ as a category of representations of a would-be quantum object, that can be also denoted by the same symbol, although this can be quite misleading.

The next ingredient is a category $\mathscr{E}$, not necessarily monoidal, but we suppose that it is additive, semisimple and indecomposable, on which the previous one $\mathscr{A}_k$ (which is monoidal) acts. Action of a monoidal category on a category has been described, under the name "module categories" by [19], and, in our context, by [36]. Using a slightly shorter description, we may say that we have such an action when we are given a (monoidal) functor from $\mathscr{A}_k$ to the (monoidal) category of endofunctors of $\mathscr{E}$. The reader can think of this situation as being an analogue of the action of a group on a given space. Actually, it may be sometimes interesting to think that $\mathscr{E}$ can be acted upon in more than one way, so that we can think of the action of $\mathscr{A}_k$ as a particular "enrichment" of $\mathscr{E}$. The word "module" being used in so many different ways, we prefer to say that we have an action, or that $\mathscr{E}$ is an actegory (another nice substantive coined by R. Street), and we shall freely use both terminologies. Irreducible objects of $\mathscr{E}$ are boundary conditions for the corresponding Conformal Field Theory specified by $\mathscr{A}_k$. It is useful to assume, from now on, that the category $\mathscr{E}$ is indecomposable (it is not equivalent to the direct sum of two non trivial categories with $\mathscr{A}_k$ action). Since $\mathscr{E}$ is additive, we have a Grothendieck group, also denoted by the same symbol. Because of the existence of an action, this (abelian) group has to be a module over the Grothendieck ring of $\mathscr{A}_k$, and it is automatically a $\mathbb{Z}_+$ module: the structure constants of the module, usually called annulus coefficients in string theory articles [20], and described by (annular) matrices $F_n = (F_n)_{ab}$, are non negative integers. Let us consider the class of a particular simple object of $\mathscr{A}_k$, namely the generator $n = 1$, for $sl(2)$ or $n = (1, 0)$ for $sl(3)$ (one of the two conjugated fundamental irreps). We interpret F_1 (or $F_{(1,0)}$) as the adjacency matrix of a graph, called the McKay graph of the category $\mathscr{E}$. The rigidity property of $\mathscr{A}_k$ implies that the module $\mathscr{E}$ is rigid (or based [36]). In other words: $(F_{\overline{n}})_{ab} = (F_n)_{ba}$. In the case of $sl(2)$, $\mathbb{Z}_+$ modules for fusion rings at level k have been classified by [16] and [18]; McKay graphs are all the Coxeter - Dynkin diagram, plus some diagrams with loops (tadpoles), and $F_1 = 2$ Cartan matrix $- \mathbb{1}$. The rigidity condition in the case of $sl(2)$ implies that the matrix F_1 is symmetric; this condition implies that non simply laced diagrams B_r, C_r, F_4 and G_2 should be rejected (this only means that they do not fit in the presented framework: see also [24] Êand references therein): we are left with the *ADE* diagrams and the tadpoles. A detailed analysis of the situation ([34], [36]) shows that the tadpole graphs do not give rise to any category endowed with an action of the monoidal categories of type $sl(2)$. As already mentioned, the category $\mathscr{E}$ is not required to be monoidal, but there are cases where it is, so that it has a tensor product, compatible with the $\mathscr{A}_k$ action. In another terminology, one says that the corresponding graphs have self - fusion (this is also related to the concept of flatness), or that they define "quantum subgroups" of $sl(2)$, whereas the others are only "quantum modules". Like in the classical situation, we have a restriction functor $\mathscr{A}_k \mapsto \mathscr{E}$ and an induction functor $\mathscr{E} \mapsto \mathscr{A}_k$. The cases where $\mathscr{E}$ is monoidal

1. Introduction

Our main and original purpose was to gather tables giving characteristic numbers for $sl(3)$ boundary conformal field theories on a torus: dimensions of the blocks of the associated weak bialgebras, global quantum dimensions ("quantum mass"), induction tables, etc. a material that is hitherto scattered in a number of publications or unavailable. Many formulae given here are new. The corresponding data for $sl(2)$ does not use much space, and we could easily summarize it, but the situation is different with $sl(3)$: to keep the size of this paper below the prescribed limits, we had sometimes to restrict ourselves to the case of Di Francesco - Zuber graphs with self-fusion (other cases will be described in [17]) and give only partial results for induction tables. Because we also needed a short introductory section discussing the underlying algebraic structures and giving our notations, we decided to describe boundary conformal field theories on a torus in terms of module categories (action of a monoidal category on a category) mostly extracting the relevant material from [36], while adding few things like the construction of weak bialgebras in terms of Hom spaces, or the description of the bimodule structure $\mathscr{A} \times \mathscr{O} \times \mathscr{A} \mapsto \mathscr{O}$, where $\mathscr{A}$ is the fusion algebra and $\mathscr{O}$ is the Ocneanu algebra of quantum symmetries. Independently of the interest of the tables of results, we hope that our presentation will provide a bridge between several mathematical or physical communities interested in those topics.

2. The stage

In this paper $\mathscr{A}_k$ is the fusion category of the affine algebra $\widehat{sl}(2)$, or $\widehat{sl}(3)$, at level k, or equivalently, the category of irreducible representations with non-zero q-dimension for the quantum groups $sl(2)$ or $sl(3)$ at roots of unity (set $q = exp(i\pi/\kappa)$, with $\kappa = k+2$ for $sl(2)$ and $\kappa = k+3$ for $sl(3)$. This category is additive (existence of $\oplus$), monoidal (existence of $\otimes : \mathscr{A}_k \times \mathscr{A}_k \mapsto \mathscr{A}_k$, with associativity constraints, unit object, etc.), tensorial ($\otimes$ is a bifunctor), complex-linear, rigid (existence of duals), finite (finitely many irreducible objects), and semisimple, with irreducible unit object. It is also modular (braided, balanced, with invertible S-matrix) and ribbon (or tortile). We refer to the literature [28], [26], [1] for a detailed description of these structures. The Grothendieck ring of this monoidal category comes with a special basis (corresponding to simple objects), it is usually called the fusion ring, or the Verlinde algebra. The corresponding structure constants, encoded by the so - called fusion matrices $(N_n)_q^p$, are therefore non - negative integers: NIM-reps in CFT terminology. The rigidity property of the category implies that $(N_{\overline{n}})_{pq} = (N_n)_{qp}$, where $\overline{n}$ refers to the dual object i.e., , in our case, to the conjugate representation, so that the fusion ring is automatically a based $\mathbb{Z}_+$ ring in the sense of [36] (maybe it would be better to call it "rigid"). In the case of $sl(2)$, this is a ring with one generator (corresponding to the fundamental representation), and fusion matrices are symmetric, because $\overline{n} = n$. In the case of $sl(3)$, it has two generators (corresponding to the two fundamental representations) that are conjugate to one another. Multiplication by a generator (choose one of the possible two in the $sl(3)$ case) is encoded by a particular fusion matrix; it is a finite size matrix of dimension $r \times r$, with $r = k+1$ for $sl(2)$ and $r = k(k+1)/2$ for $sl(3)$. Since its elements are non negative integers, it can be interpreted as the adjacency matrix of a graph, which is the Cayley graph of multiplication by this generator, that we call the McKay graph of the category. Edges are non oriented in the case of $sl(2)$ (rather, they carry

Quantum Symmetries of $sl(2)$ and $sl(3)$ graphs

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After summarizing the description, in terms of action of categories, of the algebraic structures underlying $sl(2)$ and $sl(3)$ boundary conformal field theories on a torus, in particular the associated quantum groupoid together with its fusion algebra and algebra of quantum symmetries, we provide tables of dimensions describing the semisimple and co-semisimple blocks of the weak bialgebras, tables of quantum dimensions and tables describing induction - restriction. For reasons of size, the $sl(3)$ tables of induction are only given for theories with self-fusion (existence of a monoidal structure).

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