Cold black holes and conformal continuations

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Abstract

We study Einstein gravity minimally coupled to a scalar field in a static, spherically symmetric space-time in four dimensions. Black hole solutions are shown to exist for a phantom scalar field whose kinetic energy is negative. These "scalar black holes" have an infinite horizon area and zero temperature $T_{\rm H}$ and are termed "cold black holes" (CBHs). The relevant explicit solutions are well-known in the massless case (the so-called anti-Fisher solution), and we have found a particular example of a CBH with a nonzero potential $V(\phi)$. All CBHs with $V(\phi) \not\equiv 0$ are shown to behave near the horizon quite similarly to those with a massless field. The above solutions can be converted by a conformal transformation to Jordan frames of a general class of scalar-tensor theories of gravity, but CBH horizons in one frame are in many cases converted to singularities in the other, which gives rise to a new type of conformal continuation.

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1 Introduction

The conventional idea of a black hole (BH) implies a space-time singularity hidden beyond an event horizon [1], a hypersurface which separates an external region, containing spatial infinity, from an internal region, invisible to an external observer. The well-known BH solutions of general relativity (Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman), have been generalized in many contexts, such as the presence of scalar fields of various nature, non-linear gravity theories, scalar-tensor theories etc. (see, e.g., [2] and references therein). Their different properties rise the question of an extension of the black hole notion itself. An example of "exotic" black holes are the so-called "cold black holes", obtained in scalar-tensor theories (STT) in general and in the Brans-Dicke theory in particular [3–5].

The static, spherically symmetric solutions of the Brans-Dicke theory reveal a large class of objects with black hole properties. Not all of them exhibit a singularity beyond a horizon. However, the horizon surface has in all such cases an infinite area. Moreover, all such horizons have zero surface gravity and hence zero Hawking temperature. It is for this reason that they have been named "cold black holes" (CBHs).

As other STT, the Brans-Dicke theory can be re-defined using a conformal mapping to the so-called Einstein frame, or picture, in which the nonminimal coupling between the scalar field and the curvature, which is an essential feature of an STT, is ruled out, resulting in Einstein gravity with a minimally coupled scalar field. The scalar field energy is positive (its kinetic term has its usual sign) if the Brans-Dicke coupling constant $\omega > -3/2$, and is negative if $\omega < -3/2$. In the latter case, the kinetic term has a "wrong" sign, and the theory is called anomalous, or phantom. Such kind of theories have recently become quite fashionable for both theoretical and observational reasons. The theoretical reasons are connected with the ghost condensation and tachyonic fields that result from string theories [6, 7]. From the observational viewpoint, recent analysis of the type Ia supernova and CMB data indicates that perhaps the best fit is given by *phantom fields* [8–11], of which a scalar field with the "wrong" sign of the kinetic term is the simplest example.

In the present work, we show that, as opposed to what has been believed [12], this Einstein-scalar field system, with a massless scalar field minimally coupled to gravity, admits black hole solutions, though this happens for a phantom scalar field only. These "scalar" black holes also have an infinite horizon area and zero temperature. Such solutions can be interpreted as Einstein-frame solutions of any STT and transformed to the Jordan frame by the inverse conformal mapping.

However, an interesting aspect of this procedure is the non-existence of a one-to-one correspondence between black holes in the Einstein and Jordan pictures, as we shall see using the Brans-Dicke theory as an example. The reason is that, generically, the above conformal mapping (direct or inverse) converts a CBH horizon to a singularity. Such a situation is an example of a *conformal continuation* (CC). This phenomenon was treated in some detail in the framework of STT and $f(R)$ theories of gravity in Refs. [13–16]. The point is that a conformal mapping between two manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$ comprises a one-to-one correspondence between the respective points and preserves the causal structure only if the conformal factor is everywhere smooth and finite. If, however, the conformal factor somewhere becomes infinite or zero, the mapping, in general, links only a portion of $\mathbb{M}_1$ to a portion of $\mathbb{M}_2$. In some cases, a singularity in $\mathbb{M}_1$ may be mapped into a regular surface in $\mathbb{M}_2$, then $\mathbb{M}_2$ continues beyond this surface and may have more complex global properties as compared to $\mathbb{M}_1$. The new region may, in particular, contain a singularity, a horizon or another spatial infinity.

From a more general point of view, the possible existence of CCs may mean that the observed Universe is only a region of a real, greater Universe which should be described in another, more fundamental conformal frame than the one related to our measurement instruments. Detailed discussions of the physical meaning and role of different conformal frames in the description of the Universe may be found in Refs. [17, 18].

We shall find in this study that the appearance of CCs in the context of static, spherically symmetric solutions of STT is closely related to the occurrence of a peculiar type of space-time singularities, where all curvature invariants remain finite but the analyticity of the metric is lost, which means that the manifold terminates. It is this kind of singularities that are in many cases removed by conformal mappings, leading to a CC.

We here restrict ourselves to the search and discussion of CBH solutions in the Einstein-scalar field system. A more general analysis of the same system with nonzero potentials $V(\phi)$ (but without discussing infinite-area horizons) has been performed in Ref. [19], where it was shown that phantom scalar fields with appropriate potentials can form as many as sixteen types of regular static, spherically symmetric self-gravitating configurations, including regular black holes with nonzero temperature.

The paper is organized as follows. In the next section, we reproduce the basic equations of a general STT for static, spherically symmetric metrics. Sec. 3 describes the basic properties of the static, spherically symmetric solutions to the Einstein-massless scalar field system, which simultaneously represent the Einstein-frame solutions to a general class of STT with zero scalar field potential. We single out a particular discrete family of solutions corresponding to CBHs. In Sec. 4, we compare the STT solutions in the Einstein and Jordan conformal frames, using as an example the Brans-Dicke theory, and pay special attention to CBH solutions in both frames. We discuss a new type of conformal continuations that appears in this context and make some remarks on the thermodynamical properties of CBHs, e.g., concerning the conformal invariance of the Hawking temperature. Sec. 5 is devoted to scalar-vacuum configurations with nonzero potentials $V(\phi)$. We show that the nature of CBH horizons is basically the same for both zero and nonzero $V(\phi)$ and present a specific example of a CBH with $V(\phi) \not\equiv 0$. In Sec. 5, we formulate our conclusions, and, finally, in the Appendix we present some general relations for static, spherically symmetric metrics and show, in particular, that horizons with an infinite area always possess zero Hawking temperature.

2 Scalar-tensor theory: basic equations

In the general (Bergmann-Wagoner-Nordtvedt) 4-dimensional STT, the action in the pseudo-Riemannian manifold $\mathbb{M}_{\rm J}[g]$ has the form

$$S_{\rm STT} = \int d^4x \sqrt{g} [f(\varphi)\mathcal{R} + h(\varphi)(\partial\varphi)^2 - 2U(\varphi) + L_m], \tag{1}$$

where $g_{\mu\nu}$ is the metric, $\mathcal{R} = \mathcal{R}[g]$ is the scalar curvature, $g = |\det g_{\mu\nu}|$, f, h and U are functions of the real scalar field φ, $(\partial\varphi)^2 = g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$, and L_m is the matter Lagrangian. The manifold $\mathbb{M}_{\rm J}[g]$ with the metric $g_{\mu\nu}$ comprises the so-called Jordan conformal frame.

The standard transition to the Einstein frame $\mathbb{M}_{\rm E}[\overline{g}]$ [20],

$$g_{\mu\nu} = |f(\varphi)|^{-1}\overline{g}_{\mu\nu}, \tag{2}$$

$$\frac{d\phi}{d\varphi} = \frac{\sqrt{|l(\varphi)|}}{f(\varphi)}, \qquad l(\varphi) = fh + \frac{3}{2}\Big(\frac{df}{d\varphi}\Big)^2, \tag{3}$$

removes the nonminimal scalar-tensor coupling expressed in a φ-dependent coefficient before $\mathcal{R}$. Putting $L_m = 0$ (vacuum), one can write the action (1) in terms of the new metric $\overline{g}_{\mu\nu}$ and the new scalar field ϕ as follows (up to a boundary term):

$$S_{\rm E} = \int d^4x\sqrt{\overline{g}}\Big\{(\mathrm{sign}\, f)[\overline{\mathcal{R}} + (\mathrm{sign}\, l)(\partial\phi)^2] - 2V(\phi)\Big\}, \tag{4}$$

where the determinant $\overline{g}$, the scalar curvature $\overline{\mathcal{R}}$ and $(\partial\phi)^2$ are calculated using $\overline{g}_{\mu\nu}$ and

$$V(\phi) = |f|^{-2}\, U(\varphi). \tag{5}$$

Note that $\varepsilon := \mathrm{sign}\, l = -1$ corresponds to the so-called anomalous STT, with a wrong sign of scalar field kinetic energy in the Einstein frame, while $\mathrm{sign}\, f = -1$ means that the effective gravitational constant in the Jordan frame (which can be defined as $1/f$ up to a constant factor) is negative. So the normal choice of signs is $\mathrm{sign}\, l = \mathrm{sign}\, f = 1$. Nevertheless, theories admitting $\varepsilon = -1$ and/or $f < 0$ possess many features of interest, worth studying. With $f > 0$, the action (4) describes Einstein gravity minimally coupled to a self-interacting scalar field, which is called normal for $\varepsilon = +1$ and phantom for $\varepsilon = -1$. The field equations are

$$\begin{aligned} R_{\mu\nu} &= \varepsilon\phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}V(\phi), \\ \nabla^\mu\nabla_\mu\phi &= -\varepsilon dV/d\phi. \end{aligned} \tag{6}$$

Consider the general static, spherically symmetric metric in $\mathbb{M}_{\rm E}$

$$ds^2 = \mathrm{e}^{2\gamma}dt^2 - \mathrm{e}^{2\alpha}du^2 - \mathrm{e}^{2\beta}d\Omega^2, \tag{7}$$

where $\gamma = \gamma(u)$, $\alpha = \alpha(u)$ and $\beta = \beta(u)$, u is an arbitrary radial coordinate. Let us assume $f > 0$ but admit $\varepsilon = \pm 1$. The field equations are

$$\gamma'' + \gamma'(\gamma' - \alpha' + 2\beta') = -\,\mathrm{e}^{2\alpha}V(\phi), \tag{8}$$

$$\gamma'' + 2\beta'' + \gamma'^2 + 2\beta'^2 - \alpha'(\gamma' + 2\beta') = -\varepsilon\phi'^2 - \mathrm{e}^{2\alpha}V(\phi), \tag{9}$$

$$\beta'' + \beta'(\gamma' - \alpha' + 2\beta') - \mathrm{e}^{2\alpha-2\beta} = -\,\mathrm{e}^{2\alpha}V(\phi), \tag{10}$$

$$(\,\mathrm{e}^{\gamma-\alpha+2\beta}\phi')' = \varepsilon\,\mathrm{e}^{\gamma+2\beta+\alpha}dV/d\phi, \tag{11}$$

where the prime denotes d/du. Eqs. (8)–(11) can be simplified by making a special choice of the radial coordinate u. Thus, choosing the quasiglobal coordinate $u = \rho$ defined by the condition $\alpha + \gamma = 0$ (it is particularly convenient for considering metrics with Killing horizons) and denoting

$$\mathrm{e}^{2\gamma} = \mathrm{e}^{-2\alpha} \equiv A(\rho), \qquad \mathrm{e}^{\beta} = r(\rho), \tag{12}$$

it is straightforward to bring Eqs. (8)–(11) to the form

$$(A\, r^2\, \phi')' = \varepsilon r^2\, V_\phi, \tag{13}$$

$$(A'\, r^2)' = -2V\, r^2, \tag{14}$$

$$2r''/r = -\varepsilon\, \phi'^2, \tag{15}$$

$$(r^2)''\, A - A''\, r^2 = 2. \tag{16}$$

Eq. (16) is once integrated giving

$$(A/r^2)' = 2(\rho_0 - \rho)/r^4, \qquad \rho_0 = \mathrm{const}. \tag{17}$$

Solutions to these equations with $V(\phi) \equiv 0$ are well known [21–23]. For $V(\phi) \not\equiv 0$, a general analysis has been performed in [19], establishing the existence of regular configurations including BHs which can contain one or two horizons with finite area. Complete analytical solutions can only be found for special choices of $V(\phi)$.

Given a solution to Eqs. (8)–(11) or (13)–(16), the corresponding solution in the original, Jordan picture is easily obtained using the inverse transformation (2).

3 Fisher and anti-Fisher solutions

Let us reproduce the well-known solutions to Eqs. (8)–(11) for zero potential, $V \equiv 0$. In case $\varepsilon = +1$ the solution was found by I.Z. Fisher in 1947 [21] and afterwards repeatedly re-discovered. For $\varepsilon = -1$ the corresponding solution was first obtained, to our knowledge, by Bergmann and Leipnik [22]. However, these authors used the curvature coordinates [i.e., the condition $u \equiv r$ in terms of the metric (7)] which are not well suited for the problem, and this was maybe a reason for the lack of a clear interpretation of the solutions.

3.1 General features

The solution can be written jointly for $\varepsilon = \pm 1$ if one uses the harmonic coordinate u in the metric (7), corresponding to the coordinate condition $\alpha(u) = \gamma(u) + 2\beta(u)$ [23]:

$$ds^2 = \mathrm{e}^{-2mu} dt^2 - \frac{\mathrm{e}^{2mu}}{s^2(k,u)} \left[\frac{du^2}{s^2(k,u)} + d\Omega^2\right], \qquad \phi = Cu, \tag{18}$$

where the integration constants m (the Schwarzschild mass), C (the scalar charge) and k are related by

$$2k^2 \operatorname{sign} k = 2m^2 + \varepsilon C^2, \tag{19}$$

and the function $s(k,u)$ is defined as follows:

$$s(k,u) = \begin{cases} k^{-1} \sinh ku, & k > 0 \\ u, & k = 0 \\ k^{-1} \sin ku, & k < 0. \end{cases} \tag{20}$$

The coordinate u is defined in the whole range $u > 0$ for $k \geq 0$ and in the range $0 < u < \pi/|k|$ for $k < 0$. The value $u = 0$ corresponds to flat spatial infinity, so that at small u the spherical radius is $r(u) \approx 1/u$, and the metric is approximately Schwarzschild, with $g_{tt} \approx 1 - 2m/r$.

In case $k > 0$, it is helpful to pass over to the quasiglobal coordinate ρ by the transformation

$$\mathrm{e}^{-2ku} = 1 - 2k/\rho \equiv P(\rho), \tag{21}$$

and the solution takes the form

$$ds^2 = P^a dt^2 - P^{-a} d\rho^2 - P^{1-a} \rho^2 d\Omega^2, \qquad \phi = -\frac{C}{2k} \ln P(\rho), \tag{22}$$

with the constants related by

$$a = m/k, \qquad a^2 = 1 - \varepsilon C^2/(2k^2). \tag{23}$$

Fisher's solution [21] corresponds to $\varepsilon = +1$, hence according to (19), it consists of a single branch $k > 0$ and, in (22), $|a| < 1$. It is defined in the range $\rho > 2k$, while $\rho = 2k$ is a naked central ($r = 0$) singularity which is attractive for $m > 0$ and repulsive for $m < 0$. The Schwarzschild solution is restored by making $C = 0$, $a = 1$ for $m > 0$ and by $C = 0$, $a = -1$ for $m < 0$.

The solution for $\varepsilon = -1$ (that is, for a phantom scalar field) may be conveniently termed **the anti-Fisher solution**, by analogy with de Sitter and anti-de Sitter. Its properties are more diverse and interesting. According to three variants of the function (20), this solution splits into three branches with the following main properties.

(a) $k > 0$: the solution again has the form (22), but now $|a| > 1$. For $m < 0$, that is, $a < -1$, we have, just as in the Fisher solution, a repulsive central singularity at $\rho = 2k$.

The situation is, however, drastically different for $a > 1$. Indeed, the spherical radius r then has a finite minimum at $\rho = \rho_{\rm th} = (a+1)k$, corresponding to a throat of the size

$$r(\rho_{\rm th}) = r_{\rm th} = k(a+1)^{(a+1)/2}(a-1)^{(1-a)/2}, \tag{24}$$

and tends to infinity as $\rho \to 2k$. Moreover, for $a = 2,\ 3,\ \ldots$ the metric exhibits a horizon of order a at $\rho = 2k$ and admits a continuations to smaller ρ. A peculiarity of such horizons is their infinite area. Such horizons have been termed type B [4, 5] horizons, to distinguish them from ordinary, "type A" horizons of finite area. The whole asymptotically flat configurations with type B horizons were named cold black holes (CBHs) since all of them have zero Hawking temperature.

Furthermore, all Kretschmann scalar constituents K_i (see Eqs. (A.4) in the Appendix) behave as P^{a-2} as $\rho \to 2k$ and $P \to 0$. An exception is the value $a = 1$, in which case $C = 0$, $\phi \equiv 0$, and the Schwarzschild solution is reproduced. Hence, at $\rho = 2k$ the metric has a curvature singularity if $a < 2$ (except for $a = 1$), a finite curvature if $a = 1$ and $a = 2$ and zero curvature if $a > 2$.

For non-integer $a > 2$, the qualitative behavior of the metric as $\rho \to 2k$ is the same as near a type B horizon, but a continuation beyond it is impossible due to non-analyticity of the function $P^a(\rho)$ at $\rho = 2k$. Since geodesics terminate there at a finite value of the affine parameter, this is a space-time singularity (a *singular horizon* as it is named in the Appendix) even though the curvature invariants tend there to zero.

(b) $k = 0$: the solution is defined in the range $u \in \mathbb{R}_+$ and is rewritten in terms of the quasiglobal coordinate $\rho = 1/u$ as follows:

$$ds^2 = {\rm e}^{-2m/\rho}dt^2 - {\rm e}^{2m/\rho}[d\rho^2 + \rho^2 d\Omega^2], \qquad \phi = C/\rho. \tag{25}$$

As before, $\rho = \infty$ is a flat infinity, while at the other extreme, $\rho \to 0$, the behavior is different for positive and negative mass. Thus, for $m < 0$, $\rho = 0$ is a singular center ($r = 0$ and all K_i are infinite). On the contrary, for $m > 0$, $r \to \infty$ and all $K_i \to 0$ as $\rho \to 0$. This is again a singular horizon: despite the vanishing curvature, the non-analyticity of the metric in terms of ρ makes its continuation impossible.

(c) $k < 0$: the solution describes a wormhole with two flat asymptotics at $u = 0$ and $u = \pi/|k|$. The metric has the form

$$\begin{aligned} ds^2 &= {\rm e}^{-2mu}dt^2 - \frac{k^2\,{\rm e}^{2mu}}{\sin^2(ku)}\Big[\frac{k^2\,du^2}{\sin^2(ku)} + d\Omega^2\Big] \\ &= {\rm e}^{-2mu}dt^2 - {\rm e}^{2mu}[d\rho^2 - (k^2+\rho^2)d\Omega^2], \end{aligned} \tag{26}$$

where u is expressed in terms of the quasiglobal coordinate ρ, defined on the whole real axis $\mathbb{R}$, by $|k|u = \cot^{-1}(\rho/|k|)$. If $m > 0$, the wormhole is attractive for ambient test matter at the first asymptotic ($\rho \to \infty$) and repulsive at the second one ($\rho \to -\infty$), and vice versa in case $m < 0$. For $m = 0$ one obtains the simplest possible wormhole solution, called the Ellis wormhole, although Ellis [24] actually discussed these wormhole solutions with any m.

The wormhole throat occurs at $\rho = m$ and has the size

$$r_{\rm th} = (m^2 + k^2)^{1/2} \exp\left(\frac{m}{k}\cot^{-1}\frac{m}{k}\right). \tag{27}$$

3.2 Cold black holes in the anti-Fisher solution

Among different branches of the anti-Fisher solution, of greatest interest for us is the case of CBHs. Let us briefly discuss their structure and properties.

For odd a, the principal geometric and causal properties, including the Carter-Penrose diagram, coincide with those of the Schwarzschild metric. Thus, at $\rho < 2k$, ρ is a temporal coordinate, t spatial, the space-time is homogeneous and anisotropic, corresponding to the Kantowski-Sachs type of anisotropic cosmologies. The singularity at $\rho = 0$ $(r = 0)$ is spacelike (cosmological) and is reached by all timelike geodesics in a finite time interval after crossing the horizon.

For even a, the Penrose diagram is the same as that of the extreme Reissner-Nordström space-time; however, the physical meaning of the regions where $\rho < 2k$ is quite different. Since g_{22} and g_{33} change their sign at the horizon, the metric at $\rho < 2k$ has the signature $(-\ +\ +\ +)$ instead of $(+\ -\ -\ -)$ at large ρ. The Lorentzian nature of space-time is still preserved, and one can verify that all geodesics are continued smoothly from one region to the other (the geodesic equations depend only on the Christoffel symbols and are invariant under the anti-isometry $g_{\mu\nu} \to -g_{\mu\nu}$). The time coordinate in that region is ρ since $g_{\rho\rho} < 0$ while the other diagonal components of $g_{\mu\nu}$ are positive. Thus, just as for odd a, we have there a Kantowski-Sachs type cosmology with a spacelike singularity at $\rho = 0$ $(r = 0)$. The direction of the arrow of time can be arbitrary there since timelike geodesics that penetrate from the static region become there spacelike (one cannot say for them where is the past and where is the future), and can even avoid the singularity.

The properties of the scalar field are not less exotic. According to (22), $\phi \to \infty$ as $\rho \to 2k$; this, however, does not contradict the regularity of the surface $\rho = 2k$ for $a \geq 2$ since the energy density

$$T_0^0 = -\frac{1}{2}A\phi'^2 = -\frac{C^2}{2}\frac{(\rho - 2k)^{a-2}}{\rho^{a+2}}, \tag{28}$$

as well as the other components of T_μ^ν, are finite there (recall that for $a < 2$ the curvature invariants also diverge, together with T_μ^ν). Thus the infinite value of ϕ does not prevent the continuation of the space-time manifold to smaller ρ, where the solution is valid with $\phi = -(C \ln |P|)/(2k)$. On the other hand, the total scalar field energy, calculated as the conserved quantity corresponding to the timelike Killing vector, turns out to be infinite in the static region independently of a:

$$\mathcal{E} = \int T_0^0 \sqrt{g} d^3x = -2\pi C^2 \int \frac{d\rho}{\rho(\rho - 2k)}, \tag{29}$$

and the integral logarithmically diverges at $\rho = 2k$. The divergence is related to the infinite spatial volume: the integral $\int \sqrt{{}^3g} d^3x$ diverges near $\rho = 2k$ even stronger than (29).

4 Comparison with the Brans-Dicke theory. Conformal continuations

4.1 Jordan picture in the Brans-Dicke theory

The (anti-)Fisher solution, being a solution of general relativity with a massless, minimally coupled scalar field, is simultaneously a solution of an arbitrary STT in its Einstein picture. Let us discuss the corresponding Jordan picture, for certainty, in the context of the simplest and most well-known STT, namely, the Brans-Dicke theory. The latter corresponds to the choice

$$f(\varphi) = \varphi, \qquad h(\varphi) = \omega/\varphi \tag{30}$$

in (1), ω being the Brans-Dicke coupling constant; we also take the massless version of the theory, $U(\varphi) \equiv 0$, to deal with counterparts of the (anti-)Fisher solution.

Since we are only interested in CBHs, let us restrict ourselves to solutions with $k > 0$, given by Eq. (22) with (21). Then, the Jordan-frame solution of the Brans-Dicke theory may be written in the form

$$ds_J^2 = P^{-\xi} ds_E^2 = P^{a-\xi} dt^2 - P^{-a-\xi} d\rho^2 - P^{1-\xi-a} \rho^2 d\Omega^2, \tag{31}$$

$$\varphi = \exp [\phi/\sqrt{|\omega + 3/2|}] = P^{\xi}, \tag{32}$$

where the parameter ξ is related to a and ω by

$$(3 + 2\omega)\xi^2 = 1 - a^2. \tag{33}$$

Conditions for finding black holes in this solution have been discussed in Refs. [4, 5]. Let us briefly recall them.

As in (22), a horizon in the metric (31) can occur at $\rho = 2k$ if $a > 0$. However, it has been shown [4, 5] that CBH solutions exist only when the parameters a and ξ obey the following "quantization" conditions:

$$a = \frac{m+1}{m-n}, \qquad \xi = \frac{m-n-1}{m-n}, \tag{34}$$

where m and n are positive integers satisfying the inequalities

$$m - 2 \geq n \geq 0. \tag{35}$$

The coupling constant ω should also belong to a discrete set of values,

$$2\omega + 3 = -\frac{2m(n+1) - n^2 + 1}{(m-n-1)^2} < 0. \tag{36}$$

Since, for the Brans-Dicke theory, $l(\varphi) = \omega + 3/2$ and $\varepsilon = \operatorname{sign} l$ (see (2)), we find that, just as in the Einstein picture, CBHs can only exist with a phantom scalar. The same is true for similar configurations with a nonzero electric charge [25], despite a greater number of classes of solutions.

However, the CBH existence conditions are different in the Einstein and Jordan pictures, and the global structures of the complete space-times, continued beyond the horizons, are also different [4,5]. In particular, as follows from (A.4), all K_i turn to infinity as $\rho \to 0$ in the solution (22). In other words, there is always a curvature singularity in the internal region of Einstein-frame CBHs (or, which is the same, CBHs with a minimally coupled massless phantom scalar field in general relativity). Meanwhile, many of Brans-Dicke Jordan-frame CBHs are nonsingular, and some of them have another flat asymptotic region beyond the horizon [4, 5].

4.2 Conformal continuations–III

By (34), the Jordan-frame CBHs form a discrete family with two integer parameters m and n subject to (35), while the family of Einstein-frame CBHs depends on the single integer parameter $a \geq 2$. The conformal mapping (2) that connects the two frames in some cases converts black holes into black holes, namely, when $m + 1$ is a multiple of a; according to (34), the parameter n is then expressed as $n = m - (m + 1)/a$.

In general, however, the conformal mapping (2) converts CBHs in $\mathbb{M}_\mathrm{E}$ into configurations with a singular horizon or a curvature singularity in $\mathbb{M}_\mathrm{J}$ and vice versa. Let us give some examples:

1. In case $n = 0$, $m = 2, 3, \ldots$, from (34) we obtain $a = (m+1)/m$, in which case the metric (22) in $\mathbb{M}_{\rm E}$ has a curvature singularity at $\rho = 2k$.
2. Given $m = 4$, $n = 2$, we have $a = 5/2$, a singular horizon at $\rho = 2k$ in $\mathbb{M}_{\rm E}$.
3. Given $a = 2$, i.e., a CBH in $\mathbb{M}_{\rm E}$, and $m = 2, 4, 6, \ldots$, we obtain half-integer n, hence a singular horizon at $\rho = 2k$ in $\mathbb{M}_{\rm J}$.

In all these cases and similar ones, the mapping (2) establishes a one-to-one correspondence between points of $\mathbb{M}_{\rm E}$ and $\mathbb{M}_{\rm J}$ only in the region $\rho > 2k$, which coincides with the whole manifold $\mathbb{M}_{\rm E}$ but only a portion of $\mathbb{M}_{\rm J}$ in examples 1 and 2, and vice versa in example 3. By definition [14], we are thus dealing with conformal continuations (CCs).

A conformal mapping $\mathcal{F}(\Omega) : \mathbb{M}_1 \mapsto \mathbb{M}_2$ between two (pseudo-)Riemannian manifolds $\mathbb{M}_1$ and $\mathbb{M}_2$, parametrized by the same coordinates x^μ, is a point-to-point mapping such that the respective metrics are related by $g^{(2)}_{\mu\nu} = \Omega^2(x^\mu) g^{(1)}_{\mu\nu}$, where the function $\Omega^2(x^\mu)$ (the conformal factor) is assumed to be smooth in a certain range of the arithmetic space of the coordinates $\{x^\mu\}$. Thus, in general, $\mathcal{F}(\Omega)$ connects only some regions of $\mathbb{M}_1$ and $\mathbb{M}_2$ rather than the whole manifolds, and which particular regions, depends on the analytic properties of the metrics and the conformal factor Ω^2.

Among different opportunities, a conformal continuation (CC) from $\mathbb{M}_1$ to $\mathbb{M}_2$ [14] is distinguished by the following circumstance: it maps a singular surface in $\mathbb{M}_1$ (so that $\mathbb{M}_1$ terminates there) to a regular surface $\mathbb{S}_{\rm trans} \in \mathbb{M}_2$, so that $\mathbb{M}_2$ continues beyond $\mathbb{S}_{\rm trans}$, to a region where the mapping $\mathcal{F}(\Omega)$ is not defined.

In normal STT, with $\varepsilon = +1$, the existence of CCs from $\mathbb{M}_{\rm E}$ to $\mathbb{M}_{\rm J}$ in static, spherically symmetric solutions was found to be a generic phenomenon if the function $f(\varphi)$ in (1) has a simple zero [14]. It was also concluded that the continued manifolds have generically the structure of wormholes. Explicit examples of CCs are known in the case of nonminimally coupled massless scalar fields in general relativity, treated as STT (1) with $f(\varphi) = 1 - \xi\varphi^2$ ($\xi = {\rm const} > 0$), $h(\varphi) = 1$, $U(\varphi) \equiv 0$ [14, 23, 26].

Ref. [14] classified CCs by the nature of the transition surfaces $\mathbb{S}_{\rm trans}$:

CC-I — $\mathbb{S}_{\rm trans}$ is an ordinary regular sphere in $\mathbb{M}_2$,

CC-II — $\mathbb{S}_{\rm trans}$ is a Killing horizon of finite area in $\mathbb{M}_2$.

In our case, we have a third type of conformal continuation:

CC-III — $\mathbb{S}_{\rm trans}$ is a Killing horizon of infinite area (type B) in $\mathbb{M}_2$.

One could also classify CCs by the types of singularities in $\mathbb{M}_1$ which are removed by the appropriate conformal mapping. Thus, in all cases considered in [14], the preimage of $\mathbb{S}_{\rm trans} \in \mathbb{M}_2 = \mathbb{M}_{\rm J}$ in the manifold $\mathbb{M}_1 = \mathbb{M}_{\rm E}$ was an attracting centre, being a curvature singularity like the one in Fisher's solution. Unlike that, in example 1, the surface $\rho = 2k$ is an attracting curvature singularity of infinite radius r while in example 2 it is a singular horizon, i.e., a sphere of infinite radius and zero curvature, where the analyticity of the metric is lost. The same is true in example 3, but there a singular horizon occurs in $\mathbb{M}_{\rm J}$ and a regular type B horizon in $\mathbb{M}_{\rm E}$.

To summarize, in the present study we have found CCs of a new type, which, unlike those described in Ref. [14], (i) exist in anomalous (phantom) STT only, (ii) lead to type B (infinite-area) horizons as transition surfaces $\mathbb{S}_{\rm trans}$, (iii) have other types of singularities as preimages of $\mathbb{S}_{\rm trans}$, and, finally, (iv) can occur not only from $\mathbb{M}_{\rm E}$ to $\mathbb{M}_{\rm J}$, as in examples 1 and 2, but also from $\mathbb{M}_{\rm J}$ to

$\mathbb{M}_{\rm E}$, as in example 3. The latter means that a singularity in the Jordan picture corresponds to a regular surface in Einstein's.

In the case of odd m in example 3 and other similar cases, there are CBHs in both pictures, the mapping (2) transfers a horizon to a horizon, but the global structures are different in different pictures, and there is a complicated system of one-to-one correspondences between different regions of $\mathbb{M}_{\rm E}$ and $\mathbb{M}_{\rm J}$, depending on the particular values of a, m and n. This issue may be a subject of a separate study, which is beyond our scope here.

4.3 On thermodynamics of scalar black holes

The Hawking temperature of a black hole horizon is $T_{\rm H} = (2\pi k_B)^{-1}\kappa$, where k_B is Boltzmann's constant while the surface gravity κ of the horizon is given by the expression [27]

$$\kappa = \frac{1}{2}\frac{g'_{00}}{\sqrt{|g_{00}g_{11}|}}\bigg|_{u=u_h} = \frac{1}{2}A'(\rho_h), \tag{37}$$

where $u = u_h$ is the value of an arbitrary radial coordinate u at the horizon, and after the second equality sign we give the corresponding expression in terms of the quasiglobal coordinate ρ (see the Appendix), ρ_h being its value at the horizon.

The problem of conformal invariance of the Hawking temperature has been addressed in Ref. [28]. In this work, it has been stated that $T_{\rm H}$ is the same for black holes obtained from conformally related theories under the conditions of staticity and asymptotic flatness.

Indeed, after a conformal transformation $g_{\mu\nu} = \Omega^2\tilde{g}_{\mu\nu}$, where Ω is a function of ρ, the surface gravity $\tilde{\kappa}$ at the surface $\rho = \rho_h$, defined in the new manifold with the metric $\tilde{g}_{\mu\nu}$, is

$$\tilde{\kappa} = \kappa + A(\rho_h)\frac{\Omega'}{\Omega}(\rho_h). \tag{38}$$

We have $\tilde{\kappa} = \kappa$ (i.e., invariance of Hawking's temperature under conformal mappings) if the second term in the r.h.s. of (38) is zero. And it is really the case since $A(\rho_h) = 0$ if the conformal factor Ω is regular at the horizon, i.e., if the conformal transformation is well defined on it.

However, for the presently discussed mapping between the anti-Fisher and Brans-Dicke CBHs, the question of invariance of $T_{\rm H}$ is either meaningless or trivial. Indeed, generically, as we have seen above, this transformation does not map a black hole to a black hole, and the invariance issue is meaningless. On the other hand, there is a general law saying that horizons of infinite area are always perfectly "cold", i.e., have zero temperature. Hence, in the cases where such black holes are in conformal correspondence, the invariance properties of their temperature become trivial,

Finally, the infinite horizon area of the CBHs may suggest, according to the well-known relations of black hole thermodynamics [27], that they should have infinite entropy. However, after a closer investigation, it has been argued that such black holes must in fact have zero entropy [29], which seems to be in a better agreement with a zero temperature state. This means that, for such objects, the law that relates the black hole entropy with its horizon area, is violated.

5 Cold black holes with $V \not\equiv 0$

5.1 Near-horizon behaviour of the solutions

So far we have been discussing CBHs with massless scalar fields. However, CBH solution with self-interacting scalar fields $V(\phi) \not\equiv 0)$ also exist, though under certain restrictions.

Indeed, consider Eqs. (13)–(17), let a horizon of infinite area ($r \to \infty$) be located at $\rho = 0$, and let us approximate the metric functions $A(\rho)$ and $r(\rho)$ at small ρ by

$$A \sim \rho^a, \qquad r^2 \sim \rho^{-b}, \tag{39}$$

where $a = 2, 3, \ldots$ and $b > 0$. Substitution into (15) results in $\varepsilon = -1$ (the field is necessarily phantom) and $\phi' \sim \rho^{-1}$, hence $|\phi| \sim -\log\rho \to \infty$ as $\rho \to 0$.

Furthermore, from (17) one finds

$$(a+b)\rho^{a+b-1} \approx \text{const} \cdot (\rho_0\rho^{2b} - \rho^{2b+1}). \tag{40}$$

It follows that there can be two families of solutions,

$$\mathrm{I}: \quad \rho_0 \neq 0, \qquad b = a - 1, \tag{41}$$

$$\mathrm{II}: \quad \rho_0 = 0, \qquad b = a - 2. \tag{42}$$

Substitution into (14) and (13) shows that, for family I, both V and $dV/d\phi$ behave as $o(\rho^{a-2})$ (where $a \geq 2$), while for family II both V and $dV/d\phi$ are of the order of ρ^{a-2}, where $a \geq 3$ (the value $a = 2$ is ruled out by $b > 0$, see (42)). This second family is, however, of little interest from the CBH viewpoint: an analysis of signs in Eq. (17) shows that such a horizon is only accessible from a T region, where $A(\rho) < 0$, and cannot be a black hole horizon.

In both families, $|\phi| \to \infty$ at the horizon, and, since at the same time $dV/d\phi \sim V \to 0$, the potential behaves at large $|\phi|$ as $V \sim \mathrm{e}^{-c|\phi|}$, $c = \text{const} > 0$. We conclude, in particular, that only with such potentials CBH solutions may exist.

One can verify that in all CBH solutions (they belong to family I) all functions behave near the horizon as in the anti-Fisher solution (22) with $a = 2, 3, \ldots$ (for comparison, in (22) one should move the origin of the ρ coordinate to the horizon, i.e., replace $\rho \mapsto \rho + 2k$). In fact, the anti-Fisher solution is a special case belonging to family I. The potential energy density V at small ρ is much smaller than the kinetic energy density $-A\phi'^2/2$, so that the system behaviour near the horizon is dominated by the kinetic term.

(Family II, on the contrary, can only exist in systems with nonzero $V(\phi)$, and in this case the potential and kinetic energy densities are of the same order near the horizon.)

The anti-Fisher asymptotic behavior at small ρ indicates that these solutions may be converted to the Jordan frame of any STT in the same manner as the anti-Fisher solution. Moreover, the small-ρ behavior of such Jordan-frame solutions in a given STT will be, in the main order of magnitude, the same as in the massless case, $V \equiv 0$. Therefore, one can assert that their thermodynamic properties and the nature of conformal continuations should also be the same as in the massless case.

5.2 Example

To obtain an example of an asymptotically flat CBH solution with $V \not\equiv 0$, let us use the inverse problem method (see, e.g., [19, 30]) and suppose

$$r(\rho) = \frac{\rho^2}{\sqrt{\rho^2 - b^2}}, \qquad b = \text{const}. \tag{43}$$

Substituting it into Eq. (17) and imposing $\rho_0 = 35\,b/16$, we obtain

$$A(\rho) = \frac{(\rho - b)^2(24\rho^2 + 37\rho b + 15b^2)}{24\rho^3(\rho + b)}\,. \tag{44}$$

It is easy to verify that this expression represents an asymptotically flat CBH, with $A = 0$ and $A' = 0$ at the horizon, $\rho = b$. The potential is given, from (14), by

$$V = \frac{b^3(\rho - b)(7\rho + 4b)}{12\rho^4(\rho + b)^3} \ . \tag{45}$$

The potential is zero at the horizon and at infinity. Using Eq. (15), we find an explicit expression for the scalar field:

$$\phi = \sqrt{\frac{3}{2}} \log \frac{x^2 - 1}{5 + x^2 + 2\sqrt{3(x^2 + 2)}} + 2 \log \frac{\sqrt{2} + \sqrt{x^2 + 2}}{x}, \qquad x := \frac{\rho}{b} \ . \tag{46}$$

The scalar field tends to a constant at infinity and diverges logarithmically at the horizon. However, as in the massless case, the scalar field energy density is finite at the horizon. The behavior of the potential in terms of the scalar field ϕ can only be obtained implicitly due to a complicated relation between ϕ and the radial coordinate ρ.

This solution is an explicit example of a CBH with a self-interacting scalar field. The behavior of all functions confirms our general consideration in the first part of this section.

6 Conclusions

Scalar-tensor theories, which are in general characterized by a non-minimal coupling between gravity and the scalar field, predict the existence of exotic black holes, which have an infinite horizon area and zero Hawking temperature. A well-known conformal mapping transforms any scalar-tensor theory from a large class (the Bergmann-Wagoner class) into general relativity minimally coupled to a massless scalar field. It had been thought for a long time that no black hole solution exists in this Einstein-scalar field system, at least for a massless scalar field in vacuum. We have shown here that this is not true, and we exhibit a new class of black hole solutions. However, for their existence the sign of the kinetic term of the scalar field must be reversed, leading to a negative-energy field. As in the scalar-tensor case, the "scalar" black holes have infinite horizon areas and zero temperature. However, the conditions in the parameter space for the existence of such black holes are different in Jordan's (non-minimal coupling) and Einstein's (minimal coupling) conformal frames, which leads to a new type of conformal continuations in the Einstein-frame and Jordan-frame manifolds.

The Einstein frame is common to the whole class (1) of scalar-tensor theories, whereas Jordan frames change from theory to theory together with the nonminimal coupling functions. This means that the discrete "quantization" conditions for the solution parameters, providing the existence of cold black holes, will be different in similar solutions of different theories.

The absence of continuations through certain surfaces of finite (or even zero) curvature is a peculiar property of many scalar-tensor solutions, indicating a special type of space-time singularities related to violation of analyticity, which actually means the divergence of some invariants of the metric tensor with derivatives of orders higher than two. Physical properties of such singularities and their possible regularization by taking into account more general solutions or quantum corrections may be of considerable interest.

The Hawking temperature discussed here is expressed in terms of the surface gravity κ. In a more rigorous treatement, quantum fields around such black holes must be considered. This is a delicate point, since all black holes studied in this work have zero temperature, which is, in principle, a violation of the third law of thermodynamics. For cold black holes in the Jordan frame there are anomalies in the definition of quantum fields, connected with normalization of quantum modes [31]. However, no complete study in this sense has been performed so far, mainly due to technical difficulties. It would be of interest to consider this problem in the context of the "scalar" black holes presented in this work.

Appendix

Cold black holes (CBHs) actually extend the notion of black holes to infinite horizon areas. So let us specify what we understand by a "black hole solution". For our comparatively simple case of static, spherically symmetric space-times, leaving aside more general and more rigorous definitions of horizons and black holes (see, e.g., [27]), we can rely on the following working definition. A black hole is a space-time containing (i) a static region which may be regarded external (e.g., contains a flat asymptotic), (ii) another region invisible for an observer at rest residing in the static region, and (iii) a Killing horizon of nonzero area that separates the two regions and admits an analytical extension of the metric from one region to another. This definition certainly implies that the horizon is regular, since otherwise it would be a singularity, belonging to the boundary of the space-time manifold, across which there cannot be a meaningful continuation.

We are dealing with the general metric (7),

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\beta(u)}d\Omega^2, \tag{A.1}$$

or in terms of the "quasiglobal" coordinate ρ under the condition $\alpha + \gamma = 0$, with the notations $e^{2\gamma} = e^{2\alpha} = A(\rho)$ and $e^{\beta} = r(\rho)$,

$$ds^2 = A(\rho)dt^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho)d\Omega^2. \tag{A.2}$$

A black hole horizon may be represented by a sphere $u = u_h$, or $\rho = \rho_h$, at which $g_{00} = e^{2\gamma} = A = 0$ and at which all algebraic curvature invariants are finite. To check the latter, it is sufficient to consider the behaviour of the Kretschmann invariant, given by

$$K = R^{\mu\nu\lambda\gamma}R_{\mu\nu\lambda\gamma} = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2, \tag{A.3}$$

where

$$\begin{aligned}
K_1 &= R^{01}{}_{01} = -\,e^{-\alpha-\gamma}\Big(\dot\gamma\, e^{\gamma-\alpha}\Big)^{\displaystyle\cdot} = -\frac{1}{2}A'';\\
K_2 &= R^{02}{}_{02} = R^{03}{}_{03} = -\,e^{-2\alpha}\,\dot\beta\,\dot\gamma = -\frac{1}{2}\frac{A'r'}{r};\\
K_3 &= R^{12}{}_{12} = R^{13}{}_{13} = -\,e^{-\alpha-\beta}\Big(\dot\beta\, e^{\beta-\alpha}\Big)^{\displaystyle\cdot} = A\frac{r''}{r} + \frac{1}{2}\frac{A'r'}{r};\\
K_4 &= R^{23}{}_{23} = e^{-2\beta} - e^{-2\alpha}\dot\beta^2 = \frac{1}{r^2}\,(Ar'^2 - 1),
\end{aligned} \tag{A.4}$$

where dots denote d/du and primes $d/d\rho$.

A Killing horizon (simply a horizon for short) $\rho = \rho_h$ admits a continuation to other space-time regions if and only if the function $A(\rho)$ behaves near it as $(\rho - \rho_h)^a$, $a \in \mathbb{N}$, and a is then called the order of the horizon. This restriction is related to a distinguished role of the ρ coordinate: near $\rho = \rho_h$ it varies (up to a positive constant factor) precisely as the manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric [4, 5]. Hence, using this coordinate (which was therefore termed *quasiglobal* [14]), one can "cross the horizons" preserving the formally static expression for the metric. It then also follows that ρ_h is always finite.

In cases when $A(\rho) \sim (\rho - \rho_h)^a$ and a is a fractional number, the space-time cannot be continued due to non-analyticity of the metric in terms of well-behaved coordinates. The geodesics also cannot be continued beyond the corresponding values of their canonical parameters. The sphere $\rho = \rho_h$ is thus a singularity, even if all curvature invariants are there finite. Such spheres may be referred to as *singular horizons*, to distinguish them from both regular horizons (or, simply, horizons) and curvature singularities.

In the above black hole definition, we have omitted the usual requirement that the horizon radius $r(\rho_h)$ and area $4\pi r^2(\rho_h)$ should be finite. Admitting $r(\rho_h) = \infty$, one can obtain quite a general result:

Any horizon of infinite area has zero surface gravity κ (and hence zero Hawking temperature $\kappa/(2\pi k_B)$).

Let us prove it for arbitrary static, spherically symmetric space-times. For the metric (A.1) or (A.2), the surface gravity (37) is expressed as [27]

$$\kappa = \mathrm{e}^{\gamma-\alpha}|\dot{\gamma}| = \tfrac{1}{2}A'(\rho), \tag{A.5}$$

Hence, a horizon with finite surface gravity corresponds to a simple zero of A, with $A' \neq 0$, at some finite value of ρ. On the other hand, the regularity conditions require that all K_i (A.4) should be finite at the horizon. In particular, in the same coordinates, $K_2 = -\frac{1}{2}A'r'/r$, hence, with $A' \neq 0$, $|K_2| < \infty$ is only possible in case $|r'/r| < \infty$, which in turn means that $\beta = \log r$ is finite at finite ρ. Thus a horizon with finite temperature can only occur at a sphere of finite radius $r = \mathrm{e}^{\beta}$. Hence, a horizon with an infinite area can only have zero temperature, justifying the term "cold black hole".

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