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10 June 2026Integrable \mathbb{Z}_2^2 -graded super-Liouville equation and induced \mathbb{Z}_2^2 -graded super-Virasoro algebraNaruhiko Aizawa^{1,*} , Ichi Fujii¹, Ren Ito¹ , Toshiya Tanaka¹ and Francesco Toppan² ¹ Department of Physics, Graduate School of Science, Osaka Metropolitan University, Sugimoto Campus, Osaka 558-8585, Japan² CBPF, Rua Dr. Xavier Sigaud 150, Urcae73c0 cep 22290-180, Rio de Janeiro (RJ), Brazil

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E-mail: aizawa@omu.ac.jp, sq25482d@st.omu.ac.jp, sd22709y@st.omu.ac.jp, sd23429y@st.omu.ac.jp and toppan@cbpf.brKeywords: \mathbb{Z}_2 superalgebra, super-Liouville equation, Lax operators, super-Virasoro algebra

Abstract

We present a framework for enlarging the construction of \mathbb{Z}_2^2 -graded classical Toda theory from the class of \mathbb{Z}_2^2 -graded Lie algebras to the class of \mathbb{Z}_2^2 -graded Lie superalgebras. This scheme is applied to derive a \mathbb{Z}_2^2 -graded extension of the super-Liouville equation based on a \mathbb{Z}_2^2 -graded extension of $\mathfrak{osp}(1|2)$. The mathematical tools employed in this work are a \mathbb{Z}_2^2 -graded version of the zero-curvature formalism and of the Polyakov's soldering procedure. It is demonstrated that both methods yield the same \mathbb{Z}_2^2 -graded super-Liouville equation. An algebraic construction of solutions to the resulting equations is also presented, together with their Bäcklund transformations. Furthermore, three distinct new \mathbb{Z}_2^2 -graded extensions of the super-Virasoro algebra are obtained via Hamiltonian reduction of the WZNW currents defined for \mathbb{Z}_2^2 - $\mathfrak{osp}(1|2)$.

1. Introduction

A general framework for constructing integrable \mathbb{Z}_2^2 -graded extensions ($\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2$) of classical, two-dimensional Toda and conformal affine Toda models has been presented in [1]. A key feature of this construction is that it is based on a \mathbb{Z}_2^2 -graded Lie algebra (also referred to as a \mathbb{Z}_2^2 -graded color Lie algebra) structure [2–5]; it implies that the resulting models contain fields obeying a special type of parabolic statistics [6–11]. The construction was carried out for a \mathbb{Z}_2^2 -graded extension of $\mathfrak{sl}(2)$, denoted by \mathbb{Z}_2^2 - $\mathfrak{sl}(2)$, and its affine extension, resulting in integrable graded extensions of the Liouville and sinh-Gordon equations. Furthermore, by enlarging the Polyakov's soldering procedure [12] to a \mathbb{Z}_2^2 -graded setting, an affine version of \mathbb{Z}_2^2 - $\mathfrak{sl}(2)$ with a central extension was recovered as a Poisson–Lie algebra. It was also shown that a \mathbb{Z}_2^2 -extension of the Virasoro algebra naturally arises from a Hamiltonian reduction.

In this paper we extend the analysis of [1] to a \mathbb{Z}_2^2 -graded Lie superalgebra framework. This corresponds to the more general case where not only parabosons, but also parafermionic fields are present [8, 13–17]. This is the most general setting to investigate \mathbb{Z}_2^2 -graded paraparticles because it extends, see e.g. [18–21], the notion of supersymmetry with all its physical and mathematical implications.

We stress the non-trivial significance of taking both parabosons and parafermions into account. The experimentalists learned how to simulate paraoscillators [22] and even engineer them in the laboratory [23] by using trapped ions. On the theoretical side it has been shown, see [10, 14] and also [16, 24], that certain results implied by \mathbb{Z}_2^2 -graded paraparticles cannot be reproduced by ordinary Bose–Fermi statistics. The connection of the \mathbb{Z}_2^2 -graded parastatistics with the traditional [25, 26] approach has been established in [8, 13].

In this paper the Polyakov's soldering procedure and the Lax pair formalism are applied to \mathbb{Z}_2^2 -graded extensions of Lie superalgebras. To illustrate this scheme we examine a \mathbb{Z}_2^2 -graded version of $\mathfrak{osp}(1|2)$ which allows to construct a \mathbb{Z}_2^2 -graded extension of the ordinary super-Liouville equation presented in [27, 28].

The \mathbb{Z}_2^2 -graded extension of a Lie (super)algebra is in general not unique. Many inequivalent \mathbb{Z}_2^2 -graded versions of $\mathfrak{osp}(1|2)$ have been discussed so far [3, 6, 13, 15, 17, 21, 29]. The one considered in this work is the ten-dimensional \mathbb{Z}_2^2 -graded Lie superalgebra introduced in [21] as a \mathbb{Z}_2^2 -graded extension of the superconformal mechanics.

Concerning the Lax pair formulation, the \mathbb{Z}_2^2 grading shares some feature with the $\mathcal{N} = 2$ supersymmetry; instead of a single Lax pair as in the $\mathcal{N} = 1$ models [30–34], the equations are derived by two conjugate Lax pairs like the $N = 2$ super-Toda models [35] (the $\mathcal{N} = 2$ super-Liouville theory was originally introduced in [36]).

A genuine new feature of the \mathbb{Z}_2^2 -graded superalgebra construction, with respect to the [1] \mathbb{Z}_2^2 -graded parabosonic algebra case, is that the Hamiltonian reduction of the WZNW currents induces a nontrivial, color, \mathbb{Z}_2^2 -graded extension of the super-Virasoro algebra. This is a new \mathbb{Z}_2^2 -graded superVirasoro algebra which differs from the one presented in [37]; its four component fields have respective gradings (00, 11, 10, 01) and scaling dimensions $(2, 2, \frac{3}{2}, \frac{3}{2})$. The (anti)symmetry properties of its brackets imply a nontrivial \mathbb{Z}_2^2 -graded Lie superalgebra extension of $\mathcal{N} = 1$ superVirasoro. In contrast, the Hamiltonian reduction from \mathbb{Z}_2^2 -graded $\mathfrak{sl}(2)$ produces, see [1], the (00, 11)-graded subalgebra which is isomorphic to an ordinary (not colored) Lie algebra.

It should be mentioned that the \mathbb{Z}_2^2 -graded extensions of integrable systems discussed here and in [1] reveal the existence of a broader class of integrable systems than those known so far; it is indeed obvious that the present scheme will be easily extended to more complex gradings such as $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ (n times). We remark that alternative approaches to \mathbb{Z}_2^2 -graded integrability, based on \mathbb{Z}_2^2 -graded supersymmetric Lagrangians in two-dimensional spacetime, can be found in the literature [38, 39].

This paper is organized as follows: in section 2 we recall the definition of \mathbb{Z}_2^2 -graded Lie superalgebra and introduce a \mathbb{Z}_2^2 -graded extension of $\mathfrak{osp}(1|2)$ which we denoted by \mathfrak{g} for simplicity. We also present representations of \mathfrak{g} which will be used in our analysis. In section 3 the \mathbb{Z}_2^2 -graded super-Liouville equation is derived via the Polyakov’s soldering procedure. This is achieved by considering the simplest setting sufficient to produce a nontrivial \mathbb{Z}_2^2 -graded equation. It will be shown in section 4 that the Hamiltonian reduction [12] for the current associated with \mathfrak{g} , supplemented by an additional gauge fixing inspired by [40, 41], gives rise to a \mathbb{Z}_2^2 -graded extension of the $\mathcal{N} = 1$ super-Virasoro algebra. The \mathbb{Z}_2^2 -grading admits, for the currents with nontrivial grading, three distinct (anti)periodicity conditions on a S^1 circle, leading to three distinct versions of the derived \mathbb{Z}_2^2 -graded super-Virasoro algebra. We show in section 5 that the \mathbb{Z}_2^2 -super-Liouville equation, derived by the soldering procedure, can be formulated in terms of Lax operators. This formulation enables us to construct explicit solutions in terms of chiral superfields. In section 6 two Bäcklund transformations are presented: one relating the the \mathbb{Z}_2^2 -super-Liouville equation to the free equation, and the other providing the auto-Bäcklund transformation for the \mathbb{Z}_2^2 -super-Liouville equation. A summary of the present work, together with further remarks and possible directions for future research, is given in the Conclusion.

2. \mathbb{Z}_2^2 -graded extension of $\mathfrak{osp}(1|2)$

2.1. Definitions

Let us recall the definition of the \mathbb{Z}_2^2 -graded Lie superalgebra [2, 3]. Let \mathfrak{g} be a vector space and $\hat{a} \equiv [a_1 a_2]$ an element of \mathbb{Z}_2^2 . Suppose that \mathfrak{g} is a direct sum of graded components

$$\mathfrak{g} = \bigoplus_{\hat{a} \in \mathbb{Z}_2^2} \mathfrak{g}_{\hat{a}} = \mathfrak{g}_{[00]} \oplus \mathfrak{g}_{[10]} \oplus \mathfrak{g}_{[01]} \oplus \mathfrak{g}_{[11]}. \tag{2.1}$$

If \mathfrak{g} admits a bilinear operation (the graded Lie bracket), denoted by $[[\cdot, \cdot]]$ and satisfying the identities

$$[[A_{\hat{a}}, B_{\hat{b}}]] \in \mathfrak{g}_{\hat{a}+\hat{b}}, \tag{2.2}$$

$$[[A_{\hat{a}}, B_{\hat{b}}]] = -(-1)^{\hat{a} \cdot \hat{b}} [[B_{\hat{b}}, A_{\hat{a}}]], \tag{2.3}$$

$$(-1)^{\hat{a} \cdot \hat{c}} [[A_{\hat{a}}, [[B_{\hat{b}}, C_{\hat{c}}]]]] + (-1)^{\hat{b} \cdot \hat{a}} [[B_{\hat{b}}, [[C_{\hat{c}}, A_{\hat{a}}]]]] + (-1)^{\hat{c} \cdot \hat{b}} [[C_{\hat{c}}, [[A_{\hat{a}}, B_{\hat{b}}]]]] = 0, \tag{2.4}$$

where $A_{\hat{a}}, B_{\hat{a}}, C_{\hat{a}}$ are homogeneous elements of $\mathfrak{g}_{\hat{a}}$ and

$$\hat{a} + \hat{b} = [a_1 + b_1, a_2 + b_2] \in \mathbb{Z}_2^2, \quad \hat{a} \cdot \hat{b} = a_1 b_1 + a_2 b_2 \in \mathbb{Z}_2, \tag{2.5}$$

then \mathfrak{g} is referred to as a \mathbb{Z}_2^2 -graded Lie superalgebra.

It is clear from the definition that the graded Lie brackets are realized by commutators and anticommutators as follows

$$[[A_{\hat{a}}, B_{\hat{b}}]] = \begin{cases} [A_{\hat{a}}, B_{\hat{b}}], & \hat{a} \cdot \hat{b} = 0, \\ \{A_{\hat{a}}, B_{\hat{b}}\}, & \hat{a} \cdot \hat{b} = 1. \end{cases} \tag{2.6}$$

If $[[A, B]] = 0$, we say that A and B are \mathbb{Z}_2^2 -graded commutative. It is also observed from the definition that \mathfrak{g} has \mathbb{Z}_2 -grading, too:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_0 := \mathfrak{g}_{[00]} \oplus \mathfrak{g}_{[11]}, \mathfrak{g}_1 := \mathfrak{g}_{[10]} \oplus \mathfrak{g}_{[01]}. \tag{2.7}$$

For a given Lie (super)algebra, one may consider its \mathbb{Z}_2^2 -graded extension which is generally not unique. Many inequivalent \mathbb{Z}_2^2 -graded extensions of the Lie superalgebra $\mathfrak{osp}(1|2)$ have been discussed so far [3, 6, 13, 15, 17, 21, 29]. In the present work, we consider one of them which was the ten-dimensional \mathbb{Z}_2^2 -graded Lie superalgebra introduced in [21] as a \mathbb{Z}_2^2 -extension of the superconformal mechanics. Its basis, their \mathbb{Z}_2^2 -gradings and scaling dimensions (eigenvalue of the grading operator $G := \frac{1}{2}K_0$) are summarized in the table below:

	[00]	[10]	[01]	[11]	
+1	K_+			L_+	(2.8)
$+\frac{1}{2}$		P_+	Q_+		
0	K_0			L_0	
$-\frac{1}{2}$		P_-	Q_-		
-1	K_-			L_-	

We keep using \mathfrak{g} to denote this \mathbb{Z}_2^2 -graded extension of $\mathfrak{osp}(1|2)$. The non-vanishing defining relations of \mathfrak{g} are given, in terms of (anti)commutators, as follows.

– \mathfrak{g}_0 - \mathfrak{g}_0 sector:

$$\begin{aligned} [K_0, K_{\pm}] &= \pm 2K_{\pm}, & [K_+, K_-] &= K_0, & [L_0, L_{\pm}] &= \pm 2K_{\pm}, & [L_+, L_-] &= K_0, \\ [K_0, L_{\pm}] &= \pm 2L_{\pm}, & [K_{\pm}, L_{\mp}] &= \pm L_0, & [L_0, K_{\pm}] &= \pm 2L_{\pm}, \end{aligned} \tag{2.9}$$

– \mathfrak{g}_0 - \mathfrak{g}_1 sector:

$$\begin{aligned} [K_0, P_{\pm}] &= \pm P_{\pm}, & [K_{\pm}, P_{\mp}] &= -P_{\pm}, \\ [K_0, Q_{\pm}] &= \pm Q_{\pm}, & [K_{\pm}, Q_{\mp}] &= -Q_{\pm}, \\ \{P_{\pm}, L_0\} &= \pm iQ_{\pm}, & \{P_{\pm}, L_{\mp}\} &= -iQ_{\mp}, \\ \{Q_{\pm}, L_0\} &= \mp iP_{\pm}, & \{Q_{\pm}, L_{\mp}\} &= iP_{\mp}, \end{aligned} \tag{2.10}$$

– \mathfrak{g}_1 - \mathfrak{g}_1 sector:

$$\begin{aligned} \{P_{\pm}, P_{\pm}\} &= \pm 2K_{\pm}, & \{P_+, P_-\} &= K_0, & [P_{\pm}, Q_{\pm}] &= \pm 2iL_{\pm}, & [P_{\pm}, Q_{\mp}] &= iL_0, \\ \{Q_{\pm}, Q_{\pm}\} &= \pm 2K_{\pm}, & \{Q_+, Q_-\} &= K_0. \end{aligned} \tag{2.11}$$

It is easy to see that the algebra \mathfrak{g} admits the triangular decomposition;

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-, \\ \mathfrak{g}_+ &= \{K_+, P_+, Q_+, L_+\}, \\ \mathfrak{h} &= \{K_0, L_0\}, \\ \mathfrak{g}_- &= \{K_-, P_-, Q_-, L_-\} \end{aligned} \tag{2.12}$$

where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} .

Structure and some representations of \mathfrak{g} are studied in [42–44].

2.2. Representations

In order to study integrable models associated with \mathfrak{g} , the representation of \mathfrak{g} induced from the standard (\mathbb{Z}_2 -graded) $\mathfrak{osp}(1|2)$ is useful. We denote the basis of $\mathfrak{osp}(1|2)$ by

$$[0] \quad H, E_{\pm}, \quad [1] \quad F_{\pm} \tag{2.13}$$

which subject to the non-vanishing relations

$$\begin{aligned} [H, E_{\pm}] &= \pm 2E_{\pm}, & [E_+, E_-] &= H, & [H, F_{\pm}] &= \pm F_{\pm}, \\ \{F_+, F_-\} &= H, & \{F_{\pm}, F_{\pm}\} &= \pm 2E_{\pm}, & [E_{\pm}, F_{\mp}] &= -F_{\pm}. \end{aligned} \tag{2.14}$$

Let $|v_0\rangle$ be the lowest weight vector of the fundamental representation of $\mathfrak{osp}(1|2)$, i.e.,

$$F_- |v_0\rangle = 0, \quad H |v_0\rangle = -|v_0\rangle, \quad \langle v_0 | v_0 \rangle = 1. \tag{2.15}$$

Define $|v_n\rangle := F_+^n |v_0\rangle$, then

$$\begin{aligned} H |v_n\rangle &= (n-1) |v_n\rangle, & F_+ |v_n\rangle &= |v_{n+1}\rangle, & F_+ |v_2\rangle &= 0, \\ F_- |v_1\rangle &= -|v_0\rangle, & F_- |v_2\rangle &= |v_1\rangle. \end{aligned} \tag{2.16}$$

Now we introduce the 4×4 complexified quaternionic matrices

$$M_0 := \mathbb{I}_2 \otimes \mathbb{I}_2, \quad M_1 := \mathbb{I}_2 \otimes \sigma_1, \quad M_2 := \sigma_1 \otimes \sigma_2, \quad M_3 := \sigma_1 \otimes \sigma_3 \tag{2.17}$$

which are defined by the 2×2 Identity and the Pauli matrices

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.18}$$

They satisfy for $i, j = 1, 2, 3$ the relations (the totally antisymmetric structure constant ϵ_{ijk} is normalized so that $\epsilon_{123} = 1$):

$$M_i M_j = \delta_{ij} M_0 + i \epsilon_{ijk} M_k. \tag{2.19}$$

We identify the \mathbb{Z}_2^2 -grading of the matrices as follows:

$$[00] : M_0; \quad [10] : M_1; \quad [01] : M_2; \quad [11] : M_3. \tag{2.20}$$

Then, \mathfrak{g} is realized by the matrices M_k and $\mathfrak{osp}(1|2)$ generators as follows (Cf. [1, 45]):

$$\begin{aligned} K_0 &= M_0 \otimes H, & K_{\pm} &= M_0 \otimes E_{\pm}, & P_{\pm} &= M_1 \otimes F_{\pm}, & Q_{\pm} &= M_2 \otimes F_{\pm}, \\ L_0 &= M_3 \otimes H, & L_{\pm} &= M_3 \otimes E_{\pm}. \end{aligned} \tag{2.21}$$

This realization gives the six-dimensional lowest weight representation of \mathfrak{g} . There are two linearly independent lowest weight vectors on which (2.21) act. They are given by $|00\rangle = M_0 \otimes |v_0\rangle$ and $|11\rangle = M_3 \otimes |v_0\rangle$ and have the grading indicated. Is is easy to verify that

$$\begin{aligned} P_- |00\rangle &= Q_- |00\rangle = P_- |11\rangle = Q_- |11\rangle = 0, \\ K_0 |00\rangle &= -|00\rangle, \quad K_0 |11\rangle = -|11\rangle, \quad L_0 |00\rangle = -|11\rangle, \quad L_0 |11\rangle = -|00\rangle \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} P_+ |00\rangle &= -i Q_+ |11\rangle, & Q_+ |00\rangle &= i P_+ |11\rangle, \\ P_+ Q_+ |\hat{a}\rangle &= -Q_+ P_+ |\hat{a}\rangle, & P_+^n Q_+^m |\hat{a}\rangle &= 0, \quad \hat{a} = 00, 11 \end{aligned} \tag{2.23}$$

where $n + m \geq 3$. Therefore, the \mathbb{Z}_2^2 -graded representation space induced on $|00\rangle, |11\rangle$ is six-dimensional and its basis is taken to be

$$\begin{aligned} [00]; \quad |1\rangle &:= K_+ |00\rangle, & |2\rangle &:= |00\rangle, \\ [11]; \quad |3\rangle &:= L_+ |00\rangle, & |4\rangle &:= |11\rangle, \\ [10]; \quad |5\rangle &:= P_+ |00\rangle, \\ [01]; \quad |6\rangle &:= Q_+ |00\rangle. \end{aligned} \tag{2.24}$$

The action of \mathfrak{g} on this space is readily obtained. Setting $\vec{v} := (|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle)$, we obtain the followings:

$$\begin{aligned}
 \vec{v} &\xrightarrow{K_0} (|1\rangle, -|2\rangle, |3\rangle, -|4\rangle, 0, 0), \\
 \vec{v} &\xrightarrow{K_+} (0, |1\rangle, 0, |3\rangle, 0, 0), \\
 \vec{v} &\xrightarrow{K_-} (|2\rangle, 0, |4\rangle, 0, 0, 0), \\
 \vec{v} &\xrightarrow{L_0} (|3\rangle, -|4\rangle, |1\rangle, -|2\rangle, 0, 0), \\
 \vec{v} &\xrightarrow{L_+} (0, |3\rangle, 0, |1\rangle, 0, 0), \\
 \vec{v} &\xrightarrow{L_-} (|4\rangle, 0, |2\rangle, 0, 0, 0), \\
 \vec{v} &\xrightarrow{P_+} (0, |5\rangle, 0, -i|6\rangle, |1\rangle, i|3\rangle), \\
 \vec{v} &\xrightarrow{P_-} (|5\rangle, 0, -i|6\rangle, 0, -|2\rangle, -i|4\rangle), \\
 \vec{v} &\xrightarrow{Q_+} (0, |6\rangle, 0, i|5\rangle, -i|3\rangle, |1\rangle), \\
 \vec{v} &\xrightarrow{Q_-} (|6\rangle, 0, i|5\rangle, 0, i|4\rangle, -|2\rangle).
 \end{aligned}
 \tag{2.25}$$

It follows the following matrix representation of \mathfrak{g} :

$$\begin{aligned}
 K_0 &= \text{diag}(1, -1, 1, -1, 0, 0), & K_+ &= \begin{pmatrix} \sigma_+ & 0 & 0 \\ 0 & \sigma_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 K_- &= \begin{pmatrix} \sigma_- & 0 & 0 \\ 0 & \sigma_- & 0 \\ 0 & 0 & 0 \end{pmatrix}, & L_0 &= \begin{pmatrix} 0 & \sigma_3 & 0 \\ \sigma_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 L_+ &= \begin{pmatrix} 0 & \sigma_+ & 0 \\ \sigma_+ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & L_- &= \begin{pmatrix} 0 & \sigma_- & 0 \\ \sigma_- & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 P_+ &= \begin{pmatrix} 0 & 0 & \sigma_{11} \\ 0 & 0 & i\sigma_+ \\ \sigma_+ & -i\sigma_{22} & 0 \end{pmatrix}, & P_- &= \begin{pmatrix} 0 & 0 & -\sigma_- \\ 0 & 0 & -i\sigma_{22} \\ \sigma_{11} & -i\sigma_- & 0 \end{pmatrix}, \\
 Q_+ &= \begin{pmatrix} 0 & 0 & \sigma_+ \\ 0 & 0 & -i\sigma_{11} \\ \sigma_{22} & i\sigma_+ & 0 \end{pmatrix}, & Q_- &= \begin{pmatrix} 0 & 0 & -\sigma_{22} \\ 0 & 0 & i\sigma_- \\ \sigma_- & i\sigma_{11} & 0 \end{pmatrix}
 \end{aligned}
 \tag{2.26}$$

where

$$\begin{aligned}
 \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
 \sigma_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \sigma_{22} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
 \end{aligned}
 \tag{2.27}$$

3. \mathbb{Z}_2^2 -super-Liouville equation by Polyakov's soldering

We mimick the standard procedure of soldering for deriving [12] the \mathbb{Z}_2^2 -graded version of the super-Liouville equation. We introduce the \mathbb{Z}_2^2 -graded Lie group $\mathbb{Z}_2^2\text{-Osp}(1|2)$ generated by the algebra $\mathbb{Z}_2^2\text{-osp}(1|2)$ defined in section 2. A group element of $\mathbb{Z}_2^2\text{-Osp}(1|2)$ is parametrized by

$$\begin{aligned}
 g &= \exp(\alpha_{00}K_+ + \alpha_{11}L_+) \exp(\lambda_{10}P_+ + \lambda_{01}Q_+) \exp(\beta_{00}K_0 + \beta_{11}L_0) \\
 &\times \exp(\mu_{10}P_- + \mu_{01}Q_-) \exp(\gamma_{00}K_- + \gamma_{11}L_-)
 \end{aligned}
 \tag{3.1}$$

where the group parameters $\alpha, \beta, \lambda, \mu$ and γ are also \mathbb{Z}_2^2 -commutative and their grading is indicated by the suffix. In particular, $\lambda_{10}, \lambda_{01}, \mu_{10}$ and μ_{01} are nilpotent. Throughout this article, the suffices 00, 10,

01, 11 indicate the \mathbb{Z}_2^2 -grading of the associated quantities. We assume that the group parameters are superfields which are functions of the [00] and [10]-graded \mathbb{Z}_2^2 -commutative variables

$$[00] \ x, \bar{x}, \quad [10] \ \theta, \bar{\theta}. \tag{3.2}$$

Alternatively, one could assume that the parameters are superfields on [00] and [01]-graded variables. It is obvious that both assumptions lead to the same equation, so we consider only the case of [00] and [10]-graded variables.

We introduce the [10]-graded covariant derivatives

$$D = \partial_\theta + i\theta\partial_x, \quad \bar{D} = \partial_{\bar{\theta}} + i\bar{\theta}\partial_{\bar{x}} \tag{3.3}$$

which satisfy

$$\{D, D\} = 2i\partial_x, \quad \{\bar{D}, \bar{D}\} = 2i\partial_{\bar{x}}, \quad \{D, \bar{D}\} = 0. \tag{3.4}$$

We introduce also the holomorphic and antiholomorphic WZNW-currents which are defined in terms of the group element (3.1):

$$J(x, \theta) := Dg \cdot g^{-1}, \quad \bar{J}(\bar{x}, \bar{\theta}) := g^{-1}\bar{D}g. \tag{3.5}$$

By definition, the currents $J(x, \theta), \bar{J}(\bar{x}, \bar{\theta})$ are [10]-graded and take values in $\mathbb{Z}_2^2\text{-osp}(1|2)$. Employing the matrix presentation (2.21), one may rearrange the components of the currents in terms of the $\text{osp}(1|2)$ generators. First, the group element (3.1) is given by

$$g = \exp(a \otimes E_+) \exp(b \otimes F_+) \exp(c \otimes H) \exp(d \otimes F_-) \exp(f \otimes E_-) \tag{3.6}$$

where the matrix valued fields a, b, c, d and f are defined by

$$\begin{aligned} a &:= \alpha_{00}M_0 + \alpha_{11}M_3, & b &:= \lambda_{10}M_1 + \lambda_{01}M_2, \\ c &:= \beta_{00}M_0 + \beta_{11}M_3, & d &:= \mu_{10}M_1 + \mu_{01}M_2, \\ f &:= \gamma_{00}M_0 + \gamma_{11}M_3 \end{aligned} \tag{3.7}$$

and due to the nilpotency of $\lambda_{10}, \lambda_{01}, \mu_{10}$ and μ_{01} we have the relations

$$b^2 = d^2 = \{b, d\} = 0. \tag{3.8}$$

It follows immediately that the currents (3.5) takes the following form:

$$\begin{aligned} J &= J_{++} \otimes E_+ + J_+ \otimes F_+ + J_0 \otimes H + J_- \otimes F_- + J_{--} \otimes E_-, \\ \bar{J} &= \bar{J}_{++} \otimes E_+ + \bar{J}_+ \otimes F_+ + \bar{J}_0 \otimes H + \bar{J}_- \otimes F_- + \bar{J}_{--} \otimes E_- \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} J_{++} &= -2e^{-2c}(Df - (Dd)d)a^2 - 2e^{-c}(Dd)ab - 2(Dc)a + (Db)b + Da, \\ J_+ &= -e^{-2c}(Df - (Dd)d)ab - e^{-c}(Dd)a - (Dc)b + Db, \\ J_0 &= e^{-2c}(Df - (Dd)d)a + e^{-c}(Dd)b + Dc, \\ J_- &= e^{-2c}(Df - (Dd)d)b + e^{-c}Dd, \\ J_{--} &= (Df - (Dd)d)e^{-2c} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \bar{J}_{++} &= e^{-2c}(\bar{D}a + b\bar{D}b), \\ \bar{J}_+ &= -e^{-2c}(\bar{D}a + b\bar{D}b)d + e^{-c}\bar{D}b, \\ \bar{J}_0 &= -e^{-2c}(\bar{D}a + b\bar{D}b)f - e^{-c}(\bar{D}b)d + \bar{D}c, \\ \bar{J}_- &= -e^{-2c}(\bar{D}a + b\bar{D}b)fd + e^{-c}(\bar{D}b)f - (\bar{D}c)d + \bar{D}d, \\ \bar{J}_{--} &= -e^{-2c}(\bar{D}a + b\bar{D}b)f^2 - 2e^{-c}(\bar{D}b)fd - 2(\bar{D}c)f + (\bar{D}d)d + \bar{D}f. \end{aligned} \tag{3.11}$$

According to [12] we impose Hamiltonian constraints on the currents. Taking into account the grading and the matrix nature of the components of J, \bar{J} , the appropriate constraints will be

$$J_{--} = J_0 = 0, \quad J_- = M_1, \tag{3.12}$$

$$\bar{J}_{++} = \bar{J}_0 = 0, \quad \bar{J}_+ = M_1. \tag{3.13}$$

The constraints on J_{--}, J_- and J_0 give

$$Dc = -bM_1 \tag{3.14}$$

and the constraints on \bar{J}_{++}, \bar{J}_+ gives

$$\bar{D}b = e^c M_1. \tag{3.15}$$

Acting \bar{D} on (3.14) and using (3.15), we obtain

$$D\bar{D}c = e^c. \tag{3.16}$$

Recalling the definition of c in (3.7), it follows that

$$e^c = e^{\beta_{00}M_0} e^{\beta_{11}M_3} = e^{\beta_{11}} (\cosh \beta_{11}M_0 + \sinh \beta_{11}M_3). \tag{3.17}$$

Thus we obtain the following system of equations:

$$D\bar{D}\beta_{00} = e^{\beta_{00}} \cosh \beta_{11}, \quad D\bar{D}\beta_{11} = e^{\beta_{00}} \sinh \beta_{11}. \tag{3.18}$$

Due to the nilpotency of $\theta, \bar{\theta}$ one may expand the superfields into their components:

$$\begin{aligned} \beta_{00} &= \varphi_{00}(x, \bar{x}) + \theta\psi_{10}(x, \bar{x}) + \bar{\theta}\bar{\psi}_{10}(x, \bar{x}) + \theta\bar{\theta}F_{00}(x, \bar{x}), \\ \beta_{11} &= \varphi_{11}(x, \bar{x}) + \theta\psi_{01}(x, \bar{x}) + \bar{\theta}\bar{\psi}_{01}(x, \bar{x}) + \theta\bar{\theta}F_{11}(x, \bar{x}). \end{aligned} \tag{3.19}$$

Although we started with only [10]-graded superspace, the superfield β_{11} ensures the existence of [01]-graded component fields so that the \mathbb{Z}_2^2 -super-Liouville equations (3.18) are a system of coupled equations for [00], [11], [10] and [01]-graded fields defined on two-dimensional spacetime. More explicitly, from the first equation in (3.18) we get

$$\begin{aligned} \partial\bar{\partial}\varphi_{00} &= e^{\varphi_{00}} (\cosh \varphi_{11} (\psi_{10}\bar{\psi}_{10} - \psi_{01}\bar{\psi}_{01} - F_{00}) + \sinh \varphi_{11} (\psi_{10}\bar{\psi}_{01} - \psi_{01}\bar{\psi}_{10} - F_{11})), \\ i\bar{\partial}\psi_{10} &= e^{\varphi_{00}} (-\cosh \varphi_{11} \cdot \bar{\psi}_{10} + \sinh \varphi_{11} \cdot \bar{\psi}_{01}), \\ i\partial\bar{\psi}_{10} &= e^{\varphi_{00}} (\cosh \varphi_{11} \cdot \psi_{10} - \sinh \varphi_{11} \cdot \psi_{01}), \\ F_{00} &= -e^{\varphi_{00}} \cosh \varphi_{11}, \end{aligned} \tag{3.20}$$

where $\partial = \partial_x, \bar{\partial} = \partial_{\bar{x}}$ and from the second we get

$$\begin{aligned} \partial\bar{\partial}\varphi_{11} &= e^{\varphi_{00}} (\cosh \varphi_{11} (\psi_{10}\bar{\psi}_{01} - \psi_{01}\bar{\psi}_{10} - F_{11}) + \sinh \varphi_{11} (\psi_{10}\bar{\psi}_{10} - \psi_{01}\bar{\psi}_{01} - F_{00})), \\ i\bar{\partial}\psi_{01} &= e^{\varphi_{00}} (-\cosh \varphi_{11} \cdot \bar{\psi}_{01} + \sinh \varphi_{11} \cdot \bar{\psi}_{10}), \\ i\partial\bar{\psi}_{01} &= e^{\varphi_{00}} (\cosh \varphi_{11} \cdot \psi_{01} - \sinh \varphi_{11} \cdot \psi_{10}), \\ F_{11} &= -e^{\varphi_{00}} \sinh \varphi_{11}. \end{aligned} \tag{3.21}$$

It is observed that F_{00} and F_{11} are non-propagating auxiliary fields. Elimination of them gives the following equations of motion of φ_{00} and φ_{11} :

$$\begin{aligned} \partial\bar{\partial}\varphi_{00} &= e^{2\varphi_{00}} \cosh 2\varphi_{11} \\ &\quad + e^{\varphi_{00}} (\cosh \varphi_{11} (\psi_{10}\bar{\psi}_{10} - \psi_{01}\bar{\psi}_{01}) + \sinh \varphi_{11} (\psi_{10}\bar{\psi}_{01} - \psi_{01}\bar{\psi}_{10})), \\ \partial\bar{\partial}\varphi_{11} &= e^{2\varphi_{00}} \sinh 2\varphi_{11} \\ &\quad + e^{\varphi_{00}} (\cosh \varphi_{11} (\psi_{10}\bar{\psi}_{01} - \psi_{01}\bar{\psi}_{10}) + \sinh \varphi_{11} (\psi_{10}\bar{\psi}_{10} - \psi_{01}\bar{\psi}_{01})). \end{aligned} \tag{3.22}$$

Setting φ_{11} and $\psi_{01}, \bar{\psi}_{01}$ (or $\psi_{10}, \bar{\psi}_{10}$) equal to zero, the super-Liouville equations discussed in [27] are recovered. Setting all the fields with non-trivial grading zero, the Liouville equation is recovered. Therefore, the \mathbb{Z}_2^2 -super-Liouville equations constructed here are a natural generalization of the (super-)Liouville equation. The integrability of the system of equations (3.18) is ensured by its formulation as a zero-curvature representation which we discussed in section 5. Following the method of Leznov and Saveliev, we also construct solutions to the equations presented in section 5.4.

4. \mathbb{Z}_2^2 -graded super-Virasoro algebras

In this section we consider the current algebra associated with the currents given in (3.5). Polyakov showed that the Virasoro algebra emerges from constraining the current algebra associated with $SL(2, \mathbb{R})$ [12]. This construction was extended to $Osp(1|2)$ by Sabra [40, 41], where the $\mathcal{N} = 1$ super-Virasoro algebra is obtained. In our case, a \mathbb{Z}_2^2 -graded generalization of the super-Virasoro algebra is found. This analysis is carried out within the framework of classical mechanics, i.e., making use of the Poisson brackets. Nevertheless, as in the case of \mathbb{Z}_2^2 -graded Liouville equation based on \mathbb{Z}_2^2 - $\mathfrak{sl}(2)$ discussed in [1], we observe the existence of a central term in the Poisson Lie algebra (an example, see [46], of a classical anomaly).

Only the holomorphic sector will be considered here, as the treatment of the antiholomorphic sector proceeds in a similar fashion. The transformations of the component currents defined in (3.9) are induced from the left action of the group element:

$$\begin{aligned}
 g &\rightarrow g' = g_\epsilon g, \\
 g_\epsilon &= e^{\epsilon_{++} \otimes E_+} e^{\epsilon_{+-} \otimes F_+} e^{\epsilon_0 \otimes H_+} e^{\epsilon_{-+} \otimes F_-} e^{\epsilon_{--} \otimes E_-}
 \end{aligned}
 \tag{4.1}$$

where e 's are holomorphic functions and ϵ_\pm are nilpotent and anticommute. Considering the infinitesimal transformation of g , one may obtain

$$\begin{aligned}
 \delta J_{++} &= -2\epsilon_{++} J_0 + 2\epsilon_{+-} J_+ + 2\epsilon_0 J_{++} + D\epsilon_{++}, \\
 \delta J_+ &= -\epsilon_{++} J_- + \epsilon_{+-} J_0 + \epsilon_0 J_+ - \epsilon_{-+} J_{++} + D\epsilon_+, \\
 \delta J_0 &= \epsilon_{++} J_{--} + \epsilon_{+-} J_- + \epsilon_{-+} J_+ - \epsilon_{--} J_{++} + D\epsilon_0, \\
 \delta J_- &= -\epsilon_{+-} J_{--} - \epsilon_0 J_- - \epsilon_{-+} J_0 - \epsilon_{--} J_+ + D\epsilon_-, \\
 \delta J_{--} &= -2\epsilon_0 J_{--} - 2\epsilon_{-+} J_- + 2\epsilon_{--} J_0 + D\epsilon_{--}.
 \end{aligned}
 \tag{4.2}$$

Besides the constraints given in (3.12), following [40, 41] we impose an additional gauge fixing

$$J_+ = 0.
 \tag{4.3}$$

Then, the infinitesimal transformations preserving (3.12) and (4.3) reduce to a single parameter:

$$\begin{aligned}
 \epsilon_- &= \frac{1}{2} D\epsilon_{--} M_1, & \epsilon_0 &= \frac{i}{2} \epsilon'_{--}, \\
 \epsilon_+ &= \epsilon_{--} J_{++} M_1 - \frac{i}{2} D\epsilon'_{--} M_1, \\
 \epsilon_{++} &= \frac{1}{2} (D\epsilon_{--}) J_{++} + \frac{1}{2} \epsilon_{--} D J_{++} + \frac{1}{2} \epsilon''_{--}.
 \end{aligned}
 \tag{4.4}$$

So that

$$\delta J_{++} = \frac{3i}{2} \epsilon'_{--} J_{++} + (D\epsilon_{--}) D J_{++} + i\epsilon_{--} J'_{++} + \frac{1}{2} D\epsilon''_{--}
 \tag{4.5}$$

where the prime stands for the derivative with respect to x : $a' = \partial_x a$. Recalling that J_{++} consists of \mathbb{Z}_2^2 -graded superfields and matrices M_i , one may recover the \mathbb{Z}_2^2 -graded currents J_{10} and J_{01} by

$$J_{++} = J_{10} M_0 + J_{01} M_3.
 \tag{4.6}$$

ϵ_{--} is also parametrized as:

$$\epsilon_{--} = \epsilon_{00} M_0 + \epsilon_{11} M_3.
 \tag{4.7}$$

The transformations of J_{10} and J_{01} can be readily obtained from (4.5):

$$\delta J_{10} = \frac{3i}{2} (\epsilon'_{00} J_{10} - \epsilon'_{11} J_{01}) + \frac{1}{2} [(D\epsilon_{00}) D J_{10} + (D\epsilon_{11}) D J_{01}] + i\epsilon_{00} J'_{10} - i\epsilon_{11} J'_{01} + \frac{1}{2} D\epsilon''_{00},
 \tag{4.8}$$

$$\delta J_{01} = \frac{3i}{2} (\epsilon'_{00} J_{01} - \epsilon'_{11} J_{10}) + \frac{1}{2} [(D\epsilon_{00}) D J_{01} + (D\epsilon_{11}) D J_{10}] + i\epsilon_{00} J'_{01} - i\epsilon_{11} J'_{10} + \frac{1}{2} D\epsilon''_{11}.
 \tag{4.9}$$

By assigning the scaling dimension $[x] = -1$, the scaling dimensions of the remaining quantities are determined as follows:

$$[\theta] = -\frac{1}{2}, \quad [J_{10}] = [J_{01}] = \frac{3}{2}, \quad [\epsilon_{00}] = [\epsilon_{11}] = -1. \tag{4.10}$$

One may expand the currents and the parameters in their components:

$$\begin{aligned} J_{10}(x, \theta) &= u_{10}(x) + \theta u_{00}(x), & J_{01}(x, \theta) &= u_{01}(x) + \theta u_{11}(x), \\ \epsilon_{00}(x, \theta) &= \epsilon_{00}(x) + \theta \epsilon_{10}(x), & \epsilon_{11}(x, \theta) &= \epsilon_{11}(x) + \theta \epsilon_{01}(x). \end{aligned} \tag{4.11}$$

The transformations of the components are readily obtained

$$\begin{aligned} \delta u_{00} &= i \left(2\epsilon'_{00} u_{00} + \epsilon_{00} u'_{00} + \frac{1}{2} \epsilon'''_{00} \right) + i (2\epsilon'_{11} u_{11} + \epsilon_{11} u'_{11}) \\ &\quad + i \left(\frac{3}{2} \epsilon'_{10} u_{10} + \frac{1}{2} \epsilon_{10} u'_{10} \right) - i \left(\frac{3}{2} \epsilon'_{01} u_{01} + \frac{1}{2} \epsilon_{01} u'_{01} \right), \\ \delta u_{11} &= i \left(2\epsilon'_{11} u_{00} + \epsilon_{11} u'_{00} + \frac{1}{2} \epsilon'''_{11} \right) + i (2\epsilon'_{00} u_{11} + \epsilon_{00} u'_{11}) \\ &\quad + i \left(\frac{3}{2} \epsilon'_{10} u_{01} + \frac{1}{2} \epsilon_{10} u'_{01} \right) - i \left(\frac{3}{2} \epsilon'_{01} u_{10} + \frac{1}{2} \epsilon_{01} u'_{10} \right) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \delta u_{10} &= i \left(\frac{3}{2} \epsilon'_{00} u_{10} + \epsilon_{00} u'_{10} \right) - i \left(\frac{3}{2} \epsilon'_{11} u_{01} + \epsilon_{11} u'_{01} \right) + \frac{1}{2} (\epsilon_{10} u_{00} + \epsilon_{01} u_{11} + \epsilon''_{10}), \\ \delta u_{01} &= i \left(\frac{3}{2} \epsilon'_{00} u_{01} + \epsilon_{00} u'_{01} \right) - i \left(\frac{3}{2} \epsilon'_{11} u_{10} + \epsilon_{11} u'_{10} \right) + \frac{1}{2} (\epsilon_{01} u_{00} + \epsilon_{10} u_{11} + \epsilon''_{01}). \end{aligned} \tag{4.13}$$

It can be readily verified that the scaling dimension is determined as follows:

$$[u_{00}] = [u_{11}] = 2, \quad [u_{10}] = [u_{01}] = \frac{3}{2}, \quad [\epsilon_{00}] = [\epsilon_{11}] = -1, \quad [\epsilon_{10}] = [\epsilon_{01}] = -\frac{1}{2}. \tag{4.14}$$

It follows from these results that the currents u_{00}, u_{11}, u_{10} and u_{01} can be identified with a \mathbb{Z}_2^2 -graded generalization of the Virasoro algebra. To see this explicitly, we need to introduce a correct Poisson brackets structure.

The infinitesimal transformations (4.12) and (4.13) may be reproduced by the Poisson bracket:

$$\begin{aligned} \delta u(x) &= \frac{1}{2\pi} \oint dy \{K(y), u(x)\}, \\ K(y) &:= k_1 \epsilon_{00} u_{00} + k_2 \epsilon_{11} u_{11} + k_3 \epsilon_{10} u_{10} + k_4 \epsilon_{01} u_{01}, \end{aligned} \tag{4.15}$$

where $u(x)$ stands for the \mathbb{Z}_2^2 -graded component currents and the k_i constants have to be determined. The \mathbb{Z}_2^2 -graded Poisson bracket is defined by the relations

$$\begin{aligned} \{u_{\hat{a}}, u_{\hat{b}}\} &= -(-1)^{\hat{a}\cdot\hat{b}} \{u_{\hat{b}}, u_{\hat{a}}\}, \\ \{u_{\hat{a}} u_{\hat{b}}, u_{\hat{c}}\} &= u_{\hat{a}} \{u_{\hat{b}}, u_{\hat{c}}\} + (-1)^{\hat{b}\cdot\hat{c}} \{u_{\hat{a}}, u_{\hat{c}}\} u_{\hat{b}}, \\ \{u_{\hat{a}}, \{u_{\hat{b}}, u_{\hat{c}}\}\} &= \{\{u_{\hat{a}}, u_{\hat{b}}\}, u_{\hat{c}}\} + (-1)^{\hat{a}\cdot\hat{b}} \{u_{\hat{b}}, \{u_{\hat{a}}, u_{\hat{c}}\}\}, \end{aligned} \tag{4.16}$$

where $\hat{a}, \hat{b}, \hat{c} \in \mathbb{Z}_2^2$.

By taking into account the \mathbb{Z}_2^2 -grading and the scaling dimension we propose the following Ansatz:

$$\begin{aligned} \{u_{00}(y), u_{00}(x)\} &= a_1 u'_{00}(y) \delta(y-x) + a_2 u_{00}(y) \delta'(y-x) + a_3 \delta'''(y-x), \\ \{u_{11}(y), u_{00}(x)\} &= a_4 u'_{11}(y) \delta(y-x) + a_5 u_{11}(y) \delta'(y-x), \\ \{u_{10}(y), u_{00}(x)\} &= a_6 u'_{10}(y) \delta(y-x) + a_7 u_{10}(y) \delta'(y-x), \\ \{u_{01}(y), u_{00}(x)\} &= a_8 u'_{01}(y) \delta(y-x) + a_9 u_{01}(y) \delta'(y-x), \\ \{u_{11}(y), u_{11}(x)\} &= b_1 u'_{00}(y) \delta(y-x) + b_2 u_{00}(y) \delta'(y-x) + b_3 \delta'''(y-x), \\ \{u_{10}(y), u_{11}(x)\} &= b_4 u'_{01}(y) \delta(y-x) + b_5 u_{01}(y) \delta'(y-x), \end{aligned}$$

$$\begin{aligned}
 \{u_{01}(y), u_{11}(x)\} &= b_6 u'_{10}(y) \delta(y-x) + b_7 u_{10}(y) \delta'(y-x), \\
 \{u_{10}(y), u_{10}(x)\} &= c_1 u_{00}(y) \delta(y-x) + c_2 \delta''(y-x), \\
 \{u_{01}(y), u_{10}(x)\} &= c_3 u_{11}(y) \delta(y-x), \\
 \{u_{01}(y), u_{01}(x)\} &= d_1 u_{00}(y) \delta(y-x) + d_2 \delta''(y-x),
 \end{aligned}
 \tag{4.17}$$

where the prime stands for the derivative with respect to y and our convention for the delta function is

$$\delta(x-a) = \sum_{n \in \mathbb{Z}} e^{in(x-a)},
 \tag{4.18}$$

$$\frac{1}{2\pi} \oint dx \delta(x) = \frac{1}{2\pi} \int_0^{2\pi} dx \delta(x) = 1.
 \tag{4.19}$$

The constants a_i, b_i, c_i, d_i are fixed by the requirement that (4.15) and (4.17) reproduce the current transformations (4.12) and (4.13). We note that the Poisson brackets $\{u_{10}, u_{11}\}, \{u_{01}, u_{11}\}, \{u_{10}, u_{10}\}$ and $\{u_{01}, u_{01}\}$ are symmetric, but all others are antisymmetric.

For each \mathbb{Z}_2^2 -graded current, (4.15) gives the following conditions:

$u :$	conditions
$u_{00} :$	$k_1 a_2 = k_2 a_5 = -2i, \quad k_1 a_1 = k_2 a_4 = -i, \quad k_1 a_3 = -\frac{i}{2},$ $-k_3 a_7 = k_4 a_9 = \frac{3i}{2}, \quad -k_3 a_6 = k_4 a_8 = i$
$u_{11} :$	$k_1 a_5 = k_2 b_2 = -2i, \quad k_1 a_4 = k_2 b_1 = -i, \quad k_2 b_3 = -\frac{i}{2},$ $-k_3 b_5 = k_4 b_7 = \frac{3i}{2}, \quad -k_3 b_4 = k_4 b_6 = i$
$u_{10} :$	$k_1 a_7 = k_2 b_5 = -\frac{3i}{2}, \quad k_1 a_6 = k_2 b_4 = -i, \quad k_3 c_1 = k_3 c_2 = k_4 c_3 = \frac{1}{2}$
$u_{01} :$	$k_1 a_9 = k_2 b_7 = -\frac{3i}{2}, \quad k_1 a_8 = k_2 b_6 = -i, \quad -k_3 c_3 = k_4 d_1 = k_4 d_2 = \frac{1}{2}$

Solving these conditions give the results:

$$\begin{aligned}
 k_1 &= i, & k_2 &= i, & k_3 &= i, & k_4 &= -i, \\
 a_1 &= -1, & a_2 &= -2, & a_3 &= -\frac{1}{2}, & a_4 &= -1, & a_5 &= -2, \\
 a_6 &= -1, & a_7 &= -\frac{3}{2}, & a_8 &= -1, & a_9 &= -\frac{3}{2}, \\
 b_1 &= -1, & b_2 &= -2, & b_3 &= -\frac{1}{2}, & b_4 &= -1, & b_5 &= -\frac{3}{2}, \\
 b_6 &= -1, & b_7 &= -\frac{3}{2}, \\
 c_1 &= c_2 = -\frac{i}{2}, & c_3 &= \frac{i}{2}, \\
 d_1 &= d_2 = \frac{i}{2}.
 \end{aligned}
 \tag{4.21}$$

Therefore, the \mathbb{Z}_2^2 -graded currents satisfy the relations

$$\begin{aligned}
 \{u_{00}(y), u_{00}(x)\} &= -u'_{00}(y) \delta(y-x) - 2u_{00}(y) \delta'(y-x) - \frac{1}{2} \delta'''(y-x), \\
 \{u_{11}(y), u_{00}(x)\} &= -u'_{11}(y) \delta(y-x) - 2u_{11}(y) \delta'(y-x), \\
 \{u_{10}(y), u_{00}(x)\} &= -u'_{10}(y) \delta(y-x) - \frac{3}{2} u_{10}(y) \delta'(y-x), \\
 \{u_{01}(y), u_{00}(x)\} &= -u'_{01}(y) \delta(y-x) - \frac{3}{2} u_{01}(y) \delta'(y-x), \\
 \{u_{11}(y), u_{11}(x)\} &= -u'_{00}(y) \delta(y-x) - 2u_{00}(y) \delta'(y-x) - \frac{1}{2} \delta'''(y-x), \\
 \{u_{10}(y), u_{11}(x)\} &= -u'_{01}(y) \delta(y-x) - \frac{3}{2} u_{01}(y) \delta'(y-x), \\
 \{u_{01}(y), u_{11}(x)\} &= -u'_{10}(y) \delta(y-x) - \frac{3}{2} u_{10}(y) \delta'(y-x), \\
 \{u_{10}(y), u_{10}(x)\} &= -\frac{i}{2} u_{00}(y) \delta(y-x) - \frac{i}{2} \delta''(y-x),
 \end{aligned}$$

$$\begin{aligned} \{u_{01}(y), u_{10}(x)\} &= \frac{i}{2} u_{11}(y) \delta(y-x), \\ \{u_{01}(y), u_{01}(x)\} &= \frac{i}{2} u_{00}(y) \delta(y-x) + \frac{i}{2} \delta''(y-x). \end{aligned} \tag{4.22}$$

These relations define a \mathbb{Z}_2^2 -graded extension of the $\mathcal{N} = 1$ super-Virasoro algebra which is recovered by either u_{00}, u_{10} or u_{00}, u_{01} subalgebras.

By expanding the \mathbb{Z}_2^2 -graded currents into modes, we obtain from (4.22) the infinite dimensional \mathbb{Z}_2^2 -graded super-Virasoro algebra equipped with a Poisson–Lie structure. As is well known in string theory, fermionic sectors are subject to either periodic (Ramond sector) or antiperiodic (Neveu–Schwarz sector) boundary conditions [47, 48]. Assuming, as usual, that the [00]-graded current is periodic, consistency with the current algebra (4.22) imposes specific (anti)periodicity conditions on the remaining graded sectors. One should note that even the exotic bosonic current u_{11} could satisfy either periodic or antiperiodic boundary condition. Therefore, three possible alternatives are admissible in association with [11]/[10]/[01] graded sectors:

$$(i)R/R/R \quad (ii)R/NS/NS \quad (iii)NS/NS/R \equiv NS/R/NS$$

where R and NS stand for the Ramond and Neveu–Schwarz sector, respectively. We remark that these three admissible boundary conditions already appeared in the string model induced from 1D \mathbb{Z}_2^2 -graded supersymmetry [45].

Under these periodicity conditions, the \mathbb{Z}_2^2 -graded currents may be expanded as

$$\begin{aligned} u_{00}(x) &= \sum_{n \in \mathbb{Z}} L_n e^{inx}, & u_{11}(x) &= \sum_p H_p e^{ipx}, \\ u_{10}(x) &= \sum_r G_r e^{irx}, & u_{01}(x) &= \sum_s F_s e^{isx} \end{aligned} \tag{4.23}$$

where $n \in \mathbb{Z}$ is the index for [00]-sector (Ramond sector) and p, r, s are the indices for [11], [10], [01]-sectors, respectively. The indices p, r, s take their values in \mathbb{Z} if they are in the Ramond sector or in $\mathbb{Z} + \frac{1}{2}$ if in the Neveu–Schwarz sector. Then we obtain the relations of \mathbb{Z}_2^2 -graded super-Virasoro algebra:

$$\begin{aligned} \{L_n, L_{n'}\} &= i(n' - n)L_{n+n'} + \frac{i}{2} n^3 \delta_{n+n', 0}, \\ \{H_p, L_n\} &= i(n - p)H_{p+n}, \\ \{G_r, L_n\} &= i\left(\frac{n}{2} - r\right)G_{r+n}, \\ \{F_s, L_n\} &= i\left(\frac{n}{2} - s\right)F_{s+n}, \\ \{H_p, H_{p'}\} &= i(p' - p)L_{p+p'} + \frac{i}{2} p^3 \delta_{p+p', 0}, \\ \{G_r, H_p\} &= i\left(\frac{p}{2} - r\right)F_{r+p}, \\ \{F_s, H_p\} &= i\left(\frac{p}{2} - s\right)G_{s+p}, \\ \{G_r, G_{r'}\} &= -\frac{i}{2} L_{r+r'} + \frac{i}{2} r^2 \delta_{r+r', 0}, \\ \{F_s, G_r\} &= \frac{i}{2} H_{s+r}, \\ \{F_s, F_{s'}\} &= \frac{i}{2} L_{s+s'} - \frac{i}{2} s^2 \delta_{s+s', 0} \end{aligned} \tag{4.24}$$

where the indices take the following values according to the boundary conditions. (i) all the indices are integer, (ii) $n, n', p, p' \in \mathbb{Z}$ and $r, r', s, s' \in \mathbb{Z} + \frac{1}{2}$, (iii) $n, n', s, s' \in \mathbb{Z}$ and $p, p', r, r' \in \mathbb{Z} + \frac{1}{2}$.

We derive the Lie-Poisson algebra with a [00]-graded central extension, in which the central charge takes a fixed value. The factors of the central terms n^3 and r^2 for [00]-graded and [10]-graded currents in (4.24) differ from the standard choice in CFT (note that the same applies to the [11]- and [01]-graded currents). This discrepancy arises from our convention for the mode expansion (4.23), which is defined on S^1 , instead of the punctured complex plane $\mathbb{C}^* = \mathbb{C} - \{0\}$, as customary in CFT, where the mode expansion is typically given by $u_{00}(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$. We recall that the central charges proportional to n^3, r^2 and $n(n^2 - 1), r^2 - \frac{1}{4}$ (as derived in CFT) differ by a trivial cocycle which can be reabsorbed in the redefinitions of the generators. The most general \mathbb{Z}_2^2 -graded extension of the

super-Virasoro algebra admits two central extensions: one [00]-graded and the other [11]-graded [44]. However, the extension admitting a [11]-graded central charge is not obtained within the present framework. Another open problem is the possible isomorphism between two algebras characterized by R/R/R and R/NS/NS boundary conditions. It is known that the $\mathcal{N} = 2$ Ramond and Neveu–Schwarz superalgebras are isomorphic [49]. A similar structure appears in the $\mathcal{N} = 2$ super-Virasoro algebra and the \mathbb{Z}_2^2 -graded super-Virasoro algebra derived in this section, as the latter also possesses two fermionic currents: one [10]-graded and the other [01]-graded. Therefore, it is important to clarify whether the isomorphism exists.

Finally, we comment that a different type of \mathbb{Z}_2^2 -graded extension of super-Virasoro algebra is discussed in [37] where the (00, 11, 10, 01)-graded currents have the scaling dimensions $(2, 1, \frac{3}{2}, \frac{3}{2})$, respectively. In contrast, the \mathbb{Z}_2 -graded super-Virasoro algebra presented in this work features scaling dimensions $(2, 2, \frac{3}{2}, \frac{3}{2})$ and is associated with the \mathbb{Z}_2^2 -super-Liouville equation (3.18).

The possibility of having inequivalent \mathbb{Z}_2^2 -graded extensions of superalgebras was already mentioned in Introduction. Another illustrative example is provided by the two-dimensional superPoincaré algebra, which admits \mathbb{Z}_2^2 -graded extensions with two [11]-graded translations [19], two [00]-graded translations [50] and one [00]-graded plus one [11]-graded translation [45].

5. Zero-curvature formulation of the \mathbb{Z}_2^2 -graded super-Liouville equation

In this section and the following one, we investigate the integrability of the \mathbb{Z}_2^2 -super-Liouville equation derived via the soldering procedure described in section 3. We first show that the \mathbb{Z}_2^2 -super-Liouville equation can be formulated within the zero-curvature framework. Three inequivalent Lax operators are found to be admissible for this formulation. Then, we explicitly construct a solution to the equation.

5.1. Lax operators and zero-curvature equation

We return to the superfields defined on superspace with [00] and [10]-graded coordinates. We now adopt a notation more suitable for the zero-curvature framework. The coordinate of superspace are denoted by

$$[00] \ x_+, \ x_-, \quad [10] \ \theta_+, \ \theta_- \tag{5.1}$$

and we write the corresponding covariant derivatives as

$$D_{\pm} = \partial_{\theta_{\pm}} + i\theta_{\pm}\partial_{\pm}, \quad \partial_{\pm} := \partial_{x_{\pm}}. \tag{5.2}$$

They satisfy the following relations:

$$\{D_{\pm}, D_{\pm}\} = 2i\partial_{\pm}, \quad \{D_+, D_-\} = 0. \tag{5.3}$$

Denoting the superfields by $\Phi_{00}(x_{\pm}, \theta_{\pm})$ and $\Phi_{11}(x_{\pm}, \theta_{\pm})$ we define the [10]-graded Lax operators

$$\mathcal{L}_{\pm} = \mp D_{\pm}\Phi + e^{\pm\Phi}P_{\pm}e^{\mp\Phi} = \mp D_{\pm}\Phi + A_{00}P_{\pm} + iA_{11}Q_{\pm}, \tag{5.4}$$

$$\Phi = \frac{1}{2}(\Phi_{00}K_0 + \Phi_{11}L_0) \tag{5.5}$$

where

$$A_{00} = e^{\frac{1}{2}\Phi_{00}} \cosh \frac{1}{2}\Phi_{11}, \quad A_{11} = e^{\frac{1}{2}\Phi_{00}} \sinh \frac{1}{2}\Phi_{11}. \tag{5.6}$$

Following the general construction of integrable systems, we consider the linear system

$$(D_{\pm} - \mathcal{L}_{\pm})T = 0 \tag{5.7}$$

where T is an element of the group generated by \mathfrak{g} . The compatibility condition of the system is the zero-curvature condition which is in the present case given by

$$D_+\mathcal{L}_- + D_-\mathcal{L}_+ - \{\mathcal{L}_+, \mathcal{L}_-\} = 0. \tag{5.8}$$

Using only the algebraic structure of \mathfrak{g} , one may obtain the following equations from the zero-curvature condition:

$$D_+D_-\Phi_{00} = e^{\Phi_{00}} \cosh \Phi_{11}, \quad D_+D_-\Phi_{11} = e^{\Phi_{00}} \sinh \Phi_{11} \tag{5.9}$$

and

$$\begin{aligned} D_{\pm}A_{00} &= \frac{1}{2} (D_{\pm}\Phi_{00} \cdot A_{00} + D_{\pm}\Phi_{11} \cdot A_{11}), \\ D_{\pm}A_{11} &= \frac{1}{2} (D_{\pm}\Phi_{00} \cdot A_{11} + D_{\pm}\Phi_{11} \cdot A_{00}). \end{aligned} \tag{5.10}$$

As is readily seen from (5.6), the equations in (5.10) are identities. Whereas the equations in (5.9) are non-trivial dynamical equations which are identical to (3.18).

5.2. Alternative Lax operators

We show that the \mathbb{Z}_2^2 -super-Liouville equation in the component form (3.20) and (3.21) admits an alternative Lax operator formulation. Let us consider a \mathbb{Z}_2^2 -graded superspace with coordinates

$$[00] \ x_+, \ x_-, \quad [10] \ \theta_{10}, \quad [01] \ \theta_{01} \tag{5.11}$$

where θ_{10}, θ_{01} are nilpotent and mutually commuting. The corresponding covariant derivatives are defined as

$$D_{10} = \partial_{\theta_{10}} + i\theta_{10}\partial_+, \quad D_{01} = \partial_{\theta_{01}} + i\theta_{01}\partial_- \tag{5.12}$$

and satisfy the relations:

$$\{D_{10}, D_{10}\} = 2i\partial_+, \quad \{D_{01}, D_{01}\} = 2i\partial_-, \quad [D_{10}, D_{01}] = 0. \tag{5.13}$$

We introduce the [00]-graded superfield $\tilde{\Phi}(x_{\pm}, \theta_{10}, \theta_{01})$ defined by

$$\tilde{\Phi} = \frac{1}{2} (\tilde{\Phi}_{00}K_0 + \tilde{\Phi}_{11}L_0) \tag{5.14}$$

where $\tilde{\Phi}_{00}$ and $\tilde{\Phi}_{11}$ are also superfields on the \mathbb{Z}_2^2 -graded superspace (5.11). We define the [10] and [01]-graded Lax operators:

$$\begin{aligned} \mathcal{L}_{10} &= -D_{10}\tilde{\Phi} + e^{\tilde{\Phi}}P_+e^{-\tilde{\Phi}} = -D_{10}\tilde{\Phi} + \tilde{A}_{00}P_+ + i\tilde{A}_{11}Q_+, \\ \mathcal{L}_{01} &= D_{01}\tilde{\Phi} + e^{-\tilde{\Phi}}P_+e^{\tilde{\Phi}} = D_{01}\tilde{\Phi} + \tilde{A}_{00}Q_- - i\tilde{A}_{11}P_- \end{aligned} \tag{5.15}$$

where

$$\tilde{A}_{00} = e^{\frac{1}{2}\tilde{\Phi}_{00}} \cosh \frac{1}{2}\tilde{\Phi}_{11}, \quad \tilde{A}_{11} = e^{\frac{1}{2}\tilde{\Phi}_{00}} \sinh \frac{1}{2}\tilde{\Phi}_{11}. \tag{5.16}$$

The zero-curvature condition for these Lax operators takes the form

$$D_{10}\mathcal{L}_{01} - D_{01}\mathcal{L}_{10} - [\mathcal{L}_{10}, \mathcal{L}_{01}] = 0 \tag{5.17}$$

Using only the algebraic structure of \mathfrak{g} as in the previous subsection, one may obtain the following equations from the zero-curvature condition:

$$D_{10}D_{01}\tilde{\Phi}_{00} = ie^{\tilde{\Phi}_{00}} \sinh \tilde{\Phi}_{11}, \quad D_{10}D_{01}\tilde{\Phi}_{11} = ie^{\tilde{\Phi}_{00}} \cosh \tilde{\Phi}_{11}. \tag{5.18}$$

The difference from the previous choice of Lax operators is that no equations corresponding to (5.10) are obtained in this case.

By expanding the superfields $\tilde{\Phi}_{00}$ and $\tilde{\Phi}_{11}$ as

$$\begin{aligned} \tilde{\Phi}_{00} &= \varphi_{00} + \theta_{10}\psi_{10} + i\theta_{01}\bar{\psi}_{01} - i\theta_{10}\theta_{01}F_{11}, \\ \tilde{\Phi}_{11} &= \varphi_{11} + \theta_{10}\psi_{01} + i\theta_{01}\bar{\psi}_{10} - i\theta_{10}\theta_{01}F_{00}, \end{aligned} \tag{5.19}$$

one readily verifies that the equations (5.18) are equivalent to (3.20) and (3.21).

5.3. Lax operators with spectral parameter

It is possible to formulate the component equations (see (3.20)–(3.22)) by the Lax operators with a spectral parameter and without using the superfields. Let us introduce the loop extension of \mathfrak{g} defined by

$$\begin{aligned} K_n &:= \lambda^n \otimes K_0, & K_n^\pm &:= \lambda^n \otimes K_\pm, & L_n &:= \lambda^n \otimes L_0, & L_n^\pm &:= \lambda^n \otimes L_\pm, \\ P_n^\pm &:= \lambda^n \otimes P_\pm, & Q_n^\pm &:= \lambda^n \otimes Q_\pm. \end{aligned} \tag{5.20}$$

where $n \in \mathbb{Z}$ and λ is a parameter. These form an infinite dimensional \mathbb{Z}_2^2 -graded Lie superalgebra. This algebra has central extensions and derivations with non-trivial grading [44]. However, these are not necessary for our current purpose.

We define the [00]-graded Lax operators using the elements of (5.20):

$$\begin{aligned} \mathcal{L}_+ &= -\partial_+ \varphi_{00} K_0 - \partial_+ \varphi_{11} L_0 - i(K_2^+ + K_2^-) + \bar{\psi}_{10} P_1^+ + i\bar{\psi}_{01} Q_1^+, \\ \mathcal{L}_- &= ie^{2\varphi_{00}} (\cosh 2\varphi_{11} \cdot K_{-2}^- + \sinh 2\varphi_{11} \cdot L_{-2}^-) + e^{\varphi_{00}} (\Lambda_{10} P_{-1}^- + i\Lambda_{01} Q_{-1}^-). \end{aligned} \tag{5.21}$$

It is then straightforward to verify that the zero-curvature condition

$$\partial_- \mathcal{L}_+ - \partial_+ \mathcal{L}_- + [\mathcal{L}_+, \mathcal{L}_-] = 0 \tag{5.22}$$

gives the component form of the \mathbb{Z}_2^2 -super-Liouville equation.

To show the spectral parameter dependence more explicitly, we present the matrix representation of \mathcal{L}_\pm in the space of six-dimensional representation of \mathfrak{g} given in section 2.2:

$$\mathcal{L}_+ = \left[\begin{array}{cccc|cc} -\partial_+ \varphi_{00} & -i\lambda^2 & -\partial_+ \varphi_{11} & 0 & \lambda \bar{\psi}_{10} & i\lambda \bar{\psi}_{01} \\ -i\lambda^2 & \partial_+ \varphi_{00} & 0 & \partial_+ \varphi_{11} & 0 & 0 \\ -\partial_+ \varphi_{11} & 0 & -\partial_+ \varphi_{00} & -i\lambda^2 & -\lambda \bar{\psi}_{01} & -i\lambda \bar{\psi}_{10} \\ 0 & \partial_+ \varphi_{11} & -i\lambda^2 & \partial_+ \varphi_{00} & 0 & 0 \\ \hline 0 & -\lambda \bar{\psi}_{10} & 0 & -\lambda \bar{\psi}_{01} & 0 & 0 \\ 0 & -i\lambda \bar{\psi}_{01} & 0 & -i\lambda \bar{\psi}_{10} & 0 & 0 \end{array} \right], \tag{5.23}$$

$$\mathcal{L}_- = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ i\lambda^{-2} f_{00} & 0 & i\lambda^{-2} f_{11} & 0 & -\lambda^{-1} e^{\varphi_{00}} \Lambda_{10} & -i\lambda^{-1} e^{\varphi_{00}} \Lambda_{01} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i\lambda^{-2} f_{11} & 0 & i\lambda^{-2} f_{00} & 0 & \lambda^{-1} e^{\varphi_{00}} \Lambda_{01} & i\lambda^{-1} e^{\varphi_{00}} \Lambda_{10} \\ \hline -\lambda^{-1} e^{\varphi_{00}} \Lambda_{10} & 0 & -\lambda^{-1} e^{\varphi_{00}} \Lambda_{01} & 0 & 0 & 0 \\ -i\lambda^{-1} e^{\varphi_{00}} \Lambda_{01} & 0 & -i\lambda^{-1} e^{\varphi_{00}} \Lambda_{10} & 0 & 0 & 0 \end{array} \right] \tag{5.24}$$

where $f_{00} := e^{2\varphi_{00}} \cosh 2\varphi_{11}, f_{11} := e^{2\varphi_{00}} \sinh 2\varphi_{11}$.

5.4. Reconstruction of the superfields

In this subsection, we solve the \mathbb{Z}_2^2 -super-Liouville equation (5.9) by the method developed in [30–32].

First, note that solutions of the zero curvature condition is pure gauge:

$$\mathcal{L}_\pm = (D_\pm T) T^{-1}. \tag{5.25}$$

We write T in two different ways (generalized Gauss decomposition)

$$T = e^{\mp\Phi} N_\pm B_\mp, \quad N_\pm \in \exp(\mathfrak{g}_\pm), \quad B_\mp \in \exp(\mathfrak{h} \oplus \mathfrak{g}_\mp). \tag{5.26}$$

Substituting this into (5.25) and comparing with (5.4), we obtain

$$N_\pm (D_\pm B_\mp) B_\mp^{-1} N_\pm^{-1} = -D_\pm N_\pm \cdot N_\pm^{-1} + e^{\pm 2\Phi} P_\pm e^{\mp 2\Phi}. \tag{5.27}$$

The RHS is in \mathfrak{g}_\pm , while the LHS is in $\mathfrak{h} \oplus \mathfrak{g}_\mp$ which implies the followings

$$D_\pm B_\mp = 0, \tag{5.28}$$

$$(D_\pm N_\pm) N_\pm^{-1} = e^{\pm 2\Phi} P_\pm e^{\mp 2\Phi} = e^{\Phi_{00}} (\cosh \Phi_{11} \cdot P_\pm + i \sinh \Phi_{11} \cdot Q_\pm). \tag{5.29}$$

On the other hand, substitution $T = e^{\pm\Phi} N_{\mp} B_{\pm}$ into (5.25) gives

$$\begin{aligned} (D_{\pm} B_{\pm}) B_{\pm}^{-1} &= -N_{\mp}^{-1} D_{\pm} N_{\mp} + N_{\mp}^{-1} (\mp 2D_{\pm} \Phi + P_{\pm}) N_{\mp} \\ &= P_{\pm} + (\text{terms in } \mathfrak{h} \oplus \mathfrak{g}_{\mp}). \end{aligned} \tag{5.30}$$

Now we parametrize B_{\pm} as

$$B_{\pm} = e^{f_{\pm} K_0} e^{g_{\pm} L_0} e^{q_{\pm} K_{\pm}} e^{r_{\pm} L_{\pm}} e^{\alpha_{\pm} P_{\pm}} e^{\beta_{\pm} Q_{\pm}} \tag{5.31}$$

where the parameters are superfields (functions of x_{\pm}, θ_{\pm}) with the following \mathbb{Z}_2^2 -gradings

$$[00] f_{\pm}, q_{\pm}, \quad [11] g_{\pm}, r_{\pm}, \quad [10] \alpha_{\pm}, \quad [01] \beta_{\pm} \tag{5.32}$$

and are \mathbb{Z}_2^2 -commutative. The condition (5.28) implies that all the superfields are chiral, i.e., $f_{+}(x_{+}, \theta_{+}), f_{-}(x_{-}, \theta_{-}),$ etc. From (5.30) we obtain several constraints on the superfields:

$$\begin{aligned} D_{\pm} q_{\pm} \pm (D_{\pm} \alpha_{\pm}) \alpha_{\pm} \pm (D_{\pm} \beta_{\pm}) \beta_{\pm} &= 0, \\ D_{\pm} r_{\pm} \pm 2i (D_{\pm} \beta_{\pm}) \alpha_{\pm} &= 0 \end{aligned} \tag{5.33}$$

and

$$D_{\pm} \alpha_{\pm} = e^{\mp f_{\pm}} \cosh g_{\pm}, \quad D_{\pm} \beta_{\pm} = \mp i e^{\mp f_{\pm}} \sinh g_{\pm}. \tag{5.34}$$

Let $|\rangle$ be a lowest weight vector introduced in section 2.2. The following relations follow immediately from the lowest weight nature of $|\rangle$:

$$\langle | e^{\Phi} T = \langle | B_{-}, \quad (e^{-\Phi} T)^{-1} | \rangle = B_{+}^{-1} | \rangle \tag{5.35}$$

which give the reconstruction of Φ_{00} and Φ_{11} in terms of the chiral superfields in (5.31):

$$\langle | e^{2\Phi} | \rangle = \langle | B_{-}(x_{-}, \theta_{-}) B_{+}^{-1}(x_{+}, \theta_{+}) | \rangle. \tag{5.36}$$

The LHS is easily computed to get

$$\langle 00 | e^{2\Phi} | 00 \rangle \langle 00 | e^{2\Phi} | 00 \rangle = e^{-\Phi_{00}} \cosh \Phi_{11}, \quad \langle 11 | e^{2\Phi} | 00 \rangle \langle 11 | e^{2\Phi} | 00 \rangle = -e^{-\Phi_{00}} \sinh \Phi_{11}. \tag{5.37}$$

After a lengthy computation, the corresponding RHS can be obtained, providing the solution to the \mathbb{Z}_2^2 -super-Liouville equation:

$$e^{-\Phi_{00}} \cosh \Phi_{11} = e^{f_{+} - f_{-}} (W_{00} \cosh(g_{+} - g_{-}) + W_{11} \sinh(g_{+} - g_{-})), \tag{5.38}$$

$$e^{-\Phi_{00}} \sinh \Phi_{11} = -e^{f_{+} - f_{-}} (W_{00} \sinh(g_{+} - g_{-}) + W_{11} \cosh(g_{+} - g_{-})) \tag{5.39}$$

where

$$\begin{aligned} W_{00} &= 1 + \alpha_{+} \alpha_{-} + \beta_{+} \beta_{-} - q_{+} q_{-} - R_{+} R_{-}, \\ W_{11} &= -i \alpha_{+} \beta_{-} + i \alpha_{-} \beta_{+} - q_{+} R_{-} - q_{-} R_{+} \end{aligned} \tag{5.40}$$

with $R_{\pm} = r_{\pm} \pm i \alpha_{\pm} \beta_{\pm}$.

5.5. Checking the solution

We here present a proof that (5.38) and (5.39) are indeed the solutions to the \mathbb{Z}_2^2 -super-Liouville equation. We do this by direct computation. Solving (5.38) and (5.39) for $\cosh \Phi_{11}$ and $\sinh \Phi_{11}$ and using $\cosh^2 x - \sinh^2 x = 1$, we get

$$e^{-2\Phi_{00}} = e^{2(f_{+} - f_{-})} (W_{00}^2 - W_{11}^2), \tag{5.41}$$

$$\tanh \Phi_{11} = -\frac{W_{00} \sinh(g_{+} - g_{-}) + W_{11} \cosh(g_{+} - g_{-})}{W_{00} \cosh(g_{+} - g_{-}) + W_{11} \sinh(g_{+} - g_{-})}. \tag{5.42}$$

Then, it is not difficult to compute the covariant derivatives of the superfields:

$$\begin{aligned}
 D_+D_-\Phi_{00} &= \frac{1}{(W_{00}^2 - W_{11}^2)^2} \left[(W_{00}^2 + W_{11}^2) (D_+W_{00} \cdot D_-W_{00} - D_+W_{11} \cdot D_-W_{11}) \right. \\
 &\quad - 2W_{00}W_{11} (D_+W_{00} \cdot D_-W_{11} - D_-W_{00} \cdot D_+W_{11}) \\
 &\quad \left. - (W_{00}^2 - W_{11}^2) (W_{00}D_+D_-W_{00} - W_{11}D_+D_-W_{11}) \right] \tag{5.43}
 \end{aligned}$$

and

$$\begin{aligned}
 D_+D_-\Phi_{11} &= \frac{1}{(W_{00}^2 - W_{11}^2)^2} \left[(W_{00}^2 + W_{11}^2) (D_+W_{00} \cdot D_-W_{11} - D_-W_{00} \cdot D_+W_{11}) \right. \\
 &\quad - 2W_{00}W_{11} (D_+W_{00} \cdot D_-W_{00} - D_+W_{11} \cdot D_-W_{11}) \\
 &\quad \left. - (W_{00}^2 - W_{11}^2) (W_{00}D_+D_-W_{11} - W_{11}D_+D_-W_{00}) \right]. \tag{5.44}
 \end{aligned}$$

We rewrite the above equations by using the constraints (5.33) and (5.34). Using the constraints (5.33), one may see that

$$\begin{aligned}
 D_{\pm}W_{00} &= \pm \mathcal{A}_{\mp}D_{\pm}\alpha_{\pm} \mp \mathcal{B}_{\mp}D_{\pm}\beta_{\pm}, \\
 D_{\pm}W_{11} &= \mp i(\mathcal{B}_{\mp}D_{\pm}\alpha_{\pm} + \mathcal{A}_{\mp}D_{\pm}\beta_{\pm}) \tag{5.45}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{A}_{\pm} &= \alpha_{\pm} + q_{\pm}\alpha_{\mp} + iR_{\pm}\beta_{\mp}, \quad \text{deg}(\mathcal{A}_{\pm}) = [10], \\
 \mathcal{B}_{\pm} &= \beta_{\pm} + q_{\pm}\beta_{\mp} - iR_{\pm}\alpha_{\mp}, \quad \text{deg}(\mathcal{B}_{\pm}) = [01]. \tag{5.46}
 \end{aligned}$$

It follows that

$$\begin{aligned}
 D_+W_{00} \cdot D_-W_{00} - D_+W_{11} \cdot D_-W_{11} &= (\mathcal{A}_+\mathcal{A}_- + \mathcal{B}_+\mathcal{B}_-) \mathcal{D}_{00} + (\mathcal{A}_-\mathcal{B}_+ - \mathcal{A}_+\mathcal{B}_-) \mathcal{D}_{11}, \\
 D_+W_{00} \cdot D_-W_{11} - D_+W_{11} \cdot D_-W_{00} &= -i(\mathcal{A}_+\mathcal{A}_- + \mathcal{B}_+\mathcal{B}_-) \mathcal{D}_{11} + i(\mathcal{A}_-\mathcal{B}_+ - \mathcal{A}_+\mathcal{B}_-) \mathcal{D}_{00} \tag{5.47}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{D}_{00} &:= D_+\alpha_+ \cdot D_-\alpha_- - D_+\beta_+ \cdot D_-\beta_- \stackrel{(5.34)}{=} e^{-f_+ + f_-} \cosh(g_+ - g_-), \\
 \mathcal{D}_{11} &:= D_+\alpha_+ \cdot D_-\beta_- + D_-\alpha_- \cdot D_+\beta_+ \stackrel{(5.35)}{=} -ie^{-f_+ + f_-} \sinh(g_+ - g_-) \tag{5.48}
 \end{aligned}$$

and the following relations hold true:

$$\begin{aligned}
 \mathcal{A}_+\mathcal{A}_- + \mathcal{B}_+\mathcal{B}_- &= (\alpha_+\alpha_- + \beta_+\beta_-) W_{00} + i(\alpha_-\beta_+ - \alpha_+\beta_-) W_{11}, \\
 \mathcal{A}_-\mathcal{B}_+ - \mathcal{A}_+\mathcal{B}_- &= -i(\alpha_+\alpha_- + \beta_+\beta_-) W_{11} + (\alpha_-\beta_+ - \alpha_+\beta_-) W_{00}. \tag{5.49}
 \end{aligned}$$

Then, we obtain the second order derivatives:

$$\begin{aligned}
 D_+D_-W_{00} &= -(1 - \alpha_+\alpha_- - \beta_+\beta_-) \mathcal{D}_{00} + (\alpha_-\beta_+ - \alpha_+\beta_-) \mathcal{D}_{11} \\
 D_+D_-W_{11} &= i(1 - \alpha_+\alpha_- - \beta_+\beta_-) \mathcal{D}_{11} + i(\alpha_-\beta_+ - \alpha_+\beta_-) \mathcal{D}_{00}. \tag{5.50}
 \end{aligned}$$

With these, one may compute (5.43) and (5.44) as follows:

$$\begin{aligned}
 D_+D_-\Phi_{00} &= \frac{1}{W_{00}^2 - W_{11}^2} (W_{00}\mathcal{D}_{00} + iW_{11}\mathcal{D}_{11}) \\
 &\stackrel{(5.48)}{=} \frac{e^{-f_+ + f_-}}{W_{00}^2 - W_{11}^2} (W_{00} \cosh(g_+ - g_-) + W_{11} \sinh(g_+ - g_-)) \\
 &= e^{\Phi_{00}} \cosh \Phi_{11} \tag{5.51}
 \end{aligned}$$

and

$$\begin{aligned}
 D_+D_-\Phi_{11} &= \frac{-1}{W_{00}^2 - W_{11}^2} (W_{11}\mathcal{D}_{00} + iW_{00}\mathcal{D}_{11}) \\
 &\stackrel{(5.48)}{=} \frac{-e^{-f_+ + f_-}}{W_{00}^2 - W_{11}^2} (W_{11} \cosh(g_+ - g_-) + W_{00} \sinh(g_+ - g_-)) \\
 &= e^{\Phi_{00}} \cosh \Phi_{11}. \tag{5.52}
 \end{aligned}$$

This completes the check of the solutions.

6. \mathbb{Z}_2^2 -graded Bäcklund transformations

It is known that the Liouville equation has two Bäcklund transformations, one is the transformation to the free equation and the other is the auto-Bäcklund transformation. These play important roles of integrability of the equation and the corresponding transformations for the super-Liouville equation is also known [27]. In this section, we present a \mathbb{Z}_2^2 -graded version of those transformations.

Suppose that Φ_{00} and Φ_{11} solve the \mathbb{Z}_2^2 -super-Liouville equation (5.9). We denote the transformed superfields by $\tilde{\Phi}_{00}, \tilde{\Phi}_{11}$ and write the linear combination as

$$V_{\pm} := \frac{1}{2} (\Phi_{00} \pm \tilde{\Phi}_{00}), \quad W_{\pm} := \frac{1}{2} (\Phi_{11} \pm \tilde{\Phi}_{11}). \tag{6.1}$$

To define Bäcklund transformations we need introduce two additional superfields:

$$\Lambda(x_{\pm}, \theta_{\pm}), \quad \Gamma(x_{\pm}, \theta_{\pm}), \quad \text{deg}(\Lambda) = [10], \quad \text{deg}(\Gamma) = [01]. \tag{6.2}$$

Then, a Bäcklund transformation is defined as follows

$$\begin{aligned} D_+ V_+ &= \frac{a}{2} e^{V_-} (\Lambda \cosh W_- + \Gamma \sinh W_-), \\ D_- V_- &= \frac{1}{2} e^{V_+} (\Lambda \cosh W_+ + \Gamma \sinh W_+) \\ D_+ W_+ &= \frac{a}{2} e^{V_-} (\Lambda \sinh W_- + \Gamma \cosh W_-), \\ D_- W_- &= \frac{1}{2} e^{V_+} (\Lambda \sinh W_+ + \Gamma \cosh W_+) \\ D_+ \Lambda &= e^{V_-} \cosh W_-, \quad D_- \Lambda = -\frac{1}{a} e^{V_+} \cosh W_+ \\ D_+ \Gamma &= e^{V_-} \sinh W_-, \quad D_- \Gamma = -\frac{1}{a} e^{V_+} \sinh W_+, \end{aligned} \tag{6.3}$$

where a is a non-zero constant. We claim that $\tilde{\Phi}_{00}, \tilde{\Phi}_{11}$ solve the free equations:

$$D_+ D_- \tilde{\Phi}_{00} = D_+ D_- \tilde{\Phi}_{11} = 0. \tag{6.4}$$

This can be verified straightforwardly. The relations (6.3) imply

$$2D_+ D_- V_{\pm} = e^{\Phi_{00}} \cosh \Phi_{11}, \quad 2D_+ D_- W_{\pm} = e^{\Phi_{00}} \sinh \Phi_{11}. \tag{6.5}$$

Assuming that Φ_{00} and Φ_{11} solve the \mathbb{Z}_2^2 -super-Liouville equation (5.9), these relations lead directly to the free equations (6.4).

An auto-Bäcklund transformation, which transforms the \mathbb{Z}_2^2 -super-Liouville equation to themselves, is defined by

$$\begin{aligned} D_+ V_+ &= a (\Lambda \cosh V_- \cosh W_- + \Gamma \sinh V_- \sinh W_-), \\ D_- V_- &= e^{V_+} (\Lambda \cosh W_+ + \Gamma \sinh W_+), \\ D_+ W_+ &= a (\Lambda \sinh V_- \sinh W_- + \Gamma \cosh V_- \cosh W_-), \\ D_- W_- &= e^{V_+} (\Lambda \sinh W_+ + \Gamma \cosh W_+), \\ D_+ \Lambda &= \sinh V_- \cosh W_-, \quad D_- \Lambda = -\frac{1}{a} e^{V_+} \cosh W_+, \\ D_+ \Gamma &= \cosh V_- \sinh W_-, \quad D_- \Gamma = -\frac{1}{a} e^{V_+} \sinh W_+, \end{aligned} \tag{6.6}$$

where a is a non-zero constant and V_{\pm}, W_{\pm} are given by (6.1). It is easy to verify that if Φ_{00} and Φ_{11} satisfy (5.9), then $\tilde{\Phi}_{00}$ and $\tilde{\Phi}_{11}$ do so as well. We briefly outline the computation. Using the relations in (6.6), one may obtain the followings:

$$\begin{aligned} D_+ D_- V_+ &= e^{V_+} (\cosh V_- \cosh W_+ \cosh W_- + \sinh V_- \sinh W_+ \sinh W_-), \\ D_+ D_- V_- &= e^{V_+} (\cosh V_- \sinh W_+ \sinh W_- + \sinh V_- \cosh W_+ \cosh W_-), \\ D_+ D_- W_+ &= e^{V_+} (\sinh V_- \cosh W_+ \sinh W_- + \cosh V_- \sinh W_+ \cosh W_-), \\ D_+ D_- W_- &= e^{V_+} (\sinh V_- \sinh W_+ \cosh W_- + \cosh V_- \cosh W_+ \sinh W_-). \end{aligned} \tag{6.7}$$

It follows that

$$\begin{aligned} D_+ D_- (V_+ \pm V_-) &= e^{V_+ \pm V_-} \cosh(W_+ \pm W_-), \\ D_+ D_- (W_+ \pm W_-) &= e^{V_+ \pm V_-} \sinh(W_+ \pm W_-). \end{aligned} \tag{6.8}$$

These are the \mathbb{Z}_2^2 -super-Liouville equations for Φ_{00}, Φ_{11} and $\tilde{\Phi}_{00}, \tilde{\Phi}_{11}$.

The generalized current conservation laws arise from the (auto-)Bäcklund transformations. We define the following \mathbb{Z}_2^2 -graded currents:

$$\begin{aligned} J_{00}^\pm &= D_\mp \Lambda, & J_{11}^\pm &= D_\mp \Gamma, \\ J_{10}^\pm &= \pm J_{00}^\pm \Lambda \mp J_{11}^\pm \Gamma, & J_{01}^\pm &= \pm J_{00}^\pm \Gamma \mp J_{11}^\pm \Lambda. \end{aligned} \tag{6.9}$$

Here J_{00}^\pm and J_{11}^\pm are given by (6.3) for the Bäcklund transformation and by (6.6) for the auto-Bäcklund transformation. These currents satisfy the generalized conservation laws

$$D_+ J_\alpha^+ + D_- J_\alpha^- = 0, \quad \alpha \in \{00, 11, 01, 10\} \tag{6.10}$$

which can be readily verified by using the relations

$$\begin{aligned} (D_\pm J_{00}^\pm) \Lambda - (D_\pm J_{11}^\pm) \Gamma &= 0, \\ (D_\pm J_{10}^\pm) \Gamma - (D_\pm J_{01}^\pm) \Lambda &= 0. \end{aligned} \tag{6.11}$$

For a given solution of either the free equation or the \mathbb{Z}_2^2 -super-Liouville equation, one can use the Bäcklund transformations (6.3) or (6.6) to generate another solution of the \mathbb{Z}_2^2 -super-Liouville equation. It is also expected that the Bianchi’s permutability theorem, which allows one to combine two solutions to obtain a third, can be extended to the \mathbb{Z}_2^2 -setting. However, verifying these properties is beyond the scope of the present work and is left for future investigation. The reason is that the involved computations are cumbersome and highly nontrivial due to the presence of additional superfields Λ, Γ which satisfy nontrivial relations with Φ_{00} and Φ_{11} . We point out that the alternative method described in section 5.4 is much simpler and more transparent approach to solving the \mathbb{Z}_2 -graded equation.

7. Conclusions

We have derived an integrable \mathbb{Z}_2^2 -graded extension of the super-Liouville equation and investigated its properties, along with the associated current algebra, which constitutes a new \mathbb{Z}_2^2 -graded extension of the super-Virasoro algebra. This was done within the framework of Polyakov’s soldering and the zero-curvature formulation. Explicit solutions of the derived equation were constructed by extending the method developed by Leznov and Saveliev. An auto-Bäcklund transformation was presented, along with a Bäcklund transformation to the free equation.

The graded extension of the super-Virasoro algebra was defined as a Poisson–Lie algebra, and we observed the presence of a non-vanishing [00]-graded central charge, despite the fact that the present theory is formulated within the framework of classical field theory. We pointed out that three inequivalent (anti)periodic boundary conditions are admissible in the mode expansion of the \mathbb{Z}_2^2 -graded currents. As a consequence, we obtained three distinct \mathbb{Z}_2^2 -graded extensions of the super-Virasoro algebra. However, the question of whether these algebras are equivalent remains an open problem.

The present work is based on a \mathbb{Z}_2^2 -graded extension of $\mathfrak{osp}(1|2)$, and we have considered only [00] and [10]-graded superspace coordinates. This suggests the possibility of further \mathbb{Z}_2^2 -graded extensions of the super-Liouville equation. For instance, one may consider alternative \mathbb{Z}_2^2 -graded extensions of $\mathfrak{osp}(1|2)$, as the algebra admits some inequivalent \mathbb{Z}_2^2 -graded extensions [3, 6, 13, 15, 17, 29]. It is also possible to formulate the theory on a superspace that includes [11]-graded coordinates. The square of the [11]-graded coordinate can be regarded as an emergent [00]-graded coordinate [39, 45, 51]. Therefore, incorporating [11]-graded coordinates into the superspace may lead to integrable systems formulated in higher-dimensional spacetime.

Another interesting direction for future research is the study of integrable systems based on higher-rank \mathbb{Z}_2^2 -graded superalgebras such as \mathbb{Z}_2^2 -graded version of $\mathfrak{sl}(2|1), \mathfrak{osp}(2|2)$ and their affine extensions. Repeating the present analysis within a simplified superspace, as considered in this work, may lead to integrable systems that go beyond the \mathbb{Z}_2^2 -super-Liouville equation, since higher-rank algebras allow for the introduction of additional fields and interactions associated with the increasing number of simple roots. In this way, \mathbb{Z}_2^2 -graded superalgebras are expected to give rise to a rich landscape of novel integrable systems.

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Data availability statement

No new data were created or analysed in this study.

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