

On Gauge Invariance of the Bosonic Measure in Chiral Gauge Theories

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Abstract: Gauge invariance of the measure associated with the gauge field is usually taken for granted, in a general gauge theory. We furnish a proof of this invariance, within Fujikawa's approach. To stress the importance of this fact, we briefly review gauge anomaly cancellation as a consequence of gauge invariance of the bosonic measure and compare this cancellation to usual results from algebraic renormalization, showing that there are no potential inconsistencies. Then, using a path integral argument, we show that a possible Jacobian for the gauge transformation has to be the identity operator, in the physical Hilbert space. We extend the argument to the complete Hilbert space by a direct calculation.

Keywords: gauge field theories; gauge anomalies; nonperturbative techniques

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1. Introduction

As we celebrate the 80th anniversary of Wigner's fundamental paper [1], on how to build Poincaré symmetry in the quantum domain, it seemed relevant to us to consider some aspects of the phenomenon of quantum obstruction to implementation of a given symmetry, which is called an *anomaly*. An anomaly usually manifests as an operator that prevents the expectation value of a Noether current from vanishing. Anomalies become specially critical when they refer to gauge symmetry, usually required to prove renormalization of the corresponding gauge theories. A common statement is that gauge anomalies break Ward-Takahashi (or Slavnov-Taylor) identities, necessary to reduce the number of renormalization constants to be computed, spoiling perturbative renormalization of these theories. In this way, the unavoidable presence of a gauge anomaly is usually taken as an indication that the gauge theory under analysis is not adequate.

Chiral gauge theories are examples of this kind of theory. They are defined through minimal couplings between chiral fermions and gauge fields. As they are the basis over which the standard model is built, the solution was to choose fermion representations in such a way that gauge anomalies are canceled [2]. However, when chiral gauge theories are considered as effective theories, their gauge anomalies are not always problematic. In several contexts [3–5], with the gauge field taken as external (not quantum), gauge anomalies have been used successfully to cancel unwanted boundary contributions.

The scene described above should be enough to raise a question mark on the alleged inconsistency of chiral gauge theories. There has been a lot of work [6–9] during the 80's that showed that anomalous gauge theories are not necessarily inconsistent. A more recent work [10] indicated that, in the full quantum context (i.e., integrating also over the gauge fields), the gauge anomaly had null vacuum expectation value, in an arbitrary number

of dimensions, for abelian and non-abelian theories. Its insertion in correlators of gauge invariant operators also gives a null result. These results point towards gauge symmetry restoration when the complete quantum theory is considered non-perturbatively. There are also modifications in the Ward-Takahashi identities [11], in the abelian case, which, however, do not prevent potential relations between renormalization constants, thus opening the path to prove renormalizability also in this extended context. Although this renormalizability has not yet been proved, it is clear that there is a lot of open questions concerning gauge anomalies.

This paper intends to discuss a point frequently overlooked in the literature, which concerns the gauge invariance of the *bosonic measure* in the generating functional of chiral gauge theories. In general, this is taken for granted, but no systematic analysis is found in the literature. We consider this question by using path integral arguments to show that it is indeed gauge invariant. We check our findings with known results from the renormalization of Yang-Mills theories. Given the central role played by this argument in several fundamental instances (e.g., Faddeev-Popov’s method), we believe that this study may fill an important gap in the area. We organize our discussion as follows: in Section 2, we briefly review the gauge anomaly vanishing mechanism, for the general (non-abelian) case. We also consider arguments against this vanishing, from an algebraic renormalization approach, and show that they do not apply to this context. The role of gauge invariance of the bosonic measure is stressed. In Section 3, we derive, by a path integral argument, the gauge invariance of the bosonic measure, when we restrict ourselves to the physical Hilbert space. We extend our argument to the whole Hilbert space by performing a calculation based on Fujikawa’s approach to Jacobians. In Section 4, we present our conclusions and some future perspectives.

2. Is There a Gauge Anomaly in Chiral Gauge Theories?

In order to fix our conventions and define precisely the problem under investigation, we briefly review some of the main results of reference [10], pointing the role played by the invariance of the bosonic measure when appropriate. What we call *chiral gauge theories* are described by an action $S[\psi, \bar{\psi}, A_\mu]$, given by

$$\begin{aligned} S[\psi, \bar{\psi}, A_\mu] &= S_G[A_\mu] + S_F[\psi, \bar{\psi}, A_\mu] \\ &= \int dx \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \int dx \bar{\psi} D \psi, \end{aligned} \tag{1}$$

where dx indicates integration over a d -dimensional Minkowski space. The operator D is called the Dirac operator of the theory and is given by

$$D = i\gamma^\mu (\partial_\mu \mathbf{1} - ieA_\mu) \equiv i\gamma^\mu D_\mu. \tag{2}$$

The fields ψ are *left handed Weyl fermions* ($\gamma_5 \psi = \psi$) carrying the fundamental representation of $SU(N)$. As usual, A_μ takes values in the Lie algebra of $SU(N)$ such that

$$\begin{aligned} A_\mu &= A_\mu^a T_a, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu], \end{aligned} \tag{3}$$

and the generators T_a satisfy

$$[T_a, T_b] = if_{abc} T_c, \quad \text{tr}(T_a T_b) = -\frac{1}{2} \delta_{ab}. \tag{4}$$

Considering

$$g = \exp(i\theta^a(x) T_a), \tag{5}$$

and simultaneous changes of the fields ψ and A_μ as

$$\begin{aligned} A_\mu^g &= gA_\mu g^{-1} + \frac{i}{e}(\partial_\mu g)g^{-1}, \\ \psi^g &= g\psi, \\ \bar{\psi}^g &= \bar{\psi}g^{-1}, \end{aligned} \tag{6}$$

the action S is classically gauge invariant

$$S[\psi^g, \bar{\psi}^g, A_\mu^g] = S[\psi, \bar{\psi}, A_\mu]. \tag{7}$$

2.1. The Gauge Anomaly and Its Vanishing

The quantum theory is defined by the generating functional, which is

$$Z[\eta, \bar{\eta}, j_a^\mu] = \int d\psi d\bar{\psi} dA_\mu \exp\left(iS[\psi, \bar{\psi}, A_\mu] + i \int dx [\bar{\eta}\psi + \bar{\psi}\eta + j_a^\mu A_\mu^a]\right). \tag{8}$$

We set the external sources to zero and perform a change of variables $A_\mu \rightarrow A_\mu^g$:

$$\begin{aligned} Z[0,0,0] &= \int d\psi d\bar{\psi} dA_\mu \exp(iS[\psi, \bar{\psi}, A_\mu]) \\ &= \int d\psi d\bar{\psi} dA_\mu^g \exp(iS[\psi, \bar{\psi}, A_\mu^g]). \end{aligned} \tag{9}$$

It is well known [12] that the fermion measure is not invariant under $\psi \rightarrow \psi^g, \bar{\psi} \rightarrow \bar{\psi}^g$. In what concerns the bosonic measure, its gauge invariance is universally accepted

$$dA_\mu^g = dA_\mu. \tag{10}$$

We show, below, that once Equation (10) is considered, we can promptly show that the Noether current associated to the gauge symmetry (6) is covariantly conserved. To see this, suppose $g = 1 + i\theta^a(x)T_a$, with θ^a infinitesimal, and remember that

$$A_\mu^g = A_\mu - \frac{1}{e}\mathcal{D}_\mu\theta, \tag{11}$$

with

$$\mathcal{D}_\mu\theta = T_a\left(\partial_\mu\delta_b^a + ef_{abc}A_\mu^c\right)\theta^b \equiv T_a(\mathcal{D}_\mu)^a{}_b\theta^b. \tag{12}$$

Using this expression for A_μ^g (10) we obtain:

$$\begin{aligned} Z[0,0,0] &= \int d\psi d\bar{\psi} dA_\mu^g \exp\left(iS[\psi, \bar{\psi}, A_\mu^g]\right) \\ &= \int d\psi d\bar{\psi} dA_\mu \exp\left(iS[\psi, \bar{\psi}, A_\mu] + \int dx \theta^a(x) (\mathcal{D}_\mu)^a{}_b (\bar{\psi}\gamma^\mu T_b\psi)\right) \\ &= Z[0,0,0] \\ &+ \int dx \theta^a(x) \int d\psi d\bar{\psi} dA_\mu \left[(\mathcal{D}_\mu)^a{}_b (\bar{\psi}\gamma^\mu T_b\psi)\right] \exp(iS[\psi, \bar{\psi}, A_\mu]). \end{aligned} \tag{13}$$

This means

$$\begin{aligned} &\int d\psi d\bar{\psi} dA_\mu \left[(\mathcal{D}_\mu)^a{}_b (\bar{\psi}\gamma^\mu T_b\psi)\right] \exp(iS[\psi, \bar{\psi}, A_\mu]) \\ &= \langle 0 | (\mathcal{D}_\mu)^a{}_b (\bar{\psi}\gamma^\mu T_b\psi) | 0 \rangle \\ &= 0. \end{aligned} \tag{14}$$

We could follow a different path when considering the original dependence of Z on A_μ^g , by absorbing the gauge dependence in the fermions and *using again the bosonic measure gauge invariance*:

$$\begin{aligned} Z[0, 0, 0] &= \int d\psi d\bar{\psi} dA_\mu^g \exp(iS[\psi, \bar{\psi}, A_\mu^g]) \\ &= \int d\psi d\bar{\psi} dA_\mu \exp(iS[\psi^{g^{-1}}, \bar{\psi}^{g^{-1}}, A_\mu]) \\ &= \int d\psi d\bar{\psi} dA_\mu \exp(iS[\psi, \bar{\psi}, A_\mu] - i\alpha_1 [A_\mu, g^{-1}]), \end{aligned} \tag{15}$$

where $\alpha_1 [A_\mu, g^{-1}]$ is related to the Jacobian for the gauge transformation of the fermionic measure as,

$$d\psi d\bar{\psi} = \exp(-i\alpha_1 [A_\mu, g^{-1}]) d\psi^{g^{-1}} d\bar{\psi}^{g^{-1}}. \tag{16}$$

Under an infinitesimal gauge transformation,

$$\alpha_1(A_\mu, -\theta) = i \int dx \theta^a \mathcal{A}^a(A_\mu) + \dots, \tag{17}$$

with $\mathcal{A}_a(A_\mu)$ being the gauge anomaly operator, as is well known [12]. Following this path we arrive at

$$\begin{aligned} Z[0, 0, 0] &= Z[0, 0, 0] \\ &- i \int dx \theta^a \int d\psi d\bar{\psi} dA_\mu \mathcal{A}^a(A_\mu) \exp(iS[\psi, \bar{\psi}, A_\mu]) \\ &\Rightarrow \int d\psi d\bar{\psi} dA_\mu \mathcal{A}_a(A_\mu) \exp(iS[\psi, \bar{\psi}, A_\mu]) \\ &= \langle 0 | \mathcal{A}^a(A_\mu) | 0 \rangle = 0. \end{aligned} \tag{18}$$

If the integration over the gauge fields were not performed, it is easy to see that the result would be

$$\langle 0 | (\mathcal{D}_\mu)^a_b (\bar{\psi} \gamma^\mu T_b \psi) | 0 \rangle_{A_\mu} = \mathcal{A}^a(A_\mu), \tag{19}$$

where

$$\langle 0 | (\mathcal{D}_\mu)^a_b (\bar{\psi} \gamma^\mu T_b \psi) | 0 \rangle_{A_\mu} = \int d\psi d\bar{\psi} [(\mathcal{D}_\mu)^a_b (\bar{\psi} \gamma^\mu T_b \psi)] \exp(iS[\psi, \bar{\psi}, A_\mu]), \tag{20}$$

corresponds to the situation of the chiral fermions being considered under the influence of a fixed external field A_μ .

This means that, *with all the fields being quantized*, there is no gauge anomaly preventing the conservation of the gauge current $J_a^\mu = \bar{\psi} \gamma^\mu T_b \psi$. We have to stress that this result is not in contradiction with the well-known existence and topological interpretation of the gauge anomaly (see, for example, [13,14]), since it is always present when the integration over the fields A_μ^a is not performed (i.e., they are taken as external fields). However, when quantum corrections are taken into account, the simple argument above shows that it must vanish.

If the bosonic measure dA_μ were not gauge invariant, we should have to add the contribution of the Jacobian of the gauge transformation of that measure, which would spoil the covariant conservation just obtained. We will further investigate this gauge invariance in the next section. Note also that the absence of gauge invariance of the fermionic measure plays no role in the covariant conservation of the fully quantized gauge current, as opposed to usual statements. The vanishing of the vacuum expectation value of the anomaly is to be seen as a consistency requirement, in the situation of a fully quantized gauge field.

2.2. An Algebraic Renormalization Objection

An argument, based on the technique of algebraic renormalization [15], could in principle question the findings reported above. One starts with a gauge fixed version of (1), namely

$$S_{\text{FP}}[\psi, \bar{\psi}, A_\mu, B, c, \bar{c}] = S[\psi, \bar{\psi}, A_\mu] + S_{\text{gf}}[A_\mu, B, c, \bar{c}], \tag{21}$$

where

$$S_{\text{gf}}[A_\mu, B, c, \bar{c}] = \int dx \text{tr} \left(B\partial^\mu A_\mu + \frac{1}{2}\alpha B^2 - \bar{c}\partial^\mu (i\partial_\mu + ie[c, A_\mu]) \right), \tag{22}$$

with B being an auxiliary field (the Lautrup-Nakanishi field) and c, \bar{c} being ghost fields, all of them taking values on the Lie algebra of $SU(N)$. The action (21) would be obtained from (1) by the use of the Faddeev-Popov procedure to fix the gauge. Then, one defines the linearized Slavnov-Taylor operator [16], which will test gauge invariance of the (quantum) effective action at each order of perturbation theory. When one solves the cohomology problem associated with this nilpotent operator, one finds a non-trivial solution. This means that the effective action is found to be not gauge invariant, as gauge symmetry violating terms dynamically generated at any given order can not be absorbed by suitably chosen gauge invariant counterterms. So, gauge invariance would be hopelessly lost and a gauge anomaly would be present, contradicting frontally what we found previously and putting renormalizability at risk.

There is nothing wrong with this line of reasoning, except its starting point. Gauge fixing is essential for the perturbative definition of a gauge theory, but one can not do it in a gauge anomalous theory in the same way as one does in the case of a non gauge anomalous one. If one starts with the non gauge fixed action (1) and insists in the insertion of the identity through the Faddeev-Popov's method, one ends up with a generating functional given by [8]

$$\begin{aligned} Z[0, 0, 0] &= \int d\theta d\psi d\bar{\psi} dA_\mu dBdc\bar{c} \\ &\times \exp(iS_{\text{FP}}[\psi, \bar{\psi}, A_\mu, B, c, \bar{c}] + i\alpha_1(A_\mu, \theta)) \\ &= \int d\theta d\psi d\bar{\psi} dA_\mu dBdc\bar{c} \exp(iS_{\text{full}}[\psi, \bar{\psi}, A_\mu, B, c, \bar{c}, \theta]), \end{aligned} \tag{23}$$

where α_1 was defined in (16) and is called *Wess-Zumino action*. The fields $\theta = \theta^a T_a$ are called Wess-Zumino fields and represent new quantum degrees of freedom. The action S_{full} is the true starting point from where one should restart the analysis of the cohomology of the linearized Slavnov-Taylor operator. It is gauge invariant and, thanks to Faddeev-Popov's technique, it has a well-defined gauge boson propagator. Unfortunately, as the fields θ^a have null mass dimension, it is not known how to perform this analysis up to now. In order to avoid the appearance of the Wess-Zumino fields, one is not allowed to fix the gauge. However, in doing this, there is no BRS symmetry to help one with the analysis of gauge invariance at an arbitrary perturbative order.

Thus, within the present knowledge, algebraic renormalization methods do not seem to be useful to decide if chiral gauge theories are truly gauge anomalous (and potentially inconsistent). As we pointed in the previous section, there are strong indications of the opposite.

3. On the Gauge Invariance of the Bosonic Measure

Let us now focus on the behavior of the bosonic measure under gauge transformations. To this end, let us first display a preparatory argument: consider the generating functional for correlators of gauge invariant operators in pure Yang-Mills theory (without chiral fermions). These gauge invariant operators satisfy

$$O_i(A_\mu^g) = O_i(A_\mu) \tag{24}$$

and the correlators are obtained as

$$\frac{\delta^n}{\delta\lambda^1(x_1)\dots\delta\lambda^n(x_n)}Z[\lambda^i]\Big|_{\lambda^i=0} = \langle 0|T(O_1(A_\mu)(x_1)\dots O_n(A_\mu)(x_n))|0\rangle, \tag{25}$$

with

$$Z[\lambda^i] = \int dA_\mu \exp i \int \text{tr} \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \lambda^i O_i[A_\mu] \right). \tag{26}$$

Considering the integration not over A_μ but over its gauge transformed version A_μ^g :

$$\begin{aligned} Z[\lambda^i] &= \int dA_\mu \exp i \int \text{tr} \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \lambda^i O_i(A_\mu) \right) \\ &= \int dA_\mu^g \exp i \int \text{tr} \left(\frac{1}{2}(F_{\mu\nu}F^{\mu\nu})^g + \lambda^i O_i(A_\mu^g) \right) \\ &= \int dA_\mu J[A_\mu, g] \exp \left(i \int \text{tr} \left(F_{\mu\nu}F^{\mu\nu} + \lambda^i O_i(A_\mu) \right) \right), \end{aligned} \tag{27}$$

where we allowed the potential presence of a Jacobian for the gauge transformation of the measure. What was obtained above corresponds to,

$$\begin{aligned} &\langle 0|T(J[A_\mu, g]O_1(A_\mu)(x_1)\dots O_n(A_\mu)(x_n))|0\rangle \\ &= \langle 0|T(O_1(A_\mu)(x_1)\dots O_n(A_\mu)(x_n))|0\rangle, \end{aligned} \tag{28}$$

which, translated into words, means that all correlators involving $J[A_\mu, g]$ with gauge invariant operators are the same as those involving the identity. Thus, in the physical Hilbert space of the theory, the two operators are the same.

This argument does not generalize to arbitrary, non-gauge invariant operators. However, an explicit calculation can resolve the problem. Let us use the usual prescription of defining the bosonic measure by means of a complete set of orthonormal eigenfunctions $\{\phi_n\}$ of an hermitian operator \bar{D} :

$$\begin{aligned} \bar{D}\phi_n &= \lambda_n\phi_n, \\ \int dx\phi_n^\dagger\phi_m &= \delta_{nm}, \quad \sum_n \phi_n(x)\phi_n^\dagger(y) = \delta(x-y), \\ A_\mu^a(x) &= \sum_n a_{\mu,n}^a\phi_n(x) \rightarrow dA_\mu = \prod_{a,\mu,n} da_{\mu,n}^a. \end{aligned} \tag{29}$$

Under a infinitesimal gauge transformation (11) we have

$$\begin{aligned} A_\mu^g &= \sum_n \bar{a}_{\mu,n}^a T_a\phi_n(x) = \sum_n a_{\mu,n}^a T_a\phi_n(x) - \frac{i}{e}D_\mu\theta \\ &= \left(\sum_n \left(a_{\mu,n}^a + ia_{\mu,n}^b f_{abc}\theta^c \right) \phi_n(x) - \frac{i}{e}\partial_\mu\theta^a \right) T_a. \end{aligned} \tag{30}$$

Decomposing θ^a in terms of the same eigenfunctions of \bar{D} ,

$$-\frac{i}{e}\partial_\mu\theta^a(x) = \sum_n \bar{a}_{\mu,n}^a\phi_n(x), \tag{31}$$

We obtain,

$$\bar{a}_{\mu,n}^a = \sum_m \left(\delta_{ab}\delta_{nm} + \int dx\phi_n^\dagger(x)if_{abc}\theta^c(x)\phi_m(x) \right) a_{\mu,m}^b + \bar{a}_{\mu,n}^a, \tag{32}$$

so that

$$\prod_{a,\mu,n} d\tilde{a}_{\mu,n}^a = \det \left[\delta_{ab}\delta_{nm} + \int dx \phi_n^\dagger(x) i f_{abc} \theta^c(x) \phi_m(x) \right] \prod_{a,\mu,n} da_{\mu,n}^a. \tag{33}$$

The term $\tilde{a}_{\mu,n}^a$ does not contribute because of translational invariance of each measure $da_{\mu,n}^a$. Following the steps of Fujikawa [12] we get the expression for the Jacobian:

$$J[A_\mu, \theta] = \exp \left(\sum_n \left(\text{tr} \int dx \phi_n^\dagger(x) i f_{abc} \theta^c(x) \phi_n(x) \right) \right) \tag{34}$$

where “tr” is to be computed over Lie algebra indices. It is easy to see that the expression for $J[A_\mu, \theta]$ is indefinite:

$$\begin{aligned} \sum_n \left(\text{tr} \int dx \phi_n^\dagger(x) i f_{abc} \theta^c(x) \phi_n(x) \right) &= \text{tr} \int dx i f_{abc} \theta^c(x) \sum_n \phi_n(x) \phi_n^\dagger(x) \\ &= \int dx i f_{aac} \theta^c(x) \delta(0) = 0 \times \infty. \end{aligned} \tag{35}$$

So, it must be regularized in order to make sense. It is natural to choose the eigenvalues of the operator \bar{D} to regularize the Jacobian as

$$\begin{aligned} J[A_\mu, \theta] &\equiv \exp \left(\lim_{M^2 \rightarrow \infty} \sum_n \left(\text{tr} \int dx \phi_n^\dagger(x) i f_{abc} \theta^c(x) \exp \left(-\frac{\lambda_n^2}{M^\alpha} \right) \phi_n(x) \right) \right) \\ &= \exp \left(\lim_{M^2 \rightarrow \infty} \sum_n \left(\text{tr} \int dx \phi_n^\dagger(x) i f_{abc} \theta^c(x) \exp \left(-\frac{\bar{D}^2}{M^\alpha} \right) \phi_n(x) \right) \right), \end{aligned} \tag{36}$$

where α is chosen so that the argument of the exponential is dimensionless. The choice of the operator \bar{D} is usually guided by the requisites that (a) it naturally appears in the theory; (b) it is gauge invariant; and (c) its eigenvalues are real. Besides, our choice of the coefficients $a_{\mu,n}^a$ carrying all the dependence on μ and a implies that the ϕ_n must be eigenfunctions of an scalar colorless operator. A good choice is

$$\bar{D} = \text{tr} (\mathcal{D}_\mu \mathcal{D}^\mu), \tag{37}$$

where the trace is taken only over the color indices. Under these conditions, we see that the sum is regularized and, as there is no additional dependence on color indices coming from $\exp(-\bar{D}^2/M^4)$, the trace can be immediately taken and the result is

$$\begin{aligned} J[A_\mu, \theta] &= \exp \left(\lim_{M^2 \rightarrow \infty} \sum_n \left(i f_{aac} \int dx \phi_n^\dagger(x) \theta^c(x) \exp \left(-\frac{\bar{D}^2}{M^4} \right) \phi_n(x) \right) \right) \\ &= \exp(0) = 1. \end{aligned} \tag{38}$$

Of course, one could choose other strategies and a result different from 1 could arise. However, the “gauge anomaly” coming from this “non-trivial” Jacobian could be removed by a adequate choice of counterterms. To say this more precisely, we can use what we know from the fact that Yang-Mills theories are renormalizable. In fact, ‘t Hooft’s proof [17,18] shows that it is possible to preserve gauge invariance at every order in perturbation theory and this is crucial for the demonstration that the theory is renormalizable. Algebraic renormalization results confirm this, by noticing that the cohomology of the Slavnov-Taylor operator is trivial for a Yang-Mills theory [19]. Then, even if we would regularize the theory with non-gauge invariant regulators (then obtaining a non-trivial Jacobian), a change in renormalization scheme could restore gauge invariance and set the Jacobian as 1.

4. Conclusions

Gauge invariance of the bosonic measure is essential to cancel the gauge anomaly at full quantum level in a chiral gauge theory (abelian or non-abelian). Besides this, it is crucial for the implementation of Fadeev-Popov’s technique. Although it is a commonly used feature, this gauge invariance has not deserved a careful consideration in the literature (up to our knowledge). This paper intends to furnish a detailed analysis of this useful property and, in so doing, to fill an important omission. It inserts itself into a program of investigation of the renormalization of chiral gauge theories, to see if it can really be implemented at perturbative level. Although the vacuum expectation value of the gauge anomaly vanishes, there are indications that it has non-null insertions in correlators of non-gauge invariant operators. The dynamical picture remains unclear.

An important point would be to ask what could happen to the axial anomaly, defined as

$$\langle 0 | \partial_\mu J_5^\mu | 0 \rangle_{A_\mu} = \langle 0 | \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) | 0 \rangle_{A_\mu} = \mathcal{A}_5(A_\mu).$$

It is related to the quantum violation of axial symmetry

$$\psi' = e^{\alpha \gamma_5} \psi \equiv g_5 \psi,$$

which is classically exact in the massless case. The gauge anomaly appears as the non-covariant conservation of the gauge current

$$\langle 0 | (D_\mu)^a_b J_b^\mu | 0 \rangle_{A_\mu} = \langle 0 | (D_\mu)^a_b (\bar{\psi} \gamma^\mu T_b \psi) | 0 \rangle_{A_\mu} = \mathcal{A}^a(A_\mu),$$

Whose classical conservation would be a consequence of gauge symmetry

$$\psi' = e^{i\theta^a T_a} \psi \equiv g \psi. \tag{39}$$

While the gauge transformation (39) can be transferred from the fermions to the gauge fields (with all of them being considered as quantum)

$$S[\psi^g, \bar{\psi}^g, A_\mu] = S\left[\psi, \bar{\psi}, A_\mu^{g^{-1}}\right],$$

the same can not be done with the axial transformation under the same circumstances:

$$S[\psi^{g_5}, \bar{\psi}^{g_5}, A_\mu] = S[\psi, \bar{\psi}, A_\mu],$$

which means that the gauge fields do not play any role in axial symmetry. As a consequence, we are not allowed to make the same manipulations in the functional integral and so, we can not say that the axial anomaly is canceled as well. The axial anomaly is a welcome one, as it is responsible for the explanation of the observed π^0 decay into two photons. On the other hand, the gauge anomaly is a generally undesired feature and has to be canceled (in the construction of the standard model) in order not to ruin explicit renormalizability of the theory. So, we could answer that the dynamical role of the axial anomaly is unaltered in our approach. It keeps appearing and influencing low energy effective actions of QCD, for example. However, we offer some hope that the gauge anomaly is not such a catastrophic issue.

We must also remind the reader that there must be several ways to prove gauge invariance of the bosonic measure. One of them is surely to consider the theory on the lattice, where one can define the bosonic measure as a Haar measure, which is naturally gauge invariant (see, for example, Formula (7.12) of [20]). Then, we could consider the limit of zero lattice spacing with due caution to establish this invariance in the continuum theory. However, this discussion (and other ones, using different arguments) is not usually seen in the literature and we found it could be useful to present it explicitly, by means of a Fujikawa approach.

Some big issues have to be considered in this path. Perhaps one of the most urgent would be how to define the tree level of chiral gauge theories. Since one is not allowed to fix the gauge (due to the appearance of a Jacobian for the fermion measure), we have no definition of a free gauge boson propagator, which invalidates a perturbative analysis of the theory. We also have no natural choice to choose a method of regularization to deal with loops, as gauge invariance can not be used as a guiding principle. We will continue to investigate these and other issues and our findings will be reported as they appear.

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