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Translations and reflections on the torus: identities for discrete Wigner functions and transforms

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Abstract

A finite Hilbert space can be associated to a periodic phase space, that is, a torus. Then a subgroup of operators corresponding to reflections and translations on the torus form respectively the basis for the discrete Weyl representation, including the Wigner function, and for its Fourier conjugate, the chord representation. They are invariant under Clifford transformations and obey analogous product rules to the continuous representations, so allowing for the calculation of expectations and correlations for observables. We here import new identities from the continuum for products of pure state Wigner and chord functions, involving, for instance the inverse phase space participation ratio and correlations of a state with its translate. New identities are derived involving *transition* Wigner or chord functions of transition operators $|\psi_1\rangle\langle\psi_2|$. Extension of the reflection and translation operators to a doubled torus phase space leads to the representation of superoperators and so to the construction of the propagator of Wigner functions from the Weyl representation of the evolution operator.

Keywords: phase space methods, Wigner–Weyl transform, superoperators

(Some figures may appear in colour only in the online journal)

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1. Introduction

It is well-known that any two-state quantum system can be represented on the Bloch sphere, whatever the physical context, be it a spin, the electromagnetic polarization, or a simplification of an atom down to two levels. The generalization of the Bloch sphere to a d -state system is well known and leads to the complex projective space $CP(d-1)$ for a Hilbert space \mathcal{H}_d with d dimensions (see e.g. [1, 2]). As $CP(d-1)$ carries a symplectic structure of complex dimension $d-1$ it seems natural to assign it as a phase space for such a quantum system. However a different association based on the Weyl–Heisenberg group (or on any other semi-simple group) assigns a phase space structure to a complete set of operators, similar to the classical phase space associated to the semiclassical limit of ordinary quantum mechanics. In this way \mathcal{H}_d can be represented on a torus. Indeed, identifying any orthogonal basis for \mathcal{H}_d with position states $|q_n\rangle$ on a line, we can periodically extend them, such that $|q_{n+d}\rangle = |q_n\rangle$. Then, a finite Fourier transform defines the conjugate periodic momentum basis $|p_n\rangle$. This pair of representations taken together subsume a (discretized) periodic phase space torus in analogy to the (continuous) plane phase space with periodic boundary conditions that combines the position and momentum bases for an infinite Hilbert space.

Several successful schemes have been introduced to lift quantum mechanics into a full phase space theory. Particularly, in quantum optics, Cahill and Glauber [3] constructed a continuous parametrization that includes the so called P-function, for $s = 1$, the Wigner function, for $s = 0$, and the Husimi function, also known as the Q-function, for $s = -1$. The scheme was adapted to the discrete case in [4]. Here, we are concerned exclusively with the Weyl representation of quantum operators [5], including the Wigner function [6] in the case of the density operator⁵. Position and the momentum representations of quantum states subsume a phase space only in as far as they are combined as Fourier conjugates. In contrast, the Weyl [5] representation of quantum operators, including the Wigner function [6] in the case of the density operator, are specified in the full phase space. Their Fourier conjugates, respectively, the chord representation and the chord function, are again full phase space functions. These are continuous in the case of infinite Hilbert space and are fundamentally based on the Heisenberg–Weyl group of phase space translation operators, or on their symplectic Fourier transforms, the reflection operators, also called phase point or Fano operators [7, 8]. Having chosen the torus as the phase space that corresponds to a finite Hilbert space, its portrayal as a periodic plane allows us to import periodically equivalent translation and reflection operators, now forming a finite affine subgroup. The special properties of these fundamental operators on the plane and on the torus then percolate down to the discrete representations for any finite quantum system [9].

Apparently the first construction of a discrete Wigner function was presented by Hannay and Berry [10] followed by many others [9, 12–18], based on various properties of the continuum that one wishes to emulate. However, the Weyl–Wigner representation should not be reduced to a phase space display of quantum states. It is required to provide a complete framework for their evolution and for evaluation of mean values and correlations among (Weyl represented) observables. This program is achieved by a construction based on the discrete translation and reflection operators defined on the torus that serve as complete operator bases. For instance, the product rules for calculating mean values and correlations are replaced by analogous sums [9]. Again, the simple classical propagation of arbitrary Wigner functions

⁵ Association of a torus phase space to \mathcal{H}_d is not in itself a necessity. Finite Hilbert spaces arise naturally as finite dimensional representations of compact semisimple groups, such as $SU(2)$. They are naturally associated to phase spaces on the coadjoint orbits [19] of the group, with widely different topologies [20, 21].

for a quadratic Hamiltonian also has its analogue for Clifford transformations on the torus [9, 18, 22].

In a recent publication [23], we studied the Wigner–Weyl representation of *superoperators* in the continuum case. In close analogy to the ordinary reflections and translations in ‘single’ phase space it was possible to define reflection and translation superoperators in a phase space with *doubled* dimensions, maintaining in that space all the structure of the usual Heisenberg–Weyl group. Thus center and chord functions of a superoperator could be defined as projections on these superoperator bases, retaining all the usual properties of center and chord functions of operators. The construction of these phase space distributions on the doubled phase space led to a wide range of surprising identities connecting products of pure state Wigner functions to their Fourier transforms. It is the main purpose of the present paper to extend these results to the discrete case and incorporate them to the tool box of discrete phase space methods. Indeed, we here derive new identities that relate products of different Wigner functions to *transition* Wigner functions or chord functions that represent transitions between pairs of states.

The structure of the paper is as follows. In section 2 we recall the classical affine group of translations and reflections and its quantization in terms of Heisenberg–Weyl operators in continuum quantum mechanics. The main difference with the previous [23] treatment is the labelling of translations by half-chords [24]. This has the advantage of providing very compact and symmetrical group properties, simplifying many formulae and leading in a natural way to the transition to the discrete case, which we treat in detail in section 3. The group properties are preserved, but positions and momenta on the periodic torus are now on an integer $d \times d$ grid. However centers and half chords can assume integer and half integer values and therefore translation and reflection operators take values on a larger $2d \times 2d$ grid, leading to Wigner and chord functions as discrete arrays on this grid. As operator bases, the $(2d)^2$ translations or reflections are overcomplete, as only d^2 can be linearly independent. This overcompleteness of the operator basis can only be streamlined in the case where d is odd, as explained in [9, 13, 22]. Here we will keep to the general formulae that are valid whatever the parity of d . It should be further noticed that, although in the continuum case the quantum representation of translations and reflections is essentially unique, in the discrete case there is latitude for inequivalent representations leading to a *quasi periodic* structure characterized by Floquet phases [9, 14]. These have been extensively used in the theory of the quantum Hall effect and in quantum chaos, but we do not make use of this latitude here. Section 4 is devoted to projections of centers and chords and the peculiarities arising from the redundant operator bases. In section 5 we rewrite the pure state identities of [23] identifying the differences that arise because of the discreteness. They involve quadratic and quartic relationships between Wigner and chord distributions. In section 6 we show, following [23], that reflection and translation *superoperators* can be defined and used as lagrangian coordinates in double phase space, thus allowing the representation of general superoperators either as double center or double chord discrete arrays, in exact correspondence to the representation of operators in ‘single’ phase space. A pair of appendices discuss the Choi conjugacy of superoperator bases and provide a simple application to qubits and qutrits.

2. Translation and reflection operators in continuum quantum mechanics

In this introductory section we review briefly the well known geometry of affine transformations in phase space, constituted by translations and reflections, and their well known unitary representations in continuum quantum mechanics [7, 8, 24, 26]. For simplicity of notation

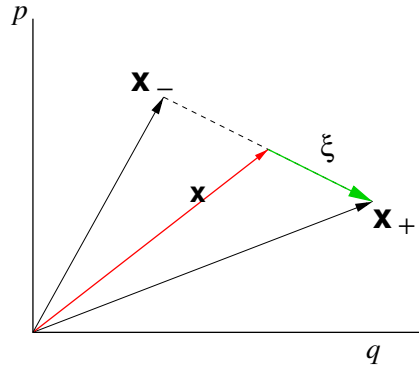


Figure 1. Definition of centers \mathbf{x} and half-chords $\boldsymbol{\xi}$.

we restrict the discussion to one degree of freedom, noting that the generalization to more degrees is straightforward. The presentation is standard, except for the fact that throughout we label translations with the half-chords [24], leading to a simplified and completely symmetric notation for reflections and translations, that stresses the complementarity of the two sets of operators.

We consider two classical phase space canonical transformations

$$T_{\boldsymbol{\xi}} : \mathbf{x}_- \rightarrow \mathbf{x}_+ = 2\boldsymbol{\xi} + \mathbf{x}_- \quad R_{\mathbf{x}} : \mathbf{x}_- \rightarrow \mathbf{x}_+ = 2\mathbf{x} - \mathbf{x}_- \quad (1)$$

where $\mathbf{x}_{\pm} = (q_{\pm}, p_{\pm})$ are two phase space points and $\mathbf{x} = \frac{1}{2}(\mathbf{x}_+ + \mathbf{x}_-)$ labels its *center* and $\boldsymbol{\xi} = \frac{1}{2}(\mathbf{x}_+ - \mathbf{x}_-)$ labels the *half-chord*. In figure 1 we show the relationship between these four vectors whose relationships will be used repeatedly in what follows. The transformations satisfy the composition rules

$$\begin{aligned} T_{\boldsymbol{\xi}} T_{\mathbf{x}} &= T_{\boldsymbol{\xi} + \mathbf{x}} & R_{\mathbf{x}} R_{\mathbf{y}} &= T_{\mathbf{x} - \mathbf{y}} \\ T_{\boldsymbol{\xi}} R_{\mathbf{x}} &= R_{\mathbf{x} + \boldsymbol{\xi}} & R_{\mathbf{x}} T_{\boldsymbol{\xi}} &= R_{\mathbf{x} - \boldsymbol{\xi}}. \end{aligned} \quad (2)$$

These operations are represented in quantum mechanics by the unitary operators

$$\hat{T}_{\boldsymbol{\xi}} = \int_{\mathbb{R}} dq |q + \xi_q\rangle \langle q - \xi_q| e^{2i/\hbar q \xi_p} \equiv \int_{\mathbb{R}} dq |q + 2\xi_q\rangle \langle q| e^{2i/\hbar (q + \xi_q) \xi_p} \quad (3)$$

$$\hat{R}_{\mathbf{x}} = \int_{\mathbb{R}} dq |q + x_q\rangle \langle x_q - q| e^{2i/\hbar q x_p} \equiv \int_{\mathbb{R}} dq |q + 2x_q\rangle \langle -q| e^{2i/\hbar (q + x_q) x_p} \quad (4)$$

where the Dirac bras and kets span the usual position basis for the Hilbert space of square integrable wave functions on the real line. We have also introduced the vector notation for centers and half-chords $\mathbf{x} = (x_q, x_p)$, $\boldsymbol{\xi} = (\xi_q, \xi_p)$. The rightmost identities result from a shift within the integral and will be useful in the transition to the discrete case. The two sets of operators are related by a symplectic Fourier transform

$$\hat{R}_{\mathbf{x}} = \frac{1}{\pi\hbar} \int_{\mathbb{R}^2} d^2\boldsymbol{\xi} e^{2i/\hbar \langle \mathbf{x}, \boldsymbol{\xi} \rangle} \hat{T}_{\boldsymbol{\xi}} \quad (5)$$

where we have introduced the symplectic form

$$\langle \boldsymbol{\xi}, \mathbf{x} \rangle = (\xi_q \quad \xi_p) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_q \\ x_p \end{pmatrix} = x_q \xi_p - x_p \xi_q. \quad (6)$$

Additional characteristic properties (besides unitarity) are

$$\hat{T}_\xi^\dagger = \hat{T}_{-\xi} \quad \hat{R}_x^\dagger = \hat{R}_x. \quad (7)$$

These operators provide a unitary (projective) representation in quantum mechanics of the respective classical compositions (2):

$$\begin{aligned} \hat{T}_\xi \hat{T}_\chi &= \hat{T}_{\xi+\chi} e^{2i/\hbar \langle \xi, \chi \rangle} & \hat{R}_x \hat{R}_y &= \hat{T}_{x-y} e^{-2i/\hbar \langle x, y \rangle} \\ \hat{T}_\xi \hat{R}_x &= \hat{R}_{\xi+x} e^{2i/\hbar \langle \xi, x \rangle} & \hat{R}_x \hat{T}_\xi &= \hat{R}_{x-\xi} e^{-2i/\hbar \langle x, \xi \rangle}. \end{aligned} \quad (8)$$

On account of the Stone–von Neumann theorem, the representation is essentially unique, up to unitary equivalence, which amounts to the choice of an arbitrary origin in phase space. The labelling of translations by half-chords leads to these very symmetric group properties, which symmetry will translate later to the discrete case. Notice the special cases

$$\hat{T}_x = \hat{R}_x \hat{R}_{0,0} \quad \hat{R}_x = \hat{T}_x \hat{R}_{0,0} \quad (9)$$

which give a special prominence to the reflection through the origin, which in quantum optics is called the parity operator

$$\hat{R}_{0,0} = \frac{1}{\pi \hbar} \int_{\mathbb{R}^2} d^2 \xi \hat{T}_\xi = \int_{\mathbb{R}} dq |q\rangle \langle -q|. \quad (10)$$

Inverting the symplectic Fourier transform in (5) an expansion of the identity is obtained

$$\hat{T}_{0,0} = \frac{1}{\pi \hbar} \int_{\mathbb{R}^2} d^2 x \hat{R}_x = \int_{\mathbb{R}} dq |q\rangle \langle q|. \quad (11)$$

Taking the trace of these relations we obtain:

$$\begin{aligned} \text{tr } \hat{T}_\xi \hat{T}_\chi^\dagger &= \pi \hbar \delta(\xi - \chi) & \text{tr } \hat{R}_x \hat{R}_y &= \pi \hbar \delta(x - y) \\ \text{tr } \hat{T}_\xi \hat{R}_x &= e^{-2i/\hbar \langle \xi, x \rangle} & \text{tr } \hat{R}_x \hat{T}_\xi &= e^{-2i/\hbar \langle x, \xi \rangle} \end{aligned} \quad (12)$$

where we have used the fact that $\text{tr } \hat{R}_x = 1$ and $\text{tr } \hat{T}_\xi = \pi \hbar \delta(\xi)$. Thus reflections and translations provide orthogonal—in the Hilbert–Schmidt sense—unitary operator bases, which are related to each other by the symplectic Fourier transform (5).

They provide two complementary bases in which any operator \hat{A} has projections $A(\mathbf{x}) = \text{tr } \hat{R}_x \hat{A}$ and $\tilde{A}(\xi) = \text{tr } \hat{T}_\xi^\dagger \hat{A}$. These constitute the well known Wigner (or center) and Weyl (or chord) [5] representations of quantum mechanical operators. They are complementary displays in phase space of the properties of the operator, in exactly the same way as the ordinary position and momentum representations are complementary. When applied to quantum states, given by positive density matrices, they yield respectively, the well known Wigner and characteristic functions. The former is a real quasi-probability distribution which displays both the classical features of the state, through its smooth positive parts and the nonclassical interferences by means of rapid ‘sub-Planck’ oscillations [25]. While possibly assuming negative values, the Wigner function has positive marginal distributions [6]. On the other hand the characteristic function is complex and does not generate marginal distributions so nicely, and is therefore less used. Here we want to stress the complete phase space symmetry and complementarity between these two representations. Just as the q and p Lagrangian coordinates provide the complementary position and momentum representations of a quantum state, reflections and translations provide Lagrangian planes in *double phase space* in which to represent operators [23, 24, 36] with the added feature that the planes are themselves *single*

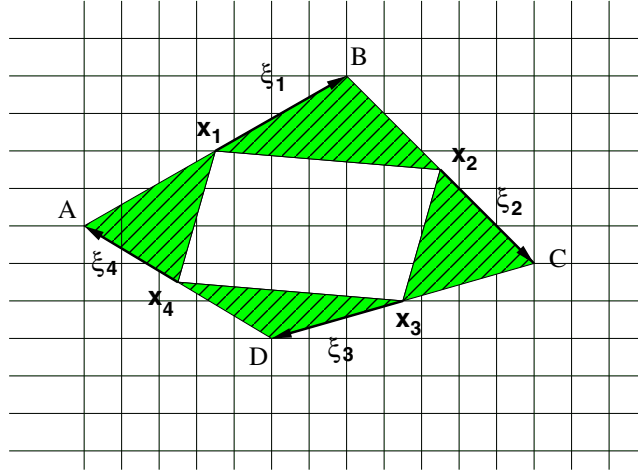


Figure 2. Geometry of centers and half-chords. The symplectic area of the inner parallelogram with corners at the centers \mathbf{x}_i is the same as the hatched areas subtended by half-chords ξ_i (bold-face arrows). In the discrete case the grid is the integer lattice described in section 3.

phase spaces. When combined to span a double phase space they provide a framework for the representation of superoperators [23], that we review in the discrete case in section 6.

We also remark that on account of the group properties (8) the trace of the product of any number of reflections and translations can be easily computed. When the product includes an odd number of reflections its trace is just a phase that can be interpreted as the area of a polygon in phase space, while a product with an even number of reflections will give a delta function times an area as a phase. As an example we compute two typical examples

$$\text{tr } \hat{T}_{\xi_1} \hat{T}_{\xi_2} \hat{T}_{\xi_3} \hat{T}_{\xi_4} = \pi \hbar \delta(\xi_1 + \xi_2 + \xi_3 + \xi_4) e^{2i/\hbar(\langle \xi_1, \xi_2 \rangle + \langle \xi_3, \xi_4 \rangle)} \quad (13)$$

$$\text{tr } \hat{R}_{\mathbf{x}_1} \hat{R}_{\mathbf{x}_2} \hat{R}_{\mathbf{x}_3} \hat{R}_{\mathbf{x}_4} = \pi \hbar \delta(\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4) e^{-2i/\hbar(\langle \mathbf{x}_1, \mathbf{x}_2 \rangle - \langle \mathbf{x}_3, \mathbf{x}_4 \rangle)}. \quad (14)$$

In figure 2 we illustrate the areas involved. The large quadrilateral ABCD has an inscribed parallelogram with corners at the centers \mathbf{x}_i whose area is the same as that of the hatched regions subtended by the half chords ξ_i (bold face arrows). A detailed description of polygons in phase space subtended by centers and chords have been used to derive center path integrals [26].

3. Translation and reflection operators for finite Hilbert space

There have been many proposals to adapt the previous well known scheme to a finite Hilbert space \mathcal{H}_d [4, 9, 11–13, 16, 17]. If periodic boundary conditions on positions and momenta are imposed, the classical phase plane \mathbb{R}^2 turns into a torus \mathbb{T}^2 , which can be assumed to be of unit area, with no loss of generality. We adopt adimensional units and unit periodicity for both position and momentum. In quantum mechanics the immediate consequence of this double periodicity is that both positions and momenta become quantized $|q\rangle \rightarrow |q_j\rangle$ and $|p\rangle \rightarrow |p_k\rangle$ with $q_j = j/d, p_k = k/d, j, k = 0, \dots, d - 1$ with periodicity $|q_{j+d}\rangle = |q_j\rangle$ and $|p_{j+d}\rangle = |p_j\rangle$. Moreover the fact that the symplectic area of the torus is finite means that $\text{Area}/(2\pi\hbar)$ is an integer d thus leading to the quantization of \hbar

$$\hbar = (2\pi d)^{-1}. \quad (15)$$

This reflects the semiclassical rule that assigns an area $2\pi\hbar$ to each orthogonal state in \mathcal{H}_d . The limit $d \rightarrow \infty$ yields the classical continuous torus. In this limit, states that are localized, such that they avoid aliasing phenomena, can be assimilated to continuous distributions in the flat phase space \mathbb{R}^2 analogous to the usual Wigner and chord functions [4]. A different approach to the transition from the plane phase space to the discrete torus can be found in [9].

The two bases are related by the finite Fourier transform

$$\langle p_k | q_j \rangle = \exp -\frac{i}{\hbar} p_k q_j = \exp -\frac{2i\pi}{d} k j. \quad (16)$$

The allowed phase space points are labeled by a periodic $d \times d$ lattice \mathbb{Z}_d^2 with integer coordinates (j, k) . On this lattice the phase space points $\mathbf{x}_{\pm} = (q_j, p_k)_{\pm}$ are integer vectors scaled by d and therefore centers and half-chords defined as

$$\mathbf{x} = \frac{\mathbf{x}_+ + \mathbf{x}_-}{2} \quad \boldsymbol{\xi} = \frac{\mathbf{x}_+ - \mathbf{x}_-}{2} \quad (17)$$

become also discretized, but on a lattice with halved spacing. This fact is clearly illustrated in figure 2. The lattice drawn has spacing $1/d$. The phase space points A, B, C, D belong to this lattice, while the centers \mathbf{x}_i and half-chords $\boldsymbol{\xi}_i$ between pairs of points assume values on a refined lattice (not drawn) with spacing $1/2d$. It is now remarkable—and this is the main advantage of the labelling by half-chords—that the transition from the continuum to the discrete case is simply done by the replacements

$$(\xi_q, \xi_p) \rightarrow (\xi_q, \xi_p)/2d \text{ with } (\xi_q, \xi_p) \in \mathbb{Z}^2$$

$$(x_q, x_p) \rightarrow (x_q, x_p)/2d \text{ with } (x_q, x_p) \in \mathbb{Z}^2.$$

Reflections can then be labeled by $\mathbf{x} = (x_q, x_p)$ and translations by $\boldsymbol{\xi} = (\xi_q, \xi_p)$ just as in the continuum but now x_q, x_p, ξ_q, ξ_p are integers. The scaling by $2d$ is taken care of in the phases as

$$\exp \left(\frac{2i}{\hbar} \langle \mathbf{x}/2d, \boldsymbol{\xi}/2d \rangle \right) = \exp \left(\frac{4i\pi d}{4d^2} \langle \mathbf{x}, \boldsymbol{\xi} \rangle \right) = \tau^{\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \quad (18)$$

where we have used (15) and the new phase $\tau = e^{i\pi/d^6}$ has been defined.

Thus labelled, the subset of discretized translations and reflections inherit the group properties from (8)

$$\begin{aligned} \hat{T}_{\boldsymbol{\xi}} \hat{T}_{\boldsymbol{\chi}} &= \hat{T}_{\boldsymbol{\xi}+\boldsymbol{\chi}} \tau^{\langle \boldsymbol{\xi}, \boldsymbol{\chi} \rangle} \\ \hat{R}_{\mathbf{x}} \hat{R}_{\mathbf{y}} &= \hat{T}_{\mathbf{x}-\mathbf{y}} \tau^{-\langle \mathbf{x}, \mathbf{y} \rangle} \\ \hat{T}_{\boldsymbol{\xi}} \hat{R}_{\mathbf{x}} &= \hat{R}_{\boldsymbol{\xi}+\mathbf{x}} \tau^{\langle \boldsymbol{\xi}, \mathbf{x} \rangle} \\ \hat{R}_{\mathbf{x}} \hat{T}_{\boldsymbol{\xi}} &= \hat{R}_{\mathbf{x}-\boldsymbol{\xi}} \tau^{-\langle \mathbf{x}, \boldsymbol{\xi} \rangle}. \end{aligned} \quad (19)$$

As expected the periodicity is now

$$\hat{T}_{\boldsymbol{\xi}+2d\boldsymbol{\chi}} = \hat{T}_{\boldsymbol{\xi}} \quad \hat{R}_{\mathbf{x}+2d\mathbf{y}} = \hat{R}_{\mathbf{x}} \quad (20)$$

⁶We should mention here that in some recent applications [22] involving the Clifford group this phase is defined with a minus sign, leading to different periodicity properties and making a distinction between the even and odd d cases.

so that the unit torus is mapped to the double periodic lattice \mathbb{Z}_{2d}^2 . On this lattice the even-even sites correspond to integer centers and half-cords, while the other sub-lattices correspond to some coordinate being half-integer. Both operator sets are still related by the symplectic discrete Fourier transform—on the double lattice—

$$\hat{R}_{\mathbf{x}} = \frac{1}{2d} \sum_{\boldsymbol{\xi} \in \mathbb{Z}_{2d}^2} \hat{T}_{\boldsymbol{\xi}} \tau^{\langle \mathbf{x}, \boldsymbol{\xi} \rangle}. \quad (21)$$

The expansions of the operators in the discretized position basis are also obtained from (3) and (4)

$$\begin{aligned} \hat{T}_{\boldsymbol{\xi}} &= \sum_{j \in \mathbb{Z}_d} |q_{j+\xi_q}\rangle \langle q_j| \tau^{(2j+\xi_q)\xi_p} \\ \hat{R}_{\mathbf{x}} &= \sum_{j \in \mathbb{Z}_d} |q_{j+x_q}\rangle \langle -q_j| \tau^{(2j+x_q)x_p}. \end{aligned} \quad (22)$$

We also remark the two relations analogous to (9)

$$\hat{T}_{\mathbf{x}} = \hat{R}_{\mathbf{x}} \hat{R}_{0,0} \quad \hat{R}_{\mathbf{x}} = \hat{T}_{\mathbf{x}} \hat{R}_{0,0} \quad (23)$$

and the normalization relationships

$$\hat{R}_{0,0} = \frac{1}{2d} \sum_{\boldsymbol{\xi} \in \mathbb{Z}_{2d}^2} \hat{T}_{\boldsymbol{\xi}} = \sum_{j \in \mathbb{Z}_d} |q_j\rangle \langle -q_j| \quad (24)$$

$$\hat{T}_{0,0} = \frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} \hat{R}_{\mathbf{x}} = \sum_{j \in \mathbb{Z}_d} |q_j\rangle \langle q_j|. \quad (25)$$

For easy comparison with other approaches we recast the definitions in terms of the Schwinger operators [29]. Notice that

$$\hat{T}_{1,0} = \sum_{j \in \mathbb{Z}_d} |q_{j+1}\rangle \langle q_j| \stackrel{\text{def}}{=} \hat{V} \quad \hat{T}_{0,1} = \sum_{j \in \mathbb{Z}_d} |q_j\rangle \tau^{2j} \langle q_j| \stackrel{\text{def}}{=} \hat{U}. \quad (26)$$

In terms of these operators we rewrite (22) as (see [16])

$$\hat{T}_{\boldsymbol{\xi}} = \hat{V}^{\xi_q} \hat{U}^{\xi_p} \tau^{\xi_q \xi_p} \quad \hat{R}_{\mathbf{x}} = \hat{V}^{x_q} \hat{U}^{x_p} \tau^{x_q x_p} \hat{R}_{0,0} \quad (27)$$

\hat{V}, \hat{U} are d -periodic, while the phase τ is what leads to the double periodicity.

In terms of the representation of the affine group, we then conclude that the transition from the continuum to the discrete case is quite straightforward and only involves the appropriate discretization on \mathbb{Z}_{2d}^2 and the scaling of the phases (18). However, when we attempt to proceed to the use of reflections and translations as operator bases we encounter the obvious fact that they are now overcomplete, as only d^2 operators can be linearly independent in \mathcal{H}_d . The over-completeness is, however, rather trivial, because of the property

$$\hat{R}_{\mathbf{x}+d\mathbf{y}} = \hat{R}_{\mathbf{x}} (-1)^{\langle \mathbf{x}, \mathbf{y} \rangle} (-1)^{d y_q y_p} \quad \hat{T}_{\boldsymbol{\xi}+d\boldsymbol{\chi}} = \hat{T}_{\boldsymbol{\xi}} (-1)^{\langle \boldsymbol{\xi}, \boldsymbol{\chi} \rangle} (-1)^{d \chi_q \chi_p} \quad (28)$$

signifying that the operators, while strictly $2d$ -periodic on the double lattice $\mathbb{Z}_{2d} \times \mathbb{Z}_{2d}$, are also periodic up to a sign in the $\mathbb{Z}_d \times \mathbb{Z}_d$ sub-lattices. The completeness of the two bases is reflected in the property

$$\text{tr } \hat{A}\hat{B} = \frac{1}{4d} \sum_{\xi \in \mathbb{Z}_{2d}^2} \text{tr } A\hat{T}_\xi \text{tr } \hat{T}_\xi^\dagger \hat{B} = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} \text{tr } A\hat{R}_\mathbf{x} \text{tr } \hat{R}_\mathbf{x} \hat{B}. \quad (29)$$

The extra factor of 4 in the normalization compensates for the overcompleteness. In fact, taking into account (28), the double lattice sums in (29) consist of four equal contributions from the four $d \times d$ quadrants. We prefer to keep this redundant normalization as it provides uniform formulae throughout, particularly in section 5.

The center and chord representations of an operator in \mathcal{H}_d are still defined as

$$A(\mathbf{x}) = \text{tr } \hat{A}\hat{R}_\mathbf{x} \quad \tilde{A}(\xi) = \text{tr } \hat{A}\hat{T}_\xi^\dagger. \quad (30)$$

They are $2d \times 2d$ periodic arrays, but, because of (28) only one fourth of the array has independent entries. The reconstruction of a general operator in \mathcal{H}_d in terms of these representations is

$$\hat{A} = \frac{1}{4d} \sum_{\xi \in \mathbb{Z}_{2d}^2} \tilde{A}(\xi) \hat{T}_\xi = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} A(\mathbf{x}) \hat{R}_\mathbf{x}. \quad (31)$$

Just as in the continuum case, some properties of the operators can be obtained as averages over their representations. From (25) and (24) we obtain

$$\text{tr } \hat{A} = \frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} A(\mathbf{x}) \quad \text{tr } \hat{A}\hat{R}_{0,0} = \frac{1}{2d} \sum_{\xi \in \mathbb{Z}_{2d}^2} \tilde{A}(\xi) \quad (32)$$

while the Hilbert–Schmidt scalar product of two operators is given by

$$\text{tr } \hat{A}\hat{B}^\dagger = \frac{1}{4d} \sum_{\xi \in \mathbb{Z}_{2d}^2} \tilde{A}(\xi) \tilde{B}(\xi)^* = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} A(\mathbf{x}) B(\mathbf{x})^*. \quad (33)$$

The important case when the operator is a quantum state $\hat{\rho}$ deserves a specific notation

$$W(\mathbf{x}) = \text{tr } R_\mathbf{x} \hat{\rho} \quad \chi(\xi) = \text{tr } T_\xi^\dagger \hat{\rho} \quad (34)$$

respectively for the centre-Wigner and chord functions. The normalization of the state then yields the properties

$$\frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W(\mathbf{x}) = 1 \quad \chi(0,0) = 1. \quad (35)$$

Moreover, from (24) we also obtain

$$\frac{1}{2d} \sum_{\xi \in \mathbb{Z}_{2d}^2} \chi(\xi) = \text{tr } \hat{\rho}\hat{R}_{0,0}. \quad (36)$$

Quadratic averages are given by

$$\frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W^2(\mathbf{x}) = \frac{1}{4d} \sum_{\xi \in \mathbb{Z}_{2d}^2} |\chi(\xi)|^2 = \text{tr } \hat{\rho}^2. \quad (37)$$

Notice that in the literature it is customary to change the normalization so that the Wigner function normalizes to unity. In view of the developments in section 5 we prefer to keep the present normalization, as it provides uniform factors in all formulae. In this context, it is important to remark that the present normalization for pure states yields for the two distributions the

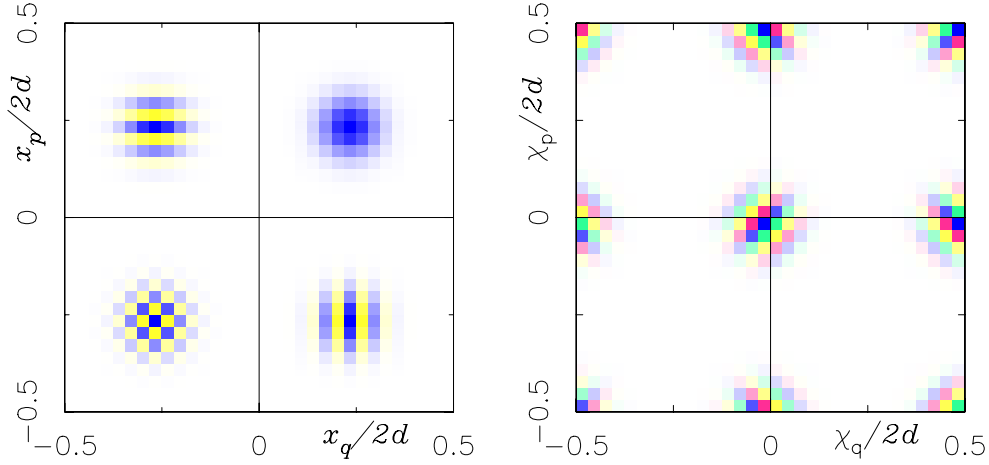


Figure 3. Center (left) and chord (right) representation of a gaussian coherent state centered at $(x_q, x_p) = (.25, .25)$ with $d = 16$. Complex amplitudes are displayed using the H.L.S. system with Hue giving the phase, Lightness giving the modulus in the interval $(0, 1)$ and Saturation = 1. In this system real positive values are colored blue (dark gray), and real negatives are colored yellow (light gray).

upper bounds $|W(\mathbf{x})| \leq 1$ and $|\chi(\xi)| \leq 1$. To exemplify the way that these distributions give complementary phase space representations of a state we plot in figure 3 the center and chord distributions for a coherent state (a gaussian wave-packet) centred at $(q, p) = (.25, .25)$ on the unit torus placed symmetrically with respect to the origin. The center distribution yields smooth positive values (blue or dark gray) peaked at $x_q/2d, x_p/2d \approx (.25, .25)$ and also oscillating positive and negative ‘images’ (blue and yellow) at a half-way distance around the torus. These images can be understood as the well known interference patterns arising from the ‘cat’ state formed by the original wave packet and its repetitions on the periodic torus. The chord representation displays the features of the state in a different way. The position of the wave-packet is reflected in the complex phases near the center (displayed with an HLS colour scheme). The modulus of the distribution is centred near zero, reflecting the fact that only small chords are relevant for a gaussian. Many more graphical examples can be found in [13, 16].

Framed in this way, the discrete theory provides a scheme valid for all d and follows the presentations in [9, 16]. Other schemes in the literature [13, 17, 22] distinguish even and odd values of d , while [12, 15] restricts d to be a prime power.

3.1. The odd- d case

In the odd- d case it is possible to eliminate completely the redundancy of the operator bases and deal only with $d \times d$ chord and center distributions. This is done simply by considering only the even-even sublattices of integer centers and half-chords $R_{2\mathbf{x}}$ and $T_{2\xi}$. The results are identical to the approach in [13]. Writing out the group properties thus restricted we obtain

$$\begin{aligned} \hat{T}_{2\xi} \hat{T}_{2\chi} &= \hat{T}_{2(\xi+\chi)} \tau^{4\langle \xi, \chi \rangle} & \hat{R}_{2\mathbf{x}} \hat{R}_{2\mathbf{y}} &= \hat{T}_{2(\mathbf{x}-\mathbf{y})} \tau^{-4\langle \mathbf{x}, \mathbf{y} \rangle} \\ \hat{T}_{2\xi} \hat{R}_{2\mathbf{x}} &= \hat{R}_{2(\xi+\mathbf{x})} \tau^{4\langle \xi, \mathbf{x} \rangle} & \hat{R}_{2\mathbf{x}} \hat{T}_{2\xi} &= \hat{R}_{2(\mathbf{x}-\xi)} \tau^{-4\langle \mathbf{x}, \xi \rangle}. \end{aligned} \quad (38)$$

All labels are integers in \mathbb{Z}_d^2 where the operators are d -periodic

$$\hat{T}_{2\xi+2d\chi} = \hat{T}_{2\xi} \quad \hat{R}_{2\mathbf{x}+2d\mathbf{y}} = \hat{R}_{2\mathbf{x}}. \quad (39)$$

Written in terms of the Schwinger operators we have

$$\hat{T}_{2\xi} = \hat{V}^{2\xi_1} \hat{U}^{2\xi_2} \tau^{4\xi_1\xi_2} \quad \hat{R}_{2\mathbf{x}} = \hat{V}^{2x_1} \hat{U}^{2x_2} \tau^{4x_1x_2} \hat{R}_{0,0}. \quad (40)$$

We remark that, for odd d , $\tau^4 = e^{4i\pi/d}$ is a primitive d -root of unity, which is not the case when d is even. For the same reason \hat{U}^2 and \hat{V}^2 are also primitive, in the sense that their powers cycle once through all powers of \hat{U} and \hat{V} —although in a different order—, thus providing a complete basis of operators. In the even- d case this scheme does not work, because the powers of \hat{U}^2 and \hat{V}^2 cycle only among half of the independent operators, and the basis then is not complete. It is important to observe that, thus restricted, the two sets of operators are still related by a symplectic Fourier transform

$$\hat{R}_{2\mathbf{x}} = \frac{1}{d} \sum_{\xi \in \mathbb{Z}_d^2} \tau^{4\langle \mathbf{x}, \xi \rangle} T_{2\xi}. \quad (41)$$

4. Projections of centers and chords

In the continuum case it is well known that, while the Wigner function of a state can assume negative values, its projections on the coordinate axes—and in fact on any pair of rotated axes—are always positive. In this section we explore how this result translates to the discrete case when the center representation is used and how a closely related result applies also to the projection of chords.

Consider first the definition of a line L in the discrete periodic lattice \mathbb{Z}_{2d}^2 . Given the integer vector $\xi = (\xi_q, \xi_p)$, a line through the origin is given by the set of points $\mathbf{x} \in \mathbb{Z}_{2d}^2$ that satisfy $\langle \xi, \mathbf{x} \rangle = 0 \pmod{2d}$. Likewise lines parallel to it are given by $\langle \xi, \mathbf{x} \rangle = a \pmod{2d}$ with $a \in \mathbb{Z}_{2d}$. When the lines are wrapped on the periodic lattice \mathbb{Z}_{2d}^2 it is important to know how many different points belong to it. This depends on the relative divisibility of ξ_q, ξ_p and $2d$: (a) when ξ_q, ξ_p are relatively prime then there are exactly $2d$ parallel lines with $2d$ points each; (b) when ξ_q, ξ_p have a common factor k , if this factor is prime to $2d$, then the same situation arises. However if k is a factor of $2d$ then the line through the origin consists of $2d \times k$ points and the parallel lines where k is relatively prime to a are empty. With these considerations the sum of reflection operators on a line can be written as

$$\hat{L}_\xi^a = \frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} \hat{R}_\mathbf{x} \delta_{2d}(\langle \mathbf{x}, \xi \rangle - a) \quad (42)$$

where the integer a labels the family of parallel lines oriented by ξ . Then, taking the Fourier transform of the δ -function

$$\hat{L}_\xi^a = \frac{1}{(2d)^2} \sum_{s=0}^{2d-1} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} \hat{R}_\mathbf{x} \tau^{s(\langle \mathbf{x}, \xi \rangle - a)} = \frac{1}{2d} \sum_{s=0}^{2d-1} \hat{T}_\xi^s \tau^{-sa} \quad (43)$$

where we have used the inverse symplectic transform in (21) to relate reflections to translations. One should note that the group properties of T_ξ imply that $T_\xi^s = T_{s\xi}$ if $s\xi$ belongs to the fundamental cell. Otherwise these vectors are pulled back periodically using (20). To

determine the nature of the \hat{L}_ξ^a operators we first notice that they are hermitian. Furthermore we compute

$$\hat{L}_\xi^a \hat{L}_\xi^b = \frac{1}{4d^2} \sum_{s,t=0}^{2d-1} \hat{T}_\xi^{(s+t)} \tau^{-(sa+tb)} = \delta_{2d}(a-b) \frac{1}{2d} \sum_{s=0}^{2d-1} \hat{T}_\xi^s \tau^{-sa} = \delta_{2d}(a-b) \hat{L}_\xi^a, \quad (44)$$

showing that they are orthogonal positive projection operators. Moreover, they also satisfy

$$\sum_{a=0}^{2d-1} \hat{L}_\xi^a = 1. \quad (45)$$

Thus each set of parallel lines in a given direction ξ gives rise in the general case to a partition of unity in terms of positive projection operators. Clearly at least half of the $2d$ operators must be zero as only d can be linearly independent. To show how this comes about we use the half-periodicity property (28) to split the sum in (43)

$$\hat{L}_\xi^a = \frac{1}{2d} \left(\sum_{s=0}^{d-1} + \sum_{s=d}^{2d-1} \right) \hat{T}_\xi^s \tau^{-sa} = \frac{1 + (-1)^a (-1)^{d\xi_q \xi_p}}{2d} \sum_{s=0}^{d-1} \hat{T}_\xi^s \tau^{-sa}. \quad (46)$$

Thus for even d the odd parallel lines project to zero, while for odd d the same thing happens except when ξ is odd-odd, in which case this happens for the even lines. The rank of the projectors is given by

$$\text{tr } \hat{L}_\xi^a = \frac{1 + (-1)^a (-1)^{d\xi_q \xi_p}}{2d} \sum_{s=0}^{d-1} \text{tr } \hat{T}_\xi^s \tau^{-sa}. \quad (47)$$

This rank is determined by the *order* of T_ξ , i.e. the minimum power r such that $\hat{T}_\xi^r = \pm \hat{1}$. When $r = d$ only one trace survives for each a and then one obtains d one-dimensional projectors, i.e. an orthonormal basis. When the order is less than d , in which case d/r is an integer k then the projectors are k dimensional when $a = 0 \pmod d$ and null otherwise.

An equivalent look at the projection properties of reflections is provided by the spectral decomposition of \hat{T}_ξ . Taking for simplicity the d even case when the order is d , the spectrum is given by the d -roots of unity

$$\hat{T}_\xi = \sum_{b=0}^{d-1} |\phi_\xi^{(b)}\rangle e^{2\pi i b/d} \langle \phi_\xi^{(b)}|. \quad (48)$$

The Fourier transform in (47) then yields $\hat{L}_\xi^{2a} = |\phi_\xi^{(a)}\rangle \langle \phi_\xi^{(a)}|$, with a similar calculation in the other cases, which needs taking care of degeneracies in the spectrum when the order is less than d . In the simplest case of vertical projections with $\xi = (0, 1)$ the eigenbasis $|\phi_{(0,1)}^a\rangle$ of $\hat{T}_{(0,1)} \equiv \hat{U}$ is simply the original discretized position basis $|q_a\rangle$. In this case we can rewrite the general result as

$$\frac{1}{2d} \sum_{x_p=0}^{2d-1} \hat{R}_{x_q, x_p} = \frac{1 + (-1)^{x_q}}{2} |x_q\rangle \langle x_q|. \quad (49)$$

We can use these considerations also to compute the projections in the chord representation. Using $\hat{T}_x = \hat{R}_x \hat{R}_{0,0}$ we rewrite (42) as

$$\hat{L}_\xi^a \hat{R}_{0,0} = \frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} \hat{T}_x \delta_{2d}(\langle \mathbf{x}, \xi \rangle - a). \quad (50)$$

Thus the projection of chords along a direction ξ , in the simplest case, yields the same projectors as before times a reflection through the origin.

When these results are applied to the distributions of a positive density matrix $\hat{\rho} = W(\mathbf{x})$ and $\chi(\xi)$ —we then obtain, for d even and for ξ_q, ξ_p relatively prime

$$\frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W(\mathbf{x}) \delta_{2d}(\langle \mathbf{x}, \xi \rangle - 2a) = \langle \phi_\xi^{(a)} | \hat{\rho} | \phi_\xi^{(a)} \rangle \quad (51)$$

$$\frac{1}{2d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} \chi(\mathbf{x}) \delta_{2d}(\langle \mathbf{x}, \xi \rangle - 2a) = \langle \phi_\xi^{(a)} | \hat{\rho} | \phi_\xi^{(-a)} \rangle. \quad (52)$$

Thus the center and chord projections yield respectively the diagonal and the skew-diagonal elements of the density matrix in the eigenbasis of \hat{T}_ξ . Considering again the vertical projections we see that the center projections provide the populations in the position basis, while the chord projections yields some information on the coherences.

We should note at this point that whenever d is a prime number (or a power of a prime) *all* operators T_ξ have order d and therefore every direction ξ defines an orthonormal basis just as in the continuum case. If this property is taken as a requirement for a proper definition of a Wigner function, then one is led to the constructions only valid in those cases, which then use in an essential way the theory of Galois fields [12, 15].

5. Center and chord identities for pure states

In a recent publication [23] we obtained in the continuum a collection of new relationships involving products and correlations between center and chord functions associated to pure states that, besides their intrinsic interest, resulted in previously unknown relationships among special functions, and in generalized pure state conditions. In this section we provide a new way to derive these relationships, more adapted to the discrete case. The fact that the group properties are identical to the continuum case result in formulae which are an almost direct transcription of the previous results in [23].

Consider the computation of the quantity $\text{tr} \hat{A} \hat{R}_x \hat{B}^\dagger \hat{R}_y$. One way to compute it [23] is to expand \hat{A}, \hat{B}^\dagger as in (31) and use the trace of the resulting quadruple product to perform the sums. An alternative is to use (29) to expand the product in the translation basis as

$$\text{tr} \hat{A} \hat{R}_x \hat{B}^\dagger \hat{R}_y = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \text{tr} \hat{A} \hat{R}_x \hat{T}_z^\dagger \text{tr} \hat{B}^\dagger \hat{R}_y \hat{T}_z = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \text{tr} \hat{A} \hat{R}_{x+\mathbf{z}} \tau^{\langle \mathbf{x}, \mathbf{z} \rangle} \text{tr} \hat{B}^\dagger \hat{R}_{y-\mathbf{z}} \tau^{-\langle \mathbf{y}, \mathbf{z} \rangle}. \quad (53)$$

The rightmost equality follows from the group rules (19). A similar expansion in the reflection basis yields

$$\text{tr} \hat{A} \hat{R}_x \hat{B}^\dagger \hat{R}_y = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \text{tr} \hat{A} \hat{R}_x \hat{R}_z \text{tr} \hat{B}^\dagger \hat{R}_y \hat{R}_z = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \text{tr} \hat{A} \hat{T}_{x-\mathbf{z}} \tau^{-\langle \mathbf{x}, \mathbf{z} \rangle} \text{tr} \hat{B}^\dagger \hat{T}_{y-\mathbf{z}} \tau^{-\langle \mathbf{y}, \mathbf{z} \rangle} \quad (54)$$

where again the group properties (19) have been used. A similar derivation for the quantity $\text{tr} \hat{A} \hat{T}_x \hat{B}^\dagger \hat{T}_y^\dagger$ and the definition (30) of the center and chord representations of \hat{A}, \hat{B} yield our two main formulae

$$\text{tr } \hat{A}R_{\mathbf{x}}\hat{B}^\dagger R_{\mathbf{y}} = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} A(\mathbf{x} + \mathbf{z})B^*(\mathbf{y} - \mathbf{z})\tau^{(\mathbf{x}-\mathbf{y}, \mathbf{z})} = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \tilde{A}(\mathbf{z} - \mathbf{x})\tilde{B}^*(\mathbf{z} - \mathbf{y})\tau^{-(\mathbf{x}+\mathbf{y}, \mathbf{z})} \quad (55)$$

$$\text{tr } \hat{A}T_{\xi}\hat{B}^\dagger T_{\omega}^\dagger = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} A(\mathbf{z} + \xi)B^*(\mathbf{z} - \omega)\tau^{(\xi-\omega, \mathbf{z})} = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \tilde{A}(\mathbf{z} - \xi)\tilde{B}^*(\mathbf{z} - \omega)\tau^{-(\xi+\omega, \mathbf{z})}. \quad (56)$$

Notice that if we define [23] the *superoperator* $\hat{A} \bullet \hat{B}^\dagger$ which acts on an operator \hat{C} as $\hat{A}\hat{C}\hat{B}^\dagger$, then the above results amount to the computation of the matrix elements of this superoperator in the reflection or translation operator bases. In this generality, formulae of this kind provide a way to represent superoperators in *double* phase space [23] in which lagrangian planes corresponding to reflections and translations are the coordinates, as will be discussed in section 6.

The adaptation of the identities derived in [23] to the discrete case now follows from these formulae in the special case in which \hat{A}, \hat{B} are both equal to the pure state projector $\hat{\rho} = |\psi\rangle\langle\psi|$. In that case the computation reduces to

$$W(\mathbf{x})W(\mathbf{y}) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W(\mathbf{x} + \mathbf{z})W(\mathbf{y} - \mathbf{z})\tau^{(\mathbf{x}-\mathbf{y}, \mathbf{z})} \quad (57)$$

$$= \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \chi(\mathbf{x} + \mathbf{z})\chi(\mathbf{y} - \mathbf{z})\tau^{(\mathbf{x}+\mathbf{y}, \mathbf{z})} \quad (58)$$

$$\chi(\xi)\chi(\omega) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \chi(\xi + \mathbf{z})\chi(\omega - \mathbf{z})\tau^{(\xi-\omega, \mathbf{z})} \quad (59)$$

$$= \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W(\xi + \mathbf{z})W(\omega - \mathbf{z})\tau^{(\xi+\omega, \mathbf{z})} \quad (60)$$

where we have used the specific notation (34) $\chi(\xi) = \langle\psi|\hat{T}_\xi^\dagger|\psi\rangle$ and $W(\mathbf{x}) = \langle\psi|\hat{R}_\mathbf{x}|\psi\rangle$ for the chord and center functions of $|\psi\rangle$, and used the fact that $W(\mathbf{x})$ is real and $\chi^*(\xi) = \chi(-\xi)$. These general formulae relate products of pure state centers and chords to their convolutions, and in some respect they generalize the pure state condition $\hat{\rho} = \hat{\rho}^2$. Restricted to special values, they lead to several identities relating quadratic and quartic products of centers and chords. For example setting $\mathbf{x} = \mathbf{y}$ in (58) and relabeling the sum we obtain

$$W^2(\mathbf{x}) = \frac{1}{4d} \sum_{\xi \in \mathbb{Z}_{2d}^2} \chi^2(\xi)\tau^{2(\mathbf{x}, \xi)}. \quad (61)$$

Thus the two distributions $W^2(\mathbf{x})$ and $\chi^2(\xi)$ constitute a symplectic Fourier transformed pair. The Parseval identity then leads to

$$M = \frac{1}{4d} \sum_{\xi \in \mathbb{Z}_{2d}^2} |\chi(\xi)|^4 = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W^4(\mathbf{x}). \quad (62)$$

Another identity of this type is obtained using (61) to show that

$$L = \frac{1}{4d} \sum_{\boldsymbol{\xi} \in \mathbb{Z}_{2d}^2} \chi(\boldsymbol{\xi})^4 = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W^2(\mathbf{x})W^2(-\mathbf{x}). \quad (63)$$

Another set of identities is obtained setting $\mathbf{x} = -\mathbf{y}$ in (57) and $\boldsymbol{\xi} = -\boldsymbol{\omega}$ in (59). We obtain

$$W(\mathbf{x})W(-\mathbf{x}) = \frac{1}{4d} \sum_{\boldsymbol{\xi} \in \mathbb{Z}_{2d}^2} \chi(\mathbf{x} + \boldsymbol{\xi})\chi^*(\mathbf{x} - \boldsymbol{\xi}) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W(\mathbf{z})W(-\mathbf{z}) \tau^{2\langle \mathbf{x}, \mathbf{z} \rangle} \quad (64)$$

$$|\chi(\boldsymbol{\xi})|^2 = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W(\mathbf{z} + \boldsymbol{\xi})W(\mathbf{z} - \boldsymbol{\xi}) = \frac{1}{4d} \sum_{\boldsymbol{\gamma} \in \mathbb{Z}_{2d}^2} |\chi(\boldsymbol{\gamma})|^2 \tau^{2\langle \boldsymbol{\xi}, \boldsymbol{\gamma} \rangle}. \quad (65)$$

The first part of these identities show that $W(\mathbf{x})W(-\mathbf{x})$ is the chord autocorrelation while $|\chi(\boldsymbol{\xi})|^2$ is the center autocorrelation. The second part shows that they are invariant under Fourier transform and center-symmetric [27, 28]. Setting $\mathbf{x} = 0$ in (64) we obtain

$$W(0)^2 = \frac{1}{4d} \sum_{\boldsymbol{\xi} \in \mathbb{Z}_{2d}^2} \chi(\boldsymbol{\xi})^2 = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W(\mathbf{x})W(-\mathbf{x}). \quad (66)$$

So far, we have specialized (55) and (56) to the case $\hat{A} = \hat{B} = |\psi\rangle\langle\psi|$, but other interesting formulae can be obtained inserting $\hat{A} = \hat{\rho}_1 = |\psi_1\rangle\langle\psi_1|$, $\hat{B} = \hat{\rho}_2 = |\psi_2\rangle\langle\psi_2|$. One obtains Fourier or convolution relationships between the center or chord functions for the two states and the *transition* center or chord functions corresponding to the transition operator $|\psi_1\rangle\langle\psi_2|$. Here we just exhibit some special cases:

$$|W_{12}(\mathbf{x})|^2 = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W_1(\mathbf{x} + \mathbf{z})W_2(\mathbf{x} - \mathbf{z}) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \chi_1(\mathbf{z})\chi_2(\mathbf{z})\tau^{-2\langle \mathbf{x}, \mathbf{z} \rangle} \quad (67a)$$

$$|\chi_{12}(\boldsymbol{\xi})|^2 = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W_1(\mathbf{z} + \boldsymbol{\xi})W_2(\mathbf{z} - \boldsymbol{\xi}) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \chi_1(\mathbf{z})^* \chi_2(\mathbf{z})\tau^{-2\langle \boldsymbol{\xi}, \mathbf{z} \rangle} \quad (67b)$$

where, with obvious notation, we have defined $W_{12}(\mathbf{x}) = \langle\psi_2|R_{\mathbf{x}}|\psi_1\rangle$ and $\chi_{12}(\boldsymbol{\xi}) = \langle\psi_2|\hat{T}_{\boldsymbol{\xi}}^\dagger|\psi_1\rangle$ as the center and chord functions of the transition operator $|\psi_1\rangle\langle\psi_2|$. The symplectic Fourier transforms in (67a) then imply the equality of the quartic sums

$$K = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} |W_{12}(\mathbf{x})|^4 = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} |\chi_{12}(\mathbf{x})|^4 \quad (68)$$

wich generalize (62).

Some of these formulae can be reinterpreted as special cases of parametrized phase space functions [4]. For example, if we set $|\psi_2\rangle$ as a gaussian reference state centered at the origin, i.e. the ground state of the Harper Hamiltonian, then the lhs of (67b) is the discrete Husimi function $Q_1(\boldsymbol{\xi})$ of $\hat{\rho}_1$, which is then computed as

$$Q_1(\boldsymbol{\xi}) = \text{tr}(\hat{\rho}_1 \hat{T}_{\boldsymbol{\xi}} \hat{\rho}_2 \hat{T}_{\boldsymbol{\xi}}^\dagger) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} \chi_1(\mathbf{z})^* \chi_2(\mathbf{z})\tau^{-2\langle \boldsymbol{\xi}, \mathbf{z} \rangle} \quad (69)$$

in terms of the symplectic transform of the product of the two characteristic functions. In the notation of [4], this is the $s = -1, x = -1$ parametrized phase space Husimi function.

Another interesting identity yields

$$W_{12}^*(\mathbf{x})W_{12}(-\mathbf{x}) = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} W_1(\mathbf{z})W_2(-\mathbf{z})\tau^{-2\langle \mathbf{x}, \mathbf{z} \rangle} \quad (70)$$

that again implies the quartic identity

$$\frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} |W_{12}(\mathbf{x})|^2 |W_{12}(-\mathbf{x})|^2 = \frac{1}{4d} \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} |W_1(\mathbf{x})|^2 |W_2(-\mathbf{x})|^2. \quad (71)$$

Evidently, these new identities that involve transition chord and center functions, and which were not included in [23], generalize immediately to the continuum case, as they are merely special cases of the general formulae (55) and (56).

All the formulae in this section are valid for all values of d , both even and odd, and they involve sums over the periodic cell of the $2d \times 2d$ periodic lattice. However the sums are redundant, and, as we saw in (29) the four $d \times d$ quadrants contribute equally, and therefore the sums can be restricted to one of the $d \times d$ quadrants, simply removing the factor of 4 in the denominator.

5.1. Localization measures in phase space

As an application of the previous identities we first consider the quantity M in (62). First of all we notice that, on account of the symplectic invariance of the Wigner function [34], M is constant not only for all translations of the state $|\psi\rangle$, but also when $|\psi\rangle$ is acted upon by the representation of linear canonical transformations. Thus M is invariant under all Clifford [22] operations on the state. Second, the fact that for a normalized pure state we have $1/(4d) \sum_{\mathbf{x} \in \mathbb{Z}_{2d}^2} W^2(\mathbf{x}) = 1$ means that $W(\mathbf{x})^2$ is a probability distribution and therefore the quartic quantity M can be considered as a kind of inverse participation ratio measuring the phase space localization of pure states [30]. Moreover M is bounded as

$$\frac{2}{d+1} \leq M \leq 1. \quad (72)$$

The upper bound is a simple consequence of the fact that $1 \geq W(\mathbf{x})^2 \geq W(\mathbf{x})^4$. The lower bound was derived by Welch [32] when considering bounds on the cross correlation of signals. Both bounds can be achieved by special classes of states of interest to the quantum information community. The upper bound obtains for pure position states $|q_j\rangle$ and for all its symplectic transformations. These constitute the set of stabilizer states, which were also called line states [13]. At the opposite lower bound, M characterizes *symmetric informationally complete (SIC) fiducial states*. In fact M is used as a cost function whose minimization leads to the numerical search for such states [22]. We remark that it is not proven that the bound can be reached for all values of d , but many numerical and some analytical results seem to show that indeed it is so. Notice also that, while the identity between chord and Wigner expressions for this quantity persists in the continuous case, we are not aware of an analogous lower bound in that case. In figure 4 we show the Wigner representation of a gaussian state and of a numerically computed [31] SIC state for $d = 10$.

The quantity M also provides a bound for L in (63). A simple application of Schwartz inequality to the definition of L shows that $L \leq M$ with equality implying that $W(\mathbf{x}) = W(-\mathbf{x})$. L

is obviously positive, and it would be interesting to know if it also has a positive lower bound for pure states.

Another interesting localization property is provided by K in (68). Consider a pure ‘cat’ state $|\psi\rangle = 1/\sqrt{2}(|\psi_1\rangle + |\psi_2\rangle)$ with normalized $|\psi_1\rangle, |\psi_2\rangle$. Its center function, for example, is given by

$$W_\psi(\mathbf{x}) = \frac{1}{2}(W_1(\mathbf{x}) + W_2(\mathbf{x}) + W_{12}(\mathbf{x}) + W_{12}^*(\mathbf{x})). \quad (73)$$

Thus the transition center function characterizes the phase space extent of the coherences between the two states. Therefore the amplitude and the phase space distribution of these coherences can be computed from (67a) in terms of W_1 and W_2 . Moreover, as

$$\frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} |W_{12}(\mathbf{z})|^2 = \frac{1}{4d} \sum_{\mathbf{z} \in \mathbb{Z}_{2d}^2} |\chi_{12}(\mathbf{z})|^2 = 1 \quad (74)$$

K can be interpreted as an inverse participation ratio for the phase space localization of the amplitude of coherences. Just as M, L it is a Clifford invariant and can be computed indifferently from center or chord transition functions.

6. Chords and centers in double phase space: superoperator representations

Superoperators determine the linear dynamics of operators, whether observables or density operators. When they are represented in the phase space operator bases \hat{R}_x, \hat{T}_ξ , the matrix elements of superoperators require a doubled phase space consisting of two centers (or two chords). The general treatment of superoperators in double phase space developed in [23] for continuous Hilbert spaces is readily adapted to doubled tori in the discrete case. We provide here the essential features with the purpose of making it more readily available to the quantum information community⁷.

We adopt a double Dirac notation $\langle\langle A||B\rangle\rangle = \text{tr } \hat{A}^\dagger \hat{B}$ for the Hilbert–Schmidt scalar product in operator space. Thus the expansion of a general superoperator \mathbf{S} in, i.e. the center–center basis, will be

$$\mathbf{S} = \frac{1}{d^2} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{2d}^2} \langle\langle R_x || \mathbf{S} || R_y \rangle\rangle |R_x\rangle \langle R_y| \quad (75)$$

acting on operators in the usual Dirac way. It will be convenient to think of $|R_x\rangle$ as a double *position* ket, while of $|T_\xi\rangle$ as a double *momentum* ket. Thus the chord–chord double matrix element would be the momentum representation of the superoperator. Associated to the superoperator basis $|R_x\rangle \langle R_y|$ we have defined in [23] the *Choi-conjugate basis* $\hat{R}_x \bullet \hat{R}_y$ whose action on operators is defined in a different way as $(\hat{A} \bullet \hat{B}) \hat{C} \stackrel{\text{def}}{=} \hat{A} \hat{C} \hat{B}$. In the appendix we reproduce from [23] how the two bases are related and how their unitary relationship generalizes the simple partial transposition of indices that underlies the Choi–Jamiołkowski homomorphism [1, 33].

With this notation we defined in [23] reflection and translation *superoperators* as Choi monomials

⁷ When comparing formulae in [23], besides the obvious discretization, one should keep in mind the differences arising from the use of half-chords, leading to simplified and more symmetrical formulae.

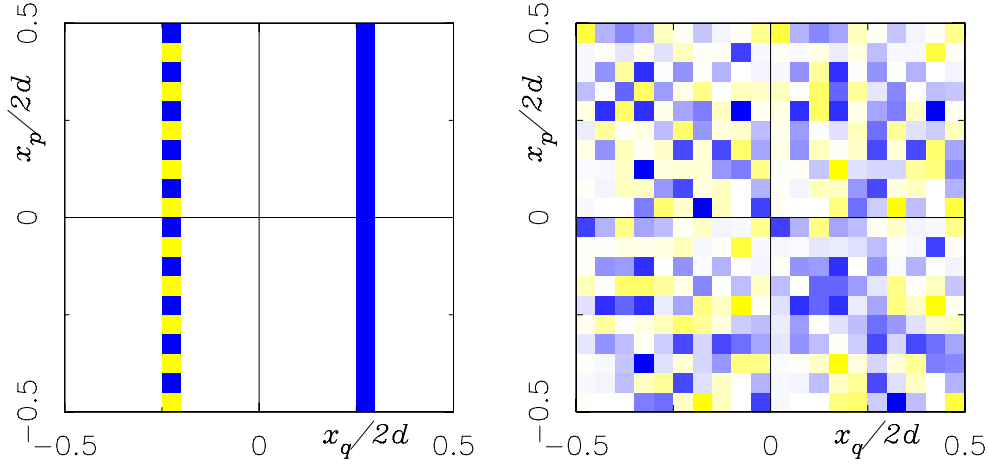


Figure 4. Center representation of position eigenstate $|q_3\rangle\langle q_3|$ with $d = 10$ (left) and a numerically computed SIC fiducial state labelled '10a' in [31] (right). The states exemplify the upper and lower bounds on the localization measure M in (72).

$$\mathbf{T}_{\mathbf{x},\xi} = \hat{T}_{\mathbf{x}+\xi} \bullet \hat{T}_{\mathbf{x}-\xi}^\dagger = \hat{T}_{\mathbf{x}+} \bullet \hat{T}_{\mathbf{x}-}^\dagger. \quad (76)$$

$$\mathbf{R}_{\mathbf{x},\xi} = \hat{R}_{\mathbf{x}+\xi} \bullet \hat{R}_{\mathbf{x}-\xi} = \hat{R}_{\mathbf{x}+} \bullet \hat{R}_{\mathbf{x}-}. \quad (77)$$

They inherit their properties from the corresponding 'single' centre and chord operators and provide a 'double' representation of the Heisenberg–Weyl group. Their basic properties are

$$\mathbf{T}_{\mathbf{x},\xi}^\dagger = \hat{T}_{\mathbf{x}+\xi}^\dagger \bullet \hat{T}_{\mathbf{x}-\xi} = \mathbf{T}_{-\mathbf{x},-\xi} \quad \mathbf{R}_{\mathbf{x},\xi}^\dagger = \hat{R}_{\mathbf{x}+\xi}^\dagger \bullet \hat{R}_{\mathbf{x}-\xi} = \mathbf{R}_{\mathbf{x},\xi}. \quad (78)$$

A pleasant surprise stems from their periodicity properties on the torus. Using (28) we derive

$$\mathbf{T}_{\mathbf{x}+d\mathbf{y},\xi+d\alpha} = \mathbf{T}_{\mathbf{x},\xi}. \quad (79)$$

Thus they only need to be defined on the \mathbb{Z}_d^4 4-dimensional periodic lattice and there is no need for redundancy as in the single case.

We next compute the composition properties

$$\begin{aligned} \mathbf{T}_{\mathbf{x},\xi} \mathbf{T}_{\mathbf{x}',\xi'} &\equiv \hat{T}_{\mathbf{x}+\xi} \hat{T}_{\mathbf{x}'+\xi'} \bullet \hat{T}_{\mathbf{x}'-\xi'}^\dagger \hat{T}_{\mathbf{x}-\xi}^\dagger \\ &= \hat{T}_{\mathbf{x}+\mathbf{x}'+\xi+\xi'} \bullet \hat{T}_{\mathbf{x}+\mathbf{x}'-\xi-\xi'}^\dagger \tau^{(\mathbf{x}+\xi,\mathbf{x}'+\xi')+(\mathbf{x}'-\xi',\mathbf{x}-\xi)} \end{aligned} \quad (80)$$

$$\begin{aligned} \mathbf{R}_{\mathbf{x},\xi} \mathbf{R}_{\mathbf{x}',\xi'} &\equiv \hat{R}_{\mathbf{x}+\xi} \hat{R}_{\mathbf{x}'+\xi'} \bullet \hat{R}_{\mathbf{x}'-\xi'} \hat{R}_{\mathbf{x}-\xi} \\ &= \hat{T}_{\mathbf{x}-\mathbf{x}'+\xi-\xi'} \bullet \hat{T}_{\mathbf{x}-\mathbf{x}'-\xi+\xi'}^\dagger \tau^{-((\mathbf{x}+\xi,\mathbf{x}'+\xi')+(\mathbf{x}'-\xi',\mathbf{x}-\xi))} \end{aligned} \quad (81)$$

$$\begin{aligned} \mathbf{T}_{\mathbf{x},\xi} \mathbf{R}_{\mathbf{x}',\xi'} &\equiv \hat{T}_{\mathbf{x}+\xi} \hat{R}_{\mathbf{x}'+\xi'} \bullet \hat{R}_{\mathbf{x}'-\xi'} \hat{T}_{\mathbf{x}-\xi}^\dagger \\ &= \hat{R}_{\mathbf{x}+\mathbf{x}'+\xi+\xi'} \bullet \hat{R}_{\mathbf{x}+\mathbf{x}'-\xi-\xi'}^\dagger \tau^{(\mathbf{x}+\xi,\mathbf{x}'+\xi')+(\mathbf{x}'-\xi',\mathbf{x}-\xi)} \end{aligned} \quad (82)$$

$$\begin{aligned} \mathbf{R}_{\mathbf{x},\xi} \mathbf{T}_{\mathbf{x}',\xi'} &\equiv \hat{R}_{\mathbf{x}+\xi} \hat{T}_{\mathbf{x}'+\xi'} \bullet \hat{T}_{\mathbf{x}'-\xi'}^\dagger \hat{R}_{\mathbf{x}-\xi} \\ &= \hat{T}_{\mathbf{x}-\mathbf{x}'+\xi-\xi'} \bullet \hat{T}_{\mathbf{x}-\mathbf{x}'-\xi+\xi'}^\dagger \tau^{-((\mathbf{x}+\xi,\mathbf{x}'+\xi')+(\mathbf{x}'-\xi',\mathbf{x}-\xi))}. \end{aligned} \quad (83)$$

The phase in the exponent is in all cases $2(\langle \xi, \mathbf{x}' \rangle + \langle \mathbf{x}, \xi' \rangle)$. Then if we define $X = (\mathbf{x}, \xi J) = (x_q, x_p, \xi_p, -\xi_q) = (Q_1, Q_2, P_1, P_2)$ as canonical coordinates in double phase space together with the corresponding symplectic form

$$\langle\langle X, X' \rangle\rangle = \langle \xi, \mathbf{x}' \rangle + \langle \mathbf{x}, \xi' \rangle = \mathbf{P}' \cdot \mathbf{Q} - \mathbf{Q}' \cdot \mathbf{P} \quad (84)$$

we obtain the composition laws in double phase space as

$$\mathbf{T}_X \mathbf{T}_Y = \mathbf{T}_{X+Y} \eta^{\langle\langle X, Y \rangle\rangle} \quad (85)$$

$$\mathbf{R}_X \mathbf{R}_Y = \mathbf{T}_{X-Y} \eta^{-\langle\langle X, Y \rangle\rangle} \quad (86)$$

$$\mathbf{T}_X \mathbf{R}_Y = \mathbf{R}_{X+Y} \eta^{\langle\langle X, Y \rangle\rangle} \quad (87)$$

$$\mathbf{R}_X \mathbf{T}_Y = \mathbf{R}_{X-Y} \eta^{-\langle\langle X, Y \rangle\rangle} \quad (88)$$

where we have absorbed the factor of 2 in the definition of $\eta = \tau^2 = e^{2\pi i/d}$. These have exactly the Weyl Heisenberg structure expected from a system with two degrees of freedom, where the positions are the centers and the momenta are the chords. The relationship between \mathbf{R}_X and \mathbf{T}_Y can be worked out using (21)

$$\mathbf{R}_X = \frac{1}{d^2} \sum_{Y \in \mathbb{Z}_d^4} \mathbf{T}_Y \eta^{\langle\langle X, Y \rangle\rangle}. \quad (89)$$

Given these properties we then identify \mathbf{T}_X and \mathbf{R}_X as complementary superoperator bases

$$\text{Tr} \mathbf{T}_X^\dagger \mathbf{T}_Y = d^2 \delta(X - Y) \quad \text{Tr} \mathbf{R}_X \mathbf{R}_Y = d^2 \delta(X - Y) \quad (90)$$

$$\text{Tr} \mathbf{T}_X^\dagger \mathbf{R}_Y = \eta^{\langle\langle X, Y \rangle\rangle} \quad \text{Tr} \mathbf{R}_X \mathbf{T}_Y = \eta^{-\langle\langle X, Y \rangle\rangle}, \quad (91)$$

where Tr is the superoperator trace $\text{Tr}(\hat{A} \bullet \hat{B}) = \text{tr} \hat{A} \text{tr} \hat{B}$. In complete analogy with the single phase space definitions, we can then define the double center and chord representations of superoperators as their projections on the two complementary bases

$$\mathbf{S}(\mathbf{x}, \xi) = \text{Tr} \mathbf{R}_{\mathbf{x},\xi} \mathbf{S} \quad \tilde{\mathbf{S}}(\mathbf{x}, \xi) = \text{Tr} \mathbf{T}_{\mathbf{x},\xi}^\dagger \mathbf{S}. \quad (92)$$

They have all the same properties of the ‘single’ variety and they provide a way to represent superoperators in phase space. The two representations are related by a double symplectic transform (89) and the superoperators are reconstructed from these c-number arrays as

$$\mathbf{S} = \frac{1}{d^2} \sum_{\mathbf{x}, \xi \in \mathbb{Z}_d^2} \mathbf{S}(\mathbf{x}, \xi) \mathbf{R}_{\mathbf{x},\xi} \quad (93)$$

$$\mathbf{S} = \frac{1}{d^2} \sum_{\mathbf{x}, \xi \in \mathbb{Z}_d^2} \tilde{\mathbf{S}}(\mathbf{x}, \xi) \mathbf{T}_{\mathbf{x},\xi}. \quad (94)$$

Properties associated to center and chord functions translate readily in this setting. Thus

$$\text{Tr} \mathbf{S} = \tilde{\mathbf{S}}(0, 0) = \frac{1}{d^2} \sum_{X \in \mathbb{Z}_d^4} \mathbf{S}(X) \quad (95)$$

$$\text{Tr}(\mathbf{S}_1 \mathbf{S}_2) = \frac{1}{d^2} \sum_{X \in \mathbb{Z}_d^4} \mathbf{S}_1(X) \mathbf{S}_2^*(X) = \frac{1}{d^2} \sum_{X \in \mathbb{Z}_d^4} \tilde{\mathbf{S}}_1(X) \tilde{\mathbf{S}}_2^*(X). \quad (96)$$

Moreover, for hermitian superoperators $\mathbf{S}(X)$ is real and $\tilde{\mathbf{S}}^*(X) = \tilde{\mathbf{S}}(-X)$.

An example of these methods is provided by unitary propagation. The superoperator $\mathbf{U} = \hat{U} \bullet \hat{U}^\dagger$, propagates unitarily the density matrix as $\hat{\rho}' = \hat{U} \hat{\rho} \hat{U}^\dagger$. Its matrix elements in the reflection basis are $\langle\langle R_{\mathbf{x}_+} \parallel \mathbf{U} \parallel R_{\mathbf{x}_-} \rangle\rangle$. Defining $W(\mathbf{x}) = \langle\langle R_{\mathbf{x}} \parallel \rho \rangle\rangle$ as the representation of the density matrix in the reflection basis, the propagator is obtained as

$$W'(\mathbf{x}_+) = \frac{1}{d} \sum_{\mathbf{x}_- \in \mathbb{Z}_d^2} \langle\langle R_{\mathbf{x}_+} \parallel \mathbf{U} \parallel R_{\mathbf{x}_-} \rangle\rangle W(\mathbf{x}_-). \quad (97)$$

On the other hand the double center or chord representations of \mathbf{U} are easily calculated in product form using (92)

$$\mathbf{U}(\mathbf{x}, \boldsymbol{\xi}) = \text{Tr} \mathbf{U} \mathbf{R}_{\mathbf{x}, \boldsymbol{\xi}} = \text{tr}(\hat{U} \hat{R}_{\mathbf{x}+\boldsymbol{\xi}}) \text{tr}(U^\dagger \hat{R}_{\mathbf{x}-\boldsymbol{\xi}}) = U(\mathbf{x} + \boldsymbol{\xi}) U^*(\mathbf{x} - \boldsymbol{\xi}) \quad (98)$$

$$\tilde{\mathbf{U}}(\mathbf{x}, \boldsymbol{\xi}) = \text{Tr} \mathbf{U} \mathbf{T}_{\mathbf{x}, \boldsymbol{\xi}}^\dagger = \text{tr}(\hat{U} \hat{T}_{\mathbf{x}+\boldsymbol{\xi}}^\dagger) \text{tr}(\hat{U}^\dagger \hat{T}_{\mathbf{x}-\boldsymbol{\xi}}) = \tilde{U}(\mathbf{x} + \boldsymbol{\xi}) \tilde{U}^*(\mathbf{x} - \boldsymbol{\xi}) \quad (99)$$

where $U(\mathbf{x}) = \langle\langle R_{\mathbf{x}} \parallel \hat{U} \rangle\rangle$ and $\tilde{U}(\mathbf{x}) = \langle\langle T_{\mathbf{x}}^\dagger \parallel \hat{U} \rangle\rangle$ are the single Weyl and chord transforms of \hat{U} , i.e. its *Weyl and chord propagators*. The matrix element can then be computed explicitly using the formulae in the appendix

$$\langle\langle R_{\mathbf{x}+\mathbf{x}_1} \parallel \mathbf{U} \parallel R_{\mathbf{x}-\mathbf{x}_1} \rangle\rangle = \frac{1}{d} \sum_{\boldsymbol{\xi} \in \mathbb{Z}_d^2} U(\mathbf{x} + \boldsymbol{\xi}) U^*(\mathbf{x} - \boldsymbol{\xi}) \eta^{(\mathbf{x}_1, \boldsymbol{\xi})}. \quad (100)$$

This is the inverse transform leading from the double Wigner function to the superoperator matrix elements. Clearly these considerations extend easily to the Kraus representation of completely positive superoperators [35]

$$\mathbf{K} = \sum_j \hat{K}_j \bullet \hat{K}_j^\dagger \quad (101)$$

where now the \mathbf{K} matrix elements are given in terms of the Weyl (or chord) functions of \hat{K}_j .

7. Discussion

Projection of translation and reflection operators onto a finite group on a torus defines a pair of bases for the representation of arbitrary operators acting on a finite set of quantum states. Discrete Weyl and chord representations are defined in this way for any Hilbert space dimension. We have stressed throughout their symmetry and complementarity which closely parallels these properties for the ordinary position and momentum representations. Many properties that hold in the continuum translate to the discrete. In particular, we have shown that surprising new identities involving quadratic and quartic relationships between the Wigner and the chord functions [23] have discrete analogues. Special cases of these identities include a measure of the inverse participation ratio for phase space localization of pure states. Clifford invariance of the Wigner function carries over to this measure, while the new

identity allows one to calculate it indifferently from either the Wigner or the chord function. Another remarkable identity establishes that the correlation of a pure state with its translation is Fourier invariant. If the Hilbert space dimension is large, this forces a duality between large scale structures of the Wigner function with its smallest scales in close analogy to the sub-Planck structures deduced for the continuum [25]. One should note that the breaking of each of these identities as a pure density operator evolves under a non unitary evolution provide delicate measures of how mixed the state has become, thus generalizing and complementing the usual von Neumann or linear entropies. Finally, the relationships that link the transition functions to products or convolutions of the individual functions provide a way to study both the magnitude and the localization properties of coherences.

Finally, we note that generalization of reflection and translation superoperators to a double torus, in strict analogy to their definition in the continuum, leads to analogous relations between the Weyl representation of the evolution operator and the propagator for the Wigner function in (100) to those that are known to hold in the continuum (see references in [23]): the propagator is recognized as the inverse of the double-Wigner transform of the evolution operator in the double phase space. A similar formula holds for the Kraus superoperator that evolves Markovian open systems.

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Appendix A

We provide here the derivation of the relationship between the Choi conjugate bases $\|R_{\mathbf{x}_+}\rangle\rangle\langle\langle R_{\mathbf{x}_-}\|$ and $\hat{R}_{\mathbf{x}_+} \bullet \hat{R}_{\mathbf{x}_-}$ that further justifies our definitions of super translations and reflections. To this end we expand the Choi basis as in (75)

$$\hat{R}_{\mathbf{x}_+} \bullet \hat{R}_{\mathbf{x}_-} = \frac{1}{4d^2} \sum_{\mathbf{y}, \mathbf{z} \in \mathbb{Z}_{2d}^2} \langle\langle R_{\mathbf{y}} \| \hat{R}_{\mathbf{x}_+} \bullet \hat{R}_{\mathbf{x}_-} \| R_{\mathbf{z}} \rangle\rangle \|R_{\mathbf{y}}\rangle\rangle\langle\langle R_{\mathbf{z}}\| = \frac{1}{d^2} \sum_{\mathbf{y}, \mathbf{z} \in \mathbb{Z}_d^2} \text{tr}(\hat{R}_{\mathbf{y}} \hat{R}_{\mathbf{x}_+ + \boldsymbol{\xi}} \hat{R}_{\mathbf{z}} \hat{R}_{\mathbf{x}_- - \boldsymbol{\xi}}) \|R_{\mathbf{y}}\rangle\rangle\langle\langle R_{\mathbf{z}}\| \quad (\text{A.1})$$

where we have used the d -periodicity to restrict the sum to \mathbb{Z}_d^2 to eliminate the factor of 4, and replaced $\mathbf{x}_{\pm} = \mathbf{x} \pm \boldsymbol{\xi}$. The trace can be evaluated using the group properties (19)

$$\begin{aligned} \hat{R}_{\mathbf{x}_+ + \boldsymbol{\xi}} \bullet \hat{R}_{\mathbf{x}_- - \boldsymbol{\xi}} &= \frac{1}{d} \sum_{\mathbf{y}, \mathbf{z} \in \mathbb{Z}_d^2} \delta(\mathbf{y} + \mathbf{z} - 2\mathbf{x}) \tau^{-\langle \mathbf{y}, \mathbf{x} + \boldsymbol{\xi} \rangle} \tau^{-\langle \mathbf{z}, \mathbf{x} - \boldsymbol{\xi} \rangle} \|R_{\mathbf{y}}\rangle\rangle\langle\langle R_{\mathbf{z}}\| \\ &= \frac{1}{d} \sum_{\mathbf{y}, \mathbf{z} \in \mathbb{Z}_d^2} \delta(\mathbf{y} + \mathbf{z} - 2\mathbf{x}) \tau^{-\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle} \tau^{-\langle \mathbf{y} - \mathbf{z}, \boldsymbol{\xi} \rangle} \|R_{\mathbf{y}}\rangle\rangle\langle\langle R_{\mathbf{z}}\| \\ &= \frac{1}{d} \sum_{\mathbf{y} \in \mathbb{Z}_d^2} \tau^{-\langle 2\mathbf{y} - 2\mathbf{x}, \boldsymbol{\xi} \rangle} \|R_{\mathbf{y}}\rangle\rangle\langle\langle R_{2\mathbf{x} - \mathbf{y}}\|. \end{aligned}$$

We now shift the summation and absorb a factor of 2 in the phase to obtain

$$\hat{R}_{\mathbf{x}_+ + \boldsymbol{\xi}} \bullet \hat{R}_{\mathbf{x}_- - \boldsymbol{\xi}} = \frac{1}{d} \sum_{\mathbf{y} \in \mathbb{Z}_d^2} \|R_{\mathbf{x}_+ + \mathbf{y}}\rangle\rangle\langle\langle R_{\mathbf{x}_- - \mathbf{y}}\| \eta^{-\langle \mathbf{y}, \boldsymbol{\xi} \rangle}. \quad (\text{A.2})$$

On the right we recognize the usual definition of a reflection operator in terms of position matrix elements analogous to (4). A similar derivation for the super translations yields

$$\hat{T}_{\mathbf{x}+\boldsymbol{\xi}} \bullet \hat{T}_{\mathbf{x}-\boldsymbol{\xi}}^\dagger = \frac{1}{d} \sum_{\mathbf{y} \in \mathbb{Z}_d^2} \|T_{\mathbf{y}+\boldsymbol{\xi}}\rangle \langle\langle T_{\mathbf{y}-\boldsymbol{\xi}} \| \eta^{-\langle \mathbf{y}, \mathbf{x} \rangle}. \quad (\text{A.3})$$

These relationships generalize to reflection and translation bases the simple partial transposition of indices that occur when transition operator bases $\hat{E}_{i,j} = |i\rangle\langle j|$ are used. In fact we have

$$\hat{E}_{k,i} \bullet \hat{E}_{l,j}^\dagger = \|E_{k,l}\rangle \langle\langle E_{i,j} \| \quad (\text{A.4})$$

as can be easily verified by acting on some operator \hat{A} . This transposition is at the basis of the relationship between the linear map and its *dynamical matrix* underlying the Choi–Jamiołkowski [1, 33] isomorphism between quantum channels and positive operators.

Appendix B. Qubits and qutrits on the torus

Consider first the case $d = 2$. The U, V matrices of (26) are simply the Pauli ones and $\tau = i$

$$\hat{U} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{V} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\sigma_x\sigma_z \quad (\text{B.1})$$

with the usual properties $\sigma_x\sigma_y = i\sigma_z$, etc. Then the translation and the reflection arrays are

$$\hat{T}_{\boldsymbol{\xi}_q, \boldsymbol{\xi}_p} = \sigma_x^{\xi_q} \sigma_z^{\xi_p} i^{\xi_q \xi_p}. \quad (\text{B.2})$$

The 4×4 arrays are then

$$\hat{T}_{\boldsymbol{\xi}_q, \boldsymbol{\xi}_p} = \begin{array}{c|cccc} 3 & \sigma_z & -\sigma_y & -\sigma_z & \sigma_y \\ 2 & 1 & -\sigma_x & 1 & -\sigma_x \\ 1 & \sigma_z & \sigma_y & -\sigma_z & -\sigma_y \\ 0 & 1 & \sigma_x & 1 & \sigma_x \\ \hline & 0 & 1 & 2 & 3 \end{array} \quad \hat{R}_{\mathbf{x}_q, \mathbf{x}_p} = \begin{array}{c|cccc} 3 & \sigma_z & -\sigma_y & -\sigma_z & \sigma_y \\ 2 & 1 & -\sigma_x & 1 & -\sigma_x \\ 1 & \sigma_z & \sigma_y & -\sigma_z & -\sigma_y \\ 0 & 1 & \sigma_x & 1 & \sigma_x \\ \hline & 0 & 1 & 2 & 3. \end{array}$$

This case is exceptional in that reflections and translations are identical. This follows from the fact that for $d = 2$ $\hat{R}_{\mathbf{0}, \mathbf{0}}$ is the identity. This is not true for other values of d .

For the qutrit case $d = 3$ we have

$$\hat{U} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau^2 \end{pmatrix} \quad \hat{R}_{\mathbf{0}, \mathbf{0}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\tau = e^{i\pi/3}$.

The translation and reflection arrays are given by

$$\hat{T}_{\boldsymbol{\xi}_q, \boldsymbol{\xi}_p} = \hat{U}^{\xi_q} \hat{V}^{\xi_p} \tau^{\xi_q \xi_p} \quad \hat{R}_{\mathbf{x}_q, \mathbf{x}_p} = \hat{U}^{x_q} \hat{V}^{x_p} \tau^{x_q x_p} \hat{R}_{\mathbf{0}, \mathbf{0}}.$$

In general translations are shifted diagonal matrices, while reflections have a shifted skew diagonal structure.

Using these operators as a basis, states and general operators are represented by doubly periodic arrays.

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