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Bekenstein inequalities and nonlinear electrodynamics

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Bekenstein and Mayo proposed a generalized bound for the entropy, which implies some inequalities between the charge, energy, angular momentum, and size of the macroscopic system. Dain has shown that Maxwell's electrodynamics satisfies all three inequalities. We investigate the validity of these relations in the context of nonlinear electrodynamics and show that Born-Infeld electrodynamics satisfies all of them. However, contrary to the linear theory, there is no rigidity statement in Born-Infeld. We study the physical meaning and the relationship between these inequalities, and in particular, we analyze the connection between the energy-angular momentum inequality and causality.

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I. INTRODUCTION

Bekenstein bounds and inequalities constitute a set of universal relations between physical quantities and fundamental constants of nature [1,2]. They were initially formulated from gedanken experiments within the scope of black hole thermodynamics (BHT), which is a formal analogy between gravitational compact systems and the three laws of thermodynamics [3–6]. This formalism is a sound effort to reconcile thermodynamics and black hole physics, an example of which is the generalized second law (GSL) [1,7,8]. The Bekenstein bounds and inequalities can be seen as necessary conditions in order to guarantee GSL and the consistency of general relativity with the laws of thermodynamics.

However, since its first proposal, there have been numerous generalizations of Bekenstein inequalities [9,10]. General arguments seem to point to a consensus of the existence, but there are still controversies on their precise formulation. The most general inequality was obtained by Bekenstein and Mayo in Ref. [11], which relates the entropy of a system with domain Σ with the size, energy, angular momentum, and charge as

$$\frac{\hbar c}{2\pi\kappa_B}S \leq \sqrt{(\mathcal{E}\mathcal{R})^2 - c^2J^2} - \frac{Q^2}{8\pi}. \tag{1}$$

In the relation above, $\mathcal{R}$ is defined as the radius of the minimum sphere, $\mathcal{B}_{\mathcal{R}}$, that circumscribes the domain Σ. It can be shown that inequality (1) is saturated for the case of a Kerr-Newman black hole [11]. This result comes as no surprise as long as the inequalities were constructed within BHT. Notwithstanding, it also shows that one should expect equality to always be reached in the most symmetric configuration. In addition, since the entropy of a system is always non-negative, Eq. (1) also implies

$$\mathcal{E}^2 \geq \frac{Q^4}{64\pi^2\mathcal{R}^2} + \frac{c^2J^2}{\mathcal{R}^2}, \tag{2}$$

where equality happens if $S = 0$. Contrary to the first inequality, the only fundamental constant appearing in (2) is the speed of light, which makes theories of electrodynamics particularly appropriate to test it. Along this line, Dain [12] has proven that the above inequality holds for any field configuration of Maxwell electrodynamics.

We can still decompose (2) in two particular cases: one for vanishing angular momentum and the other for neutral objects. For vanishing angular momentum, $J = 0$, the energy and charge of a system have to satisfy the inequality

$$\mathcal{E} \geq \frac{Q^2}{8\pi\mathcal{R}}. \tag{3}$$

The equality in this case states that the total energy of the system equals the electrostatic energy of a spherical thin shell of radius $\mathcal{R}$ and constant surface charge density in Maxwell's theory. Thus, the equality is associated with the most symmetric case in the linear electrodynamics theory.

For neutral objects, $Q = 0$, we obtain a quasilocal inequality that relates the energy of the electromagnetic field and its angular momentum for the region Σ as

$$\mathcal{E}(\Sigma) \geq \frac{c|J(\Sigma)|}{\mathcal{R}}. \tag{4}$$

For Maxwell electrodynamics, the total energy $\mathcal{E}$ is always greater than $\mathcal{E}(\Sigma)$, and hence inequality (4) implies (2) with $Q = 0$. However, there is no such guarantee for nonlinear electrodynamics. Besides, there seems to have no straightforward interpretation for inequality (4). We can gain some insight by looking again to the case of a rigid slowly rotating spherical thin shell in Maxwell electrodynamics. Within this approximation, it can be shown that

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$$\mathcal{E}(\Sigma) \geq \frac{2}{3}\frac{J^2}{2I_s}, \tag{5}$$

where I_s is the moment of inertia of a thin shell. Thus, in the linear theory, the quasilocal energy of a thin spherical shell $\mathcal{E}(\Sigma)$ is bounded from below by two-thirds of its minimum rotational energy. This result suggests that the inequality (4) could be strengthened. However, the fact that the complete inequality holds for the Kerr-Newman black hole is a strong constraint to any attempt to modify it. In addition, Dain has proven [12] that the inequality between energy and angular momentum is a direct consequence of the dominant energy condition (DEC) and, moreover, that the equality in (4) is reached in Maxwell electrodynamics only for radiation fields, i.e., $E_\alpha E^\alpha = B_\alpha B^\alpha = B_\alpha E^\alpha = 0$.

There are many examples of nonlinear electrodynamics (NLED) in the literature [13–18]. Up to now, Maxwell electrodynamics has never been seriously challenged by any experiment. Nevertheless, there are interesting theoretical arguments [19–25] that prompt us to investigate NLED. In addition, NLED naturally appears as the effective action for quantum electrodynamics if we consider vacuum polarization effects [26,27].

Bekenstein bounds and inequalities are supposed to have universal validity. Therefore, it is reasonable to use these inequalities as a possible test for NLED candidates. This criterion can be understood as complementary to already known theoretical [28–30] and experimental [31–34] criteria in the literature. The minimum requirement for a NLED is to recover Maxwell electrodynamics in the appropriate regime. However, there are physical arguments based on causality that restrict the form of NLED Lagrangians. In this paper, we shall use inequalities (2), (3), and (4) as a physical argument to test NLED.

The paper is organised as follows. In the next section, in order to fix notation, we briefly review the covariant formalism of linear and nonlinear electrodynamics. In Sec. III A, we explicitly show that Born-Infeld electrodynamics, similarly to Maxwell electrodynamics, also satisfies all three inequalities. In addition, in Sec. III B, we present counterexamples showing that NLED in general does not satisfy Bekenstein inequalities. In Sec. IV, we investigate the relation of the angular momentum inequality (4) with causality and show that, even though it is a consequence of the DEC, this inequality cannot be strictly associated with causality. Finally, in Sec. V, we conclude with some general remarks.

II. ELECTRODYNAMICS

In this short review, we shall define some relevant objects and fix our notation. Throughout our development, we shall use Heaviside-Lorentz units with $\kappa_B = \hbar = c = 1$. Let us start by fixing spacetime as the flat Minkowski metric that in Cartesian coordinates reads $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Electromagnetism is understood as the vector gauge theory with symmetry group $U(1)$ and is hence described by the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This automatically guarantees, for any electromagnetic theory, the validity of the second pair of Maxwell's equations given by $\partial_{[\alpha} F_{\mu\nu]} = 0$ where brackets mean total antisymmetry in the indices.

The electric and magnetic fields are defined as the projection of the Faraday tensor and its dual along the observer's worldline. The dual of the Faraday tensor is given by $\tilde{F}^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu\alpha\beta}F_{\alpha\beta}$ where $\eta^{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Cività tensor. Thus, consider an observer with normalized velocity v^μ, i.e., $v^\mu v_\mu = 1$. The electric and magnetic fields are defined, respectively, as

$$E^\mu = F^\mu{}_\nu v^\nu, \qquad B^\mu = \tilde{F}^\mu{}_\nu v^\nu. \tag{6}$$

Both electromagnetic vectors are spacelike with negative norms, i.e., $E^\mu E_\mu = -E^2$ and $B^\mu B_\mu = -B^2$. Furthermore, by definition, they are perpendicular to the velocity field $E^\mu v_\mu = B^\mu v_\mu = 0$. We can construct two Lorentz-invariant quantities with the Faraday tensor and its dual, namely,

$$F \equiv \frac{1}{2}F^{\mu\nu}F_{\mu\nu} = E_\alpha E^\alpha - B_\alpha B^\alpha \tag{7}$$

$$G \equiv \frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = B_\alpha E^\alpha. \tag{8}$$

These Lorentz invariants constitute the only linearly independent scalars that can be constructed from $F^{\mu\nu}$ and its dual [35]. Indeed, a direct calculation shows the following algebraic relations:

$$\tilde{F}^{\mu\alpha}\tilde{F}_{\alpha\nu} - F^{\mu\alpha}F_{\alpha\nu} = F\delta^\mu{}_\nu \tag{9}$$

$$\tilde{F}^{\mu\alpha}F_{\alpha\nu} = -G\delta^\mu{}_\nu \tag{10}$$

$$F^\mu{}_\alpha F^\alpha{}_\beta F^\beta{}_\nu = -G\tilde{F}^\mu{}_\nu - FF^\mu{}_\nu \tag{11}$$

$$F^\mu{}_\alpha F^\alpha{}_\beta F^\beta{}_\lambda F^\lambda{}_\nu = G^2\delta^\mu{}_\nu - FF^\mu{}_\alpha F^\alpha{}_\nu. \tag{12}$$

Therefore, one can construct rank-2 objects only up to the second power of the Faraday tensor; i.e., any power of the electromagnetic tensor and its dual is a combination of the identity $\delta^\mu{}_\nu$, $F^\mu{}_\nu$, $\tilde{F}^\mu{}_\nu$, and $F^\mu{}_\alpha F^\alpha{}_\nu$.

The source of electrodynamics is charged particles. We shall denote Σ as the region that contains all charges.

Definition 1.—The size of the region Σ can be characterized by the radius $\mathcal{R}$, which we define as the radius of the smallest sphere $\mathcal{B}_\mathcal{R}$ that encloses Σ. Additionally, we shall designate the center of this sphere by x_0.

The total electric charge contained in Σ is given by

$$Q(\Sigma) = \int_{\Sigma} \rho.$$

Two other important quantities for our analysis are the energy and angular momentum of the distribution of charges. These quantities are defined as the integral of combinations of the energy-momentum tensor components. We shall define our energy-momentum tensor through the variation of the matter action with respect to the metric tensor. Thus, we have

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g}\mathcal{L}_{\mathrm{mat}}) = 2\frac{\delta \mathcal{L}_{\mathrm{mat}}}{\delta g^{\mu\nu}} - \mathcal{L}_{\mathrm{mat}} g_{\mu\nu}. \tag{13}$$

A. Maxwell electrodynamics

Maxwell's electrodynamics is described by a set of four differential equations. The two source-free equations allow us to define the Faraday tensor as the exterior derivative of the vector potential 1-form, i.e., $F = \mathrm{d}A$. The other two equations are associated with the source terms. Defining the current vector $j^\mu = (\rho, \mathbf{j})$, the second set of Maxwell's equations reads

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{14}$$

These equations can be derived from a variational principle for the vector potential A_μ in which the appropriate action is defined with the Larmor Lagrangian, i.e., $\mathcal{L} = -\frac{1}{2}F$. The invariant G represents a total divergence and hence does not contribute to the dynamics equations. Thus, up to a multiplicative constant, the Larmor Lagrangian is the unique linear electromagnetic Lagrangian.

As usual, the energy-momentum tensor is defined as the variation of the matter action with respect to the spacetime metric, which for Maxwell's theory gives

$$T^\mu{}_\nu = F^{\mu\alpha} F_{\alpha\nu} + \frac{F}{2} \delta^\mu{}_\nu. \tag{15}$$

In particular, the Maxwellian electromagnetic energy density u_M is the time-time components of the energy-momentum tensor, i.e., $u_M = -\frac{1}{2}(E_\alpha E^\alpha + B_\alpha B^\alpha)$, and the total energy reads

$$\mathcal{E}_M = -\frac{1}{2} \int_{\mathbb{R}^3} (E_\alpha E^\alpha + B_\alpha B^\alpha). \tag{16}$$

Similarly, the angular momentum of a region Σ with respect to a point x_0 projected along the direction $\mathbf{k}$ is defined as

$$J(\Sigma) = \int_{\Sigma} \epsilon_{ijk} \epsilon^{iab} E_a B_b k^j x^k. \tag{17}$$

B. Nonlinear electrodynamics

The most general Lorentz-invariant electromagnetic Lagrangian is a function of the two scalar invariants F and G, i.e., $\mathcal{L} = \mathcal{L}(F, G)$. Given an arbitrary Lagrangian, its energy-momentum tensor reads

$$T^\mu{}_\nu = -F^{\mu\alpha} E_{\alpha\nu} - \mathcal{L}\delta^\mu{}_\nu, \tag{18}$$

where the excitation tensor is defined as $E_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = 2(\mathcal{L}_F F_{\mu\nu} + \mathcal{L}_G \tilde{F}_{\mu\nu})$ and $\mathcal{L}_x$ stands for the partial derivative of the Lagrangian with respect to x. The field equations can be written in terms of $E_{\mu\nu}$ as

$$\partial_\mu E^{\mu\nu} = -J^\nu. \tag{19}$$

In the same way as before, the energy density reads

$$u = 2(\mathcal{L}_F E^\alpha E_\alpha + \mathcal{L}_G G) - \mathcal{L}, \tag{20}$$

and the angular momentum is

$$J(\Sigma) = -2 \int_{\Sigma} \mathcal{L}_F \epsilon_{ijk} \epsilon^{iab} E_a B_b k^j x^k. \tag{21}$$

It is worth noting that, contrary to the energy density, the angular momentum depends only on the first derivative of the Lagrangian with respect to the invariant F. Evidently, both expressions recover the linear case for $\mathcal{L} = -\frac{1}{2}F$.

III. BEKENSTEIN BOUNDS AND INEQUALITIES WITHIN NLED

There are two possible ways to approach the interplay of NLED and Bekenstein bounds and inequalities. From one point of view, since the latter is supposed to be valid for an arbitrary physical system, they can be used to test possible NLED candidates. On the other hand, NLED is a fertile framework that allows us to investigate different theoretical situations, which can provide us with deeper insight into the physical meaning of the Bekenstein inequalities.

In this section, we begin by proving that Born-Infeld electrodynamics satisfies all three inequalities. Our development follows closely the analysis done by Dain in Ref. [12] for Maxwell's electrodynamics. Next, we show a concrete example of NLED that violates the inequality between energy and charge but still respects the inequality between energy and angular momentum.

A. Born-Infeld electrodynamics

Maxwell's electrodynamics is a classical theory that suffers from divergences such as the value of the electromagnetic energy as one approaches a charged particle. This, and other similar problems, motivated finite-size models for the electron, which would give an upper bound for its self-energy as one probes the radius goes to zero limit.

An alternative context is to modify the electrodynamics to include nonlinear effects. The first attempt along these lines, due to Mie [36], was to introduce a model in which there is an upper limit for the value of the electric field, but this formulation was not Lorentz covariant.

Following Mie, in 1933, Born and Infeld proposed a nonlinear modification of Maxwell electrodynamics [19,20,37] that also has an upper limit for the electromagnetic fields. Born-Infeld electrodynamics is a special nonlinear theory due to its theoretical features. By construction, it is a gauge-invariant theory with finite electromagnetic mass pointlike sources. The energy is positive definite, and the Poynting vector is everywhere nonspacelike. In addition, it has no birefringence phenomena.

In general, photon propagation in nonlinear theories depends on the value of the electromagnetic fields. As a consequence, different polarization states propagate along different light cones [38–42]. Notwithstanding, Boillat [43,44] showed that Born-Infeld is unique in the sense that it is the only NLED without birefringence phenomena and shock waves can occur only across characteristic surfaces of the field equations as is the case for the linear theory.

Besides trying to eliminate the classical divergences, the proposal by Born and Infeld was inspired by the theory of general relativity. They argued that the diffeomorphism invariance of the action can be obtained by taking the square root of the determinant of a tensor field $|a_{\mu\nu}|$. In particular, they identified its symmetric part with the metric tensor and its antisymmetric part with the Faraday tensor, i.e., $a_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu}$. To recover Maxwell electrodynamics in the weak field limit, the desired combination is

$$S = \int \mathrm{d}\tau \beta^2 \left(\sqrt{-|g_{\mu\nu}|} - \sqrt{-|g_{\mu\nu} + \beta^{-1} F_{\mu\nu}|} \right), \qquad (22)$$

where β constitutes a maximum field parameter. Assuming Cartesian coordinates in a flat spacetime, the above action reads

$$S = \int \mathrm{d}\tau \beta^2 (1 - \sqrt{U}), \qquad (23)$$

where $U = 1 + F/\beta^2 - G^2/\beta^4$.

The Born-Infeld field equations read

$$\partial_\mu \left[\frac{1}{\sqrt{U}} \left(-F^{\mu\nu} + \frac{G}{\beta^2} \tilde{F}^{\mu\nu} \right) \right] = -j^\nu, \qquad (24)$$

which constitutes the generalization for the Ampère-Maxwell and Gauss equations. These equations can be recast in vector notation as

$$\nabla \cdot \mathbf{D} = \rho \qquad (25a)$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j} \qquad (25b)$$

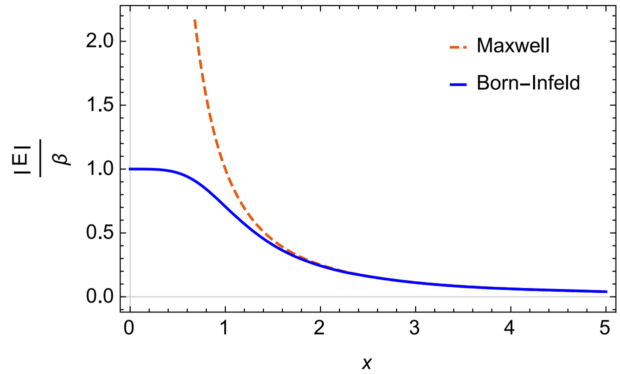

FIG. 1. The electrostatic field of a pointlike charged particle as a function of $x = r\sqrt{\beta/e}$ for Born-Infeld (solid line) and Maxwell (dashed line) electromagnetism.

with

$$\mathbf{D} \equiv \frac{1}{\sqrt{U}} \left(\mathbf{E} + \frac{(\mathbf{E} \cdot \mathbf{B})}{\beta^2} \mathbf{B} \right),$$
$$\mathbf{H} \equiv \frac{1}{\sqrt{U}} \left(\mathbf{B} - \frac{(\mathbf{E} \cdot \mathbf{B})}{\beta^2} \mathbf{B} \right), \qquad (26)$$

which resemble Maxwell's equations inside matter with nonlinear permittivity and permeability. For the electrostatic case, Eq. (25a) allows us to calculate the electric field for a pointlike charged particle. The value of the electric field in the limit $r \to 0$ gives the maximum electrostatic field β. Figure 1 shows the difference between the Born-Infeld and Maxwell electrostatic fields.

Using the Born-Infeld Lagrangian equation (23) in the definition of equation (18), we obtain the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{\sqrt{U}} \left(F_{\mu\alpha} F^{\alpha}{}_{\nu} + \frac{G^2}{\beta^2} g_{\mu\nu} \right) + g_{\mu\nu} \beta^2 (\sqrt{U} - 1). \qquad (27)$$

A similar calculation gives the angular momentum of the distribution of charged particles in the region Σ as

$$J_{BI}(\Sigma) = \int_\Sigma \frac{1}{\sqrt{U}} \epsilon_{ijk} \epsilon^{iab} E_a B_b k^j x^k, \qquad (28)$$

and its energy density is

$$u_{BI} = \frac{\beta^2}{\sqrt{U}} \left(1 - \sqrt{U} - \frac{B_\alpha B^\alpha}{\beta^2} \right). \qquad (29)$$

Born-Infeld electrodynamics has a maximum value for both fields given by the parameter β. Thus, for future analysis, it is convenient to normalize the electric and magnetic fields; i.e., we define the two parameters $\alpha \equiv \beta^{-1} |\mathbf{E}|$ and $\gamma \equiv \beta^{-1} |\mathbf{B}|$ that give, respectively, the normalized strength of the electric and magnetic fields and such that $(\alpha, \gamma) \in [0, 1]$. In terms of these parameters, the Born-Infeld function reads

$$U = 1 + \gamma^2 - \alpha^2 - \gamma^2\alpha^2\cos^2\theta, \tag{30}$$

with $\cos\theta \equiv \mathbf{E}.\mathbf{B}/(|\mathbf{E}||\mathbf{B}|)$. Note that $U \in [0, 2]$ but can be divided in two distinct domains. As a fact, $U \in [0, 1)$ implies $\alpha > \gamma$, and $U \in [1, 2]$ implies $\gamma \geq \alpha$.

1. Inequality between charge and energy

We will begin by examining the inequality between charge and energy (3) in Born-Infeld electrodynamics. We shall prove the following theorem.

Theorem 1.—Assume that the charge density ρ has compact support contained in the region Σ and Born-Infeld electrodynamics holds. Then, the total charge Q contained in Σ and the total electromagnetic energy $\mathcal{E}_{BI}$ of the system satisfy the inequality

$$\mathcal{E}_{BI} > \frac{Q^2}{8\pi\mathcal{R}}, \tag{31}$$

where $\mathcal{R}$ is defined as in Definition 1.

Proof.—In Theorem 2.2 of Ref. [12], Dain has shown[1] that Maxwell's electrodynamics satisfies a similar inequality, namely, that

$$\mathcal{E}_{Ms} \geq \frac{Q^2}{8\pi\mathcal{R}}, \tag{32}$$

where $\mathcal{E}_{Ms}$ is the Maxwell electrostatic energy of the system and Q and $\mathcal{R}$ have the same meaning as here. Furthermore, there is a rigidity condition. Equality in (32) holds if and only if the electric field is the one produced by a spherical thin shell of constant surface charge density and radius $\mathcal{R}$. As a consequence, for the equality to hold in (32), the electric field has to vanish inside Σ.

To prove the inequality (31), it is sufficient to show that the Born-Infeld energy is always greater than its electrostatic counterpart and then show that the Born-Infeld electrostatic energy is always greater than the Maxwell electrostatic energy.

In Maxwell's linear theory, the electromagnetic energy is always greater than or equal to the electrostatic case, but this is no longer the case for a generic NLED. The Born-Infeld theory is a special case in which this property is indeed valid. Note that, since $1 - \alpha^2\cos^2\theta \geq 0$, Born-Infeld energy density is an increasing function of the parameter γ, and hence

$$u_{BI}(\alpha, \gamma) \geq u_{BI}(\alpha, 0). \tag{33}$$

As a consequence the Born-Infeld energy is always greater than its electrostatic version. Thus, it suffices to show that Born-Infeld electrostatic energy density is always greater than Maxwell electrostatic energy density. Their difference reads

$$u_{BI}(\alpha, 0) - u_{Ms} = \frac{\beta^2}{\sqrt{U}}\left[1 - \left(1 + \frac{\alpha^2}{2}\right)\sqrt{1 - \alpha^2}\right] \tag{34}$$

$$\geq \frac{\beta^2}{\sqrt{U}}[1 - (1 + \alpha^2)\sqrt{1 - \alpha^2}] \tag{35}$$

$$= \frac{\beta^2}{\sqrt{U}}\left[1 - \sqrt{\frac{1 - \alpha^4}{1 + \alpha^2}}\right] \geq 0. \tag{36}$$

The equality above holds only when the electric field vanishes everywhere. Therefore, the Born-Infeld electrostatic energy is always greater than the Maxwell electrostatic energy. ■

There is no rigidity statement for Born-Infled electrodynamics because its energy density is always greater than Maxwell. As we have mentioned before, in NLED, the nonlinearity of the theory allows for a nontrivial dependence of its energy density with the strength of the electromagnetic fields. Thus, it is possible to have NLED with energy density lower than the Maxwell energy density. We will explore this scenario later on.

2. Inequality between energy and angular momentum

Our next step is to prove the inequality between the energy and angular momentum. The main difference to Theorem 1 is that inequality (4) relates two quasilocal quantities. In addition, in this section, we shall consider the case in which $Q = 0$ and $J \neq 0$ but with otherwise arbitrary electromagnetic field's configurations. We want to prove the following theorem.

Theorem 2.—Consider a distribution of charged particles in the region σ with no net charge, i.e., $Q = 0$. Let the radius $\mathcal{R}$ be defined as in 1 and x_0 be the center of the corresponding sphere. If Born-Infeld electrodynamics equations hold, then

$$\mathcal{E}_{BI}(\Sigma)\mathcal{R} \geq |J_{BI}(\Sigma)|, \tag{37}$$

where $J_{BI}(\Sigma)$ is the angular momentum of the electromagnetic field given by Eq. (28) with respect to the point x_0. Furthermore, the equality in Eq. (37) holds if and only if the electromagnetic fields vanish in Σ.

Proof.—To prove the above theorem, we shall calculate the difference between the energy and angular momentum in the region Σ. Using the definitions (28) and (29), we have

[1]There is a factor 4π of difference due to our choice of units. Namely, Eq. (4) in Ref. [12] has a factor 4π that does not appear in our Eq. (25).

$$\begin{aligned}\mathcal{E}_{BI}(\Sigma)-\frac{1}{\mathcal{R}}|J_{BI}(\Sigma)| \\ =\int_\Sigma \frac{\beta^2}{\sqrt{U}}(1+\gamma^2-\sqrt{U}) \\ +-\frac{1}{\mathcal{R}}\left|\int_\Sigma\left(\frac{1}{\sqrt{U}}\epsilon_{ijk}\epsilon^{iab}E_aB_bk^jx^k\right)\right| \\ \geq \int_\Sigma \frac{\beta^2}{\sqrt{U}}\left(1+\gamma^2-\sqrt{U}-\frac{x}{\mathcal{R}}\alpha\gamma\right).\end{aligned} \tag{38}$$

In the last line, we have used the inequality $|\int f(x)| \leq \int |f(x)|$ and the fact that $|(\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))\cdot\hat{\mathbf{k}}| \leq |\mathbf{x}\times(\mathbf{E}\times\mathbf{B})| \leq |\mathbf{x}||\mathbf{E}||\mathbf{B}|$. Recalling Eq. (30), we can rearrange the above expression as

$$\begin{aligned}\mathcal{E}_{BI}(\Sigma)-\frac{1}{\mathcal{R}}|J_{BI}(\Sigma)| \\ \geq \int_\Sigma \frac{\beta^2}{2\sqrt{U}}\left[(1-\sqrt{U})^2+\left(\gamma-\frac{x}{\mathcal{R}}\alpha\right)^2\right. \\ \left.+\alpha^2\gamma^2\cos^2\theta+\alpha^2\left(1-\frac{x^2}{\mathcal{R}^2}\right)\right].\end{aligned} \tag{39}$$

It is obvious that all the integrands in the above equation are non-negative and hence the integral is greater than or equal to zero. Thus, we have proven inequality (37). In the above form, it can also be seen that the equality can only be achieved when the integrand in (39) is zero; hence, every term in the integrand has to identically vanish. Thus, equality holds if and only if the electric and magnetic fields vanish in Σ, proving the rigidity condition. ■

3. Inequality between charge, energy, and angular momentum

Finally, we shall prove the full inequality (2) involving the charge, angular momentum, and total energy of the system.

Theorem 3.—Assume that the charge density $\rho(x,t_0)$, for some time t_0, has compact support contained in the region Σ. Consider a solution of Born-Infeld dynamics equations that decays at infinity. Then, at t_0, the total charge Q contained in Σ, the total electromagnetic energy $\mathcal{E}_{BI}$, and the angular momentum $J_{BI}(\Sigma)$ with respect to x_0 satisfy the inequality

$$\mathcal{E}_{BI} > \frac{Q^2}{8\pi\mathcal{R}}+\frac{|J_{BI}(\Sigma)|}{\mathcal{R}}, \tag{40}$$

where $\mathcal{R}$ and x_0 are defined as in Definition 1.

Proof.—Let us express the electric and magnetic fields in the Coulomb gauge,

$$\mathbf{B}=\nabla\times\mathbf{A}, \qquad \mathbf{E}=-\boldsymbol{\nabla}\Phi-\partial_t\mathbf{A}, \tag{41}$$

where the vector potential satisfies the Coulomb gauge condition

$$\nabla\cdot\mathbf{A}=0. \tag{42}$$

It is convenient to decompose the scalar potential using an auxiliary potential Φ_1. Thus, we define

$$\Phi=\Phi_0+\Phi_1, \tag{43}$$

where

$$\Phi_0=\begin{cases} Q/4\pi r, & \text{if } r\geq\mathcal{R}, \\ Q/4\pi\mathcal{R}, & \text{if } r\leq\mathcal{R}\end{cases} \tag{44}$$

is the potential of a spherical shell of radius $\mathcal{R}$ and the same total charge Q as contained in Σ. Note that $\nabla\Phi_0=0$ inside $\mathcal{B}_\mathcal{R}$ and by construction we have

$$\Delta\Phi_1=\begin{cases} 0 & \text{if } r>\mathcal{R} \\ -\rho & \text{if } r<\mathcal{R}\end{cases} \tag{45}$$

and

$$\oint_{\partial\mathcal{B}_\mathcal{R}}\partial_r\Phi_1=0. \tag{46}$$

Before calculating the total energy $\mathcal{E}_{BI}$, let us consider the integral over all space of the modulus squared of the electric field,

$$\int_{\mathbb{R}^3}\mathbf{E}^2=\int_{\mathbb{R}^3}\{|\nabla\Phi|^2+|\partial_t\mathbf{A}|^2+2\nabla\Phi\cdot\partial_t\mathbf{A}\}. \tag{47}$$

The last term gives no contribution since it can be recast as a surface term. Indeed, we can rewrite it as

$$\int_{\mathbb{R}^3}\nabla\Phi\cdot\partial_t\mathbf{A}=\int_{\mathbb{R}^3}[\nabla\cdot(\Phi\partial_t\mathbf{A})-\Phi\partial_t(\nabla\cdot\mathbf{A})]=0, \tag{48}$$

where the first term on the right is zero due to Gauss's theorem and the falloff condition of Φ, and the last term vanishes since the vector potential satisfies the Coulomb gauge condition.

Now, we shall use the auxiliary scalar potential to rewrite the first term of Eq. (47),

$$\int_{\mathbb{R}^3}|\nabla\Phi|^2=\int_{\mathbb{R}^3}|\nabla\Phi_0|^2+|\nabla\Phi_1|^2+2\nabla\Phi_0\cdot\nabla\Phi_1. \tag{49}$$

Again, the last term does not contribute. We can decompose the integral in two regions: inside and outside of the sphere $\mathcal{B}_\mathcal{R}$. Inside the sphere,

$$\int_{\mathcal{B}_\mathcal{R}}\nabla\Phi_0\cdot\nabla\Phi_1=0, \tag{50}$$

since the potential Φ_0 is constant and hence $\nabla\Phi_0=0$. Outside the sphere, we have

$$\int_{\mathbb{R}^3\setminus\mathcal{B}_\mathcal{R}} \nabla\Phi_0\cdot\nabla\Phi_1 = \int_{\mathbb{R}^3\setminus\mathcal{B}_\mathcal{R}} [\nabla\cdot(\Phi_0\nabla\Phi_1) - \Phi_0\Delta\Phi_1] = \int_{\mathbb{R}^3\setminus\mathcal{B}_\mathcal{R}} \nabla\cdot(\Phi_0\nabla\Phi_1), \quad (51)$$

where we have used Eq. (45). Gauss's theorem now gives

$$\int_{\mathbb{R}^3\setminus\mathcal{B}_\mathcal{R}} \nabla\cdot(\Phi_0\nabla\Phi_1) = \lim_{r\to\infty}\oint_{\partial\mathcal{B}_r}\Phi_0\partial_r\Phi_1 - \oint_{\partial\mathcal{B}_\mathcal{R}}\Phi_0\partial_r\Phi_1 = -\oint_{\partial\mathcal{B}_\mathcal{R}}\Phi_0\partial_r\Phi_1 = -\Phi_0\oint_{\partial\mathcal{B}_\mathcal{R}}\partial_r\Phi_1 = 0. \quad (52)$$

From the first to the second lines, we have used the falloff condition of the potential. From the second to the third lines, we have used the fact that Φ_0 is constant on the sphere $\mathcal{B}_\mathcal{R}$, and finally in the last line, Eq. (46) shows that the cross-term does not contribute to Eq. (49).

Our last step is to combine $|\nabla\Phi_1|^2$ with $|\partial_t\mathbf{A}|^2$,

$$\int_{\mathbb{R}^3} |\nabla\Phi_1|^2 + |\partial_t\mathbf{A}|^2 = \int_{\mathbb{R}^3} |\nabla\Phi_1 + \partial_t\mathbf{A}|^2 - 2\nabla\Phi_1\cdot\partial_t\mathbf{A} = \int_{\mathbb{R}^3} |\nabla\Phi_1 + \partial_t\mathbf{A}|^2, \quad (53)$$

where we have discarded the cross-term with the same arguments as used in Eq. (48). Combining all these results, the integral of the modulus squared of the electric fields yields

$$\int_{\mathbb{R}^3} \mathbf{E}^2 = \int_{\mathbb{R}^3} |\nabla\Phi_0|^2 + |\nabla\Phi_1 + \partial_t\mathbf{A}|^2 = \frac{Q^2}{4\pi\mathcal{R}} + \int_{\mathbb{R}^3} |\nabla\Phi_1 + \partial_t\mathbf{A}|^2. \quad (54)$$

The Born-Infled total energy reads

$$\mathcal{E}_{BI} = \int_{\mathbb{R}^3} \frac{\beta^2}{\sqrt{U}}(1+\gamma^2-\sqrt{U}). \quad (55)$$

We can sum and subtract the integral (47) to obtain

$$\mathcal{E}_{BI} = \int_{\mathbb{R}^3}\frac{\mathbf{E}^2}{2} + \int_{\mathbb{R}^3}\frac{\beta^2}{\sqrt{U}}\left(1+\gamma^2-\sqrt{U}\left(1+\frac{\alpha^2}{2}\right)\right) = \frac{Q^2}{8\pi\mathcal{R}} + \int_{\mathbb{R}^3}\frac{\beta^2}{\sqrt{U}}\left(1+\gamma^2-\sqrt{U}\left(1+\frac{\alpha^2}{2}\right)\right) + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\Phi_1+\partial_t\mathbf{A}|^2. \quad (56)$$

Now, we can split the limits of integration to separate the space in inside and outside Σ. Thus, we have

$$\mathcal{E}_{BI} = \frac{Q^2}{8\pi\mathcal{R}} + \mathcal{E}_{BI}(\Sigma) + \int_{\mathbb{R}^3\setminus\Sigma}\left[f(\alpha,\gamma) + \frac{|\nabla\Phi_1+\partial_t\mathbf{A}|^2}{2}\right], \quad (57)$$

where

$$f(\alpha,\gamma) \equiv \frac{\beta^2}{\sqrt{U}}\left(1+\gamma^2-\sqrt{U}\left(1+\frac{\alpha^2}{2}\right)\right). \quad (58)$$

Theorem 2 allows us to write

$$\mathcal{E}_{BI} - \frac{Q^2}{8\pi\mathcal{R}} - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_{\mathbb{R}^3\setminus\Sigma}\left[f(\alpha,\gamma) + \frac{|\nabla\Phi_1+\partial_t\mathbf{A}|^2}{2}\right]. \quad (59)$$

Note that the function $f(\alpha,\gamma)$ is non-negative. Rearranging the terms, we have

$$1+\gamma^2-\sqrt{U}\left(1+\frac{\alpha^2}{2}\right) = \frac{1}{2}\left[(1-\sqrt{U})^2 + \alpha^2\gamma^2\cos^2\theta + \alpha^2\left(1-\sqrt{U}+\frac{\gamma^2}{\alpha^2}\right)\right]. \quad (60)$$

The only term that is not explicitly non-negative on the right-hand side of the above equation is the last one. This comes from the fact that $0\leq U\leq 2$. However, one can check that $U>1$ only if $\gamma>\alpha$. Thus, we have $1-\sqrt{U}+\gamma^2/\alpha^2>0$, which in turn implies $f(\alpha,\gamma)>0$. Therefore, the integrand on the right-hand side of inequality (59) is non-negative. Furthermore, the integral has to be greater than zero since in order to have $\nabla\Phi_1=0$ we need a nonzero electric. Indeed, either $\alpha=0$ and $\nabla\Phi_1=-\nabla\Phi_0$ or $\nabla\Phi_1=0$ and $\alpha\neq 0$. ■

Born-Infeld electrodynamics is an important example of NLED. This theory, besides respecting the Maxwell limit, has many unique features. We have proven that Born-Infeld electrodynamics satisfies all three Bekenstein inequalities. Assuming the universal validity of these inequalities, Born-Infeld can include this extra feature in its theoretical motivations.

B. Breaking Bekenstein's inequality

In this section, we want to show a specific NLED that violates the simplest of the three inequalities, namely, the one relating the total energy and total charge of the system.

In Maxwell's theory, this inequality can be interpreted as showing that the total energy is always greater than the minimum electrostatic energy given by a thin spherical shell charge distribution. Thus, it might seem that any NLED would also satisfy this inequality. Indeed, many NLED do satisfy it [13,15,37]. To violate inequality (3), we need a NLED that has a minimum electrostatic energy lower than the minimum Maxwellian electrostatic energy.

Thus, consider the NLED $\mathcal{L}(F)$ given by the logarithmic function

$$\mathcal{L} = \beta^2 \ln\left(1 - \frac{F}{2\beta^2}\right), \tag{61}$$

which is a modification of the logarithmic Lagrangian introduced by Gaete and Helayël-Neto in Ref. [13]. This NLED has the correct Maxwellian limit for weak fields and hence can be considered as a physically reasonable theory. Its energy-momentum tensor reads

$$T_{\mu\nu} = \frac{F_{\mu\alpha}F^{\alpha}{}_{\nu}}{1 - \frac{F}{2\beta^2}} - g_{\mu\nu}\beta^2 \ln\left(1 - \frac{F}{2\beta^2}\right), \tag{62}$$

and its electrostatic energy density is

$$u_{\log} = \beta^2\left[\frac{2\alpha^2}{2+\alpha^2} - \ln\left(1 + \frac{\alpha^2}{2}\right)\right]. \tag{63}$$

Contrary to the Born-Infeld case, in logarithmic electrodynamics, there is no upper limit for the electric field, i.e., $\alpha \in [0, \infty)$. However, this NLED is well defined only if we assume that charged particles have finite size. This is due to the fact that there is a minimum allowed radius in order to guarantee the reality of the electric field. In the electrostatic regime, the electric displacement $\mathbf{D}$ is related with the electric field by

$$\mathbf{D} = \frac{\mathbf{E}}{1 + |\mathbf{E}|^2/2\beta^2}. \tag{64}$$

For a static spherically symmetric distribution with total charge Q, Gauss's theorem shows that $\mathbf{D} = Q/4\pi r^2 \hat{\mathbf{r}}$. We can invert Eq. (64) and write

$$|\mathbf{E}| = \frac{\sqrt{2}\beta}{r_0}\left(r^2 \pm \sqrt{r^4 - r_0^4}\right), \tag{65}$$

where $r_0^2 \equiv \sqrt{2}Q/4\pi\beta$ is the minimum size of charged particles. A comparison between the logarithmic and Maxwell energy density is plotted in Fig. 2.

The binding energy for a spherical shell of radius r_0 and total charge Q within Maxwell electrodynamics is

$$\mathcal{E}_{\mathrm{M}} = \frac{Q^2}{8\pi r_0} \sim 0.1186\sqrt{Q^3\beta}. \tag{66}$$

The binding energy for the same charge distribution in the logarithmic electrodynamics can be calculated by the integral of the energy density

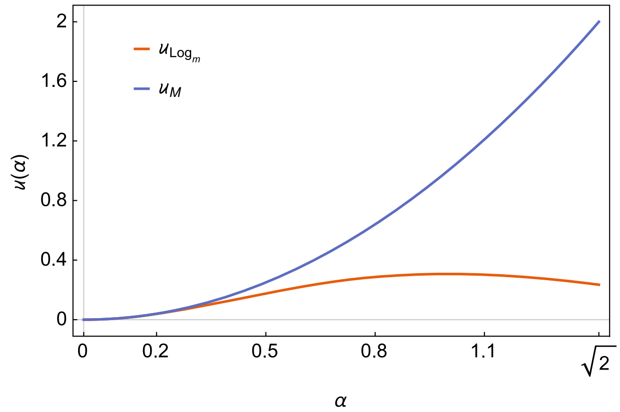

FIG. 2. The electrostatic energy density for logarithmic electrodynamics and the electrostatic energy density for Maxwell electrodynamics.

$$\begin{aligned}\mathcal{E}_{\log} &= 4\pi\int_{r_0}^{\infty} r^2 dr\left\{\frac{E^2}{1+\frac{E^2}{2\beta^2}} - \beta^2\ln\left(1+\frac{E^2}{2\beta^2}\right)\right\}\\ &= \sqrt{\frac{Q^3\beta}{8\pi\sqrt{2}}}\int_{\sqrt{2}}^{\infty} dy y^2\left\{\frac{2}{1+y^4} - \ln\left(1+\frac{1}{y^4}\right)\right\}\\ &\sim 0.1108\sqrt{Q^3\beta}.\end{aligned} \tag{67}$$

Note that Maxwell's binding energy is greater than the logarithmic binding energy

$$\mathcal{E}_{\mathrm{M}} > \mathcal{E}_{\log}. \tag{68}$$

Since the equality in (3) is reached only by $\mathcal{E}_{\mathrm{M}}$, we can conclude that the electrostatic configuration $\mathcal{E}_{\log}$ violates the inequality between energy and charge. Furthermore, following the same reasoning used in Sec. III A 2, one can show that the logarithmic electrodynamics does satisfy the inequality between energy and angular momentum. This result proves that the validities of each partial inequality are independent of each other.

IV. CAUSALITY AND THE ANGULAR-MOMENTUM INEQUALITY

There is an interesting connection between Theorem 2 and the spacetime causal structure. Dain has shown [12], and we reproduce the argument in Appendix A, that the DEC is a sufficient condition for inequality (37). The DEC is a physically motivated condition on the energy-momentum tensor, which prohibits superluminal propagation [45,46].

On the other hand, inequality (37) relates the quasilocal total energy of the system with the quasilocal angular momentum with respect to the origin of the minimum sphere that surrounds the system. In a sense, this inequality shows that the total energy has to be greater than or equal to the angular kinetic energy of the system. Furthermore, their ratio is proportional to a mean angular velocity of the system, and hence inequality (37) can also be interpreted as saying that this angular velocity has to be smaller than unit (or that the mean velocity is smaller than c). Thus, it indeed seems reasonable to associate this inequality with the causal structure of the theory.

To study the connection between causality and inequality (37), we shall analyze if the DEC is not only a sufficient but also a necessary condition. In fact, we want to show the opposite, namely, that a noncausal NLED can satisfy inequality (37). In particular, we examine an example of $\mathcal{L}(F)$ NLED, in which case causality can be expressed as the condition [28,29]

$$\mathcal{L}_F \leq 0, \tag{69}$$

which together with unitarity is equivalent to imposing the DEC. Thus, consider the Lagrangian introduced by Kruglov in Ref. [17] as a modification of exponential electrodynamics,

$$\mathcal{L} = -\frac{F}{2} e^{-\frac{F}{2\beta^2}}. \tag{70}$$

Then, the causality condition reads

$$B^2 \leq 2\beta^2 + E^2, \tag{71}$$

which can be seen as an upper bound for the magnetic field with respect to the electric field. Since the DEC guarantees the validity of inequality (37), we shall focus only on the noncausal configurations, i.e., $\mathcal{L}_F > 0$. The difference between the energy and angular momentum reads

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} = \int_\Sigma e^{-\frac{F}{2\beta^2}} \left(\frac{E^2 + B^2}{2} - \frac{F}{2\beta^2} E^2 \right) - \frac{1}{\mathcal{R}} \left| \int_\Sigma e^{-\frac{F}{2\beta^2}} \left(\frac{F}{2\beta^2} - 1 \right) \epsilon_{ijk} \epsilon^{iab} B_a E_b k^j x^k \right|. \tag{72}$$

Following the same approach used in Sec. III A, the right-hand side of the above expression can be majored as

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma e^{-\frac{F}{2\beta^2}} \left\{ \frac{E^2 + B^2}{2} + \frac{x}{\mathcal{R}} EB|\sin\theta| - \frac{F}{2\beta^2} \left(E^2 + \frac{x}{\mathcal{R}} EB|\sin\theta| \right) \right\}. \tag{73}$$

It is straightforward to check that certain fields' configurations can, indeed, satisfy the inequality between the energy and angular momentum. The crucial point is to check if certain field's configurations that violate the causal condition can simultaneously satisfy the inequality.

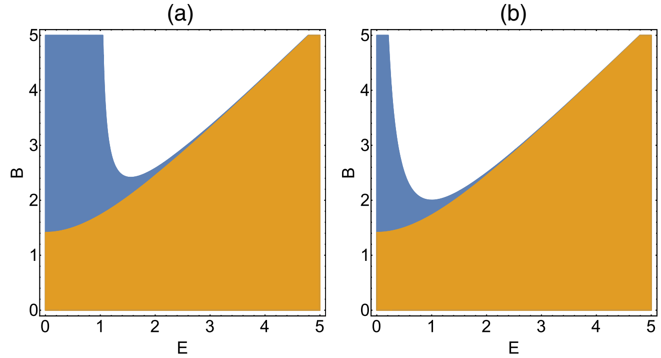

FIG. 3. The values for which the integrand in (73) is positive for $x = \mathcal{R}$ (blue region) compared to the values for which the causality condition (71) holds (orange region) for (a) $\theta = 0$ and (b) $\theta = \pi/2$.

Figure 3 shows that, indeed, there are field configurations that violate the causal condition (71) but satisfy inequality (37). Therefore, the DEC is only a sufficient and not a necessary condition. In fact, one should be very careful to regard inequality (37) as a causal condition.

V. CONCLUSION AND PERSPECTIVES

In the present work, we investigated some physically motivated inequalities relating the charge, energy, and angular momentum in the context of NLED. These inequalities are a direct consequence of the generalized Bekenstein-Mayo inequality (1). We have proven that, similarly to Maxwell's theory, Born-Infeld electrodynamics satisfies all three inequalities, but there is no rigidity statement inasmuch as Born-Infeld total energy is always greater than Maxwell electromagnetic energy.

We have also shown that the inequality between the charge and energy is independent of the quasilocal inequality relating energy and angular momentum by presenting a counterexample in which only one of these inequalities is violated. Furthermore, this result suggest that these inequalities can be used, apart from the obvious Maxwell limit condition for weak fields, as a physically motivated criteria to select between NLED.

The fact that the DEC is a sufficient condition to prove Theorem 2 indicates a possible relationship between the inequality between the energy and angular momentum and causality. Notwithstanding, we have shown that the DEC is only a sufficient and not a necessary condition to this inequality, hence obscuring its physical content. It would be interesting to find a modified inequality that is a necessary and sufficient condition of the DEC.

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APPENDIX: DEC AND THE INEQUALITY BETWEEN ENERGY AND ANGULAR MOMENTUM

In this Appendix, we shall reproduce the argument of Dain [12], showing that the dominant energy condition is a sufficient condition to prove quasilocal inequality (37) between the energy and angular momentum.

Consider an arbitrary energy-momentum tensor $T_{\mu\nu}$ associated with some field theory. Given a timelike congruence v^μ, the three-dimensional hypersurface $\mathcal{V}$ orthogonal to the congruence defines its rest space. The energy associated with the observer's worldline is defined as

$$\mathcal{E} = \int_{\mathcal{V}} T_{\mu\nu} v^\mu v^\nu. \tag{A1}$$

If ω^μ is a Killing vector field associated to space rotations, the angular momentum can be defined as

$$J(\mathcal{V}) = \frac{1}{c}\int_{\mathcal{V}} T_{\mu\nu} v^\mu \omega^\nu. \tag{A2}$$

Since the background is a flat Minkowski spacetime, we can choose Cartesian coordinates in which x^i expand $\mathcal{V}$ and $v^\mu = (1,0,0,0)$. In these coordinates, the rotation vector defined with respect to the direction $\hat{n}$ reads

$$\omega_i = \epsilon_{ijk} n^j x^k. \tag{A3}$$

The norm of the rotation vector reads $\omega \equiv \sqrt{-\omega^\mu \omega_\mu} = \sqrt{\omega_i \omega^i}$, and hence we can define a spacelike unitary vector as $\hat{\omega}^\mu = \omega^\mu/\omega$.

The dominant energy condition implies that for all future-directed timelike ξ^μ or null k^μ vectors the energy-momentum tensor satisfies

$$T_{\mu\nu}\xi^\mu k^\nu \geq 0. \tag{A4}$$

To prove inequality (37), we can choose a timelike vector $\xi^\mu = v^\mu$ and a null vector $k^\mu = v^\mu - \hat{\omega}^\mu$. From Eq. (A4), we have

$$T_{\mu\nu} v^\mu v^\nu \geq T_{\mu\nu} v^\mu \hat{\omega}^\nu. \tag{A5}$$

The radius $\mathcal{R}$ of the minimum sphere $\mathcal{B}_{\mathcal{R}}$ (Definition 1) encloses all region Σ, and hence by definition, we have $\omega \leq \mathcal{R}$. Therefore, we have

$$\begin{aligned} \mathcal{E}(\Sigma) &= \int_\Sigma T_{\mu\nu} v^\mu v^\nu \\ &\geq \int_\Sigma T_{\mu\nu} v^\mu \hat{\omega}^\nu \\ &\geq \frac{1}{\mathcal{R}}\int_\Sigma T_{\mu\nu} v^\mu \omega^\nu = \frac{cJ(\Sigma)}{\mathcal{R}}. \end{aligned} \tag{A6}$$

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