AIP | Journal of Mathematical Physics

ℓ-oscillators from second-order invariant PDEs of the centrally extended conformal Galilei algebras

N. Aizawa, Z. Kuznetsova, and F. Toppan

Citation: Journal of Mathematical Physics **56**, 031701 (2015); doi: 10.1063/1.4908232
View online: http://dx.doi.org/10.1063/1.4908232
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/56/3?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **56**, 031701 (2015)

ℓ-oscillators from second-order invariant PDEs of the centrally extended conformal Galilei algebras

N. Aizawa,[1,a)] Z. Kuznetsova,[2,b)] and F. Toppan[3,c)]
[1]*Department of Mathematics and Information Sciences, Graduate School of Science, Osaka Prefecture University, Nakamozu Campus, Sakai, Osaka 599-8531, Japan*
[2]*UFABC, Av. dos Estados 5001, Bangu, cep 09210-580, Santo André (SP), Brazil*
[3]*CBPF, Rua Dr. Xavier Sigaud 150, Urca, cep 22290-180, Rio de Janeiro (RJ), Brazil*

(Received 7 January 2015; accepted 2 February 2015; published online 5 March 2015)

We construct, for any given $\ell = \frac{1}{2} + \mathbb{N}_0$, the second-order, linear partial differential equations (PDEs) which are invariant under the centrally extended conformal Galilei algebra. At the given ℓ, two invariant equations in one time and $\ell + \frac{1}{2}$ space coordinates are obtained. The first equation possesses a continuum spectrum and generalizes the free Schrödinger equation (recovered for $\ell = \frac{1}{2}$) in 1 + 1 dimension. The second equation (the "ℓ-oscillator") possesses a discrete, positive spectrum. It generalizes the 1 + 1-dimensional harmonic oscillator (recovered for $\ell = \frac{1}{2}$). The spectrum of the ℓ-oscillator, derived from a specific $osp(1|2\ell+1)$ h.w.r., is explicitly presented. The two sets of invariant PDEs are determined by imposing (representation-dependent) *on-shell invariant conditions* both for *degree* 1 operators (those with continuum spectrum) and for *degree* 0 operators (those with discrete spectrum). The on-shell condition is better understood by enlarging the conformal Galilei algebras with the addition of certain second-order differential operators. Two compatible structures (the algebra/superalgebra duality) are defined for the enlarged set of operators. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4908232]

I. INTRODUCTION

In this paper, four results are presented. At first, we enlarge the one-dimensional, centrally extended, Conformal Galilei Algebras (CGAs) of the half-integer, $\ell = \frac{1}{2} + \mathbb{N}_0$, series. The $sl(2)$ subalgebra elements and the central charge have an integer grading with respect to the Cartan element of $sl(2)$. The remaining generators have a half-integer grading. The enlargement consists in adding to the set of generators the anticommutators of the generators with half-integer grading. Two compatible structures are defined on the enlarged set of generators,[1] namely, a (finite, non-semi-simple) Lie algebra of commutators and a $\mathbb{Z}_2$-graded Lie algebra of (anti)commutators. This compatibility is referred to as *algebra/superalgebra duality*.

The enlarged algebra/superalgebra possesses a differential realization induced by the differential realization of the original CGA (the differential realizations of the latter have been computed, e.g., in Refs. 2 and 3). The differential realization suitable to the present work requires one time and $\ell + \frac{1}{2}$ space coordinates. The first algebra of the series, obtained at $\ell = \frac{1}{2}$, is the Schrödinger algebra; since it is realized by just one space coordinate, this whole CGA series is referred to as "one-dimensional."

The second result consists in imposing, for the given differential realization, separate *on-shell invariant conditions* for the operators of grading ± 1 (*degree* ± 1 operators) and grading 0 (*degree*

a)E-mail: aizawa@mi.s.osakafu-u.ac.jp
b)E-mail: zhanna.kuznetsova@ufabc.edu.br
c)E-mail: toppan@cbpf.br

0 operators) of the enlarged algebra. By construction, these operators are second-order differential operators. The operators which solve the *on-shell* condition induce an invariant partial differential equation (PDE) for the centrally extended CGA. At degree 1 (or −1), we recover the invariant PDEs obtained in Ref. 3. The novel feature is the invariant operator at degree 0. The degree 1 invariant equation, in one time and $\ell + \frac{1}{2}$ space coordinates, generalizes the free Schrödinger equation (recovered for $\ell = \frac{1}{2}$) in 1 + 1 dimension. Its associated "static" equation possesses a continuum spectrum. For the degree 0 operator, a transformation (which generalizes the one described in Refs. 4 and 5 for the Schrödinger case) of the space and time coordinates (our third result) allows to recast the associated invariant PDE as a second-order differential equation in the new variables. In the new form, it generalizes the 1 + 1-dimensional harmonic oscillator (recovered for $\ell = \frac{1}{2}$). The associated "static" equation is given by a Hamiltonian which possesses a positive, discrete spectrum of eigenvalues. The last and main result of our paper is the presentation of this invariant "ℓ-oscillator" Hamiltonian and the computation of its spectrum. The latter can be derived from a given highest weight (or lowest weight, depending on the conventions) representation of $osp(1|2\ell + 1)$, which plays the role of the spectrum-generating, *off-shell* symmetry subalgebra of the invariant PDE.

The constructions that we employed are explained at length in the text.

For convenience, we present here the main results. The ℓ-oscillator invariant equation is

$$i\partial_\tau \Psi(\tau, \mathbf{x}_j) = \mathbf{H}^{(\ell)}\Psi(\tau, \mathbf{x}_j) \tag{1}$$

for the Hamiltonian $\mathbf{H}^{(\ell)}$ which, in a canonical form and suitably normalized, is given by

$$\mathbf{H}^{(\ell)} = -\frac{1}{2m}\partial^2_{\mathbf{x}_1} + \frac{m}{2}{\mathbf{x}_1}^2 + \sum_{j=1}^{\ell-\frac{1}{2}} \Big((2j+1)\mathbf{x}_{j+1}\partial_{\mathbf{x}_{j+1}} - (2\ell - 2j + 1)\mathbf{x}_j\partial_{\mathbf{x}_{j+1}}\Big) + \frac{1}{8}(2\ell - 1)(2\ell + 3). \tag{2}$$

τ is the time coordinate (of scaling dimension $[\tau] = -1$), while the $\mathbf{x}_j$'s, $j = 1, \ldots, \ell + \frac{1}{2}$, are (anisotropic) space coordinates of scaling dimension $[\mathbf{x}_j] = -j + \frac{1}{2}$.

The spectrum of the $\mathbf{H}^{(\ell)}$ Hamiltonian, given by the energy eigenvalues $E_{\vec{n}}$, is

$$E_{\vec{n}} = \sum_{j=1}^{\ell+\frac{1}{2}} \omega_j n_j + \omega_0, \quad \text{with} \quad \omega_j = (2j - 1), \quad \omega_0 = \frac{1}{8}(2\ell + 1)^2. \tag{3}$$

In the above formula, the n_j's are non-negative integers, ω_j is the energy of the jth oscillatorial mode, and ω_0 is the vacuum energy. The spectrum coincides with the spectrum of $\ell + \frac{1}{2}$ decoupled oscillators of appropriate frequency.

The Hamiltonian is a second-order differential operator with respect to $\mathbf{x}_1$ and a first-order differential operator with respect to the remaining space coordinates. One should note that the space coordinates are coupled. The Hamiltonian of the harmonic oscillator is recovered not only at $\ell = \frac{1}{2}$ but also from the consistent set of restrictions $\partial_{\mathbf{x}_j}\Psi(\tau, \vec{\mathbf{x}}) = 0$ for $j = 2, \ldots, \ell + \frac{1}{2}$.

It should be noted that $\mathbf{H}^{(\ell)}$, despite having a real, positive spectrum, is not, for $\ell \geq \frac{3}{2}$, a Hermitian operator (a large class of non-Hermitian operators with real spectrum, PT-symmetric operators,[6,7] pseudo-Hermitian operators,[8] are currently very actively investigated).

The challenging problem of finding dynamical systems and invariant PDEs for the conformal Galilei algebras is much studied. In a series of papers,[9–13] it was shown that the Pais-Uhlenbeck oscillators are invariant under the $\ell = \frac{1}{2} + \mathbb{N}_0$ CGAs. These systems, however, unlike ℓ-oscillator (1), are defined by higher-derivatives equations. The ℓ-oscillator induces a second-order differential equation.

The scheme of the paper is as follows. In Sec. II, we introduce the one-dimensional centrally extended CGAs and present the $\ell = \frac{3}{2}$ differential realization. In Sec. III, we introduce the enlarged algebras and discuss the algebra/superalgebra duality. In Sec. IV, we introduce the on-shell condition to derive the invariant PDEs. For $\ell = \frac{3}{2}$, we derive the explicit invariant operators at degree 1 and 0. The transformation relating their suitable differential realizations, applied to the $\ell = \frac{3}{2}$ case, is presented in Sec. V. In Sec. VI, we explicitly discuss the spectrum-generating subalgebra for the $\ell = \frac{3}{2}$-invariant oscillator. Section VII summarizes the previous constructions for the most general, $\ell = \frac{1}{2} + \mathbb{N}_0$, case. For convenience, a table with the first ℓ-oscillator Hamiltonians up to $\ell = \frac{9}{2}$ is given. In the Conclusions, we present some open questions and the lines of future investigations.

II. THE CENTRALLY EXTENDED, $\ell \in \frac{1}{2} + \mathbb{N}_0$, CGAS

We introduce here the CGAs in $d = 1$ space dimensions.[2] In the following we denote them as "$\mathfrak{cga}_\ell$." They are labelled by a parameter ℓ which is either a non-negative integer or a positive half-integer number.

The non-centrally extended CGAs consist of an $sl(2)$ subalgebra with generators $z_{\pm1}, z_0$ (z_0 being the Cartan generator), acting on an abelian subalgebra ($2\ell + 1$ generators w_j, $j = -\ell, -\ell + 1, \ldots, \ell$, which span a spin-$\ell$ representation of $sl(2)$). The generator z_0 induces a grading, so that $[z_{\pm1}] = \pm1, [z_0] = 0, [w_j] = j$. For ℓ half-integer ($\ell \in \frac{1}{2} + \mathbb{N}_0$), the corresponding CGA admits a central extension[14] (the centrally extended algebra will be denoted as "$\widehat{\mathfrak{cga}}_\ell$"). In the presence of the central extension c, the abelian subalgebra is replaced by $\ell + \frac{1}{2}$ pairs of Heisenberg subalgebras given by $w_{\pm j}$. The $\ell = \frac{1}{2}$ case corresponds to the one-dimensional Schrödinger algebra, spanned by 6 generators, including the central charge c. For reasons discussed in Sec. III, we focus here on the centrally extended CGAs; therefore, ℓ is a half-integer number.

The $\widehat{\mathfrak{cga}}_\ell$ algebra contains two generators of grading 0 (z_0 and c), $\ell + \frac{3}{2}$ generators in the positive sector (z_{+1} and w_j, with $j > 0$) and an equal number of generators in the negative sector (z_{-1} and w_{-j}, with $j > 0$).

With standard techniques (see, e.g., Refs. 15–17), one can construct a D-module rep realized by first-order differential operators induced by a coset construction. The differential operators depend on the space and time coordinates t, x_j, which are dual to the positive generators. The time t is dual to z_{+1} and has dimension $[t] = -1$, while the x_j's are dual to the w_j's. Therefore, $[x_j] = -j$. One should note the anisotropy of the space coordinates, as well as the fact that, for $\ell > \frac{1}{2}$ and in presence of the central extension, more than two coordinates are required to realize the conformal Galilei algebra $\widehat{\mathfrak{cga}}_\ell$.

The $\ell = \frac{1}{2}$ Schrödinger algebra has been directly constructed as the symmetry algebra of the free Schrödinger equation in 1 + 1-dimensions (see, e.g., Ref. 18); its explicit presentation can be found in that work and several other papers in the literature. The next interesting and much less studied case corresponds to $\ell = \frac{3}{2}$. The algebra $\widehat{\mathfrak{cga}}_{\ell=\frac{3}{2}}$ is spanned by the 8 generators $z_0, z_{\pm1}, w_{\pm\frac{1}{2}}, w_{\pm\frac{3}{2}}, c$. Explicitly, its non-vanishing commutators are given by

$$
\begin{aligned}
[z_{+1}, z_{-1}] &= 2z_0,\\
[z_0, z_{\pm1}] &= \pm z_{\pm1},\\
[z_0, w_{\pm\frac{3}{2}}] &= \pm\frac{3}{2} w_{\pm\frac{3}{2}},\\
[z_0, w_{\pm\frac{1}{2}}] &= \pm\frac{1}{2} w_{\pm\frac{1}{2}},\\
[z_{\pm1}, w_{\pm\frac{1}{2}}] &= w_{\pm\frac{3}{2}},\\
[z_{\pm1}, w_{\mp\frac{1}{2}}] &= 2w_{\pm\frac{1}{2}},\\
[z_{\pm1}, w_{\mp\frac{3}{2}}] &= 3w_{\mp\frac{1}{2}},\\
[w_{+\frac{1}{2}}, w_{-\frac{1}{2}}] &= c,\\
[w_{+\frac{3}{2}}, w_{-\frac{3}{2}}] &= -3c.
\end{aligned}
\tag{4}
$$

Its D-module representation, constructed, as explained above, in Ref. 3, is realized by differential operators depending on $t, x_{\frac{1}{2}}, x_{\frac{3}{2}}$. For simplicity, we will denote $x_{\frac{1}{2}} \equiv x$ and $x_{\frac{3}{2}} \equiv y$. Their respective scaling dimensions are $[t] = -1$, $[x] = -\frac{1}{2}, [y] = -\frac{3}{2}$.

We overline the generators in the D-module rep. They are given by

$$\begin{aligned}
\overline{z}_{+1} &= \partial_t,\\
\overline{z}_0 &= -t\partial_t - \frac{1}{2}x\partial_x - \frac{3}{2}y\partial_y - 1,\\
\overline{z}_{-1} &= -t^2\partial_t - tx\partial_x - 3ty\partial_y - 3y\partial_x - cx^2 - 2t,\\
\overline{w}_{+\frac{3}{2}} &= \partial_y,\\
\overline{w}_{+\frac{1}{2}} &= t\partial_y + \partial_x,\\
\overline{w}_{-\frac{1}{2}} &= t^2\partial_y + 2t\partial_x + cx,\\
\overline{w}_{-\frac{3}{2}} &= t^3\partial_y + 3t^2\partial_x + 3tcx - 3cy,\\
\overline{c} &= c.
\end{aligned} \tag{5}$$

This construction naturally extends to the $\widehat{\mathfrak{cga}}_\ell$ algebras for $\ell > \frac{3}{2}$.

III. ENLARGED ALGEBRAS: ALGEBRA/SUPERALGEBRA DUALITY

For the conformal Galilei algebras in the half-integer series ($\ell = \frac{1}{2} + \mathbb{N}_0$), z_0 induces an integer grading on the generators $z_{\pm1}, z_0, c$ and a half-integer grading on the w_j's generators ($\forall g \in \widehat{\mathfrak{cga}}_\ell$, $[z_0, g] = n_g g$). It is therefore quite natural to interpret the grading induced by z_0 as discriminating the even sector $\mathcal{G}_0$ (generators g s.t. $n_g \in \mathbb{Z}$) from the odd sector $\mathcal{G}_1$ (generators g s.t. $n_g \in \frac{1}{2} + \mathbb{Z}$) of an associated superalgebra.

By recalling that the w_j's generators induce $\ell + \frac{1}{2}$ pairs of Heisenberg subalgebras, this possibility is made concrete by the existence (see Ref. 19) of an oscillatorial representation of the $B(0,n) = osp(1|2n)$ superalgebra in terms of n bosonic generators. Here, $n = \ell + \frac{1}{2}$.

For consistency, a closed superalgebra structure requires that the anticommutators of the odd generators belong to the even sector $\mathcal{G}_0$. Therefore, we need to add to $\mathcal{G}_0$ the $(2\ell+1)(\ell+1)$ generators $w_{i,j} = \{w_i, w_j\}$ of grading $i + j$ (we have $w_{i,j} = w_{j,i}$).

The enlarged superalgebra spanned by $z_0, z_{\pm1}, c, w_{i,j} \in \mathcal{G}_0$ and $w_j \in \mathcal{G}_1$ is a non-semisimple, finite, closed, Lie superalgebra endowed with graded commutators and satisfying the graded Jacobi identities. It will be denoted as "$\mathfrak{s}\widehat{\mathfrak{cga}}_\ell$." It contains $2\ell^2 + 3\ell + 5$ even generators and $2\ell + 1$ odd generators. The generators $w_{i,j}$ close the $sp(2\ell+1)$ bosonic subalgebra, while the set of $w_{i,j}, w_j$ generators close the $osp(1|2\ell+1)$ subalgebra. We can therefore write

$$\mathfrak{s}\widehat{\mathfrak{cga}}_\ell = u(1) \oplus sl(2) \oplus_S osp(1|2\ell+1), \tag{6}$$

where "$\oplus_S$" denotes the semidirect sum.
In the superalgebra framework, the central charge c entering $\widehat{\mathfrak{cga}}_\ell$ is an extra $u(1)$ generator. It is worth pointing out that the construction of the superalgebra $\mathfrak{s}\widehat{\mathfrak{cga}}_\ell$ is based on a different viewpoint and requires a different procedure from the construction of the supersymmetric extensions of the CGAs which have been discussed in Refs. 20–30. Indeed, these supersymmetric extensions can be identified as symmetries of supersymmetric models, while $\mathfrak{s}\widehat{\mathfrak{cga}}_\ell$ is the symmetry superalgebra of a purely bosonic system.

The D-module representation of $\widehat{\mathfrak{cga}}_\ell$ in terms of first-order differential operators (as given in (5) for $\ell = \frac{3}{2}$) is naturally extended to a realization of $\mathfrak{s}\widehat{\mathfrak{cga}}_\ell$ in terms of differential operators, with the operators $\overline{w}_{i,j} = \{\overline{w}_i, \overline{w}_j\}$ being of second order.

Once constructed the anticommutators $w_{i,j}$ from the set of w_j's generators in $\widehat{\mathfrak{cga}}_\ell$, we can pose the question whether the $\widehat{\mathfrak{cga}}_\ell$ algebra, enlarged with the addition of the $w_{i,j}$ generators, closes as a non-semisimple finite Lie algebra (the brackets being defined by the ordinary commutators). The answer is positive. We denote as "$\mathfrak{e}\widehat{\mathfrak{cga}}_\ell$" the bosonic Lie algebra spanned by $z_{\pm1}, z_0, c, w_j, w_{i,j}$. The algebra $\mathfrak{e}\widehat{\mathfrak{cga}}_\ell$ contains $2\ell^2 + 5\ell + 6$ generators. They are all even. We have, explicitly,

$$\mathfrak{e}\widehat{\mathfrak{cga}}_\ell = \widehat{\mathfrak{cga}}_\ell \oplus_S sp(2\ell+1). \tag{7}$$

Both $\widehat{\mathfrak{scga}}_\ell$ and $\widehat{\mathfrak{ecga}}_\ell$ are obtained from $\widehat{\mathfrak{cga}}_\ell$ by adding an extra set of generators, expressed as anticommutators. In terms of the differential realizations given by (5) and its $\ell > \frac{3}{2}$ counterparts, the second order differential operators $\overline{w}_{i,j}$ entering both $\widehat{\mathfrak{scga}}_\ell$ and $\widehat{\mathfrak{ecga}}_\ell$ are identical. Stated otherwise, on the same set of first and second-order differential operators, two mutually compatible structures can be defined. The first structure is the $\mathbb{Z}_2$-graded superalgebra $\widehat{\mathfrak{scga}}_\ell$. The second structure is the ordinary Lie algebra $\widehat{\mathfrak{ecga}}_\ell$. We refer to this property of conformal Galilei algebras with half-integer ℓ as the algebra/superalgebra duality. It can be schematically expressed by the correspondence

$$\widehat{\mathfrak{scga}}_\ell \Leftrightarrow \widehat{\mathfrak{ecga}}_\ell. \tag{8}$$

The symbol "$\Leftrightarrow$" denotes this duality relation.

IV. INVARIANT PDES FROM THE ON-SHELL CONDITION

We address here the problem of constructing PDEs admitting the centrally extended conformal Galilei algebras as their symmetry algebras. One should note that this is an inverse problem with respect to the derivation of the Schrödinger algebra ($\widehat{\mathfrak{cga}}_\ell$, for $\ell = \frac{1}{2}$) from the (free) Schrödinger equation. In that case the invariant equation is assumed and its symmetry algebra, induced by first-order differential operators, is derived with standard techniques (see Refs. 31 and 32). For $\ell \geq \frac{3}{2}$, a reverse problem has to be solved. The first-order differential operators generating $\mathfrak{cga}_\ell$ are known. We have instead to determine the invariant PDEs associated with the given D-module reps.

One possibility is offered by the method, based on Lie symmetries, which produces non-linear invariant PDEs, see Refs. 33 and 34. Another possibility, leading to linear invariant PDEs, is based on the construction of singular vectors of the given representation[17] (see Ref. 3 for the case of $\widehat{\mathfrak{cga}}_\ell$). We discuss in this section, a different approach to determine linear invariant PDEs, based on imposing an on-shell invariant condition (see Refs. 18 and 35). Applied to the $\widehat{\mathfrak{cga}}_\ell$ algebras, this approach naturally leads to second-order linear PDEs (invariant PDEs with higher derivatives can, in principle, also be constructed).

Let us consider, for a given ℓ, the enlarged algebras introduced in Sec. III (either the superalgebra $\widehat{\mathfrak{scga}}_\ell$ or its dual bosonic counterpart $\widehat{\mathfrak{ecga}}_\ell$). We can consider the most general generators Ω_r of grading r (also called the *degree* and defined by the commutator $[z_0, \Omega_r] = r\Omega_r$), with $r = 0, \pm 1$. The Ω_r's are even generators entering both $\widehat{\mathfrak{scga}}_\ell$ and $\widehat{\mathfrak{ecga}}_\ell$. For $\ell = \frac{3}{2}$, we have, e.g.,

$$\begin{aligned} \Omega_{\pm1} &= a_1 z_{\pm1} + a_2 w_{\frac{\pm1}{2},\frac{\pm1}{2}} + a_3 w_{\frac{\pm3}{2},\frac{\mp1}{2}}, \qquad a_{1,2,3} \in \mathbb{C}, \\ \Omega_0 &= b_0 c + b_1 z_0 + b_2 w_{\frac{1}{2},\frac{-1}{2}} + b_3 w_{\frac{3}{2},\frac{-3}{2}}, \qquad b_{0,1,2,3} \in \mathbb{C}. \end{aligned} \tag{9}$$

By construction, $\forall g \in \widehat{\mathfrak{scga}}_\ell \Leftrightarrow \widehat{\mathfrak{ecga}}_\ell$, the commutators

$$[g, \Omega_r] = \omega_r^g \tag{10}$$

close on the elements $\omega_r^g \in \widehat{\mathfrak{scga}}_\ell \Leftrightarrow \widehat{\mathfrak{ecga}}_\ell$.

In a given differential realization of $\widehat{\mathfrak{scga}}_\ell \Leftrightarrow \widehat{\mathfrak{ecga}}_\ell$, the generators Ω_r are expressed as $\overline{\Omega}_r$, while the r.h.s. elements ω_r^g are expressed as $\overline{\omega}_r^g$.

We have now all the ingredients to define a cohomological problem. To determine whether it is possible to choose $\overline{\Omega}_r$ in the r-graded sector of $\widehat{\mathfrak{scga}}_\ell \Leftrightarrow \widehat{\mathfrak{ecga}}_\ell$ such that, $\forall g$, the differential operators $\overline{\omega}_r^g$ are expressed as

$$\overline{\omega}_r^g = f^g \overline{\Omega}_r, \tag{11}$$

where, for a given g, f^g is a specific function of the space and time coordinates (for $\ell = \frac{3}{2}$, $f^g \equiv f^g(t, x, y)$) and f^g can also be vanishing.

As shown below, this cohomological problem admits non-trivial solutions in the presence of a non-vanishing central extension c.

The requirement that Eq. (11) should be satisfied for any g will be called the *on-shell condition* for an invariant PDE. Indeed, if (11) is satisfied for a given r, we obtain, applied to the solutions

$\Psi(t,\vec{x})$ of the partial differential equation

$$\overline{\Omega}_r \Psi(t,\vec{x}) = 0, \tag{12}$$

the symmetries of the equation of motion

$$[\overline{g}, \overline{\Omega}_r]\Psi(t,\vec{x}) = 0. \tag{13}$$

We can express this property through the position

$$[\overline{g}, \overline{\Omega}_r] = f^g \overline{\Omega}_r \Rightarrow [\overline{g}, \overline{\Omega}_r] \approx 0. \tag{14}$$

The existence of the on-shell symmetry implies that explicit solutions (eigenfunctions and eigenvalues) of the invariant PDE can be constructed in terms of the associated spectrum generating subalgebras (either belonging to $\mathfrak{e}\widehat{\mathfrak{cga}}_\ell$ or to $\mathfrak{s}\widehat{\mathfrak{cga}}_\ell$).

Let us present now the construction, from the on-shell condition, of the invariant PDE at $r = 1$ induced by differential realization (5) of the $\ell = \frac{3}{2}$ case.

For $\ell = 3/2$, we have

$$\mathfrak{s}\widehat{\mathfrak{cga}}_{\frac{3}{2}} = u(1) \oplus sl(2) \oplus_S osp(1|4), \quad \mathfrak{e}\widehat{\mathfrak{cga}}_\ell = \widehat{\mathfrak{cga}}_{\frac{3}{2}} \oplus_S sp(4). \tag{15}$$

A non-trivial solution of (11) on-shell condition is guaranteed if we take the following linear combinations of generators:

$$\begin{aligned} \Omega_0 &= z_0 + \frac{1}{4c} w_{\frac{1}{2},\frac{-1}{2}} - \frac{1}{4c} w_{\frac{3}{2},\frac{-3}{2}}, \\ \Omega_1 &= z_{+1} + \frac{1}{2c} w_{\frac{3}{2},\frac{-1}{2}} - \frac{1}{2c} w_{\frac{1}{2},\frac{1}{2}}. \end{aligned} \tag{16}$$

D-module rep (5) of $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ induces the second order differential operators $\overline{\Omega}_0$ and $\overline{\Omega}_1$, explicitly given by

$$\overline{\Omega}_0 = -t\overline{\Omega}_1, \quad \overline{\Omega}_1 = \partial_t + x\partial_y - \frac{1}{c}{\partial_x}^2. \tag{17}$$

In the given differential realization, $\overline{\Omega}_1$ satisfies on-shell condition (11). Indeed, $\forall g \in \mathfrak{s}\widehat{\mathfrak{cga}}_{\frac{3}{2}} \Leftrightarrow \mathfrak{e}\widehat{\mathfrak{cga}}_{\frac{3}{2}}$, its only non-vanishing commutators are given by

$$\begin{aligned} [\overline{z}_{-1}, \overline{\Omega}_1] &= -2\overline{\Omega}_0 \ = 2t\overline{\Omega}_1, \\ [\overline{z}_0, \overline{\Omega}_1] \ &= \overline{\Omega}_1. \end{aligned} \tag{18}$$

Therefore, $f^{z_{-1}} = 2t$, $f^{z_0} = 1$, $f^g = 0$ otherwise.

For what concerns the $\overline{\Omega}_0$ operator, its commutators are vanishing when applied to the solutions of the

$$\overline{\Omega}_1 \Psi(t,x,y) = 0 \equiv \Psi_t(t,x,y) + x\Psi_y(t,x,y) - \frac{1}{c}\Psi_{xx}(t,x,y) = 0 \tag{19}$$

invariant PDE.

We have, indeed, that its only non-vanishing commutators in the basis expressed by $\overline{z}_{\pm1}, \overline{z}_0, \overline{w}_j, \overline{w}_{i,j}, \overline{c}$ are given by

$$\begin{aligned} [\overline{z}_{+1}, \overline{\Omega}_0] &= -\overline{\Omega}_1 \quad\ = t^{-1}\overline{\Omega}_0, \\ [\overline{z}_{-1}, \overline{\Omega}_0] &= -t^2\overline{\Omega}_1 \ = t\overline{\Omega}_0. \end{aligned} \tag{20}$$

We have, furthermore,

$$[\overline{\Omega}_1, \overline{\Omega}_0] = -\overline{\Omega}_1. \tag{21}$$

It follows, from the last equations in the r.h.s. of (20), that $\overline{\Omega}_0$ satisfies the on-shell condition for a singular choice of space-time functions, namely, $f^{z_{+1}} = t^{-1}$, $f^{z_{-1}} = t$, and $f^g = 0$ otherwise.

Equation (19) invariant PDE is the counterpart, at $\ell = \frac{3}{2}$, of the free Schrödinger equation in $1 + 1$ dimensions. The centralizer algebra for the operator $\overline{\Omega}_1$ (induced by the operators $\overline{g}$ which strictly satisfy $[\overline{g}, \overline{\Omega}_1] = 0$), will be called the *off-shell* invariant algebra for (19) PDE. This subalgebra is obtained by disregarding the $sl(2)$ Borel generators $\overline{z}_0, \overline{z}_{-1}$. In the superalgebra framework, the off-shell invariant algebra can be presented as

$$u(1) \oplus u(1) \oplus_S osp(1|4) \subset \widehat{\mathfrak{scga}}_{\frac{3}{2}}, \tag{22}$$

with the $u(1)$ generator acting on $osp(1|4)$ given by $\overline{z}_{+1}$.

PDE (19), derived from the on-shell condition, is identical to the lowest member of the hierarchy, whose construction is based on singular vectors, given in Ref. 3. The on-shell condition allowed us to identify here another invariant operator ($\overline{\Omega}_0$, at degree 0), whose importance is discussed in Sec. V.

V. THE INVARIANT PDE OF DEGREE 0

Besides the free case, in 1 + 1 dimensions, the Schrödinger algebra is derived as the symmetry algebra of the Schrödinger equation for two other choices of the potential, the linear potential, and the quadratic potential of the harmonic oscillator.[4,36,37] The difference between the free case and the oscillator case lies in the fact that the time-derivative operator is, in the first case, associated with a positive root of the $sl(2)$ subalgebra, as in formula (5). In the second case, it is associated with the Cartan generator (for the linear potential, the time-derivative operator is a symmetry generator which does not coincide with a generator of the $sl(2)$ subalgebra[1,37]). In the framework of the on-shell condition, the free Schrödinger equation is the invariant PDE at degree 1, while the equation of the harmonic oscillator is the invariant PDE at degree 0.[1] The appearance of a discrete spectrum for the harmonic oscillator in contrast to the continuous spectrum of the free particle can be traced to these differences.

At $\ell = \frac{1}{2}$, the D-module rep associated with the harmonic oscillator can be recovered from the original D-module rep via a transformation (this point has been discussed in Refs. 4 and 5). This transformation can be extended to other values of ℓ entering $\widehat{\mathfrak{gca}}_\ell$. We present it for $\ell = \frac{3}{2}$. Essentially, the transformation requires presenting z_0 as the time-derivative operator with respect to a new time variable (in a related context, see Ref. 38).

We recall that we denoted as $\overline{g}$, a $g \in \widehat{\mathfrak{gca}}_{\frac{3}{2}}$ generator in D-module rep (5). We denote as "$\widetilde{g}$" the generator in the new D-module rep. We can write $\overline{g} \in \overline{V}$, $\widetilde{g} \in \widetilde{V}$, where $\overline{V}, \widetilde{V}$ are the corresponding D-module reps. The transformation τ mapping

$$\tau : \overline{V} \to \widetilde{V}, \quad \tau : \overline{g} \mapsto \widetilde{g}, \tag{23}$$

can be realized in three steps:

(i) at first, any given generator $\overline{g}$ in (5) is dressed by the similarity transformation $\overline{g} \mapsto \check{g} = t\overline{g}t^{-1}$;

(ii) next, the $\check{g}$ generators are re-expressed as differential operators in the new variables s, u, v, related to the previous variables t, x, y through the positions

$$\begin{aligned} t &= e^s, \\ x &= e^{\frac{s}{2}} u, \\ y &= e^{\frac{3s}{2}} v, \end{aligned} \tag{24}$$

(s plays now the role of the new "time" variable);

(iii) finally, the $\check{g}$ generators are dressed by a similarity transformation which preserves $\check{z}_0$ as the time-derivative operator. There is an arbitrariness in the choice of the similarity transformation. In this and Sec. VII, it is convenient to work with the choice $\check{g} \mapsto \widetilde{g} = e^{\frac{cu^2}{2}} \check{g} e^{-\frac{cu^2}{2}}$. A different similarity transformation, leading to the "canonical" form of the ℓ-oscillator Hamiltonian given by (2), is introduced in Sec. VII.

The result of the three combined operations produces a D-module rep of $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$, given by the first-order differential operators in s, u, v,

$$\begin{aligned}
\widetilde{z}_{+1} &= e^{-s}(\partial_s - \frac{1}{2}u\partial_u - \frac{3}{2}v\partial_v - 1 + \frac{1}{2}cu^2),\\
\widetilde{z}_0 &= -\partial_s,\\
\widetilde{z}_{-1} &= e^{s}(-\partial_s - \frac{1}{2}u\partial_u - \frac{3}{2}v\partial_v - 3v\partial_u - \frac{1}{2}cu^2 - 1 + 3cuv),\\
\widetilde{w}_{+\frac{3}{2}} &= e^{-\frac{3s}{2}}\partial_v,\\
\widetilde{w}_{+\frac{1}{2}} &= e^{-\frac{s}{2}}(\partial_v + \partial_u - cu),\\
\widetilde{w}_{-\frac{1}{2}} &= e^{\frac{s}{2}}(\partial_v + 2\partial_u - cu),\\
\widetilde{w}_{-\frac{3}{2}} &= e^{\frac{3s}{2}}(\partial_v + 3\partial_u - 3cv).\\
\widetilde{c} &= c.
\end{aligned} \tag{25}$$

In this differential realization, the generators Ω_0, Ω_1 introduced in (16) are expressed as

$$\begin{aligned}
\widetilde{\Omega}_0 &= -\partial_s - u\partial_v - \frac{3}{2}u\partial_u + \frac{3}{2}v\partial_v + \frac{1}{c}{\partial_u}^2 + \frac{1}{2}cu^2,\\
\widetilde{\Omega}_1 &= -e^{-s}\widehat{\Omega}_0.
\end{aligned} \tag{26}$$

Both $\widetilde{\Omega}_0$ (at degree $r = 0$) and $\widetilde{\Omega}_1$ (at degree $r = 1$) satisfy on-shell condition (11). Indeed, their respective non-vanishing commutators with the operators in $\widetilde{V}$ are given by

$$\begin{aligned}
[\widetilde{z}_{+1}, \widetilde{\Omega}_0] &= e^{-s}\widetilde{\Omega}_0,\\
[\widetilde{z}_{-1}, \widetilde{\Omega}_0] &= e^{s}\widetilde{\Omega}_0,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
[\widetilde{z}_0, \widetilde{\Omega}_1] &= \widetilde{\Omega}_1,\\
[\widetilde{z}_{-1}, \widetilde{\Omega}_1] &= 2e^{s}\widetilde{\Omega}_1.
\end{aligned} \tag{28}$$

We have, furthermore, the relation

$$[\widetilde{\Omega}_0, \widetilde{\Omega}_1] = \widetilde{\Omega}_1. \tag{29}$$

One should note that the degree 0 on-shell invariant operator $\widetilde{\Omega}_0$, in this D-module rep does not present an explicit dependence on the time coordinate s. Its associated invariant PDE is given by

$$\widetilde{\Omega}_0\Psi(s,u,v) = 0 \equiv -\Psi_s - u\Psi_v - \frac{3}{2}u\Psi_u + \frac{3}{2}v\Psi_v + \frac{1}{c}\Psi_{uu} + \frac{1}{2}cu^2\Psi = 0. \tag{30}$$

It is a second-order partial differential equation, containing a term proportional to u^2, which implements the $\ell = \frac{3}{2}$ counterpart of the harmonic oscillator in $1 + 1$ dimensions. Its off-shell invariant algebra is obtained by disregarding the root generators $\widetilde{z}_{\pm 1}$. In the superalgebra framework, it can be expressed as a subalgebra

$$u(1) \oplus u(1) \oplus_S osp(1|4) \subset \widehat{\mathfrak{scga}}_{\frac{3}{2}}. \tag{31}$$

It is a different $\widehat{\mathfrak{scga}}_{\frac{3}{2}}$ subalgebra with respect to (22). In that case the $u(1)$ generator acting on $osp(1|4)$ is the root generator z_{+1}, while here it is the Cartan generator z_0.

VI. EIGENFUNCTIONS AND EIGENVALUES FROM THE SPECTRUM GENERATING SUBALGEBRAS

Both the $\ell = \frac{3}{2}$ degree 1 (19) and degree 0 (30) invariant PDEs admit the $osp(1|4)$ superalgebra as a subalgebra of their respective off-shell symmetry algebras.

We focus here in the degree 0 case. As already recalled, the superalgebras of the $osp(1|2n)$ series present a realization in terms of n bosonic oscillators. In our case, we implemented concretely this realization with $w_{\pm\frac{1}{2}}$, $w_{\pm\frac{3}{2}}$ expressed as first-order differential operators, see (25). The superalgebra $osp(1|4)$ can be used as the spectrum generating algebra to construct eigenfunctions and eigenvalues of (30) PDE. Based on the algebra/superalgebra duality discussed in Sec. III and in analogy with the construction for the ordinary harmonic oscillator in $(1+1)$ dimensions, see Ref. 1, two compatible viewpoints can be adopted here. In the bosonic viewpoint, the construction of eigenfunctions and eigenvalues is derived from the Fock space of two bosonic oscillators. In the superalgebra viewpoint, the same result is recovered from a highest weight representation of the $osp(1|4)$ superalgebra.

The operator $\widetilde{\Omega}_0$ given in (26) commutes with $\widetilde{z}_0$. If we set

$$\begin{aligned}\widetilde{\Omega}_0 &= \widetilde{z}_0 + \widetilde{H},\\ \widetilde{H} &= \frac{1}{2c}(\widetilde{w}_{-\frac{1}{2}}\widetilde{w}_{+\frac{1}{2}} - \widetilde{w}_{-\frac{3}{2}}\widetilde{w}_{+\frac{3}{2}}) + 1 = -u\partial_v - \frac{3}{2}u\partial_u + \frac{3}{2}v\partial_v + \frac{1}{c}{\partial_u}^2 + \frac{1}{2}cu^2,\end{aligned} \tag{32}$$

we have that $[\widetilde{z}_0, \widetilde{H}] = 0$. Therefore, the equation $\widetilde{\Omega}_0\Psi(s,u,v) = 0$ is solved by the common eigenfunctions $\Psi_E(s,u,v)$ such that

$$\widetilde{z}_0\Psi_E(s,u,v) = -E\Psi_E(s,u,v) \quad , \quad \widetilde{H}\Psi_E(s,u,v) = E\Psi_E(s,u,v). \tag{33}$$

$\widetilde{H}$ plays the role of an effective Hamiltonian and E is the energy level. Eigenstates and eigenvalues are obtained from a highest weight representation by applying the creation operators $\widetilde{w}_{\frac{-3}{2}}, \widetilde{w}_{\frac{-1}{2}}$ on the vacuum solution $\psi_{vac}(s,u,v)$, defined by the conditions

$$\widetilde{w}_{\frac{1}{2}}\psi_{vac}(u,v) = \widetilde{w}_{\frac{3}{2}}\psi_{vac}(u,v) = 0 \tag{34}$$

and

$$-\widetilde{z}_0\psi_{vac}(s,u,v) = \widetilde{H}\psi_{vac}(s,u,v) = E_{vac}\psi_{vac}(s,u,v). \tag{35}$$

Due to (25), the unnormalized solution of (34) is

$$\psi_{vac} = \chi(s)e^{\frac{1}{2}cu^2} \tag{36}$$

for an arbitrary function $\chi(s)$. The vacuum energy is given by $E_{vac} = 1$.

The first equation in (35) constraints $\chi(s)$ to be $\chi(s) \propto e^s$.

One can easily verify that the solutions of (33) can be expressed as $\psi_E(s,u,v) \propto e^{Es}\varphi_E(u,v)$. The ground state function $\varphi_{vac}(u,v) = e^{\frac{1}{2}cu^2}$ turns out to be independent of v and normalizable in $(-\infty,\infty)$ if the central charge is restricted to $c < 0$.

The higher energy eigenstates

$$\psi_{m,n}(s,u,v) = e^{sE_{m,n}}\varphi_{m,n}(u,v) \tag{37}$$

of (33) are given by

$$\psi_{m,n} = (\widetilde{w}_{-\frac{3}{2}})^m(\widetilde{w}_{-\frac{1}{2}})^n\psi_{vac} \tag{38}$$

(we can therefore also set $\psi_{vac} \equiv \psi_{0,0}$, as well as $\varphi_{vac} \equiv \varphi_{0,0}$).

Due to (34) and to (4) commutators, the associated $E_{m,n}$ eigenvalues are

$$E_{m,n} = \frac{3}{2}m + \frac{1}{2}n + 1. \tag{39}$$

One should note that they do not depend on the value of the central charge. Furthermore, the eigenvalues $E_{m,n}$ are degenerate for $E_{m,n} \geq \frac{5}{2}$.

Since $\widetilde{H}$ does not depend on s, the two-variable functions $\varphi_{m,n}(u,v)$ are solutions of the "static" equation

$$\widetilde{H}\varphi_{m,n}(u,v) = E_{m,n}\varphi_{m,n}(u,v). \tag{40}$$

Oscillatorial solutions are recovered if s is assumed to be imaginary. In terms of the new time variable τ, we have

$$\psi_{m,n}(\tau,u,v) = e^{-i\tau E_{m,n}}\varphi_{m,n}(u,v), \qquad (s = -i\tau). \tag{41}$$

The first few (unnormalized) eigenstates corresponding to the lowest energy eigenvalues (for $E_{m,n} \leq 3$) are given by

$$
\begin{aligned}
E = 1 :&\ \psi_{(0,0)} = e^{s} e^{\frac{1}{2}cu^2},\\
E = \frac{3}{2} :&\ \psi_{(0,1)} = e^{\frac{3}{2}s}(cue^{\frac{1}{2}cu^2}),\\
E = 2 :&\ \psi_{(0,2)} = e^{2s}(c(2+cu^2)e^{\frac{1}{2}cu^2}),\\
E = \frac{5}{2} :&\ \psi_{(1,0)} = e^{\frac{5}{2}s}(3c(u-v)e^{\frac{1}{2}cu^2}),\\
E = \frac{5}{2} :&\ \psi_{(0,3)} = e^{\frac{5}{2}s}(c^2u(6+cu^2)e^{\frac{1}{2}cu^2}),\\
E = 3 :&\ \psi_{(1,1)} = e^{3s}(3c(1+cu(u-v))e^{\frac{1}{2}cu^2}),\\
E = 3 :&\ \psi_{(0,4)} = e^{3s}(c^2(12+12cu^2+c^2u^4)e^{\frac{1}{2}cu^2}).
\end{aligned}
\tag{42}
$$

By construction, the two-variable functions $\varphi_{m,n}(u,v)$ which solve static Eq. (40) are expressed as products of polynomials in u, v which multiply the ground state function $\varphi_{0,0} = e^{\frac{1}{2}cu^2}$.

VII. THE ℓ-OSCILLATOR FOR ANY $\ell = \frac{1}{2} + \mathbb{N}_0$

Differential realizations of $\widehat{\mathfrak{cga}}_\ell$ for any half-integer ℓ have been computed in Ref. 3. In that paper, the second-order differential operators which, in our language, are on-shell invariant operators of degree 1, were presented. These operators possess a continuum spectrum.

On the other hand, as shown in Secs. V–VI, a discrete spectrum can be obtained if we impose the on-shell condition for degree 0 operators. We discussed at length the derivation of (39) discrete spectrum for $\ell = \frac{3}{2}$.

We present here the general case of second-order invariant operators with discrete spectrum (the ℓ-oscillators) for any $\ell = \frac{1}{2} + \mathbb{N}_0$. Since the derivation is a straightforward generalization of the $\ell = \frac{3}{2}$ case, we can keep the discussion short.

The differential realizations of $\widehat{\mathfrak{cga}}_\ell$ given in Ref. 3 require the differential operators to depend on t and the $\ell + \frac{1}{2}$ variables $x_{\frac{1}{2}}, x_{\frac{3}{2}}, \ldots, x_\ell$. The scaling dimension of the variables is $[t] = -1$, $[x_j] = -j$. A convenient change of notation, $x_j \rightarrow y_{j+\frac{1}{2}}$, is here introduced. The new variables are (t, y_a), with $a = 1, 2, \ldots, \ell + \frac{1}{2}$. In these new variables, the generators given in Ref. 3 read as follows:

$$
\begin{aligned}
\overline{z}_{+1} &= \partial_t,\\
\overline{z}_0 &= -t\partial_t - \sum_{a=1}^{\ell+\frac{1}{2}} (a - \frac{1}{2}) y_a \partial_{y_a} - \delta,\\
\overline{z}_{-1} &= 2t\overline{z}_0 + t^2\partial_t - \sum_{a=1}^{\ell-\frac{1}{2}} (\ell + a + \frac{1}{2}) y_{a+1}\partial_{y_a} - (\ell + \frac{1}{2})\frac{c}{2} y_1^2,\\
\overline{w}_j &= \sum_{k=0}^{\ell-j} \binom{\ell-j}{k} t^{\ell-j-k} \partial_{y_{\ell+\frac{1}{2}-k}},\\
\overline{w}_{-j} &= \sum_{k=0}^{\ell-\frac{1}{2}} \binom{\ell+j}{k} t^{\ell+j-k}\partial_{y_{\ell+\frac{1}{2}-k}} - \frac{(\ell+j)!c}{(\ell-\frac{1}{2})!(\ell+\frac{1}{2})!} \sum_{a=1}^{j+\frac{1}{2}} (-1)^a \frac{(\ell+\frac{1}{2}-a)!}{(j+\frac{1}{2}-a)!} t^{j+\frac{1}{2}-a} y_a,\\
\overline{c} &= c,
\end{aligned}
\tag{43}
$$

where $j = \frac{1}{2}, \frac{3}{2}, \ldots, \ell$ and $\delta = \frac{1}{4}(\ell + \frac{1}{2})^2$.

The on-shell invariant second order operator $\overline{\Omega}_1$ is given by

$$\overline{\Omega}_1 = \partial_t + \sum_{a=1}^{\ell-\frac{1}{2}}(\ell + \frac{1}{2} - a)y_a\partial_{y_{a+1}} - \frac{1}{2c}(\ell + \frac{1}{2})\partial^2_{y_1}. \tag{44}$$

The corresponding degree 0 operator can be defined by $\overline{\Omega}_0 = -t\overline{\Omega}_1$. It is straightforward to check that these two operators satisfy the same on-shell relations as (18)–(21).

For any half-integer ℓ, the new differential realization, allowing to conveniently express the ℓ-oscillator Hamiltonian, is obtained by performing the following three-step transformation on operators entering (43) and its associated enlarged algebra:

(i) the similarity transformation $\overline{g} \mapsto \breve{g} = t^\delta\,\overline{g}\,t^{-\delta}$ is applied to any such operator $\overline{g}$;

(ii) a change of variables, $(t, y_a) \mapsto (s, u_a)$, is performed on the differential operators $\breve{g}$,

$$t = e^s, \qquad y_a = e^{(a-\frac{1}{2})s}u_a; \tag{45}$$

(iii) the similarity transformation $\breve{g} \mapsto \widetilde{g} = \exp(-\frac{\lambda}{2}u_1^2)\breve{g}\exp(\frac{\lambda}{2}u_1^2)$, with $\lambda = -\dfrac{c}{2\ell+1}$, is applied (for this choice of λ, one eliminates in the final ℓ-oscillator Hamiltonian a term proportional to $u_1\partial_{u_1}$).

The combination of these three operations produces the following differential realization of $\widehat{\mathfrak{cga}}_\ell$ for any half-integer ℓ:

$$\begin{aligned}
\widetilde{z}_{+1} &= e^{-s}\Big(\partial_s - \sum_{a=1}^{\ell+\frac{1}{2}}(a-\frac{1}{2})u_a\partial_{u_a} + \frac{c}{2}\frac{1}{2\ell+1}u_1^2 - \delta\Big),\\
\widetilde{z}_0 &= -\partial_s,\\
\widetilde{z}_{-1} &= e^s\Big(-\partial_s - \sum_{a=1}^{\ell+\frac{1}{2}}(a-\frac{1}{2})u_a\partial_{u_a} - \sum_{a=1}^{\ell-\frac{1}{2}}(\ell+\frac{1}{2}+a)u_{a+1}\partial_{u_a} + \frac{c}{2}(\frac{1}{2\ell+1} - \ell - \frac{1}{2})u_1^2\\
&\quad + \frac{c}{2}\frac{2\ell+3}{2\ell+1}u_1u_2 - \delta\Big),\\
\widetilde{w}_j &= e^{-js}\sum_{k=0}^{\ell-j}\binom{\ell-j}{k}\partial_{u_{\ell+\frac{1}{2}-k}} - \delta_{j,\frac{1}{2}}\frac{c}{2\ell+1}e^{-\frac{s}{2}}u_1,\\
\widetilde{w}_{-j} &= e^{js}\Bigg(\sum_{k=0}^{\ell-\frac{1}{2}}\binom{\ell+j}{k}\partial_{u_{\ell+\frac{1}{2}-k}} - \frac{(\ell+j)!c}{(\ell-\frac{1}{2})!(\ell+\frac{1}{2})!}\sum_{a=1}^{j+\frac{1}{2}}(-1)^a\frac{(\ell+\frac{1}{2}-a)!}{(j+\frac{1}{2}-a)!}u_a\\
&\quad - \frac{c}{2\ell+1}\binom{\ell+j}{\ell-\frac{1}{2}}u_1\Bigg),\\
\widetilde{c} &= c.
\end{aligned} \tag{46}$$

The on-shell invariant operators $\overline{\Omega}_0, \overline{\Omega}_1$ are mapped into

$$\begin{aligned}
\widetilde{\Omega}_0 &= -\partial_s + \sum_{j=2}^{\ell+\frac{1}{2}}(j-\frac{1}{2})u_j\partial_{u_j} - \sum_{j=1}^{\ell-\frac{1}{2}}(\ell+\frac{1}{2}-j)u_j\partial_{u_{j+1}}\\
&\quad + \frac{1}{2c}(\ell+\frac{1}{2})\partial^2_{u_1} - \frac{c}{4}\frac{1}{2\ell+1}u_1^2 + \frac{1}{16}(2\ell-1)(2\ell+3),\\
\widetilde{\Omega}_1 &= -e^{-s}\widetilde{\Omega}_0.
\end{aligned} \tag{47}$$

The relations (27)–(29) hold true for any half-integer ℓ.

We are able to re-express, with the further change of variable $s = -2i\tau$ and after setting $c = -(2\ell+1)m$, the on-shell invariant PDE $\widetilde{\Omega}_0\psi(s,u_a) = 0$ as

$$
\begin{aligned}
&i\partial_\tau\psi(\tau,u_a) = \mathbf{H}^{(\ell)}\psi(\tau,u_a),\\
&\mathbf{H}^{(\ell)} = -\frac{1}{2m}\partial_{u_1}^2 + \frac{m}{2}u_1^2 + \sum_{a=2}^{\ell+\frac{1}{2}}(2a-1)u_a\partial_{u_a} - \sum_{a=1}^{\ell-\frac{1}{2}}(2\ell+1-2a)u_a\partial_{u_{a+1}}\\
&\qquad + \frac{1}{8}(2\ell-1)(2\ell+3).
\end{aligned}
\tag{48}
$$

It coincides (after setting $u_a = \mathbf{x}_a$ to improve readability) with the ℓ-oscillator equation that we introduced in Eqs. (1) and (2).

Since the differential operator $\mathbf{H}^{(\ell)}$ is independent of τ, PDE (48) admits solutions of the form

$$
\psi(\tau,u_a) = e^{-iE\tau}\varphi(u_a), \qquad \mathbf{H}^{(\ell)}\varphi(u_a) = E\varphi(u_a). \tag{49}
$$

The eigenvalue problem for the operator $\mathbf{H}^{(\ell)}$ is solved via an algebraic method. The vacuum solution $\varphi_{vac}(u_a)$ is defined to satisfy

$$
\widetilde{w}_j\,\varphi_{vac}(u_a) = 0, \quad j = \frac{1}{2},\frac{3}{2},\ldots,\ell. \tag{50}
$$

We get, as a consequence,

$$
\varphi_{vac}(u_a) = \exp(-\frac{m}{2}u_1^2), \qquad \mathbf{H}^{(\ell)}\varphi_{vac}(u_a) = 2\delta\varphi_{vac}(u_a). \tag{51}
$$

The relations

$$
[\widetilde{\Omega}_0,\widetilde{w}_{\pm j}] = 0, \qquad [\widetilde{z}_0,\widetilde{w}_{\pm j}] = \pm j\widetilde{w}_{\pm j}, \qquad 2\widetilde{\Omega}_0 = 2\widetilde{z}_0 + \mathbf{H}^{(\ell)}, \tag{52}
$$

imply that

$$
[\mathbf{H}^{(\ell)},\widetilde{w}_{\pm j}] = \mp 2j\widetilde{w}_{\pm j}. \tag{53}
$$

Therefore, the eigenvalues and eigenfunctions are, respectively, given by

$$
\begin{aligned}
E_{\vec{n}} &= \sum_{a=1}^{\ell+\frac{1}{2}}(2a-1)n_a + \frac{1}{2}(\ell+\frac{1}{2})^2,\\
\varphi_{\vec{n}} &= \widetilde{w}_{-\frac{1}{2}}^{n_1}\widetilde{w}_{-\frac{3}{2}}^{n_2}\cdots\widetilde{w}_{-\ell}^{n_{\ell+\frac{1}{2}}}\varphi_{vac}(u_a),
\end{aligned}
\tag{54}
$$

where $\vec{n} = (n_1,n_2,\ldots,n_{\ell+\frac{1}{2}})$ is a $(\ell+\frac{1}{2})$-component vector with entries $n_a \in \mathbb{N}_0$.

This concludes the derivation of the discrete spectrum that we introduced in (3).

The table of the first ℓ-oscillator Hamiltonians for $\ell \le \frac{9}{2}$ is here reported for convenience. With the same conventions as in the Introduction (namely, $u_a = \mathbf{x}_a$), we have

$$
\begin{aligned}
\mathbf{H}^{(\frac{1}{2})} &= -\frac{1}{2m}\partial_{\mathbf{x}_1}^2 + \frac{m}{2}{\mathbf{x}_1}^2,\\
\mathbf{H}^{(\frac{3}{2})} &= -\frac{1}{2m}\partial_{\mathbf{x}_1}^2 + \frac{m}{2}{\mathbf{x}_1}^2 + 3\mathbf{x}_2\partial_{\mathbf{x}_2} - 2\mathbf{x}_1\partial_{\mathbf{x}_2} + \frac{3}{2},\\
\mathbf{H}^{(\frac{5}{2})} &= -\frac{1}{2m}\partial_{\mathbf{x}_1}^2 + \frac{m}{2}{\mathbf{x}_1}^2 + 3\mathbf{x}_2\partial_{\mathbf{x}_2} + 5\mathbf{x}_3\partial_{\mathbf{x}_3} - 4\mathbf{x}_1\partial_{\mathbf{x}_2} - 2\mathbf{x}_2\partial_{\mathbf{x}_3} + 4,\\
\mathbf{H}^{(\frac{7}{2})} &= -\frac{1}{2m}\partial_{\mathbf{x}_1}^2 + \frac{m}{2}{\mathbf{x}_1}^2 + 3\mathbf{x}_2\partial_{\mathbf{x}_2} + 5\mathbf{x}_3\partial_{\mathbf{x}_3} + 7\mathbf{x}_4\partial_{\mathbf{x}_4} - 6\mathbf{x}_1\partial_{\mathbf{x}_2} - 4\mathbf{x}_2\partial_{\mathbf{x}_3} - 2\mathbf{x}_3\partial_{\mathbf{x}_4} + \frac{15}{2},\\
\mathbf{H}^{(\frac{9}{2})} &= -\frac{1}{2m}\partial_{\mathbf{x}_1}^2 + \frac{m}{2}{\mathbf{x}_1}^2 + 3\mathbf{x}_2\partial_{\mathbf{x}_2} + 5\mathbf{x}_3\partial_{\mathbf{x}_3} + 7\mathbf{x}_4\partial_{\mathbf{x}_4} + 9\mathbf{x}_5\partial_{\mathbf{x}_5}\\
&\quad -8\mathbf{x}_1\partial_{\mathbf{x}_2} - 6\mathbf{x}_2\partial_{\mathbf{x}_3} - 4\mathbf{x}_3\partial_{\mathbf{x}_4} - 2\mathbf{x}_4\partial_{\mathbf{x}_5} + 12.
\end{aligned}
\tag{55}
$$

VIII. CONCLUSIONS

For $\ell \geq \frac{3}{2}$, the ℓ-oscillator Hamiltonians fit into the class of non-Hermitian operators with real spectrum (see, e.g., Refs. 6 and 8). Besides being real, their spectrum is positive and discrete. It coincides with the spectrum of a set of decoupled harmonic oscillators of appropriate ($\nu_j \propto 2j-1$) frequency. The full quantum theory of the ℓ-oscillators requires the introduction of an inner product, norm, orthogonality conditions for the polynomials such as those entering (42). It deserves being fully scrutinized in a separate paper.

The spectrum is recovered from the $\ell + \frac{1}{2}$ harmonic oscillators realizing the $osp(1|2\ell+1)$ off-shell invariant subalgebra. In the $\ell \to \infty$ limit, an infinite tower of oscillatorial modes is created. The connection of the $sp(2m)$ algebras and their orthosymplectic extensions with the higher spin theories (see, e.g., Refs. 39 and 40) is well-established. It is tempting, in the light of the AdS/CFT correspondence and non-relativistic holography, to conjecture that ℓ-oscillators could appear in the dual, CFT side of higher-spin theories (possibly, in some non-relativistic contraction limit). This is an important point for future investigations.

The Pais-Uhlenbeck oscillators (see, e.g., Ref. 11) are invariant under the CGAs. Moreover, up to a normalization factor, their frequencies coincide with the ℓ-oscillator spectrum of frequencies. The two systems on the other hand are quite different. The Pais-Uhlenbeck oscillators describe higher-derivatives theories, while the ℓ-oscillators are second-order (no higher derivative) PDEs. A possible connection can arise from the on-shell condition. We investigated it for the differential realizations belonging to the enlarged $\widehat{\mathfrak{cga}}_\ell$ algebras. This is the natural setup for second-order invariant PDEs. One can of course search for solutions of the on-shell condition for differential realizations of $\mathcal{U}(\widehat{\mathfrak{cga}}_\ell)$, the $\widehat{\mathfrak{cga}}_\ell$ universal enveloping algebra. If such solutions are encountered, then the associated invariant PDEs are of higher order.

Our construction can be straightforwardly extended to more general cases. It can be applied to d-dimensional CGAs with half-integer ℓ. The first invariant PDEs of the series, at $\ell = \frac{1}{2}$, are either the free Schrödinger or the harmonic oscillator equation in $1+d$-dimensions. For generic $\ell = \frac{1}{2} + \mathbb{N}_0$, one recovers, as off-shell invariant subalgebras, both $so(d)$ and $osp(1|d(2\ell+1))$ (realized by $d(\ell+\frac{1}{2})$ oscillators). The fact that the oscillatorial modes can be accommodated into spin representations of $so(d)$ makes quite interesting to investigate the possible arising of Regge trajectories for ℓ-oscillators with $d > 1$.

The last comment regards supersymmetry. The supersymmetric extensions of $\widehat{\mathfrak{cga}}_\ell$ possess a $\mathbb{Z}_2$-grading. Their enlarged superalgebras require a second $\mathbb{Z}_2$-grading induced by the Cartan element of the $sl(2)$ subalgebra. The *algebra/superalgebra* duality of the enlarged bosonic algebra is replaced, in the supersymmetric case, by the *superalgebra/$\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebra* duality of the enlarged superalgebra. The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebras have been investigated, albeit to a less extent than superalgebras, in the mathematical literature, see, e.g., Refs. 41 and 42. A work on the *superalgebra/$\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebra* duality is currently under finalization.

ACKNOWLEDGMENTS

N. Aizawa is grateful to UFABC, where this research was conducted, for invitation and warm hospitality. This research was supported by FAPESP under Grant No. 2014/03560-0 and by CNPq under PQ Grant No. 306333/2013-9.

[1] F. Toppan, "Symmetries of the Schrödinger equation and algebra/superalgebra duality," e-print arXiv:1411.7867 [math-ph], to appear in the Proceedings of the 30th ICGTMP, Ghent 2014.
[2] J. Negro, M. A. del Olmo, and A. Rodríguez-Marco, J. Math. Phys. **38**, 3786 (1997).
[3] N. Aizawa, Y. Kimura, and J. Segar, J. Phys. A: Math. Theor. **46**, 405204 (2013); e-print arXiv:1308.0121 [math-ph].
[4] U. Niederer, Helv. Phys. Acta **46**, 191 (1973).
[5] G. Bluman and V. Shtelen, J. Phys. A: Math. Gen. **29**, 4473 (1996).
[6] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998); e-print arXiv:physics/9712001.
[7] C. M. Bender, S. Boettcher, and P. Meisinger, J. Math. Phys. **40**, 2201 (1999); e-print arXiv:quant-ph/9809072.
[8] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002); e-print arXiv:math-ph/0107001; **43**, 2814 (2002); e-print arXiv:math-ph/0110016; **43**, 3944 (2002); e-print arXiv:math-ph/0203005.
[9] J. Gomis and K. Kamimura, Phys. Rev. D **85**, 045023 (2012); e-print arXiv:1109.3773 [hep-th].
[10] A. Galajinsky and I. Masterov, Phys. Lett. B **723**, 190 (2013); e-print arXiv:1303.3419 [hep-th].

[11] K. Andrzejewski, A. Galajinsky, J. Gonera, and I. Masterov, Nucl. Phys. B **885**, 150 (2014); e-print arXiv:1402.1297 [hep-th].
[12] K. Andrzejewski, Phys. Lett. B **738**, 405 (2014); e-print arXiv:1409.3926 [hep-th].
[13] K. Andrzejewski, Nucl. Phys. B **889**, 333 (2014); e-print arXiv:1410.0479 [hep-th].
[14] D. Martelli and Y. Tachikawa, J. High Energy Phys. **05**, 091 (2010); e-print arXiv:0903.5184 [hep-th].
[15] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, 1978).
[16] A. W. Knapp, *Representation Theory of Semisimple Groups (An Overview Based on Examples)* (Princeton University Press, 1986).
[17] V. K. Dobrev, Rep. Math. Phys. **25**, 159 (1988).
[18] U. Niederer, Helv. Phys. Acta **45**, 802 (1972).
[19] L. Frappat, A. Sciarrino, and P. Sorba, *Dictionary on Lie algebras and superalgebras* (Academic Press, 2000).
[20] C. Duval and P. A. Horváthy, J. Math. Phys. **35**, 2516 (1994).
[21] M. Sakaguchi and K. Yoshida, J. Math. Phys. **49**, 102302 (2008); e-print arXiv:0805.2661 [hep-th].
[22] M. Sakaguchi and K. Yoshida, J. High Energy Phys. **0808**, 049 (2008); e-print arXiv:0806.3612 [hep-th].
[23] J. A. de Azcárraga and J. Lukierski, Phys. Lett. B **678**, 411 (2009); e-print arXiv:0905.0141 [math-ph].
[24] M. Sakaguchi, J. Math. Phys. **51**, 042301 (2010); e-print arXiv:0905.0188 [hep-th].
[25] A. Bagchi and I. Mandal, Phys. Rev. D **80**, 086011 (2009); e-print arXiv:0905.0580 [hep-th].
[26] S. Fedoruk and J. Lukierski, Phys. Rev. D **84**, 065002 (2011); e-print arXiv:1105.3444 [math-ph].
[27] I. Mandal, J. High Energy Phys. **1011**, 018 (2010); e-print arXiv:1003.0209 [hep-th].
[28] I. Masterov, J. Math. Phys. **53**, 072904 (2012); e-print arXiv:1112.4924 [hep-th].
[29] N. Aizawa, J. Phys. A: Math. Theor. **45**, 475203 (2012); e-print arXiv:1206.2708 [math-ph].
[30] N. Aizawa, Z. Kuznetsova, and F. Toppan, J. Math. Phys. **54**, 093506 (2013); e-print arXiv:1307.5259 [hep-th].
[31] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, 1982).
[32] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, 1986).
[33] W. I. Fushchych and R. M. Cherniha, J. Phys. A: Math. Gen. **18**, 3491 (1985).
[34] R. M. Cherniha and M. Henkel, J. Math. Anal. Appl. **369**, 120 (2010); e-print arXiv:0910.4822 [math-ph].
[35] W. I. Fushchych and A. G. Nikitin, *Symmetries of Equations of Quantum Mechanics* (Allerton Press, 1994).
[36] U. Niederer, Helv. Phys. Acta **47**, 167 (1974).
[37] C. P. Boyer, Helv. Phys. Acta **47**, 589 (1974).
[38] N. L. Holanda and F. Toppan, J. Math. Phys. **55**, 061703 (2014); e-print arXiv.org:1402.7298 [hep-th].
[39] C. Fronsdal, "Essays on supersymmetry," Math. Phys. Stud. **8**, 163 (1986).
[40] M. A. Vasiliev, Fortschr. Phys. **52**, 702 (2004).
[41] V. Rittenberg and D. Wyler, J. Math. Phys. **19**, 2193 (1978).
[42] M. Scheunert, J. Math. Phys. **20**, 712 (1979).