Generation of zonal flows by ion-temperature-gradient and related modes in the presence of neoclassical viscosity

A. B. Mikhailovskii
Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia and Nonlinear Physics Laboratory, Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi 141700, Moscow Region, Russia

A. I. Smolyakov
University of Saskatchewan, 116 Science Place, Saskatoon, Canada S7N 5E2 and Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia

E. A. Kovalishen
Nonlinear Physics Laboratory, Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi 141700, Moscow Region, Russia and Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia

M. S. Shirokov
Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia and Moscow Engineering Physics Institute, Kashirskoe Shosse 31, Moscow 115409, Russia

V. S. Tsypin
Physics Institute, University of São Paulo, Cidade Universitaria, 05508-900 São Paulo, Brazil

R. M. O. Galvão
Physics Institute, University of São Paulo, Cidade Universitaria, 05508-900 São Paulo, Brazil and Brazilian Center for Research in Physics, Rua Xavier Sigaud, 150, 22290-180 Rio de Janeiro, Brazil

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Generation of zonal flows by primary waves that are more complex than those considered in the standard drift-wave model is studied. The effects of parallel ion velocity and ion perturbed temperature and the part of the nonlinear mode interaction proportional to the ion pressure are taken into account. This generalization of the standard model allows the analysis of generation of zonal flows by a rather wide variety of primary modes, including ion temperature gradients, ion sound, electron drift, and drift-sound modes. All the listed effects, which are present in the slab geometry model, are complemented by effects of neoclassical viscosity inherent to toroidal geometry. We show that the electrostatic potential of secondary small-scale modes is expressed in terms of a nonlinear shift of the mode frequency and interpret this shift in terms of the perpendicular and parallel Doppler, nonlinear Kelvin-Helmholtz (KH), and nonlinear ion-pressure-gradient effects. A basic assumption of our model is that the primary modes form a nondispersive monochromatic wave packet. The analysis of zonal-flow generation is performed following an approach similar to that of convective-cell theory. Neoclassical zonal-flow instabilities are separated into fast and slow ones, and these are divided into two varieties. The first of them is independent of the nonlinear KH effect, while the second one is sensitive to it. © *2006 American Institute of Physics*.
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I. INTRODUCTION

The ion-temperature-gradient (ITG) instability[1] is one of the main instabilities that contribute to anomalous losses in tokamaks[2] and is taken into account in the ITER program.[3] It is then important to study not only its direct consequences, i.e., the ITG turbulence,[4] but also indirect ones revealed as generation of secondary modes,[5] particularly the possibility of generation of zonal flows.[6] Such an analysis was the subject of several recent papers, including Refs. 7–9. The main goal of the present paper is to develop a more complete theoretical model for this mechanism.

Considerable efforts have been undertaken in the recent theory of zonal-flow generation for studying the case when the primary (generating) modes are drift waves in a plasma with cold ions, allowing for the perpendicular ion inertia (see Refs. 6 and 10–12 and works cited therein). These modes can be referred to as the standard drift modes. The perpendicular ion inertia causes their radial dispersion, i.e., dependence of their mode frequency ω on the radial wave vector k_x. Such a dispersion is mathematically described by a term of order $k_x^2\rho_s^2$ in the dispersion relation, where ρ_s is the ion sound Larmor radius, i.e., the ion Larmor radius defined by the electron temperature. As a rule, the terms with finite $k_x\rho_s$ prove to be very important for the zonal-flow theory,[6,10–12] although particular results of such a theory can be used for the limiting transition $k_x\rho_s \rightarrow 0$. As a whole, the approach of zonal-flow theory developed for the case of standard drift waves can be called the dispersive one.

In contrast to the standard drift modes, the ITG modes can be studied neglecting their radial dispersion. Actually, this is the approximation made in the original paper on these modes, Ref. 1. Therefore, it seems reasonable, as a first step, to develop the theory of zonal flows generated by the ITG modes in the scope of the *nondispersive approach*, i.e., neglecting the radial dispersion of both the ITG modes and generated zonal flows. This approach is used in the present paper. Such a simplification of the general theory is motivated also by the consideration of the ion temperature being comparable to the electron temperature, as an approach for ITG modes. Therefore, assuming the parameter $k_x^2\rho_s^2$ to be finite, one should also allow for finiteness of the parameter $k_x^2\rho_i^2$, where ρ_i is the ion Larmor radius, and deal essentially with ion equations that are more complete than those in the limit $k_x^2\rho_i^2 \to 0$.

The study of zonal-flow generation by the standard drift waves was performed mainly by means of the wave kinetic equation (WKE), initially formulated for arbitrary types of waves in Ref. 13. One of the results significant for our topic, obtained by means of the WKE, is that a nondispersive monochromatic drift-wave packet can generate nondispersive poloidal zonal flow, as pointed out in Ref. 11 and reproduced in Ref. 12.

Actually, this fact should not be considered as unexpected if one supposes that the poloidal zonal flows are a particular case of the convective cells and that, according to Ref. 14, the convective cells can be generated by a monochromatic drift-wave packet. One can see from that work that such a generation takes place independently of whether this wave packet is dispersive or nondispersive. The approach used in Ref. 14 goes back to the idea of secondary instabilities.[5] It does not use the WKE and can be applied for the cases of both dispersive and nondispersive primary and secondary waves. In Ref. 12, it was emphasized that the above generation of nondispersive zonal flows by the nondispersive drift-wave packets is a rather fundamental nonlinear phenomenon and it was called the standard zonal-flow instability of hydrodynamic type. In addition to the WKE approach, the dispersion relation for this instability was derived in Ref. 12 also by means of the convective-cell formalism.

Taking the above arguments into consideration, we will perform our analysis following an approach similar to Ref. 14, i.e., without using the WKE. Our first goal is to elucidate whether the nondispersive ITG modes can generate the nondispersive poloidal zonal flow by the mechanism pointed out in Ref. 11.

It has been shown in Ref. 12 that, if the neoclassical viscosity is taken into account, the nondispersive drift-wave packet can generate not only the poloidal zonal flows but also the toroidal ones. In this connection, it is of interest to elucidate whether the toroidal zonal flows can be generated by ITG modes. This constitutes a further objective of the present paper.

There is a difficulty in the mathematical formulation of the current problem, which is related to ITG modes being essentially a kinetic phenomenon in high-temperature plasmas.[1,15] However, the analytical study of the nonlinear behavior of the kinetic ITG modes seems to be rather difficult. Therefore, we use the hydrodynamic model of nonlinear ITG modes, referring back to Ref. 16 and then to Ref. 15. In the scope of this model, we allow for the Reynolds stresses necessary for analysis of the zonal-flow instabilities. In order to complement the hydrodynamic equations of those references, we include terms corresponding to the neoclassical viscosity, using the formalism described in Refs. 17 and 18 and works quoted therein.

The equations are given in Sec. II. The main variables entering these equations are the electrostatic potential, the parallel plasma velocity, and the ion pressure. In addition, we explain in this section our approach to treat the variables describing the primary and secondary modes. In contrast to the theory of nonlinear generation of convective cells,[14,19] we do not specify the time-spatial dependence of the variables characterizing the zonal flows. Therefore, we cannot introduce the side-band harmonics of the secondary small-scale perturbations. Instead of them, we deal with the amplitudes of these perturbations, which are slow functions of the time and radial coordinate. It follows from a comparison with Ref. 12 that our approach leads to the same results as the approach of the theory of convective cells.

According to Ref. 20 (see also Ref. 15), the hydrodynamic model of ITG modes[16] shows that these modes are unstable for parallel wave vector $k_\parallel$ smaller than the critical value $k_{\parallel\mathrm{crit}}$, i.e., $k_\parallel < k_{\parallel\mathrm{crit}}$. At the same time, since the characteristic growth rates of zonal-flow instabilities are sufficiently small, one needs to consider almost marginally stable primary modes. Therefore, in order to avoid analyzing a rather complicated scenario of nonlinear stationary ITG modes, we consider stable ITG modes, i.e., those for $k_\parallel \approx k_{\parallel\mathrm{crit}}$. The dispersion relation for them is derived in Sec. III.

In the scope of the hydrodynamic model, the ITG modes look as a particular case of a family of nondispersive modes described by a common dispersion relation derived in Ref. 20 (see Sec. III). This family includes the ion-sound modes, electron drift modes (the nondispersive limit of the standard drift waves), and the drift-sound modes. In addition to the ITG modes, we analyze also the generation of zonal flows by these modes, introducing their dispersion relations in Sec. III. We derive the equations characterizing the small-scale secondary modes in the same section and express the amplitudes of the parallel velocity and the ion pressure in terms of the electrostatic potential. Then we obtain an equation for the amplitude of the electrostatic potential of these modes in terms of the electrostatic potential of the primary modes, and show that the electrostatic potential of the small-scale secondary modes can be expressed in terms of the nonlinear shift of the small-scale mode frequency. This nonlinear shift of the frequency is governed by the nonlinear Doppler shift of the primary mode frequency, the nonlinear Kelvin-Helmholtz (KH) effect, as well as the nonlinear ion-pressure-gradient effect. Introducing the nonlinear shift of the mode frequency, we thereby span a bridge between our approach and the WKE formalism[10–12] that deals with this frequency shift.

In Sec. IV we derive equations describing the large-scale secondary modes, i.e., the zonal flows. The main one is an equation for the zonal-flow poloidal velocity, Eq. (58). This equation differs from that derived in Ref. 12 as follows. First, it allows for the effects of finite ion temperature. Second, effects of order $(k_\parallel/\omega)^2$ are taken into account in Eq. (58). Third, the above-mentioned effects of finite $k_x^2\rho_s^2$ are not allowed for.

In addition to the poloidal zonal-flow velocity, Eq. (58) includes also the large-scale parts of the ion pressure and the parallel velocity. The evolution equations for them are also derived in Sec. IV. From that we prove that the averaged Reynolds stresses in these equations vanish. Therefore, the large-scale part of the ion pressure also vanishes, while the evolution of the large-scale part of the parallel velocity is governed solely by the neoclassical viscosity. As a result, we arrive at two coupled evolution equations for the averaged poloidal and parallel velocities. In Sec. IV we separate the large-scale perturbations into fast and slow ones. The fast perturbations do not depend on the averaged parallel velocity, i.e., they are characterized only by the poloidal velocity.

As a whole, Secs. II–IV describe a mathematical formalism for studying zonal-flow generation by drift-type and sound-type modes in the nondispersive approximation. Analysis of such a generation is the goal of Sec. V. Discussions of the results obtained in the paper are given in Sec. VI.

II. STARTING EQUATIONS AND VARIABLES DESCRIBING THE PRIMARY AND SECONDARY MODES

A. Starting equations

One of the main starting equations of our analysis is the ion continuity equation. We take it in the following dimensionless form:

$$\frac{\partial N}{\partial t} + V_\parallel \nabla_\parallel N + \nabla_\parallel V_\parallel + V_{*e}\frac{\partial \Phi}{\partial y} + [\Phi, N] = I_\perp + I_{\mathrm{NC}}. \tag{1}$$

Here $N=n/n_0$ and $\Phi=e\phi/T_{0e}$, where n_0 and T_{0e} are the equilibrium number density and the equilibrium electron temperature, e is the ion charge, ϕ is the electrostatic potential, and n is the perturbed plasma number density. The time derivative is normalized on the ion cyclotron frequency ω_{Bi}. The spatial derivatives are taken in units of the ion sound Larmor radius ρ_s, defined by $\rho_s=c_{se}/\omega_{Bi}$, where $c_{se}=(T_{0e}/M_i)^{1/2}$ is the electron part of the sound speed (see in detail below) and M_i is the ion mass. The parallel gradient $\nabla_\parallel$ is defined by $\nabla_\parallel=\rho_s^{-1}\mathbf{h}\cdot\boldsymbol{\nabla}$, where $\mathbf{h}$ is the unit vector along the equilibrium magnetic field directed along the Cartesian coordinate z. The value $V_\parallel$ is the perturbed ion parallel velocity normalized on c_{se}. The value V_{*e} is the electron drift velocity determined by the density gradient and normalized on c_{se}. The Poisson brackets $[a,b]$ are defined by

$$[a,b] = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}, \tag{2}$$

where x is the dimensionless radial coordinate and y is the "drift direction."

The value $I_\perp$ describes the perpendicular inertia and is defined by

$$I_\perp = \frac{\partial}{\partial x}\left\{(1+2q^2)\frac{\partial V_p}{\partial t} + [\Phi, V_p]\right\}. \tag{3}$$

Here V_p is the perturbed poloidal ion velocity given by

$$V_p = \frac{\partial}{\partial x}(\Phi + p) + \frac{\epsilon}{q}V_\parallel, \tag{4}$$

ϵ is the inverse aspect ratio, q is the safety factor, p is the normalized perturbed ion pressure given by

$$p = \tau N + T, \tag{5}$$

where $\tau=T_{0i}/T_{0e}$, T_{0i} is the equilibrium ion temperature, and T is the perturbed ion temperature normalized on the equilibrium electron temperature T_{0e}. Note that Eq. (4) is a consequence of the radial motion equation of the ions (see Refs. 17 and 18 for details).

The value I_{NC} describes the contribution of the neoclassical viscosity. It is defined by

$$I_{\mathrm{NC}} = \kappa_\theta \frac{\partial}{\partial x}\left(V_p - k\frac{\partial T}{\partial x}\right). \tag{6}$$

Here κ_θ is the neoclassical viscosity coefficient and k is a numerical factor dependent on the collisionality regime.[17,18]

The parallel velocity $V_\parallel$ satisfies the normalized equation of ion parallel motion

$$\frac{\partial V_\parallel}{\partial t} + V_\parallel\nabla_\parallel V_\parallel + \nabla_\parallel(p+N) + [\Phi, V_\parallel] = -\gamma_\parallel\left(V_p - k\frac{\partial T}{\partial x}\right), \tag{7}$$

where $\gamma_\parallel=\epsilon\kappa_\theta/q$. The right-hand side of this equation allows for the effect of neoclassical viscosity. The perturbed ion temperature T is governed by the ion heat-balance equation

$$\frac{\partial T}{\partial t} + V_\parallel\nabla_\parallel T + \frac{2}{3}\tau\nabla_\parallel V_\parallel + [\Phi, T] - V_{*T_i}\frac{\partial \Phi}{\partial y} = 0. \tag{8}$$

Here V_{*T_i} is the ion diamagnetic drift velocity determined by the ion temperature gradient and normalized to c_{se}.

The ion Eqs. (1), (7), and (8) are complemented by the equation for the electron parallel motion, of the form

$$\nabla_\parallel(\Phi - N) = 0. \tag{9}$$

Let us relate these equations to the system of equations derived in Ref. 16. Equation (1), without the terms $I_\perp$ and I_{NC}, corresponds to the first equation of that system, written for the ions. Equation (7), without the contribution of neoclassical viscosity, is obtained by summing the parallel components of the equations of motion for electrons and ions, multiplied by their respective masses. Equation (8) corresponds to the third equation of the system, taken for the ions. Finally, Eq. (9) is a particular case of the second equation of the system, taken for electrons but neglecting their inertia, the perturbed temperature, and the parallel friction force.

In order to obtain the value $I_\perp$ given by Eq. (3), one can turn to Eq. (4.44) of Ref. 15 and to Eq. (19.9) of Ref. 18. The value I_{NC} given by Eq. (6) can be found turning to Eqs. (19.9), (19.18), and (19.19) of Ref. 18 and to series of formulas of Ref. 17. Finally, for obtaining the term with neoclassical viscosity in Eq. (7), it is useful to turn to Eqs. (19.20)–(19.22) of Ref. 18 and to Ref. 17.

B. Variables describing the primary and secondary modes

We separate each perturbed value X into

$$X=\tilde{X}+\hat{X}+\bar{X}. \tag{10}$$

Here $\tilde{X}$ corresponds to the primary small-scale modes, while $\hat{X}$ and $\bar{X}$ characterize the small-scale and large-scale parts of the secondary modes, respectively. Evidently, the variables $\bar{X}$ concern the zonal flows.

We describe the primary small-scale modes in terms of the functions $\tilde{X}=(\tilde{\Phi},\tilde{V}_{\|},\tilde{p},\tilde{T})$, taken in the form

$$\tilde{X}=\tilde{X}_0\exp(ik_xx+ik_yy+ik_{\|}z-i\omega t)+\text{c.c.} \tag{11}$$

In the general case of the ITG modes, the mode frequency ω is assumed to be complex, $\omega=\omega_R+i\gamma$, where ω is the radial part of the mode frequency and γ is the growth rate. However, as noted in Sec. I, we will consider these modes at their stability boundary, so that only the case $\omega=\omega_R$, $\gamma=0$ will be considered.

The secondary small-scale modes are characterized by the functions $\hat{X}=(\hat{\Phi},\hat{V}_{\|},\hat{p},\hat{T})$ given by

$$\hat{X}=\hat{X}_1(t,x)\exp(ik_xx+ik_yy+ik_{\|}z-i\omega t)+\text{c.c.} \tag{12}$$

At last, the large-scale perturbations are described by the functions

$$\bar{X}(t,x)=[\bar{\Phi}(t,x),\bar{V}_{\|}(t,x),\bar{p}(t,x),\bar{T}(t,x)]. \tag{13}$$

One can see that our approach to present the functions $\hat{X}$ and $\bar{X}$ differs from that used in the theory of nonlinear generation of convective cells.[14,19] Then, in accordance with Sec. I, it is reasonable to demand that both these approaches lead to the same results. This is accomplished in detail in Ref. 12.

In order to facilitate understanding the correspondence between our approach and the convective-cell theory, note that the functions $\bar{X}(t,x)$ describing the large-scale perturbations are similar to the variables characterizing the convective cells. In the simplest case of monochromatic large-scale perturbations, these functions have a time-space dependence of the form $\exp(-i\Omega t+iq_xx)$, which is physically the same as a monochromatic convective cell. Also, the functions $\hat{X}_1(t,x)$, characterizing the secondary small-scale modes, with the accuracy of the multiplier $\exp(-i\Omega t+iq_xx)$, play the role of the sideband amplitudes of the convective-cell theory.

III. ANALYSIS OF SMALL-SCALE MODES

A. Primary modes

Let us consider the primary modes neglecting the perpendicular inertia and neoclassical viscosity. Then Eqs. (1), (7), and (8) reduce to

$$(\omega-\omega_{*e})\tilde{\Phi}_0-k_{\|}\tilde{V}_{\|0}=0, \tag{14}$$

$$\omega\tilde{V}_{\|0}-k_{\|}(\tilde{p}_0+\tilde{\Phi}_0)=0, \tag{15}$$

$$\omega\tilde{p}_0+\omega_{*pi}\tilde{\Phi}_0-(5/3)\tau k_{\|}\tilde{V}_{\|0}=0, \tag{16}$$

where $\omega_{*e}=k_yV_{*e}$, $\omega_{*pi}=k_yV_{*pi}$, and $V_{*pi}=V_{*Ti}-\tau V_{*e}$ is the ion diamagnetic drift velocity determined by the pressure gradient. We find from Eqs. (15) and (16) that the amplitudes $\tilde{V}_{\|0}$ and $\tilde{p}_0$ are related to $\tilde{\Phi}_0$ by

$$\tilde{V}_{\|0}=k_{\|}(\omega-\omega_{*pi})\tilde{\Phi}_0/f_{\|}, \tag{17}$$

$$\tilde{p}_0=[-\omega\omega_{*pi}+(5/3)\tau k_{\|}^2]\tilde{\Phi}_0/f_{\|}, \tag{18}$$

where

$$f_{\|}\equiv\omega^2-(5/3)\tau k_{\|}^2. \tag{19}$$

Substituting Eqs. (17) and (18) into Eq. (16), we arrive at the dispersion relation

$$D_0(\omega)\equiv\omega^2(\omega-\omega_{*e})-k_{\|}^2\{\omega[1+(5/3)\tau]-\omega_{*p_i}-(5/3)\tau\omega_{*e}\}=0. \tag{20}$$

This rather complicated dispersion relation describes a family of simple modes. Let us consider several of the relevant particular cases.

1. ITG modes

Let us assume for simplicity $\tau=1$ and introduce the parameter δ_T by

$$\delta_T\equiv\eta-2/3, \tag{21}$$

where $\eta\equiv\partial\ln T_{0i}/\partial\ln n_0$. Then Eq. (20) reduces to

$$1-\frac{\omega_{*e}}{\omega}-\frac{\omega_s^2}{\omega^2}\left(1+\frac{3}{8}\frac{\omega_{*e}}{\omega}\delta_T\right)=0, \tag{22}$$

where ω_s is the sound frequency introduced by $\omega_s^2=(8/3)k_{\|}^2$. It is known[20] that, for $\delta_T>0$, Eq. (22) describes the ITG instability. If $\delta_T\simeq1$, the root of this dispersion relation, corresponding to the unstable mode, has $\gamma\simeq\omega_R$. In order to exclude such a situation, we take $\delta_T\ll1$ and consider the modes with

$$\omega\ll\omega_{*e}. \tag{23}$$

In addition, we assume the ratio $(\omega_s/\omega_{*e})^2$ to be a small parameter of the same order of magnitude as the parameter δ_T,

$$(\omega_s/\omega_{*e})^2\simeq\delta_T. \tag{24}$$

Then Eq. (22) reduces to

$$\omega^2+\omega_s^2\omega/\omega_{*e}+(3/8)\omega_s^2\delta_T=0. \tag{25}$$

It hence follows that, for

$$\delta_T>\delta_T^0\equiv(2/3)(\omega_s/\omega_{*e})^2, \tag{26}$$

the ITG modes are characterized by

$$\omega_R=-\omega_s^2/(2\omega_{*e}), \tag{27}$$

$$\gamma = \frac{|\omega_s|}{2}\left[\frac{3}{2}\delta_T - \left(\frac{\omega_s}{\omega_{*e}}\right)^2\right]^{1/2}. \tag{28}$$

For

$$\delta_T \leqslant \delta_T^0, \tag{29}$$

Eq. (25) yields $\gamma=0$, while

$$\omega_R = -\frac{\omega_s^2}{2\omega_{*e}} \pm \frac{|\omega_s|}{2}\left[\left(\frac{\omega_s}{\omega_{*e}}\right)^2 - \frac{3}{2}\delta_T\right]^{1/2}. \tag{30}$$

Only such a case of ITG modes will be considered in the sequel.

2. Ion-sound modes

Neglecting the drift terms in Eq. (20), we arrive at the dispersion relation

$$\omega^2 = k_\parallel^2[1+(5/3)\tau], \tag{31}$$

which describes the ion-sound modes in a homogeneous plasma. For $\tau=1$ one has from Eq. (31)

$$\omega^2 = \omega_s^2. \tag{32}$$

The correspondence between the ion-sound modes and the ITG modes can be seen from Eq. (22).

3. Nondispersive electron drift modes

For $k_\parallel \to 0$, Eq. (20) reduces to

$$\omega = \omega_{*e}. \tag{33}$$

The modes described by this dispersion relation can be called nondispersive electron drift modes. They are the limiting case of the standard (dispersive) drift modes for $k_x^2\rho_s^2 \to 0$, studied in Refs. 10–12 as the primary modes generating the zonal flows.

4. Drift-sound modes for cold ions

Assuming $\tau \to 0$, Eq. (20) yields

$$\omega(\omega-\omega_{*e}) - k_\parallel^2 = 0. \tag{34}$$

Hence we find

$$\omega = \frac{\omega_{*e}}{2} \pm \left(\frac{\omega_{*e}^2}{4} + k_\parallel^2\right)^{1/2}. \tag{35}$$

These frequencies characterize the drift-sound modes in an inhomogeneous plasma with cold ions. According to Ref. 15, they are one of the fundamental types of waves in a weakly inhomogeneous plasma.

B. Small-scale secondary modes

Using Eqs. (1), (7), and (8), and neglecting the perpendicular inertia and the neoclassical viscosity, we obtain the following system of equations for the secondary modes:

$$[\partial/\partial t - i(\omega-\omega_{*e})]\hat{\Phi}_1 + ik_\parallel \hat{V}_{\parallel 1} + i(k_y\bar{V}_0 + k_\parallel \bar{V}_\parallel)\tilde{\Phi}_0 = 0, \tag{36}$$

$$\left(\frac{\partial}{\partial t} - i\omega\right)\hat{V}_{\parallel 1} + ik_\parallel(\hat{p}_1 + \hat{\Phi}_1) + i(k_y\bar{V}_0 + k_\parallel\bar{V}_\parallel)\tilde{V}_{\parallel 0} - ik_y\bar{V}_\parallel'\tilde{\Phi}_0 = 0, \tag{37}$$

and

$$(\partial/\partial t - i\omega)\hat{p}_1 + i(5/3)\tau k_\parallel \hat{V}_{\parallel 1} - i\omega_{*pi}\hat{\Phi}_1 + i(k_y\bar{V}_0 + k_\parallel\bar{V}_\parallel)\tilde{p}_0 - ik_y\bar{p}'\tilde{\Phi}_0 = 0, \tag{38}$$

where $\bar{V}_0 = \bar{\Phi}'$ is the averaged cross-field drift velocity and the prime denotes the x derivative. Similarly to Eqs. (17) and (18), it follows from Eqs. (37) and (38) that

$$\hat{V}_{\parallel 1} = \frac{k_\parallel}{f_\parallel}\left\{(\omega-\omega_{*pi})\hat{\Phi}_1 - \frac{i}{f_\parallel}\left[\omega(\omega-2\omega_{*pi}) + \frac{5}{3}\tau k_\parallel^2\right]\frac{\partial\hat{\Phi}_1}{\partial t}\right\} + \frac{\tilde{\Phi}_0}{f_\parallel}\left\{\frac{k_\parallel}{f_\parallel}\left[\omega(\omega-2\omega_{*pi}) + \frac{5}{3}\tau k_\parallel^2\right] \times (k_y\bar{V}_0 + k_\parallel\bar{V}_\parallel) - k_y(\omega\bar{V}_\parallel' + k_\parallel\bar{p}')\right\} \tag{39}$$

and

$$\hat{p}_1 = \frac{1}{f_\parallel}\left\{\left(-\omega\omega_{*pi} + \frac{5}{3}\tau k_\parallel^2\right)\hat{\Phi}_1 - \frac{i}{f_\parallel} \times \left[-\omega^2\omega_{*pi} + \frac{5}{3}\tau k_\parallel^2(2\omega-\omega_{*pi})\right]\frac{\partial\hat{\Phi}_1}{\partial t}\right\} + \frac{\tilde{\Phi}_0}{f_\parallel}\left\{\frac{1}{f_\parallel}\left[-\omega^2\omega_{*pi} + \frac{5}{3}\tau k_\parallel^2(2\omega-\omega_{*pi})\right](k_y\bar{V}_0 + k_\parallel\bar{V}_\parallel) - k_y\left(\omega\bar{p}' + \frac{5}{3}\tau k_\parallel\bar{V}_\parallel'\right)\right\}. \tag{40}$$

Substituting Eq. (39) into Eq. (36), we find

$$\partial\hat{\Phi}_1/\partial t = -i\tilde{\Phi}_0\delta\omega^{\mathrm{NL}}. \tag{41}$$

Here $\delta\omega^{\mathrm{NL}}$ is given by

$$\delta\omega^{\mathrm{NL}} = k_y\bar{V}_0 + k_\parallel\bar{V}_\parallel - \frac{k_\parallel k_y}{\partial D_0/\partial\omega}(\omega\bar{V}_\parallel' + k_\parallel\bar{p}'). \tag{42}$$

In accordance with Eq. (20),

$$\partial D_0/\partial\omega = 3\omega^2 - 2\omega\omega_{*e} - k_\parallel^2[1+(5/3)\tau]. \tag{43}$$

The value $\delta\omega^{\mathrm{NL}}$ can be interpreted as the nonlinear shift of the small-scale mode frequency. Such an interpretation is substantiated in Sec. III C.

Note that we have neglected the terms with $\partial/\partial t$ in the right-hand side of Eq. (41), because they are important only for extremely large amplitudes of the primary modes [see also discussion after Eq. (76)].

C. Nonlinear small-scale mode frequency

In contrast to Eq. (10), let us separate the variable X in the form

$$X = X^{>} + \bar{X}, \tag{44}$$

where, in accordance with Eq. (10), $X^{>} = \tilde{X} + \hat{X}$. The time dependence of $X^{>}$ is taken in the form $\exp(-i\omega^{\rm NL}t)$, where $\omega^{\rm NL}$ is the nonlinear small-scale mode frequency. Turning to Eqs. (1), (7), and (8), we arrive at the system of equations for $X^{>} = (\Phi^{>}, V_{\parallel}^{>}, p^{>})$ [cf. Eqs. (14)–(16) and (36)–(38)],

$$(\tilde{\omega} - \omega_{*e})\Phi^{>} - k_{\parallel} V_{\parallel}^{>} = 0,$$

$$\tilde{\omega} V_{\parallel}^{>} - (k_{\parallel} - k_y \partial \bar{V}_{\parallel}/\partial x)\Phi^{>} - k_{\parallel} p^{>} = 0, \tag{45}$$

and

$$\tilde{\omega} p^{>} + (\omega_{*pi} + k_y \partial \bar{p}/\partial x)\Phi^{>} - (5/3)\tau k_{\parallel} V_{\parallel}^{>} = 0,$$

where

$$\tilde{\omega} = \omega^{\rm NL} - k_y \bar{V}_0 - k_{\parallel} \bar{V}_{\parallel}. \tag{46}$$

This system leads to the nonlinear dispersion relation for the small-scale modes of the form

$$D_0(\tilde{\omega}) + k_y k_{\parallel}\left(\tilde{\omega}\frac{\partial \bar{V}_{\parallel}}{\partial x} + k_{\parallel}\frac{\partial \bar{p}}{\partial x}\right) = 0, \tag{47}$$

where the function $D_0(\tilde{\omega})$ is defined by Eq. (20) with the substitution $\omega \to \tilde{\omega}$.

Allowing for the zero-order dispersion relation, Eq. (20), Eq. (47) reduces to

$$(\tilde{\omega} - \omega)\frac{\partial D_0}{\partial \omega} + k_y k_{\parallel}\left(\omega\frac{\partial \bar{V}_{\parallel}}{\partial x} + k_{\parallel}\frac{\partial \bar{p}}{\partial x}\right) = 0. \tag{48}$$

It hence follows that

$$\tilde{\omega} - \omega = -\frac{k_y k_{\parallel}}{\partial D_0/\partial \omega}\left(\omega\frac{\partial \bar{V}_{\parallel}}{\partial x} + k_{\parallel}\frac{\partial \bar{p}}{\partial x}\right). \tag{49}$$

Substituting here Eq. (46), we arrive at

$$\omega^{\rm NL} = \omega + \delta\omega^{\rm NL}, \tag{50}$$

where $\delta\omega^{\rm NL}$ is given by Eq. (42).

If one considers the linear modes in a plasma flowing with an equilibrium parallel velocity $V_{\parallel 0}$, one arrives at Eqs. (45) with the substitution $\bar{V}_{\parallel} \to V_{\parallel 0}$ and without the term proportional to $\partial \bar{p}/\partial x$. Then, for sufficiently large $\partial V_{\parallel 0}/\partial x$, the electrostatic Kelvin-Helmholtz (KH) instability appears (see in detail Sec. 3.5 of Ref. 15). Therefore, the effect related to the term with $\partial \bar{V}_{\parallel}/\partial x$ can be called the nonlinear KH effect. On the other hand, according to the last Eq. (45), the term with $\partial \bar{p}/\partial x$ describes the nonlinear shift of the ion-pressure-gradient drift frequency ω_{*pi}.

Thus, the first two terms on the right-hand side of Eq. (42) represent the Doppler shifts of the mode frequency due to the averaged cross-field drift and the averaged parallel velocity. The term with $\bar{V}_{\parallel}'$ describes the nonlinear KH effect, while the term with $\bar{p}'$ corresponds to the nonlinear ion-pressure-gradient effect similar to that leading to the linear ITG instability.

IV. DERIVATION OF EQUATIONS FOR LARGE-SCALE PERTURBATIONS

A. Transformation of the averaged ion continuity equation

Averaging Eq. (1) over the small-scale oscillations, we obtain

$$\langle I_{\perp}\rangle + \langle I_{\rm NC}\rangle = 0, \tag{51}$$

where $\langle \ldots \rangle$ denotes average. By means of Eqs. (3), (6), and (10)–(13), we find

$$\langle I_{\perp}\rangle = (1+2q^2)\frac{\partial^2 \bar{V}_p}{\partial x \partial t} + k_y \frac{\partial^2}{\partial x^2}[i\tilde{\Phi}_0^* \hat{V}_{p1} - i\hat{\Phi}_1 \tilde{V}_{p0}^* + \text{c.c.}], \tag{52}$$

$$\langle I_{\rm NC}\rangle = \kappa_\theta \frac{\partial}{\partial x}\left(\bar{V}_p - k\frac{\partial \bar{T}}{\partial x}\right). \tag{53}$$

Using Eqs. (4), (17), (18), (39), and (40), we express in Eq. (52) the amplitudes $\tilde{V}_{p0}$ and $\hat{V}_{p1}$ in terms of $\tilde{\Phi}_0$ and $\hat{\Phi}_1$:

$$\tilde{V}_{p0} = c_{\perp}\tilde{\Phi}_0\left(ik_x + \frac{\epsilon}{q}\frac{k_{\parallel}}{\omega}\right), \tag{54}$$

$$\hat{V}_{p1} = c_{\perp}\left(ik_x + \frac{\partial}{\partial x} + \frac{\epsilon}{q}\frac{k_{\parallel}}{\omega}\right)\hat{\Phi}_1, \tag{55}$$

where

$$c_{\perp} = \omega(\omega - \omega_{*pi})/f_{\parallel}. \tag{56}$$

On the right-hand side of Eq. (55), we have neglected terms with $\tilde{\Phi}^{(0)}$; this is motivated by the same reason for neglecting the terms with $\partial/\partial t$ on the right-hand side of Eq. (41). Then we arrive at

$$\langle I_{\perp}\rangle = (1+2q^2)\frac{\partial^2 \bar{V}_p}{\partial x \partial t} + c_{\perp} k_y \frac{\partial^2}{\partial x^2} \times \left[i\tilde{\Phi}_0^*\left(2ik_x + \frac{\partial}{\partial x}\right)\hat{\Phi}_1 + \text{c.c.}\right]. \tag{57}$$

We act on Eq. (51) with the operator $\partial/\partial t$ and allow for Eqs. (57) and (53) for $\langle I_{\perp}\rangle$ and $\langle I_{\rm NC}\rangle$ and Eq. (41) for $\hat{\Phi}_1$. Then, integrating over x, we obtain

$$(1+2q^2)\frac{\partial^2 \bar{V}_p}{\partial t^2} + c_{\perp} k_y^2 I_{\mathbf{k}} \times \left[\bar{V}_0'' + \frac{k_{\parallel}}{k_y}\bar{V}_{\parallel}'' - \frac{k_{\parallel}}{\partial D_0/\partial \omega}(k_{\parallel}\bar{p}''' + \omega \bar{V}_{\parallel}''')\right] + \kappa_\theta \frac{\partial}{\partial t}\left(\bar{V}_p - k\frac{\partial \bar{T}}{\partial x}\right) = 0, \tag{58}$$

where

$$I_{\mathbf{k}} = 2\tilde{\Phi}_0^* \tilde{\Phi}_0. \tag{59}$$

The terms in the square brackets of Eq. (58) can be interpreted in the same manner as the terms on the right-hand side of Eq. (41) [see also Eq. (42)].

B. Excluding the zonal-flow parts of ion pressure and ion temperature

Allowing for Eqs. (5) and (1), we find from Eq. (8)

$$\frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial x}\left\langle \Phi \frac{\partial p}{\partial y}\right\rangle = 0. \tag{60}$$

By means of Eqs. (10)–(12), this equation reduces to

$$\frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial x}[ik_y(\tilde{\Phi}_0^* \hat{p}_1 + \hat{\Phi}_1^* \tilde{p}_0) + \text{c.c.}] = 0. \tag{61}$$

Using Eqs. (18) and (40), we express here the amplitudes of oscillatory pressure in terms of the amplitude of electrostatic potential arriving at

$$\partial \bar{p}/\partial t = 0. \tag{62}$$

Therefore, we can consider $\bar{p}=0$ in the following. In addition, it follows from Eq. (1) that, approximately, $\bar{N}=0$. Then, in accordance with Eq. (5), $\bar{T}=0$. As a result, Eq. (58) reduces to

$$(1+2q^2)\frac{\partial^2 \bar{V}_p}{\partial t^2} + c_\perp k_y^2 I_{\mathbf{k}}\left(\bar{V}_0'' + \frac{k_\parallel}{k_y}\bar{V}_\parallel'' - \frac{k_\parallel \omega \bar{V}_\parallel'''}{\partial D_0/\partial \omega}\right) + \kappa_\theta \frac{\partial \bar{V}_p}{\partial t} = 0, \tag{63}$$

where, in accordance with Eq. (4),

$$\bar{V}_p = \bar{V}_0 + (\epsilon/q)\bar{V}_\parallel. \tag{64}$$

Equation (63) governs the evolution of the averaged poloidal velocity $\bar{V}_p$ due to the Reynolds stress and the neoclassical viscosity. Since the Reynolds stress is defined by the averaged cross-field drift velocity $\bar{V}_0$ and (for $k_\parallel \neq 0$) the averaged parallel velocity $\bar{V}_\parallel$, allowing for Eq. (64), one can see that the parallel velocity, which is close to the averaged toroidal velocity $\bar{V}_t$ ($\bar{V}_t \approx \bar{V}_\parallel$ for $\epsilon \ll 1$) also can evolve. As a whole, Eq. (63) can be called the main evolution equation.

The term with $\bar{V}_0$ in Eq. (63) characterizes the standard part of the Reynolds stress, i.e., the $\bar{V}_\parallel$-independent part of the Reynolds stress. It is related to the standard Doppler shift of the nonlinear mode frequency given by the first term on the right-hand side of Eq. (42). In contrast to this, the terms with $\bar{V}_\parallel$ represent the $\bar{V}_\parallel$-dependent part of the Reynolds stress. In accordance with Eq. (42), the term with $\bar{V}_\parallel''$ is due to the parallel Doppler shift of the nonlinear mode frequency, while the term with $\bar{V}_\parallel'''$ is due to the nonlinear KH effect.

C. Transformation of the averaged ion parallel equation of motion

Similarly to Eqs. (60) and (51), we find from Eq. (7)

$$\frac{\partial \bar{V}_\parallel}{\partial t} + \frac{\partial}{\partial x}\left\langle \Phi \frac{\partial V_\parallel}{\partial y}\right\rangle = -\gamma_\parallel \bar{V}_p. \tag{65}$$

By analogy with Eq. (61), it hence follows that

$$\frac{\partial \bar{V}_\parallel}{\partial t} + \frac{\partial}{\partial x}[ik_y(\tilde{\Phi}_0^* \hat{V}_{\parallel 1} + \hat{\Phi}_1^* \tilde{V}_{\parallel 0}) + \text{c.c.}] = -\gamma_\parallel \bar{V}_p. \tag{66}$$

The terms in the square brackets do not contribute to this equation, cf. Eq. (62). Then, Eq. (66) reduces to

$$\partial \bar{V}_\parallel/\partial t = -\gamma_\parallel \bar{V}_p. \tag{67}$$

Thus, the evolution of the mean parallel velocity takes place only in the presence of the neoclassical viscosity.

D. Separation of large-scale perturbations into fast and slow ones

It is clear from Eqs. (63) and (67) that, similarly to Ref. 12, the large-scale perturbations can be separated into the fast and slow ones. In the case of fast perturbations, one can take $V_\parallel=0$ in Eq. (63). Then this equation reduces to

$$(1+2q^2)\frac{\partial^2 \bar{V}_0}{\partial t^2} + c_\perp k_y^2 I_{\mathbf{k}} \bar{V}_0'' + \kappa_\theta \frac{\partial \bar{V}_0}{\partial t} = 0. \tag{68}$$

It describes evolution of the averaged cross-field drift velocity.

In the case of slow perturbations, one can neglect the perpendicular inertia in Eq. (63), i.e., the term with $\partial^2/\partial t^2$. Then we have

$$c_\perp k_y^2 I_{\mathbf{k}}\left(\bar{V}_0'' + \frac{k_\parallel}{k_y}\bar{V}_\parallel'' - \frac{k_\parallel \omega}{\partial D_0/\partial \omega}\bar{V}_\parallel'''\right) + \kappa_\theta \frac{\partial \bar{V}_p}{\partial t} = 0. \tag{69}$$

This equation is complemented by Eqs. (67) and (64). These equations describe an evolution process in which all three velocities, $\bar{V}_0$, $\bar{V}_p$, and $\bar{V}_\parallel$, take part.

V. ANALYSIS OF ZONAL-FLOW INSTABILITIES

A. Fast zonal-flow instabilities neglecting neoclassical viscosity (ideal nondispersive instabilities)

In this subsection, we analyze perturbations described by Eq. (68) neglecting the neoclassical viscosity. Then Eq. (68) reduces to

$$(1+2q^2)\partial^2 \bar{V}_0/\partial t^2 + c_\perp k_y^2 I_{\mathbf{k}} \bar{V}_0'' = 0. \tag{70}$$

The only difference with the evolution of various primary modes considered in Sec. III is related to the factor $c_\perp$. Therefore, the analysis of Eq. (70) consists mathematically in elucidation of the role of this factor in particular cases of these modes. As a whole, the instabilities described by Eq. (70) can be called the ideal nondispersive ones.

1. The standard drift modes

Let us take $k_\parallel \to 0$ and $\tau \to 0$. In this case we deal with the standard drift modes described by Eq. (33). At the same time, according to Eqs. (19) and (56), we have

$$c_\perp = 1. \tag{71}$$

Using Eq. (71), Eq. (70) becomes

$$(1+2q^2)\partial^2\bar{V}_0/\partial t^2 + k_y^2 I_{\mathbf{k}}\bar{V}_0'' = 0. \tag{72}$$

We take

$$\bar{V}_0 = V_0^{(0)}\exp(iq_x x - i\Omega t) + \text{c.c.} \tag{73}$$

and obtain from Eq. (72) the zonal-flow dispersion relation in the form

$$\Omega^2 = -\Gamma^2, \tag{74}$$

where

$$\Gamma^2 = \Gamma_0^2/(1+2q^2) \tag{75}$$

and

$$\Gamma_0^2 = q_x^2 k_y^2 I_{\mathbf{k}}. \tag{76}$$

One can see that $\Omega^2 < 0$, i.e., the roots of Eq. (74) are imaginary, $\text{Re}\,\Omega = 0$. One of them, $\text{Im}\,\Omega > 0$, corresponds to unstable perturbations. This result is in agreement with that obtained in Refs. 11 and 12.

We have used the approximation $\omega \gg \Omega$. Taking for estimations $\omega \simeq k_y V_{*e}$ and $\Omega \simeq \Gamma_0$ and using Eq. (59), one finds that this estimation means $\tilde{V}_{Ey} \ll V_{*e}$, where $\tilde{V}_{Ey} \simeq k_x\tilde{\Phi}$ is the cross drift velocity caused by the primary modes. This approximation is violated for $\tilde{V}_{Ey} \simeq V_{*e}$, which corresponds to the case of extremely large amplitudes of the primary modes mentioned at the end of Sec. III B.

2. Drift modes for arbitrary ion temperature

Taking $k_\parallel = 0$ and τ to be arbitrary, the dispersion relation for the primary modes, Eq. (33), remains valid. This shows the known fact that the ion dynamics does not affect the linear electron drift modes. However, the expression for $c_\perp$ is now modified as follows:

$$c_\perp = 1 + \tau(1+\eta). \tag{77}$$

For $\bar{V}_0$ given by Eq. (73), we find, instead of Eq. (74),

$$\Omega^2 = -c_\perp\Gamma^2. \tag{78}$$

Assuming

$$\eta > -(1+1/\tau), \tag{79}$$

it hence follows that the zonal-flow instability is generated not only for $\tau = 0$ but also for arbitrary ion temperature.

3. Drift-sound modes for cold ions

Now we consider the case $k_\parallel \neq 0$ and $\tau \to 0$, corresponding to the drift-sound modes for cold ions described by the dispersion relation given by Eq. (35). In this case, Eq. (71) for $c_\perp$ remains in force. Then we arrive again at Eqs. (72)–(76). Thereby, we have shown that the fast zonal-flow instability, inherent to the standard drift modes, takes place also in the case of the drift-sound modes and is characterized by the same zonal-flow dispersion relation, Eq. (74).

4. Ion-sound modes for finite ion temperature

Now we turn to the ion-sound modes for finite ion temperature described by Eq. (31). In this case, according to Eqs. (19) and (56),

$$c_\perp = 1 + (5/3)\tau. \tag{80}$$

Then one arrives at the roots for the zonal-flow dispersion relation given by Eq. (78) with $c_\perp$ defined by Eq. (80).

5. ITG modes

In the case of ITG modes described by Eqs. (27) and (30), similarly to Eqs. (71), (77), and (80), we find

$$c_\perp = 4/3. \tag{81}$$

It can be seen from Eqs. (78) and (81) that the ITG modes do generate the fast zonal-flow perturbations.

B. Nondispersive fast dissipative neoclassical zonal-flow instabilities

Starting from Eq. (68) and keeping the term with neoclassical viscosity, we arrive at the neoclassical zonal-flow dispersion relation

$$\Omega^2 + i\gamma_p\Omega + c_\perp\Gamma_0^2 = 0. \tag{82}$$

Here $\gamma_p \equiv \kappa_\theta/(1+2q^2)$ is the poloidal decay rate.[17,18] The roots of this dispersion relation are

$$\Omega = \Omega_\pm \equiv -\frac{i\gamma_p}{2} \mp i\left(\frac{\gamma_p^2}{4} + c_\perp\Gamma^2\right)^{1/2}. \tag{83}$$

One can see that, if $c_\perp > 0$, for arbitrary ratio between Γ and γ_p, the root $\Omega = \Omega_+$ describes the damping perturbations, while the root $\Omega = \Omega_-$ describes the growing ones.

For $\gamma_p \gg 2\Gamma$ it hence follows that

$$\Omega = \Omega_- = ic_\perp\Gamma_0^2/\kappa_\theta = i\Gamma_d, \tag{84}$$

where

$$\Gamma_d = c_\perp\Gamma_0^2/\kappa_\theta, \tag{85}$$

and the subscript "d" denotes "dissipative." This root as well as the root $\Omega = \Omega_-$ given by Eq. (83) corresponds to the nondispersive fast dissipative neoclassical zonal-flow instabilities. One sees that these instabilities take place not only in the case of standard drift waves considered in Ref. 12, but also in all the cases discussed in Sec. V A, including the case of ITG modes, when the ideal zonal-flow instabilities are driven.

C. Nondispersive slow neoclassical zonal-flow instabilities

1. The approximation of vanishing $\bar{V}_\parallel$-independent part of Reynolds stress

Let us neglect the terms with $\bar{V}_\parallel$ in the expression for the Reynolds stress, i.e., the terms with $k_\parallel$ in Eq. (63). Then, allowing for Eq. (64), Eq. (69) reduces to

$$c_\perp k_y^2 I_{\mathbf{k}}[\bar{V}_p'' - (\epsilon/q)\bar{V}_\parallel''] + \kappa_\theta\partial\bar{V}_p/\partial t = 0. \tag{86}$$

Combining Eq. (86) with Eq. (67), we arrive at

$$\frac{\partial^2 \bar{V}_\parallel}{\partial t^2} + c_\perp k_y^2 I_{\mathbf{k}} \left(\frac{1}{\kappa_\theta} \frac{\partial \bar{V}_\parallel''}{\partial t} + \frac{\epsilon^2}{q^2} \bar{V}_\parallel'' \right) = 0. \tag{87}$$

Neglecting the term with $(\epsilon/q)^2$, Eq. (87) reduces to

$$\frac{\partial \bar{V}_\parallel}{\partial t} + \frac{c_\perp k_y^2 I_{\mathbf{k}}}{\kappa_\theta} \bar{V}_\parallel'' = 0. \tag{88}$$

Taking $\bar{V}_\parallel$ in the form similar to Eq. (73), one can see that Eq. (88) describes the dissipative instability characterized by the growth rate given by Eq. (85). Then we find that neglecting the term with $(\epsilon/q)^2$ in Eq. (87) is reasonable for

$$\Gamma_d \gg \epsilon \gamma_\parallel / q. \tag{89}$$

In the opposite case of sufficiently small Γ_d,

$$\Gamma_d \ll \epsilon \gamma_\parallel / q, \tag{90}$$

Eq. (87) reduces to

$$\partial^2 \bar{V}_\parallel / \partial t^2 + (\epsilon/q)^2 c_\perp k_y^2 I_{\mathbf{k}} \bar{V}_\parallel'' = 0. \tag{91}$$

Mathematically, this equation coincides with Eq. (70) if one substitutes

$$1 + 2q^2 \to (q/\epsilon)^2. \tag{92}$$

This is the neoclassical inertia renormalization discussed in Ref. 12. Physically, Eq. (91) describes the toroidal flow generation with growth rate

$$\mathrm{Im}\, \Omega \simeq (\epsilon/q) \Gamma_0. \tag{93}$$

It follows from Eq. (67) that, for the process with the growth rate given by Eq. (93), the averaged poloidal velocity is sufficiently small, so that Eq. (64) can be approximated by

$$\bar{V}_0 = -(\epsilon/q) \bar{V}_\parallel. \tag{94}$$

It then can be seen that Eq. (91) describes also generation of the averaged cross-field drift velocity with the growth rate given by Eq. (93).

2. The role of the $\bar{V}_\parallel$-dependent part of the Reynolds stress

Allowing for the $\bar{V}_\parallel$-dependent part of the Reynolds stress, instead of Eq. (91), we obtain

$$\frac{\partial^2 \bar{V}_\parallel}{\partial t^2} + \left(\frac{\epsilon}{q} \right)^2 c_\perp k_y^2 I_{\mathbf{k}} \left[\left(1 + \frac{q}{\epsilon} \frac{k_\parallel}{k_y} \right) \bar{V}_\parallel'' - \frac{q}{\epsilon} \frac{k_\parallel \omega}{\partial D_0 / \partial \omega} \bar{V}_\parallel''' \right] = 0. \tag{95}$$

Let us now estimate the terms with $k_\parallel$ in this equation. For the drift-wave range of the primary mode frequency, $\omega \simeq \omega_{*e}$, we have the estimate

$$k_\parallel / k_y \simeq \rho_s / L_n, \tag{96}$$

where $L_n = (\partial \ln n_0 / \partial r)^{-1}$ is the characteristic scale length of the equilibrium density gradient. Then, for $\rho_s / L_n < \epsilon/q$ the term $k_\parallel \bar{V}_\parallel''$, related to the parallel nonlinear Doppler shift of the mode frequency, can be neglected in Eq. (95). As a result, Eq. (95) reduces to

$$\frac{\partial^2 \bar{V}_\parallel}{\partial t^2} + \left(\frac{\epsilon}{q} \right)^2 c_\perp k_y^2 I_{\mathbf{k}} \left[\bar{V}_\parallel'' - \frac{q}{\epsilon} \frac{k_\parallel \omega}{\partial D_0 / \partial \omega} \bar{V}_\parallel''' \right] = 0. \tag{97}$$

Now we turn to the estimate of the term with $k_\parallel \bar{V}_\parallel'''$ in Eq. (97), related to the nonlinear KH effect. According to Eq. (42), the estimation for $\partial D_0 / \partial \omega$ is ω^2, so that $\omega / (\partial D_0 / \partial \omega) \simeq 1/\omega$. Then we arrive at the estimates

$$\frac{\omega k_y q_x}{\partial D_0 / \partial \omega} \simeq \hat{q}_x L_n, \tag{98}$$

where $\hat{q}_x \equiv \rho_s q_x$ is the dimensional zonal-flow radial wave vector, and

$$\frac{\omega k_\parallel q_x}{\partial D_0 / \partial \omega} \simeq \hat{q}_x \rho_s. \tag{99}$$

One can see that neglecting the terms with $k_\parallel$ in Eq. (97) is reasonable for

$$\rho_s \hat{q}_x < \epsilon / q. \tag{100}$$

In this case, Eq. (97) reduces to Eq. (91). In the opposite case,

$$\rho_s \hat{q}_x > \epsilon / q, \tag{101}$$

Eq. (97) reduces to

$$\frac{\partial^2 \bar{V}_\parallel}{\partial t^2} - \frac{\epsilon}{q} c_\perp k_y^2 I_{\mathbf{k}} \frac{k_\parallel \omega}{\partial D_0 / \partial \omega} \bar{V}_\parallel''' = 0. \tag{102}$$

Taking $\bar{V}_\parallel$ in the form of Eq. (73), Eq. (102) yields

$$\Omega^2 + i c_\perp \frac{\epsilon}{q} \Gamma_0^2 \frac{\omega k_\parallel q_x}{\partial D_0 / \partial \omega} = 0. \tag{103}$$

One of the roots of Eq. (103) describes an instability with the growth rate of the order of

$$\mathrm{Im}\, \Omega \simeq \mathrm{Re}\, \Omega \simeq \Gamma_0 \left(\frac{\epsilon}{q} \right)^{1/2} |\hat{q}_x \rho_s|^{1/2}. \tag{104}$$

Comparing Eq. (104) with Eq. (93), we conclude that the nonlinear KH effect, for the condition given by Eq. (101), results in increasing the growth rate of the toroidal-flow instability.

VI. DISCUSSIONS AND CONCLUSIONS

We have studied generation of zonal flows by the primary waves more complex than those used in the standard drift-wave model,[10,11] allowing for the effects of the parallel ion motion and the ion perturbed temperature [Eqs. (1), (7), and (8)]. In addition, we have allowed for the part of the nonlinear mode interaction proportional to the ion pressure [Eqs. (3) and (4)]. Such a generalization of the standard model allows one to analyze generation of zonal flows by a rather wide variety of primary modes, including ITG modes as well as the ion-sound modes, electron drift modes, and the drift-sound modes. All the listed effects inherent to the slab geometry have been complemented by the effects of neoclassical viscosity inherent to the toroidal geometry; see Eq. (6) for I_{NC} and the term with $\gamma_\parallel$ in Eq. (7).

One of our significant results is Eq. (41) for the electrostatic potential of the secondary small-scale modes. It is remarkable that this potential is expressed in terms of the nonlinear shift of the small-scale mode frequency. We have shown that this shift is defined not only by the well-known perpendicular Doppler effect due to the mean cross-field drift velocity entering the standard drift-wave theory,[10] but also by the following three effects. The first is the parallel Doppler effect due to the mean parallel velocity, the second is defined by the nonlinear Kelvin-Helmholtz effect, and the third is caused by the nonlinear ion-pressure-gradient effect [see in detail Eq. (42) and following explanations].

We have restricted ourselves to the case of the monochromatic wave packet of the primary modes. This has allowed us to derive evolution equations for the large-scale perturbations by an approach close to that of the theory of nonlinear generation of a convective cell (the direct approach).

We have followed the nondispersive approximation, i.e., neglected the radial dispersion of both the primary modes and zonal flows. In this approximation, the zonal part of the ion pressure vanishes, while the zonal part of the parallel velocity evolves only in the presence of neoclassical viscosity coupling this velocity with the poloidal flow one [Eq. (67)]. At the same time, the neoclassical viscosity and the perpendicular inertia couple the poloidal velocity with the mean cross-field drift velocity, which enters the Reynolds stress. Such a coupling is given by Eq. (63), which is called the main evolution equation. In addition, all the three velocities are interrelated by the radial equation of motion for the ions [Eq. (64)]. The above-mentioned equation system is a basis for our analysis of zonal-flow instabilities.

Before currying out the analysis, we have simplified the above-mentioned system of equations separating the zonal-flow perturbations into the fast and slow ones. It has been assumed that in the case of fast perturbations, the parallel velocity is negligible. Then, instead of the three intercoupled equations, we have turned to a single one, Eq. (66), describing evolution of the mean cross-field drift velocity. In the case of slow perturbations, we have neglected the perpendicular inertia in the main evolution equation, Eq. (63), substituting it by Eq. (69). Thereby, as in the general problem, in this case we have dealt with all three velocities intercoupled by three evolution equations.

Our first problem in analyzing the fast zonal-flow instabilities was to find out the picture of these instabilities driven by the above-listed primary modes in the absence of the neoclassical viscosity. We have shown that this picture is similar to that obtained in Refs. 11 and 12 with the only difference that the squared growth rate of generated zonal flows is modified by the factor $c_\perp$ given by Eq. (56), describing the renormalization of nonlinear interaction due to finite ion temperature. For all the listed primary modes, this factor turns out to be positive besides the case of strong inverse ion temperature gradient, only, when Eq. (79) is violated. As a whole, the fast zonal-flow instabilities in the absence of neoclassical viscosity can be called the ideal ones. Here, as in Ref. 18, we emphasize their analogy with the linear ideal magnetohydrodynamic (MHD) instabilities.

According to our analysis, the effect of neoclassical viscosity on the fast zonal-flow instabilities leads to their transition to a new family of instabilities, which are called the dissipative ones. As was noted in Ref. 18, they are similar to the linear ideal-viscous instabilities. Their difference from the dissipative instability, studied in Ref. 12 for the case of the standard drift waves, consists in their dependence on the nonlinear renormalization coefficient $c_\perp$, see Eqs. (82)–(85).

Turning to the slow neoclassical zonal-flow instabilities, we have considered two of their cases. In the first case, the Reynolds stress can be calculated neglecting its $V_\parallel$-dependent part. This case is realized for the sufficiently small radial wave vector of the zonal flows. The picture of corresponding zonal-flow instabilities proves to be similar to that found in Ref. 12. The second case is realized for the not too small zonal-flow radial wave vector. In this case, the Reynolds stress should be calculated allowing for the nonlinear KH effect. This effect leads to increasing the growth rate of the slow zonal-flow instabilities.

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