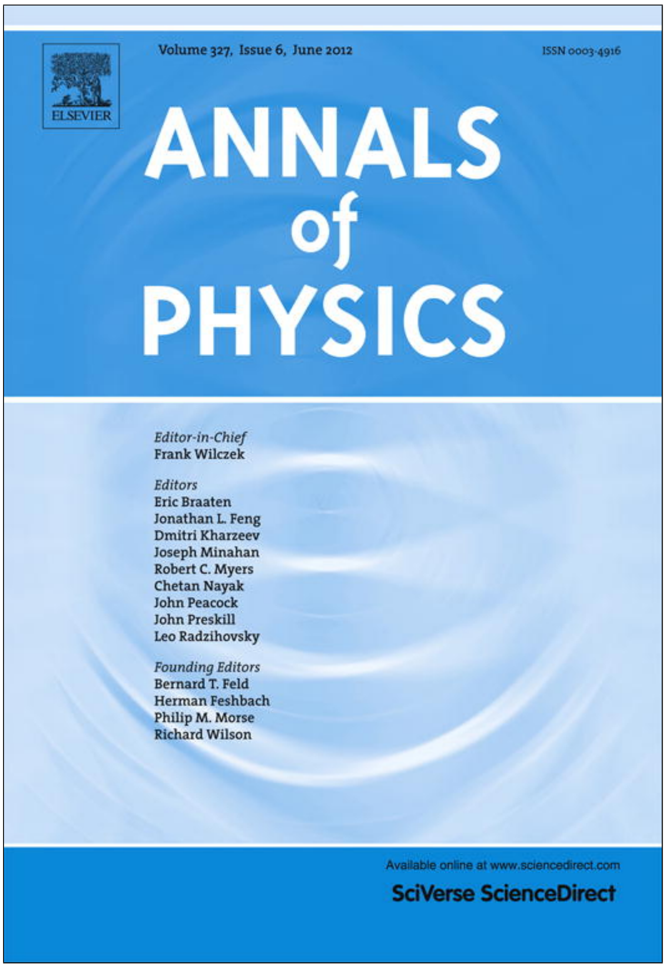
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Gauge anomaly cancellation in chiral gauge theories

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ABSTRACT

We consider chiral fermions interacting minimally with abelian and non-abelian gauge fields. Using a path integral approach and exploring the consequences of a mechanism of symmetry restoration, we show that the gauge anomaly has null expectation value in the vacuum for both cases (abelian and non-abelian). We argue that the same mechanism has no possibility to cancel the chiral anomaly, what eliminates competition between chiral and gauge symmetry at full quantum level. We also show that the insertion of the gauge anomaly in arbitrary gauge invariant correlators gives a null result, which points towards anomaly cancellation in the subspace of physical state vectors.

1. Introduction

A gauge anomaly is the quantum breakdown of gauge invariance [1–3]. It manifests itself through a non-null expectation value of the divergence of the gauge current. It appears in a great variety of contexts, from superstrings [4], passing through quantum gravity [5] to the description of the fractional quantum Hall effect [6]. In this work we will consider the important example of gauge theories of Weyl fermions minimally coupled to gauge fields. In this situation, the appearance of a gauge anomaly is viewed as unavoidable due to the necessarily simultaneous occurrence of chiral and gauge symmetry at classical level and their quantum competition [7–10]. It is usually said that the gauge anomaly destroys Slavnov–Taylor identities, crucial for renormalization, and turns unitarity uncertain. This is enough to discard theories where gauge anomalies appear, when it is not possible to cancel them through any other means [11].

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In the discussions mentioned above, it must be noticed that while the anomaly is a *quantum* phenomenon, it is usually computed as a functional of the gauge fields, which are then considered as *classical*. This means, in a path integral context, that one does not usually integrate over them. However, during the 80's, some works considered the full quantum nature of the theory (integrating also over the gauge fields) and gave support to the idea that anomalous gauge theories are not necessarily inconsistent. The pioneering work was that of Jackiw and Rajaraman [12] in which it was shown that a two-dimensional gauge anomalous theory was unitary and had massive photons in its spectrum. It was soon followed by the one of Faddeev and Shatashvili [13] who noticed that the gauge anomaly could be dealt with by the introduction of new quantum degrees of freedom, that transformed second class constraints (correlated to the gauge anomaly) into first class ones. Integrating over these extra fields and over the chiral fermions, one was led to an effective action, a functional of the gauge field, which was a *gauge invariant* one. Then, it was understood independently by Harada and Tsutsui [14] and Babelon, Schaposnik and Viallet [15] that the application of Faddeev–Popov's method to an anomalous theory introduced these new degrees of freedom naturally, associated to the non-factorization of the integration over the gauge group. These last arguments were no longer restricted to two dimensions.

In the context of abelian theories in two dimensions, one could see more recently [16] that important issues such as renormalizability and unitarity could be achieved in gauge anomalous models, both in the vector (Dirac fermions) and in the chiral case (Weyl fermions), when the gauge field is also quantized. The fact that the anomaly was a trivial (in the vector case) or a non-trivial cocycle (in the chiral case) did not seem to matter for the consistency of the model.

It would be natural to consider what happens to the gauge anomaly in this new context, of gauge invariant effective actions, in an arbitrary number of dimensions d. One would expect, on the basis of the results obtained in the 80's, that the gauge anomaly should vanish after considering the gauge field as a *quantum* field. Following this line of reasoning, in this work we briefly review the approach mentioned above to the gauge anomaly through the use of functional methods, that incorporate in a natural way the extra degrees of freedom. Then, using this formalism, we show the vanishing of the vacuum expectation value (v.e.v.) of the gauge anomaly and of its insertions in arbitrary gauge invariant correlators for chiral gauge theories in d dimensions.

We organize the discussion as follows: in the second section, we review the arguments contained in [14] to obtain gauge invariant effective actions for anomalous gauge theories, fixing our conventions and definitions in the process. This is a review section, intended to recall the methods and procedures used to show symmetry restoration. The third section is devoted to the consideration of the abelian case in an arbitrary number of dimensions: we derive an expression for the gauge transform of the gauge anomaly and we show that the v.e.v. of the gauge anomaly has to vanish as a consequence of it. In the fourth section, we show that the same argument cannot be used for the non-abelian case, again in d dimensions. By employing a different line of reasoning we consider the covariant divergence of the gauge current (in terms of the matter fields) in the fifth section and we show that its v.e.v. has to vanish. Consistence of our results is checked by indicating how to show independently that the expression of the v.e.v. of the gauge anomaly in terms of the effective action (that is, as a functional of the gauge field) also vanishes. Arbitrary gauge invariant correlators with insertions of the gauge anomaly are shown to be zero in the sixth section. We discuss the fate of the chiral anomaly in the seventh section, where we indicate that it remains possibly different from zero. We present our conclusions in the eighth section. A small Appendix is dedicated to reviewing the proof that the v.e.v. of an abelian gauge field vanishes.

2. Quantum restoration of gauge symmetry

In this section, we briefly review the appearance of a gauge anomaly and we show the way to restore gauge invariance on an anomalous chiral gauge theory, along the lines of the work of Harada and Tsutsui [14]. This will fix our definitions and conventions, which will be used along the body of our work.

We consider theories described by an action $I[\psi, \bar{\psi}, A_\mu]$, given by

$$I[\psi, \bar{\psi}, A_\mu] = I_G[A_\mu] + I_F[\psi, \bar{\psi}, A_\mu] = \int dx \, \frac{1}{2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + \int dx \, \bar{\psi} D \psi, \tag{1}$$

where dx indicates integration over a d-dimensional Minkowski space. The fields ψ are Weyl fermions carrying the fundamental representation of $SU(N)$. As usual, A_μ takes values in the Lie algebra of $SU(N)$ such that

$$A_\mu = A_\mu^a T_a, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu], \tag{2}$$

and the generators T_a satisfy

$$[T_a, T_b] = i f_{abc} T_c, \quad \mathrm{tr}(T_a T_b) = -\frac{1}{2} \delta_{ab}. \tag{3}$$

The operator D is the covariant derivative, and is called the Dirac operator of the theory. It is given by

$$D = i\gamma^\mu (\partial_\mu \mathbf{1} + ieA_\mu) \equiv i\gamma^\mu D_\mu. \tag{4}$$

Under gauge transformations,

$$g = \exp(i\theta^a(x) T_a), \tag{5}$$

and simultaneous changes of the fields ψ and A_μ as

$$\begin{aligned} A_\mu^g &= g A_\mu g^{-1} + \frac{i}{e} (\partial_\mu g) g^{-1}, \\ \psi^g &= g\psi, \\ \bar{\psi}^g &= \bar{\psi} g^{-1}, \end{aligned} \tag{6}$$

the action I is classically gauge invariant

$$I[\psi^g, \bar{\psi}^g, A_\mu^g] = I[\psi, \bar{\psi}, A_\mu]. \tag{7}$$

Invariance of the action under gauge transformations leads to the classical covariant conservation of the *gauge current*

$$(D_\mu)_{ab} J_b^\mu = 0, \tag{8}$$

with

$$J_a^\mu \equiv \bar{\psi} \gamma^\mu T_a \psi \tag{9}$$

and

$$(D_\mu)_{ab} = \delta_{ab} \partial_\mu + e f_{abc} A_\mu^c. \tag{10}$$

The quantum theory is defined by the generating functional, which is

$$Z[\eta, \overline{\eta}, j_a^\mu] = \int d\psi d\bar{\psi} dA_\mu \exp\left(iI[\psi, \bar{\psi}, A_\mu] + i \int dx [\overline{\eta}\psi + \bar{\psi}\eta + j_a^\mu A_\mu^a] \right). \tag{11}$$

Non-invariance of the fermion measure under gauge transformations leads to a potential quantum violation of the classical conservation law (8). To see this, we perform the following infinitesimal change of variables

$$\begin{aligned} \psi^g &= g\psi \approx (1 + i\delta\theta^a T_a)\psi, \\ \bar{\psi}^g &= \bar{\psi} g^{-1} \approx \bar{\psi}(1 - i\delta\theta^a T_a). \end{aligned} \tag{12}$$

In this way

$$Z_g = \int d\psi^g d\bar{\psi}^g dA_\mu \exp\left(iI[\psi^g, \bar{\psi}^g, A_\mu] + i\int dx[\overline{\eta}\psi^g + \bar{\psi}^g\eta + j_a^\mu A_\mu^a]\right)$$
$$= \int d\psi d\bar{\psi} dA_\mu J[A_\mu, g] \exp\left(iI[\psi, \bar{\psi}, A_\mu] + i\int dx[\overline{\eta}\psi + \bar{\psi}\eta + j_a^\mu A_\mu^a]\right.$$
$$\left. + \; i\int dx(i\delta\theta^a[i(D_\mu)_{ab}J_b^\mu + \overline{\eta}T_a\psi - \bar{\psi}T_a\eta])\right). \tag{13}$$

We notice the appearance of a Jacobian $J[A_\mu, g] = J[A_\mu, \delta\theta] \equiv \exp(i\alpha_1[A_\mu, \delta\theta])$. Given the infinitesimal character of the transformation, it can be functionally expanded to first order in $\delta\theta$

$$J[A_\mu, \delta\theta] = 1 + \int dx\, \delta\theta^a \mathcal{A}_a(A_\mu) + \cdots, \tag{14}$$

or, in terms of α_1

$$\alpha_1[A_\mu, \delta\theta] = -i\int dx\, \delta\theta^a \mathcal{A}_a(A_\mu) + \cdots.$$

Imposing that the result of the integral should be the same for both variables, we have

$$Z = Z_g, \tag{15}$$

which means that

$$\int d\psi d\bar{\psi} dA_\mu dx \exp\left(iI[\psi, \bar{\psi}, A_\mu] + i\int dx[\overline{\eta}\psi + \eta\bar{\psi} + j_a^\mu A_\mu^a]\right)$$
$$\times\, i\delta\theta^a[i(D_\mu)_{ab}J_b^\mu - i\mathcal{A}_a(A_\mu) + \overline{\eta}T_a\psi - \bar{\psi}T_a\eta] = 0. \tag{16}$$

Then, setting the external sources to zero, we see that

$$\int d\psi d\bar{\psi} dA_\mu \{(D_\mu)_{ab}J_b^\mu\} \exp(iI[\psi, \bar{\psi}, A_\mu])$$
$$= \int d\psi d\bar{\psi} dA_\mu \{\mathcal{A}_a(A_\mu)\} \exp(iI[\psi, \bar{\psi}, A_\mu]), \tag{17}$$

or, in terms of v.e.v.s,

$$\langle 0|(D_\mu)_{ab}J_b^\mu|0\rangle = \langle 0|\mathcal{A}_a(A_\mu)|0\rangle. \tag{18}$$

So, from the functional integral point of view, it has long been clear that a possible anomaly in gauge symmetry is intrinsically related to the non-invariance of the fermionic measure [17]. However, we notice that *there is still an expectation value to be taken, before we definitely say that current conservation is violated*.

Before computing $\langle 0|\mathcal{A}_a(A_\mu)|0\rangle$ it is instructive to look closely at the symmetry structure of the theory in the absence of gauge invariance of the fermionic measure. It is well known [18], in the context of a non-anomalous theory, that integration over the field A_μ has to be restricted to configurations that are not physically equivalent, due to the gauge symmetry of the action. The Faddeev–Popov technique exposes the factorization of the gauge group volume and restricts integration over non-equivalent representatives of gauge orbits. Coming back to (11) we notice that, if we proceed applying Faddeev–Popov's method, the gauge volume does not factor out, since there is an additional dependence on the group elements coming from the Jacobian,

$$d\psi d\bar{\psi} = \exp(i\alpha_1[A_\mu, g^{-1}])d\psi^g d\bar{\psi}^g. \tag{19}$$

Introducing the famous "1" of Faddeev–Popov,

$$1 = \Delta_{\rm FP}[A_\mu]\int dg\, \delta(f[A_\mu^g]) \tag{20}$$

in the vacuum amplitude $Z[0]$ (with $\Delta_{FP}[A_\mu]$ being the Faddeev–Popov determinant and $f(A_\mu) = 0$ being the gauge fixing condition) we see that

$$\begin{aligned} Z[0] &= \int d\psi d\bar{\psi} dA_\mu dg\ \Delta_{FP}[A_\mu]\delta(f[A_\mu^g]) \exp(iI[\psi, \bar{\psi}, A_\mu]) \\ &= \int d\psi d\bar{\psi} dA_\mu^{g^{-1}} dg\ \Delta_{FP}[A_\mu^{g^{-1}}]\delta(f[(A_\mu^{g^{-1}})^g]) \exp(iI[\psi, \bar{\psi}, A_\mu^{g^{-1}}]) \\ &= \int d\psi d\bar{\psi} dA_\mu dg\ \Delta_{FP}[A_\mu]\delta(f[A_\mu]) \exp(iI[\psi^g, \bar{\psi}^g, A_\mu]) \\ &= \int d\psi d\bar{\psi} dA_\mu dg\ \Delta_{FP}[A_\mu]\delta(f[A_\mu]) \exp(iI[\psi, \bar{\psi}, A_\mu] + i\alpha_1[A_\mu, g^{-1}]) \\ &= \int d\psi d\bar{\psi} \mathcal{D}A_\mu dg\ \exp(iI[\psi, \bar{\psi}, A_\mu] + i\alpha_1[A_\mu, g^{-1}]), \end{aligned} \tag{21}$$

where we defined

$$\mathcal{D}A_\mu = dA_\mu \Delta_{FP}[A_\mu]\delta(f[A_\mu]).$$

The non-factorization of the gauge volume naturally generates new degrees of freedom, the *Wess–Zumino fields* $\theta^a(x)$, that come from the integration over the local group element $g(x)$

$$dg = \prod_a d\theta_a(x). \tag{22}$$

Following the spirit of Ref. [14], we show below that these degrees of freedom produce a new gauge invariant action. To see this, we define

$$\exp(iW[A_\mu]) := \int d\psi d\bar{\psi} \exp(iI[\psi, \bar{\psi}, A_\mu]). \tag{23}$$

The Jacobian can be related to $W[A_\mu]$ in the following way:

$$\begin{aligned} \exp(iW[A_\mu^g]) &= \int d\psi d\bar{\psi} \exp\left(i\int dx\, \bar{\psi} D(A_\mu^g)\psi\right) \\ &= \int d\psi d\bar{\psi} \exp\left(i\int dx\, \bar{\psi}^{g^{-1}} D(A_\mu)\psi^{g^{-1}}\right) \\ &= J[A_\mu, g] \exp(iW[A_\mu]) \\ &\to J[A_\mu, g] = \exp(i[W[A_\mu^g] - W[A_\mu]]). \end{aligned} \tag{24}$$

So, we see that α_1 is given by

$$\alpha_1[A_\mu, g] = W[A_\mu^g] - W[A_\mu], \tag{25}$$

and exhibits clearly its behaviour under gauge transformations (cocycle property) [19]

$$\begin{aligned} \alpha_1[A_\mu^h, g] &= W[A_\mu^{hg}] - W[A_\mu^h] \\ &= \alpha_1[A_\mu, hg] - \alpha_1[A_\mu, h]. \end{aligned} \tag{26}$$

In particular, we find a familiar expression [3] for the anomaly in terms of $W[A_\mu]$:

$$\begin{aligned} \mathcal{A}_a(x) &= i\frac{\delta\alpha_1[A_\mu, g]}{\delta\theta_a(x)}\bigg|_{\theta=0} = i\frac{\delta W[A_\mu^g]}{\delta\theta_a(x)}\bigg|_{\theta=0} \\ &= i\int dz \frac{\delta W[A_\mu^g]}{\delta A_{\mu,b}^g(z)} \frac{\delta A_{\mu.b}^g(z)}{\delta\theta_a(x)}\bigg|_{\theta=0} \\ &= i\int dz \frac{\delta W[A_\mu]}{\delta A_{\mu,b}(z)} \left(\frac{\delta A_{\mu,b}^g(z)}{\delta\theta_a(x)}\bigg|_{\theta=0}\right). \end{aligned} \tag{27}$$

Using that

$$\left.\frac{\delta A^{g}_{\mu,b}(z)}{\delta\theta_a(x)}\right|_{\theta=0} = -\frac{1}{e}(D_\mu)_{ab}\delta(z-x) \tag{28}$$

we obtain[1]

$$\mathcal{A}_a(x) = \frac{i}{e}\left(D_\mu \frac{\delta W[A_\mu]}{\delta A_\mu(z)}\right)_a . \tag{29}$$

Now we define an effective action integrating only over the fermions and Wess–Zumino fields:

$$\begin{aligned}\exp(iI_{\text{eff}}[A_\mu]) &:= \int d\psi d\bar{\psi} dg \exp(iI[\psi,\bar{\psi},A_\mu] + i\alpha_1[A_\mu, g^{-1}]) \\ &= \int d\psi d\bar{\psi} dg \exp(iI[\psi,\bar{\psi},A_\mu] + iW[A_\mu^{g^{-1}}] - iW[A_\mu]) \\ &= \int dg \exp(iW[A_\mu^{g^{-1}}]).\end{aligned} \tag{30}$$

The original vacuum amplitude is written in terms of it as

$$Z[0] = \int \mathcal{D}A_\mu \exp(iI_{\text{eff}}[A_\mu]). \tag{31}$$

This new effective action is gauge invariant, as is shown below:

$$\begin{aligned}\exp(iI_{\text{eff}}[A_\mu^h]) &= \int dg \exp(iW[(A_\mu^h)^{g^{-1}}]) = \int dg \exp(iW[A_\mu^{(gh^{-1})^{-1}}]) \\ &= \int d(gh^{-1}) \exp(iW[A_\mu^{(gh^{-1})^{-1}}]) = \exp(iI_{\text{eff}}[A_\mu]).\end{aligned} \tag{32}$$

Expression (31) is the usual one that corresponds to a gauge theory in which one chooses a gauge fixing condition. Gauge invariance of the effective action strongly indicates the cancellation of the anomaly, as long as gauge symmetry is restored at quantum level, through the introduction of the Wess–Zumino fields. We will verify that this is possibly true in the next sections.

3. Weyl fermions interacting with an abelian gauge field in *d* dimensions

Now we will focus in the v.e.v. of the anomaly. We restrict ourselves to the abelian case (gauge group $U(1)$, only one generator $T=1$, one parameter $\theta(x)$, $g=\exp(i\theta(x))$, $f_{abc}=0$) in an arbitrary number of dimensions in this section. The v.e.v. of the anomaly can be written as

$$\begin{aligned}\langle 0|\mathcal{A}(A_\mu)|0\rangle &= \int d\psi d\bar{\psi} dA_\mu (\mathcal{A}(A_\mu)) \exp(iI[\psi,\bar{\psi},A_\mu]) \\ &= \int d\psi d\bar{\psi} dA_\mu d\theta \Delta_{\text{FP}}[A_\mu]\delta(f(A_\mu^g))(\mathcal{A}(A_\mu)) \exp(iI[\psi,\bar{\psi},A_\mu]) \\ &= \int d\psi d\bar{\psi} \mathcal{D}A_\mu d\theta (\mathcal{A}(A_\mu^{g^{-1}})) \exp(iI[\psi,\bar{\psi},A_\mu] + i\alpha_1[A_\mu, g^{-1}]),\end{aligned} \tag{33}$$

[1] This expression of the gauge anomaly in terms of the effective action means that we are considering the *consistent* anomaly, as defined, for example, in Section 14.2 of [20].

where we performed the usual steps of the Faddeev–Popov method but took into consideration that the fermionic measure is not invariant under a gauge transformation. We notice that the gauge transform of the anomaly can be written as a functional derivative

$$\begin{aligned}\mathcal{A}(A_\mu^{g^{-1}}) &= \frac{i}{e}\partial_\mu \frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(x)} = -\frac{i}{e}\int dz \frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(z)}\partial_\mu \delta(z-x) \\ &= i\int dz \frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(z)}\frac{\delta A_\mu^{g^{-1}}(z)}{\delta\theta(x)} = i\frac{\delta W[A_\mu^{g^{-1}}]}{\delta\theta(x)} = i\frac{\delta\alpha_1[A_\mu, g^{-1}]}{\delta\theta(x)}. \end{aligned} \tag{34}$$

Now we can proceed, using the result just derived

$$\begin{aligned}\langle 0|\mathcal{A}(A_\mu)|0\rangle &= i\int d\psi d\bar{\psi}\mathcal{D}A_\mu d\theta \left(i\frac{\delta}{\delta\theta(x)}\alpha_1[A_\mu, g^{-1}]\right) \\ &\quad\times \exp(iI[\psi,\bar{\psi},A_\mu] + i\alpha_1[A_\mu, g^{-1}]) \\ &= \int d\psi d\bar{\psi}\mathcal{D}A_\mu d\theta \frac{\delta}{\delta\theta(x)}[\exp(iI[\psi,\bar{\psi},A_\mu] + i\alpha_1[A_\mu, g^{-1}])] \\ &= \int d\theta \frac{\delta}{\delta\theta(x)}F[\theta] = 0. \end{aligned} \tag{35}$$

which shows that the anomaly vanishes because of the translational invariance of the functional measure [21].

We would like to briefly comment on the special case $d = 2$, where the gauge anomaly is [12]

$$\mathcal{A}(A_\mu) = -\frac{e}{4\pi}\{(a-1)\partial_\mu A^\mu + \varepsilon^{\mu\nu}\partial_\mu A_\nu\}. \tag{36}$$

It is a linear function of the gauge field and so it is obvious that its v.e.v. vanishes, as a consequence of Poincaré invariance of the vacuum:

$$\langle 0|A_\mu(x)|0\rangle = 0. \tag{37}$$

In fact, again considering the 2-dimensional case, this is true also for the non-abelian case, because the anomaly is given by

$$\mathcal{A}_a(A_\mu) = -\frac{e}{4\pi}\{(a-1)\partial_\mu A_a^\mu + \varepsilon^{\mu\nu}\partial_\mu A_{a,\nu}\}. \tag{38}$$

For completeness of the argument, we will briefly present a demonstration of Eq. (37) in an Appendix.

4. The non-abelian case

Inspired by the mechanism of anomaly cancellation in the abelian case, we investigate if it is possible to generalize it for the non-abelian situation [22]. Again, the focus is the v.e.v. of the anomaly, which is given, as before, by the expression

$$\begin{aligned}\langle 0|\mathcal{A}(A_\mu)|0\rangle &= \int d\psi d\bar{\psi} dA_\mu (\mathcal{A}(A_\mu)) \exp(iI[\psi,\bar{\psi},A_\mu]) \\ &= \int d\psi d\bar{\psi} dA_\mu d\theta \Delta_{\rm FP}[A_\mu]\delta(f(A_\mu))(\mathcal{A}(A_\mu^{g^{-1}})) \exp(iI[\psi,\bar{\psi},A_\mu] \\ &\quad + i\alpha_1[A_\mu, g^{-1}]). \end{aligned} \tag{39}$$

Starting again from the expression

$$\mathcal{A}_a(x) = \frac{i}{e}\left(D_\mu \frac{\delta W[A_\mu]}{\delta A_\mu(z)}\right)_a. \tag{40}$$

it is easy to write

$$\mathcal{A}_a(A_\mu^{g^{-1}}) = \frac{i}{e}\left(D_\mu^{g^{-1}}\frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(x)}\right)_a = -\frac{i}{e}\int dz\frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_{\mu,b}^{g^{-1}}(z)}(D_\mu^{g^{-1}})_{ba}\delta(z-x). \tag{41}$$

In order to proceed along the same lines, we have to investigate if the following equation is true or not

$$-\frac{i}{e}(D_\mu^{g^{-1}})_{ba}\delta(z-x) = \lambda\frac{\delta A_{\mu,b}^{g^{-1}}(z)}{\delta\theta^a(x)}, \tag{42}$$

with λ being a constant (independent of θ^a, to be determined). If it is true, then the v.e.v. of the non-abelian anomaly cancels with an argument which is parallel to that of the abelian case. This is not an easy question to be answered, in the general case ($SU(N)$). We will analyse the case in which the gauge group is $SU(2)$. In the fundamental representation, the generators T_a are given by

$$T_1 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad T_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{43}$$

The gauge group element has the well known closed form

$$g(\theta) = \exp\left(i\theta^a T_a\right) = \cos\left(\frac{\theta}{2}\right) + 2in^aT_a\sin\left(\frac{\theta}{2}\right),$$

with

$$\begin{aligned}\theta &= \sqrt{(\theta^1)^2+(\theta^2)^2+(\theta^3)^2},\\ n^a &= \frac{\theta^a}{\theta}.\end{aligned} \tag{44}$$

Eq. (42) *is valid only for* $\theta \neq 0$. This is an important observation and we will return to it in the end of the calculation. Some auxiliary results will be useful during this computation:

$$\begin{aligned}\partial_\mu\theta &= n^a\partial_\mu\theta^a,\\ \partial_\mu n^a &= \frac{1}{\theta}(\delta^{ab}-n^an^b)\partial_\mu\theta^b,\\ (n^aT_a)(\partial_\mu n^bT_b) &= \frac{1}{2\theta}(\vec{n}\times\partial_\mu\vec{\theta})^cT_c,\\ \frac{\delta\theta(z)}{\delta\theta^a(x)} &= n^a(z)\delta(z-x),\\ \frac{\delta n^b(z)}{\delta\theta^a(x)} &= \frac{1}{\theta}(\delta^{ba}-n^bn^a)(z)\delta(z-x),\end{aligned}$$

The gauge transform of the gauge field can be put into a completely analytical form. It is not difficult to obtain the two contributions for the gauge transform of the gauge field:

$$A_\mu^{g^{-1}} = g^{-1}A_\mu g + \frac{i}{e}(\partial_\mu g^{-1})g, \tag{45}$$

$$g^{-1}A_\mu g = A_\mu + (\vec{n}\times\vec{A}_\mu)^a\sin\theta T_a - 2\left((\vec{n}\times\vec{A}_\mu)\times\vec{n}\right)^a\sin^2\left(\frac{\theta}{2}\right)T_a, \tag{46}$$

$$\begin{aligned}\frac{i}{e}(\partial_\mu g^{-1})g \;=\; &\frac{1}{e}n^a(\partial_\mu\theta)T_a - \frac{1}{e\theta}(\partial_\mu\vec{n}\times\vec{n})^a\\ &+\frac{1}{e}(\partial_\mu n^a)\sin\theta T_a + \frac{1}{e\theta}(\partial_\mu\vec{n}\times\vec{n})^a\cos\theta T_a.\end{aligned} \tag{47}$$

To proceed, we should functionally differentiate the expression above with respect to $\theta^a(x)$ and compare the result with $(D_\mu^{g^{-1}})_{ba}\delta(z-x)$. This is a long and tedious task. We can circumvent this calculation if we perform a consistency check. By computing

$$\frac{\delta A_{\mu,b}^{g^{-1}}(z)}{\delta\theta^a(x)} = \frac{\delta}{\delta\theta^a(x)}\left[(g^{-1}A_\mu^c T_c g)^b + \frac{i}{e}((\partial_\mu g^{-1})g)^b\right]$$

one should be able to obtain, among other terms, something proportional to the first term of the covariant derivative

$$\delta^{ba}\partial_\mu\delta(z-x).$$

So, we can investigate the possible contributions to a term like this in the functional derivative. The term $g^{-1}A_\mu g$ clearly cannot contribute as it contains no derivatives of θ^a. The last two terms in the expression of $(\partial_\mu g^{-1})g$ will always contain $\sin\theta$ and $\cos\theta$ and so they are not candidates. All that remains is to compute the contributions of the first two terms:

$$\begin{aligned}
\frac{\delta}{\delta\theta^a(x)}\left(\frac{1}{e}n^b(\partial_\mu\theta)(z)\right) &= \frac{1}{e}n^b(z)\left(\partial_\mu\frac{\delta\theta(z)}{\delta\theta^a(x)}\right)+\cdots\\
&= \frac{1}{e}(n^a n^b)(z)\partial_\mu\delta(z-x)+\cdots,\\
\frac{\delta}{\delta\theta^a(x)}\left(-\frac{\varepsilon_{bcd}}{e\theta}(\partial_\mu n^c)n^d(z)\right) &= -\frac{\varepsilon_{bcd}}{e\theta}\left(\partial_\mu\frac{\delta n^c(z)}{\delta\theta^a(x)}\right)n^d(z)+\cdots\\
&= -\frac{\varepsilon_{bcd}}{e\theta}\partial_\mu\left(\frac{1}{\theta}(\delta^{ca}-n^c n^a)(z)\delta(z-x)\right)n^d(z)+\cdots\\
&= -\frac{\varepsilon_{bcd}}{e\theta^2}(n^d\delta^{ca}-n^c n^a n^d)(z)\partial_\mu\delta(z-x)+\cdots
\end{aligned}$$

This reasoning proves that $\delta^{ba}\partial_\mu\delta(z-x)$ is not contained in the expression for $\delta A_{\mu,b}^{g^{-1}}(z)/\delta\theta^a(x)$, so it cannot be proportional to $(D_\mu^{g^{-1}})_{ba}\delta(z-x)$. This means that the v.e.v. of the gauge anomaly, in the non-abelian case, does not cancel in the same way as in the abelian case. We stress that we did not proved that the v.e.v. of the gauge anomaly is non-zero for the non-abelian case. We proved that it is not cancelled by a generalization of the argument given in Section 3. However, in the next sections, we will give more general arguments to show that it cancels also in the non-abelian case.

Before ending this section we want to compare our findings with Eq. (28). It is precisely the relation we were seeking (changing g for g^{-1}), but computed for $\theta^a = 0$. A generalization of this equation would be natural (and would be Eq. (42)) and then one could think to have arrived at a contradiction. In fact, the exact analytic form for $A_\mu^{g^{-1}}$ (expressions (45)–(47)), is valid only under the assumption that some of the θ^a are distinct from zero. So, (28) would not be obtainable from (42) (if it would be true) and, in this way, we could suspect that the generalization of (28) for $\theta^a \neq 0$ would not be trivial. The exact analytic form of $\delta A_{\mu,b}^{g^{-1}}(z)/\delta\theta^a(x)$ can be directly (but patiently) obtained from the elements we gave and we see that the expression is far more complicated than the (relatively) simple form $(D_\mu^{g^{-1}})_{ba}\delta(z-x)$. What makes the non-abelian case so different from the abelian one remains to be discovered.

5. Null divergence of the gauge current

Now we will give general arguments to the cancellation of v.e.v. of the gauge anomaly even in the non-abelian case, in d dimensions. We will consider the v.e.v. of the covariant divergence of the current and we will show very simply that it has to vanish. To see this, we consider a bosonic change of variables in the functional integral:

$$Z[0] = \int d\psi d\bar\psi dA_\mu \exp(iI[\psi,\bar\psi,A_\mu]) = \int d\psi d\bar\psi dA_\mu^g \exp(iI[\psi,\bar\psi,A_\mu^g])$$

$$= \int d\psi d\bar{\psi} dA_\mu \exp(iI[\psi, \bar{\psi}, A^g_\mu]), \tag{48}$$

where we used again the invariance of the bosonic measure $dA^g_\mu = dA_\mu$. The functional integral does not contain a gauge group volume, as it would happen in a non-anomalous gauge theory, because fermion integration produces a gauge non-invariant $W[A_\mu]$. So, the Faddeev–Popov trick is unnecessary here, as it has already been stressed. This way of facing the problem is known as *gauge non-invariant representation* [20,23].

Next, we consider an infinitesimal gauge transformation, characterized by $g \approx 1 + i\delta\theta^a T_a$,

$$\begin{aligned} I[\psi, \bar{\psi}, A^g_\mu] &= I\left[\psi, \bar{\psi}, A_\mu + \frac{1}{e} D_\mu \delta\theta\right] \\ &= I[\psi, \bar{\psi}, A_\mu] + \frac{1}{e}\int dx\, (D_\mu \delta\theta(x))_a \frac{\delta I}{\delta A^a_\mu(x)} \\ &= I[\psi, \bar{\psi}, A_\mu] - \frac{1}{e}\int dx\, \delta\theta^a(x) \left(D_\mu \frac{\delta I}{\delta A_\mu(x)}\right)_a . \end{aligned} \tag{49}$$

This gives

$$\begin{aligned} Z[0] =& \int d\psi d\bar{\psi} dA_\mu \exp(iI[\psi, \bar{\psi}, A^g_\mu]) \\ \approx& \int d\psi d\bar{\psi} dA_\mu \exp(iI[\psi, \bar{\psi}, A_\mu]) \\ &- \frac{1}{e}\int dx\, \delta\theta^a(x) \int d\psi d\bar{\psi} dA_\mu \left(D_\mu \frac{\delta I}{\delta A_\mu(x)}\right)_a \exp(iI[\psi, \bar{\psi}, A_\mu]) \\ \Rightarrow& \int d\psi d\bar{\psi} dA_\mu \left(D_\mu \frac{\delta I}{\delta A_\mu(x)}\right)_a \exp(iI[\psi, \bar{\psi}, A_\mu]) = 0. \end{aligned} \tag{50}$$

Remembering that

$$\frac{\delta I}{\delta A^a_\nu(x)} = (D_\mu F^{\mu\nu})_a - e\bar{\psi}\gamma^\nu T_a \psi, \tag{51}$$

and that $(D_\mu D_\nu F^{\mu\nu})_a = 0$ identically,

$$\left(D_\mu \frac{\delta I}{\delta A_\mu(x)}\right)_a = (D_\mu)_{ab} \bar{\psi}\gamma^\mu T_b \psi, \tag{52}$$

we conclude that

$$\langle 0|(D_\mu)_{ab}\bar{\psi}\gamma^\mu T_b \psi|0\rangle = \langle 0|(D_\mu)_{ab} J^\mu_b|0\rangle = 0. \tag{53}$$

We notice that this result was reached without making any fermionic change of variables. In fact, it does not even matter if the fermionic measure is gauge invariant or not. The above equation is also completely consistent with our previous conclusions in the abelian case, but it is also valid for the non-abelian case as well.

We can face this result as a general argument (i.e. independent of the consideration of Weyl or Dirac fermions) for the vanishing of the expectation value of the covariant divergence of the gauge current. It relies only on the existence of classical gauge symmetry and gauge invariance of the gauge field functional measure. It means that, if the theory is to be consistent, the right-hand side of Eq. (18) *must vanish accordingly*. In the case where we are considering Dirac fermions, this is seen as a trivial consequence of the invariance of the fermion measure under gauge transformations (which says that the Jacobian $J[A_\mu, g] = 1$ and thus that $\mathcal{A}_a = 0$). However, in the Weyl fermion case, the Jacobian is not trivial and we must investigate in further detail if the right hand side vanishes or if we simply ended at an inconsistency. This can be easily achieved in two steps: first, gauge invariance of I_{eff} (Eq. (30)) implies

$$\int dg(\delta\theta^{g^{-1}})^a(x)\left[D_\mu^{g^{-1}}\left(\frac{\delta W[A_\mu^{g^{-1}}]}{\delta A_\mu^{g^{-1}}(x)}\right)\right]_a \exp(iW[A_\mu^{g^{-1}}])=0,$$

with $\delta\theta^{g^{-1}}=g^{-1}\delta\theta g$ and $\delta\theta=\delta\theta^a T_a$. Integrating this equation over the gauge field and using again the gauge invariance of the bosonic measure we obtain

$$\begin{aligned}0 = &\int dA_\mu\left[D_\mu\left(\frac{\delta W[A_\mu]}{\delta A_\mu(x)}\right)\right]_a \exp(iW[A])\\ &\Rightarrow \langle 0|\mathcal{A}_a(A_\mu)|0\rangle = 0.\end{aligned} \tag{54}$$

This works as a consistency check of the anomaly equation, as this vanishing is obtained on a independent basis of the vanishing of the divergence of the fermion current (although both proofs use gauge invariance of dA_μ).

6. The gauge anomaly and the Hilbert space

A very important question is if the behaviour found above for the anomaly is repeated for its insertion on an arbitrary correlation function. If this would be true it would be a very dangerous situation for the theory, for it could imply that the gauge field operator has to be zero itself, what could mean that the theory is trivial (or inconsistent). We found no means to answer completely this question. However, it can be answered in the case of arbitrary gauge invariant correlators, as we show below.

Any gauge invariant operators can be expressed in terms of ψ, $\bar\psi$ and A_μ. The basic property that they must obey is

$$O(\psi^g,\bar\psi^g,A_\mu^g)=O(\psi,\bar\psi,A_\mu). \tag{55}$$

This property, by its turn, implies that

$$O(\psi,\bar\psi,A_\mu^g)=O(\psi^{g^{-1}},\bar\psi^{g^{-1}},A_\mu). \tag{56}$$

That is all we need to analyse gauge invariant correlators. We consider the generating functional for correlators of general composite operators $O_i(\psi,\bar\psi,A_\mu)$,

$$Z_O[\lambda_i]=\int d\psi d\bar\psi dA_\mu \exp\left(iI[\psi,\bar\psi,A_\mu]+i\int dx\lambda_i O_i(\psi,\bar\psi,A_\mu)\right), \tag{57}$$

Arbitrary correlators are obtained from $Z_O[\lambda_i]$ in a standard way:

$$\langle 0|T(O_{i_1}(x_1)\dots O_{i_n}(x_n))|0\rangle = (-i)^n \left.\frac{\delta^n Z_O[\lambda_i]}{\delta\lambda_{i_1}(x_1)\dots\delta\lambda_{i_n}(x_n)}\right|_{\lambda=0}. \tag{58}$$

Now, we can begin integrating over A_μ^g instead of A_μ (as before, a dummy integration variable):

$$\begin{aligned}Z_O[\lambda_i] &= \int d\psi d\bar\psi dA_\mu^g \exp\left(iI[\psi,\bar\psi,A_\mu^g]+i\int dx\lambda_i O_i(\psi,\bar\psi,A_\mu^g)\right)\\ &= \int d\psi d\bar\psi dA_\mu \exp\left(iI[\psi^{g^{-1}},\bar\psi^{g^{-1}},A_\mu]+i\int dx\lambda_i O_i(\psi^{g^{-1}},\bar\psi^{g^{-1}},A_\mu)\right)\\ &= \int d\psi d\bar\psi dA_\mu \exp\left(iI[\psi,\bar\psi,A_\mu]+i\int dx\lambda_i O_i(\psi,\bar\psi,A_\mu)-i\alpha_1(A_\mu,g^{-1})\right),\end{aligned} \tag{59}$$

where we took into account the gauge invariance of the gauge field measure and the gauge non-invariance of the fermion measure

$$dA_\mu = dA_\mu^g,$$
$$d\psi d\bar{\psi} = \exp(-i\alpha_1[A_\mu, g^{-1}])d\psi^{g^{-1}} d\bar{\psi}^{g^{-1}}. \tag{60}$$

Restricting ourselves to an infinitesimal gauge transformation,

$$\alpha_1(A_\mu, -\delta\theta) = i\int dx\, \delta\theta^a \mathcal{A}_a(A_\mu) + \cdots, \tag{61}$$

we obtain

$$\begin{aligned} Z_O[\lambda_i] = {} & Z_O[\lambda_i] - i\int dx\,\delta\theta^a \int d\psi d\bar{\psi} dA_\mu \mathcal{A}_a(A_\mu)\exp\Big(iI[\psi,\bar{\psi},A_\mu] \\ & + i\int dx\lambda_i O_i(\psi,\bar{\psi},A_\mu)\Big) \\ & \Rightarrow \int d\psi d\bar{\psi} dA_\mu \mathcal{A}_a(A_\mu)\exp\Big(iI[\psi,\bar{\psi},A_\mu] + i\int dx\lambda_i O_i(\psi,\bar{\psi},A_\mu)\Big) = 0. \end{aligned} \tag{62}$$

Taking arbitrary functional derivatives with respect to λ_i and setting them to zero we obtain

$$\langle 0|T(\mathcal{A}_a(A_\mu)O_{i_1}(x_1)\dots O_{i_n}(x_n))|0\rangle = 0. \tag{63}$$

We showed that the gauge anomaly has null insertion into arbitrary gauge invariant correlators. This fact indicates that the anomaly is null in the subspace of the Hilbert space consisting of physical vectors (those who are annihilated by the BRST charge, obeying physical restrictions such as gauge conditions [24]). It must be remarked that the technique above cannot say anything about the case of gauge non-invariant correlators. Other techniques (such as lattice calculations, for example) can bring some light to this question.

7. Chiral symmetry versus gauge symmetry

Another important question concerns the chiral anomaly. When the gauge field is not quantized, it is well known [7–10] that chiral symmetry is in competition with gauge symmetry. Then, we must investigate what happens with the chiral anomaly in a context where the gauge anomaly cancels. Classical chiral symmetry is expressed by the transformations [11]

$$\psi^c = c\psi,$$
$$\bar{\psi}^c = \bar{\psi}c, \tag{64}$$
$$A_\mu^c = A_\mu, \tag{65}$$

where $c = \exp(i\alpha\gamma_{d+1})$, with $\partial_\mu\alpha = 0$ and[2] $\gamma_{d+1} = i\gamma^0\gamma^1\dots\gamma^{d-1}$. The classical action (1) is symmetric with respect to this transformation

$$I[\psi^c, \bar{\psi}^c, A_\mu] = I[\psi, \bar{\psi}, A_\mu], \tag{66}$$

and, as a consequence,

$$\partial_\mu J_{d+1}^\mu \equiv \partial_\mu(\bar{\psi}\gamma^\mu\gamma_{d+1}\psi) = 0.$$

[2] The essential property of γ_{d+1} is $\{\gamma_{d+1}, \gamma^\mu\} = 0$, which is not satisfied in odd dimensions. In order to avoid this difficulty, we will restrict our discussion, in this section, to the case in which d is even.

Following the same standard reasoning of section II (an infinitesimal chiral transformation, with $\partial_\mu \delta\alpha \neq 0$) we conclude that

$$\begin{aligned} Z_c &= \int d\psi^c d\bar{\psi}^c dA_\mu \exp\left(iI[\psi^c, \bar{\psi}^c, A_\mu] + i\int dx[\overline{\eta}\psi^c + \bar{\psi}^c\eta + j^\mu_a A^a_\mu] \right) \\ &= \int d\psi d\bar{\psi} dA_\mu J[A_\mu, c] \exp\left(iI[\psi, \bar{\psi}, A_\mu] + i\int dx[\overline{\eta}\psi + \bar{\psi}\eta + j^\mu_a A^a_\mu] \right. \\ &\quad \left. + \, i\int dx(i\delta\alpha[\partial_\mu J^\mu_{d+1} + \overline{\eta}\gamma_{d+1}\psi + \bar{\psi}\gamma_{d+1}\eta]) \right). \end{aligned} \tag{67}$$

Expanding the Jacobian

$$\begin{aligned} J[A_\mu, c] &= J[A_\mu, \delta\alpha] \equiv \exp(i\beta_1[A_\mu, \delta\alpha]) \\ &\approx 1 + i\int dx\, \delta\alpha(x) \left.\frac{\delta\beta_1}{\delta\alpha(x)}(A_\mu)\right|_{\delta\alpha=0} \\ &\equiv 1 + \int dx\, \delta\alpha(x) \mathcal{A}_{d+1}(A_\mu), \end{aligned} \tag{68}$$

and imposing $Z = Z_c$, we find the usual equation that indicates the possible presence of the chiral anomaly:

$$\langle 0|\partial_\mu J^\mu_{d+1}|0\rangle = \langle 0|\mathcal{A}_{d+1}(A_\mu)|0\rangle \tag{69}$$

We can try to follow the same path as in Sections 5 and 6, in order to see if we can prove that one or both of the two sides of (69) is zero. In Section 5 we began by making a change of variables in the expression for $Z[0]$ (Eq. (48)): it was a gauge transformation on A_μ. Then we explored gauge invariance of the bosonic measure dA_μ to show that the gauge anomaly vanished. As the chiral transformation affects only the fermion fields (because $A^c_\mu = A_\mu$), it is not possible to apply this procedure here. The same reasoning invalidates the extension of the procedure followed in the end of Section 5 to the case of the chiral anomaly, because there is no sense in asking if $I_{\text{eff}}[A_\mu]$ is or is not chiral invariant. So, we see that one cannot conclude that the chiral anomaly vanishes on the basis of the same arguments that we used to the case of gauge symmetry.

8. Conclusion

We considered the anomaly equation

$$\langle 0|(D_\mu)_{ab} J^\mu_b|0\rangle = \langle 0|\mathcal{A}_a(A_\mu)|0\rangle$$

and showed arguments to support the vanishing of both sides, independently. In one case (l.h.s.) we considered the expectation value of a fermionic/bosonic operator $(D_\mu)_{ab}J^\mu_b$ and showed that it vanishes. On the other side of the equation, we had the expectation value of a completely bosonic operator $\mathcal{A}_a(A_\mu)$, which was shown to be also zero. In both demonstrations we have something in common: gauge invariance of the *bosonic* measure. So, gauge invariance at quantum level seems to be determined by the behaviour of the fluctuations of the A_μ field. The conclusion is that gauge invariance of the fermionic measure does not seems to be important to guarantee gauge invariance of the full theory. It seems that gauge invariance is much more resistant than it is usually supposed.

However, we showed that there are signs indicating that the precise mechanisms behind the cancelling of the gauge anomaly in the abelian and non-abelian cases are very distinct. The cancellation of the anomaly in the abelian case seems to be a consequence of the Dyson–Schwinger equations extended to the Wess–Zumino fields. This could indicate that the anomaly should be dealt with as a subsidiary condition (similar to what happens in the case of a Proca model, when we obtain $\partial_\mu A^\mu$ as a consistency condition). This seems not to be the situation in the non-abelian case. What is the true mechanism behind the cancellation of the non-abelian anomaly is a question that deserves to be investigated further.

The chiral anomaly was shown not to be cancelled on the basis of the arguments that we presented. Although the proof is quite trivial, it is very important to stress this fact explicitly, because this anomaly has deep phenomenological implications (in an opposite way to the gauge anomaly, which is only used to find inconsistencies in the theory). It remains possibly different from zero, both in the vector and in the chiral cases.

The functional formalism points out the origin of the gauge anomaly and a road for its formal cancellation. Restoration of gauge symmetry implies a null expectation value for the anomaly. This cancellation suggests that anomalies are not an obstacle to the quantization of theories involving chiral fermions. The usual argument is that anomalies destroy Slavnov–Taylor identities, necessary to relate renormalization constants and prove the renormalizability of the theory. On the basis of our results, there is no reason to believe that Slavnov–Taylor identities are not preserved in a gauge anomalous theory. A detailed analysis of the perturbative renormalization procedure under this new perspective would be very important and will be considered in detail in the future. We already began to investigate it by considering insertions of the gauge anomaly in arbitrary gauge invariant correlators. Their vanishing points in the direction of a null anomaly in the physical subspace. However, we must also obtain progress in the gauge non-invariant case, if we want to understand the full dynamics of the gauge field in a so called gauge anomalous theory.

One can see that the main difference between the vector case (coupling to both chiralities) and the chiral case (coupling to just one of them) is that gauge invariance can be maintained at all steps of the quantization in the vector case, even before quantizing the gauge field. In the chiral case, this is not so. One has to go through the complete quantization of the theory to see gauge invariance again. But it seems to be present there, in the end.

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Appendix A

We present a simple demonstration of the vanishing of the v.e.v. of the gauge field. First we show that the derivative of a scalar operator has vanishing v.e.v.: let $U(\Lambda, a)$ denote the operator that represents a Poincaré transformation in Hilbert space (Λ is the Lorentz transformation and a is the vector that indicates translation). Then

$$\langle 0|\partial_\mu \Omega(x)|0\rangle = \langle 0|U^\dagger(1,x)\partial_\mu\Omega(0)U(1,x)|0\rangle = \langle 0|\partial_\mu\Omega(0)|0\rangle, \tag{70}$$

as expected from Poincaré invariance of the vacuum. Now, performing a pure Lorentz transformation,

$$\langle 0|\partial_\mu \Omega(0)|0\rangle = \langle 0|U^\dagger(\Lambda,0)\partial_\mu\Omega(0)U(\Lambda,0)|0\rangle = {\Lambda_\mu}^\nu\langle 0|\partial_\nu\Omega(0)|0\rangle. \tag{71}$$

This can only be true (for general Λ) if $\langle 0|\partial_\mu\Omega(0)|0\rangle = 0$. Now, we can consider the gauge field: using a similar argument for translations we find

$$\langle 0|A_\mu(x)|0\rangle = \langle 0|A_\mu(0)|0\rangle. \tag{72}$$

However, under a general Lorentz transformation, this field does not behaves as a tensor (see [25])

$$U^\dagger(\Lambda,0)A_\mu(x)(0)U(\Lambda,0) = {\Lambda_\mu}^\nu A_\nu(\Lambda x) + \partial_\mu\Omega(\Lambda,x), \tag{73}$$

with $\Omega(\Lambda,x)$ being a scalar field dependent on the particular Lorentz transformation under consideration. So,

$$\begin{aligned}\langle 0|A_\mu(0)|0\rangle &= \langle 0|U^\dagger(\Lambda,0)A_\mu(0)U(\Lambda,0)|0\rangle \\ &= {\Lambda_\mu}^\nu\langle 0|A_\nu(0)|0\rangle + \langle 0|\partial_\mu\Omega(\Lambda,0)|0\rangle \\ &= {\Lambda_\mu}^\nu\langle 0|A_\nu(0)|0\rangle = 0.\end{aligned} \tag{74}$$

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