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The limit distribution in the q-CLT for $q \geqslant 1$ is unique and can not have a compact support

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Abstract

In a paper by Umarov *et al* (2008 *Milan J. Math.* **76** 307–28), a generalization of the Fourier transform, called the q-Fourier transform, was introduced and applied for the proof of a q-generalized central limit theorem (q-CLT). Subsequently, Hilhorst illustrated (2009 *Braz. J. Phys.* **39** 371–9; 2010 *J. Stat. Mech.* P10023) that the q-Fourier transform for $q > 1$, is not invertible in the space of density functions. Indeed, using an invariance principle, he constructed a family of densities with the same q-Fourier transform and noted that 'as a consequence, the q-CLT falls short of achieving its stated goal'. The distributions constructed there have compact support. We prove now that the limit distribution in the q-CLT is unique and can not have a compact support. This result excludes all the possible counterexamples which can be constructed using the invariance principle and fills the gap mentioned by Hilhorst.

Keywords: q-central limit theorem, q-Fourier transform, q-Gaussian, invariance principle

1. Introduction

The q-central limit theorem (q-CLT), proved in [1] (see also [2]), deals with sequences of random variables of the form

$$Z_N = \frac{S_N - N\mu_q}{N^{\frac{1}{2(2-q)}}}, \tag{1}$$

where $S_N = X_1 + \cdots + X_N$, the random variables $X_1, \ldots X_N$ being identically distributed and q-independent, $\mu_q = \int x[f(x)]^q \mathrm{d}x$, and $1 \leqslant q < 2$. Here $f(x)$ is the density function of the random variable X_1. Without loss of generality one can assume that $\mu_q = 0$. Three types of q-independence were discussed in paper [1]. Namely, identically distributed random variables X_N are q-independent (see [1]) if

$$\text{Type I:} \qquad F_q[X_1 + \cdots + X_N](\xi) = F_q[X_1](\xi) \otimes_q \ldots \otimes_q F_q[X_N](\xi); \tag{2}$$

$$\text{Type II:} \qquad F_q[X_1 + \cdots + X_N](\xi) = F_q[X_1](\xi) \otimes_{q_1} \ldots \otimes_{q_1} F_q[X_N](\xi); \tag{3}$$

$$\text{Type III:} \qquad F_q[X_1 + \cdots + X_N](\xi) = F_{q_1}[X_1](\xi) \otimes_{q_1} \ldots \otimes_{q_1} F_{q_1}[X_N](\xi), \tag{4}$$

if these relationships hold for all $N \geqslant 2$ and $\xi \in (-\infty, \infty)$; $q_1 = \frac{1+q}{3-q}$. Here the operator F_q is the q-Fourier transform (q-FT) defined as

$$F_q[X_1](\xi) = \tilde{f}_q(\xi) \coloneqq \int_{-\infty}^{\infty} \frac{f(x)\,\mathrm{d}x}{[1 + \mathrm{i}(1-q)x\xi f^{q-1}(x)]^{\frac{1}{q-1}}}, \tag{5}$$

with $q > 1$. If $q \to 1 + 0$, then $F_q[X_1](\xi) \to F[X_1](\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}x\xi}\mathrm{d}x$, coinciding with the Fourier transform of f.

The q-CLT states that if $X_1, \ldots X_N$ are identically distributed and q-independent random variables, then the sequence Z_N in (1) weakly converges to a random variable with the q_{-1}-Gaussian density; see [1] for details.

The invertibility of q-FT in the class of q-Gaussian densities is established in [3] and in the space of hyper-functions in [4, 5]. However, using a specific invariance principle Hilhorst [6, 7] showed that q-FT is not invertible in the entire space of densities. He constructed a family of densities containing the q-Gaussian and with the same q-FT. Any density of this family except the q-Gaussian has a compact support. In the present note we establish that a limit distribution in q-CLT can not have a compact support. This fact implies that all the distributions with compact support in Hilhorst's counterexamples can not be a limiting distribution in the q-CLT, except the q-Gaussian density. However, deformations used in the invariance principle with functions $H(\xi)$, $H(0) = 0$, lead to distributions, which have non-compact support and share the same asymptotic behavior at infinity as the q_{-1}-Gaussian. We prove that the limit distribution Z_∞ and any of its H-deformation has the same $(2q-1)$-variance if and only if the deforming function $H(\xi)$ is identically zero. Using this fact and intrinsic properties of q-independent random variables we prove the uniqueness of the distribution of the scaling limit of q-independent random variables. This fact rules out all the possible counterexamples indicated by Hilhorst in his paper [6]. Thus, the q-FT is used only for the existence of limiting distribution, while intrinsic properties of q-independent random variables supply the uniqueness of this limiting distribution. We note that the inverse q-FT is nowhere required in the present proof.

Now let us recall some facts about q-algebra, q-exponential and q-logarithmic functions. By definition, the *q-sum* of two numbers is defined as $x \oplus_q y = x + y + (1-q)xy$. The q-sum is commutative, associative, recovers the usual summing operation if $q = 1$ (i.e. $x \oplus_1 y = x + y$), and preserves 0 as the neutral element (i.e. $x \oplus_q 0 = x$). The *q-product* for x, y is defined by the binary relation $x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$. This operation also commutative, associative, recovers the usual product when $q = 1$, and preserves 1 as the unity. The q-product is defined only when $x^{1-q} + y^{1-q} \geqslant 1$. The q-exponential and q-logarithmic functions are respectively defined as (see for instance [1])

$$\exp_q(x) = [1 + (1-q)x]_+^{\frac{1}{1-q}} \quad (\exp_1(x) = e^x),$$

and

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q},\ x > 0 \quad (\ln_1(x) = \ln x).$$

It is easy to see (see [1]) that for q-exponential, the relations $\exp_q(x \oplus_q y) = \exp_q(x)\exp_q(y)$ and $\exp_q(x+y) = \exp_q(x) \otimes_q \exp_q(y)$ hold. In terms of q-log these relations can be equivalently rewritten as follows: $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$, and $\ln_q(xy) = \ln_q x \oplus_q \ln_q y$. It follows from the definition of q-logarithm that if $1 < q_1 < q_2$, then

$$\ln_{q_2}(x) \geqslant \frac{q_1 - 1}{q_2 - 1} \ln_{q_1}(x) \quad \text{for all } x > 1. \tag{6}$$

For $q > 1$ the q-exponential is defined for all $x < \frac{1}{q-1}$ and blows up at the point $x = \frac{1}{q-1}$. The q-exponential can also be extended to the complex plane and it is bounded on the imaginary axis: $|\exp_q(\mathrm{i}y)| \leqslant 1$. Moreover, $|\exp_q(\mathrm{i}y)| \to 0$ if $|y| \to \infty$. Using the q-exponential function, the q-FT of f can be represented in the form

$$\tilde{f}_q(\xi) = \int_{-\infty}^{\infty} f(x) \exp_q(\mathrm{i}x\xi [f(x)]^{q-1})\, \mathrm{d}x. \tag{7}$$

We refer the reader to the papers [1, 2, 8–16] for various properties and applications of the q-FT. Also, functions of the form $\exp_q(-\beta x^2)$ $(\beta > 0)$ will be hereafter referred to as q-Gaussians.

At this point, before addressing the technical aspects of the present problem, let us remind why the q-CLT may be very relevant in physics and other disciplines. It is common belief that the ubiquity of Gaussians in nature and elsewhere is due to the classical CLT. Indeed, this theorem provides a mathematical basis for observing the Gaussian attractors under quite general circumstances involving many *independent* (or quasi-independent) random variables. Analogously, also q-Gaussians emerge ubiquitously in nature and elsewhere, which strongly suggests the existence of a wide class of many *correlated* random variables whose corresponding attractors are q-Gaussians instead of Gaussians. Such experimental and theoretical examples include anomalous diffusion in type-II superconductors [17] and granular matter [18], non-Gaussian momenta distributions for cold atoms in optical lattices [19–21], dirty plasma [22], trapped atoms [23], area-preserving maps [24], high-energy physics [25], probabilistic models [26], to mention but a few (see [27]).

2. On the support of the limit distribution

For the sake of simplicity we consider a continuous and symmetric about zero density function f of a random variable X_1. Other cases can be considered in a similar manner with appropriate care. Denote $\lambda(x) = x[f(x)]^{q-1}$, where $1 \leqslant q < 2$. Since f is symmetric, it suffices to consider $\lambda(x)$ only for positive x. Suppose the maximum value of λ is m and $x_m > 0$ is the rightmost point where λ attains its maximum, i.e. $m = \max_{0<x\leqslant a}\{x[f(x)]^{q-1}\} = x_m[f(x_m)]^{q-1}$. Let

$$\tau_* = \begin{cases} \frac{1}{m(q-1)}, & \text{if } 0 < q < 2, \\ \infty, & \text{if } q = 1. \end{cases}$$

Proposition 2.1. *Let f be a continuous symmetric density function with* $\operatorname{supp} f \subseteq [-a, a]$. *Then the q-FT of f satisfies the following estimate*

$$|\tilde{f}_q(\eta - \mathrm{i}\tau)| \leqslant \exp_q(x_m M_q \tau), \tag{8}$$

where $\eta \in (-\infty, \infty)$, $\tau < \tau_*$, $M_q = \max_{[0,a]}\{[f(x)]^{q-1}\}$, *and x_m is the rightmost point where xf^{q-1} attains its maximum m.*

Proof. For f with $\operatorname{supp} f \subseteq [-a, a]$, equation (5) takes the form

$$\tilde{f}_q(\xi) = \int_{-a}^{a} \frac{f(x)\mathrm{d}x}{[1 + \mathrm{i}(1-q)x\xi f^{q-1}(x)]^{\frac{1}{q-1}}}. \tag{9}$$

Let $\xi = \eta - \mathrm{i}\tau$ where $\eta = \Re(\xi)$ is the real part of ξ and $-\tau = \Im(\xi)$ is its imaginary part. We assume that $\eta \in (-\infty, \infty)$ and $|\tau| < \frac{1}{m(q-1)}$. Then for the denominator of the integrand in (9) one has

$$\begin{aligned}
&[1 + \mathrm{i}(1-q)x(\eta - \mathrm{i}\tau)f^{q-1}(x)]^{\frac{1}{q-1}} = [1 + \mathrm{i}(1-q) \times \eta f^{q-1}(x) + (1-q)\tau x f^{q-1}(x)]^{\frac{1}{q-1}} \\
&= [1 + (1-q)\tau x f^{q-1}(x)]^{\frac{1}{q-1}} \left[1 + \mathrm{i}\frac{(1-q) \times \eta f^{q-1}(x)}{1 + (1-q)\tau x f^{q-1}(x)}\right]^{\frac{1}{q-1}} \\
&= (\exp_q(\tau x f^{q-1}(x)))^{-1} \left(\exp_q(\mathrm{i}\frac{(1-q) \times \eta f^{q-1}(x)}{1 + (1-q)\tau x f^{q-1}(x)})\right)^{-1}.
\end{aligned} \tag{10}$$

Using the inequality $|\exp_q(\mathrm{i}y)| \leqslant 1$ valid for all $y \in (-\infty, \infty)$ if $q \geqslant 1$, it follows from (10) that

$$|1 + \mathrm{i}(1-q)x(\eta - \mathrm{i}\tau)f^{q-1}(x)|^{\frac{1}{q-1}} \geqslant (\exp_q(\tau x f^{q-1}(x)))^{-1},$$

or

$$|1 + \mathrm{i}(1-q)x(\eta - \mathrm{i}\tau)f^{q-1}(x)|^{\frac{1}{1-q}} \leqslant (\exp_q(\tau x f^{q-1}(x))). \tag{11}$$

Now, (9) together with (11) and $f(x)$ being a density function, yield (8). □

Remark 2.2. Proposition (2.1) can be viewed as a generalization of the well known Paley–Wiener theorem. Indeed, if $q = 1$ then (8) takes the form

$$|\tilde{f}(\eta - \mathrm{i}\tau)| \leqslant \exp(a|\tau|),\ \eta + \mathrm{i}\tau \in \mathcal{C}, \tag{12}$$

which represents the Paley–Wiener theorem for continuous density functions.

Inequality (12) can be used for estimation of the size of the support of f. Consider an example with $f(x) = (2a)^{-1}\mathcal{I}_{[-a,a]}(x)$, where $\mathcal{I}_{[-a,a]}(x)$ is the indicator function of the interval $[-a, a]$. The Fourier transform of this function is $\tilde{f}(\xi) = (a\xi)^{-1}\sin(a\xi)$, $M_q = M_1 = 1$, and $x_m = a$. Therefore, we have $|\tilde{f}(-\mathrm{i}\tau)| \leqslant e^{a\tau}$, $\tau > 0$. The latter yields

$$2a \geqslant 2\sup_{\tau>0} \frac{\ln|\tilde{f}(-\mathrm{i}\tau)|}{\tau},$$

which gives an estimate from below for the size $d(f) = 2a$ of the support of f. This idea can be used to estimate the size of the support of f using the q-FT and proposition 2.1. Namely, inequality (8) with $\eta = 0$ implies

$$d(f) = 2a \geqslant 2x_m \geqslant \frac{2}{M_q} \sup_{0<\tau<\tau_*} \frac{\ln_q |\tilde{f}_q(-\mathrm{i}\tau)|}{\tau}. \tag{13}$$

We notice that the integrand in the integral

$$\tilde{f}_q(-\mathrm{i}\tau) = \int_{-a}^{a} \frac{f(x)\mathrm{d}x}{[1-(q-1)\tau x f^{q-1}(x)]^{\frac{1}{q-1}}}$$

is strictly grater than $f(x)$ if $\tau > 0$, implying $|\tilde{f}_q(-\mathrm{i}\tau)| > 1$, since f is a density function. Therefore, the right-hand side of (13) is positive and gives indeed an estimate of the size of the support of f from below.

Let $f_N(x) = f_{S_N}(x)$ be the density function of $S_N = X_1 + \cdots + X_N$, where $X_1, \ldots, X_N$ are q-independent random variables with the same density function $f = f_{X_1}$ whose support is $[-a, a]$. We show that the q-independence condition can not reduce the support of f_N to an interval independent of N. More precisely, $d(f_N)$ increases at the rate of N when $N \to \infty$.

Theorem 2.3. *Let $X_1, \ldots, X_N$ be q-independent of any type I–III random variables all having the same density function f with* $\operatorname{supp} f \subseteq [-a, a]$. *Then, for the size of the density f_N of S_N, there exists a constant $K_q > 0$ such that the estimate*

$$d(f_N) \geqslant K_q N \sup_{0<\tau<\tau_*} \frac{\ln_q |\tilde{f}_q(-\mathrm{i}\tau)|}{\tau} \tag{14}$$

holds.

Proof. Using formula (13) one has

$$d(f_N) \geqslant \frac{2}{M_{q,N}} \sup_{0<\tau<\tau_*} \frac{\ln_q |\widetilde{(f_N)}_q(-\mathrm{i}\tau)|}{\tau}, \tag{15}$$

where $M_{q,N} = \max_{x\in[-Na,Na]} f_N^{q-1}(x)$. It is clear from probabilistic arguments that $M_{q,N} \leqslant M_q$ for all $N \geqslant 2$. Therefore, it follows from (15) that

$$d(f_N) \geqslant \frac{2}{M_q} \sup_{0<\tau<\tau_*} \frac{\ln_q |\widetilde{(f_N)}_q(-\mathrm{i}\tau)|}{\tau}, \tag{16}$$

Let X_N be q-independent of type I (see (2)). Making use of the inequality $|z - r| \geqslant |z| - r$, which holds true for any complex z and positive integer number r, one has

$$\begin{aligned} |\widetilde{(f_N)}_q(-\mathrm{i}\tau)| &= |\tilde{f}_q(-\mathrm{i}\tau) \otimes_q \ldots \otimes_q \tilde{f}_q(-\mathrm{i}\tau)| \\ &= \left| [N(\tilde{f}_q(-\mathrm{i}\tau))^{1-q} - (N-1)]^{\frac{1}{1-q}} \right| \\ &\geqslant [N|\tilde{f}_q(-\mathrm{i}\tau)|^{1-q} - (N-1)]^{\frac{1}{1-q}} \\ &= |\tilde{f}_q(-\mathrm{i}\tau)| \otimes_q \ldots \otimes_q |\tilde{f}_q(-\mathrm{i}\tau)|. \end{aligned}$$

Taking q-logarithm of both sides in this inequality and using the property $\ln_q(g \otimes_q h) = \ln_q g + \ln_q h$, one obtains

$$\ln_q |\widetilde{(f_N)}_q(-\mathrm{i}\tau)| \geqslant N \ln_q |\tilde{f}_q(-\mathrm{i}\tau)|. \tag{17}$$

Now estimate (14) follows from inequalities (16) and (17).

Similarly, for random variables independent of type III, we have

$$\ln_q |\widetilde{(f_N)}_q(-\mathrm{i}\tau)| \geqslant N \ln_{q_1} |\tilde{f}_{q_1}(-\mathrm{i}\tau)|. \tag{18}$$

For random variables X_N independent of type II, equation (17) takes the form

$$\ln_{q_1} |\widetilde{(f_N)}_q(-\mathrm{i}\tau)| \geqslant N \ln_{q_1} |\tilde{f}_q(-\mathrm{i}\tau)|. \tag{19}$$

Since $1 < q < q_1$ and $\frac{q-1}{q_1-1} = \frac{3-q}{2}$, making use of inequalitiy (6), and taking into account that $|\widetilde{(f_N)}_q(\mathrm{i}\tau)| \geqslant 1$, one has

$$\ln_{q_1} |\widetilde{(f_N)}_q(-\mathrm{i}\tau)| \geqslant \frac{(3-q)}{2} \ln_q |\widetilde{(f_N)}_q(-\mathrm{i}\tau)|,$$

which implies

$$\ln_q |\widetilde{(f_N)}_q(-\mathrm{i}\tau)| \geqslant \frac{2N}{3-q} \ln_q |\tilde{f}_q(-\mathrm{i}\tau)|. \tag{20}$$

Both (18) and (20) obviously imply estimate (14). □

Corollary 2.4. *Let $X_1, \ldots, X_N$ be q-independent of any type I–III random variables all having the same density function f with* $\operatorname{supp} f \subseteq [-a, a]$. *If the sequence Z_N has a distributional limit random variable in some sense, then this random variable can not have a density with compact support. Moreover, due to the scaling present in Z_N, the support of the limit variable is the entire set of real numbers.*

The proof obviously follows immediately from (14) upon letting $N \to \infty$.

3. On the variance and quasivariance of a limit distribution

Let $1 \leqslant q < 2$ and a random variable X with a density function $f(x)$ has zero q-mean ($\mu_q(X) = 0$) and a finite quasivariance

$$QV(X) = \nu_{2q-1}(X)\sigma^2_{2q-1}(X) = \int (x - \mu_q)^2 [f(x)]^{2q-1} \mathrm{d}x, \tag{21}$$

where $\nu_{2q-1}(X)$, μ_q, and $\sigma_{2q-1}(X)$ are defined as

$$\nu_q = \nu_q(X) = \int [f(x)]^q \mathrm{d}x, \; \mu_q = \int x [f(x)]^q \mathrm{d}x, \tag{22}$$

and

$$\sigma^2_{2q-1} = \sigma^2_{2q-1}(X) = \int (x - \mu_q)^2 \frac{[f(x)]^{2q-1}}{\nu_{2q-1}} \mathrm{d}x. \tag{23}$$

Note that if $q = 1$ then $\nu_q = 1$ and $\sigma^2_{2q-1} = \mathrm{Var}(X)$, the variance of X, implying $QV(X) = \mathrm{Var}(X)$. As above, without loss of generality, we assume that $\mu_q = 0$. If X and Y are q-independent (of any type I–III) random variables, then for their quasivariances the relation

$$QV(X + Y) = QV(X) + QV(Y) \tag{24}$$

holds. To see the validity of this fact one can use the formula $(F_q[X])''(0) = -qQV(X)$ (see [28]) and the definition of q-independence (2) and (3). Taking into account the fact that the

density of aX for a constant $a > 0$ is $a^{-1}f(x/a)$, one can easily verify that [1]

$$\sigma_{2q-1}(aX) = a^2\sigma_{2q-1}(X). \tag{25}$$

Let X_1 be a random variable with the q-Gaussian density

$$G_q(\beta, x) = \frac{\sqrt{\beta}}{C_q}e_q^{-\beta x^2},\ \beta > 0,$$

where C_q is the normalizing constant [1]. The direct calculation shows that $\sigma^2_{2q-1}(X_1) = \beta^{-1}$. The sequence of identically distributed q-Gaussian variables $X_1, \ldots, X_N$ is q-independent of type II if the density of $X_1 + \cdots + X_N$ is the $G_{(N^{-\frac{1}{2-q}}\beta,\ x)}$; (see [1]). Using (25), one can see that

$$\sigma^2_{2q-1}(X_1 + \cdots + X_N) = N^{\frac{1}{2-q}}\sigma^2_{2q-1}(X_1), \tag{26}$$

and

$$\sigma^2_{2q-1}(Z_N) = \frac{1}{N^{\frac{1}{2-q}}}\sigma^2_{2q-1}(X_1 + \cdots + X_N) = \sigma^2_{2q-1}(X_1), \quad \text{for all } N \geqslant 1. \tag{27}$$

If $q = 1$ then (26) reduces to the known relationship $\mathrm{Var}(X_1 + \cdots + X_N) = N\mathrm{Var}(X_1)$ valid for variances of independent and identically distributed (i.i.d.) random variables $X_1, \ldots, X_N$. In this case (27) becomes $\mathrm{Var}(Z_N) = \mathrm{Var}(X_1)$, valid for rescaled sums of i.i.d. random variables. In other words the equality

$$QV(Z_N) = QV(X_1) \tag{28}$$

holds if $q = 1$.

We notice that relations (24) and (26) imply

$$\nu_{2q-1}(X_1 + \cdots + X_N) = \frac{\nu_{2q-1}(X_1)}{N^{\frac{q-1}{2-q}}}. \tag{29}$$

Indeed, due to (24)

$$\begin{aligned}\sigma^2_{2q-1}(X_1 + \cdots + X_N) &= \frac{QV(X_1 + \cdots + X_N)}{\nu_{2q-1}(X_1 + \cdots + X_N)}\\ &= \frac{QV(X_1) + \cdots + QV(X_N)}{\nu_{2q-1}(X_1 + \cdots + X_N)}\\ &= \frac{NQV(X_1)}{\nu_{2q-1}(X_1 + \cdots + X_N)}\\ &= N\frac{\nu_{2q-1}(X_1)\sigma^2_{2q-1}(X_1)}{\nu_{2q-1}(X_1 + \cdots + X_N)}.\end{aligned}$$

The latter and equality (26) imply (29).

In the case $q = 1$ for any i.i.d. random variables $\nu_{2q-1}(X_1 + \cdots + X_N) = \int f_N(x)\mathrm{d}x = 1$, where f_N is the density function of the sum $X_1 + \cdots + X_N$. However, if $q > 1$ then $\nu_{2q-1}(X_1 + \cdots + X_N)$ does depend on N, and as relation (29) shows, the most natural dependence on N can be given by the condition

$$\nu_{2q-1}(X_1 + \cdots + X_N) \sim O\left(N^{\frac{1-q}{2-q}}\right),\ N \to \infty. \tag{30}$$

Let us consider some examples. If $q = 1$ then for any i.i.d. sequence of random variables the relation (29) is reduced to the identity $1 = 1$, thus satisfying condition (30). As the above example states, for the type II q-i.i.d. q-Gaussian random variables relation (29) is valid, thus again satisfying condition (30). One can verify that for q-Gaussians independent of type I or III condition (30) is also verified. Random variables studied in [29] also satisfy condition (30) since they are asymptotically equivalent to q-indpendent random variables (see [29]). As is shown in [30] random variables in [29] are variance mixtures of normal densities. This gives a strong evidence of the fact that the subclass of variance mixtures of normal densities leading to q-Gaussians will also satisfy (30). For connection of variance mixtures to superstatistics developed by Beck and Cohen [31] see [30]. In our further considerations we assume condition (30) for q-i.i.d. random variables $X_1, \ldots, X_N$.

The asymptotic expansion of the q-exponential function $\exp_q(x)$ near zero implies that (see proposition II.3 in [2], case $\alpha = 2$)

$$F_q[X](\xi) = 1 - \frac{q}{2} QV(X)\xi^2 + o(\xi^2),\ \xi \to 0. \tag{31}$$

Making use of properties of the q-Fourier transform one can see that (see details in [1])

$$F_q[Z_N](\xi) = 1 - \frac{q}{2} QV(X_1)\xi^2 + o\left(\frac{\xi^2}{N}\right),\ N \to \infty, \tag{32}$$

which shows that $QV(Z_N) = QV(X_1)$, $N \geqslant 1$. Hence, relation (28) is valid not only for $q = 1$, but for all $1 < q < 2$, as well. This immediately implies that if the limit distribution $Z_\infty = \lim_{N\to\infty} Z_N$ exists in some sense, then its quasivariance must be equal to $QV(X_1)$, i.e.

$$QV(Z_\infty) = QV(X_1). \tag{33}$$

The lemma below will be used in section 5.

Lemma 3.1. *Let $\omega(x)$ be a continuous function defined on $[0, \infty)$ such that*

(a) $\omega(1) = 0$,
(b) $\omega(x) > 0$ *on* $(0, 1)$, *and* $\omega(x) < 0$ *on* $(1, \infty)$,
(c) $\int_0^1 \omega(x)\mathrm{d}x = \int_1^\infty |\omega(x)|\mathrm{d}x$.

Then

$$\int_0^1 x^2\omega(x)\mathrm{d}x < \int_1^\infty x^2|\omega(x)|\mathrm{d}x. \tag{34}$$

Proof. Since $x^2 < 1$ for $x \in (0, 1)$, one has

$$\int_0^1 x^2\omega(x)\mathrm{d}x < \int_0^1 \omega(x)\mathrm{d}x. \tag{35}$$

Similarly, for $x > 1$,

$$\int_1^\infty |\omega(x)|\mathrm{d}x < \int_1^\infty x^2|\omega(x)|\mathrm{d}x. \tag{36}$$

Now condition (c) and estimates (35) and (36) imply (34). □

4. On the invariance principle and Hilhorst's counterexamples

In this section first we recall the invariance principle used by Hilhorst [6] to construct counterexamples which show that q-FT is not invertible. Then we apply the invariance principle to the q-Gaussian and study properties of densities produced by the invariance principle in this case. Let $f(x)$, $x \in (-\infty, \infty)$, be a symmetric density function, such that $\lambda(x) = x[f(x)]^{q-1}$ restricted to the semiaxis $[0, \infty)$ has a unique (local) maximum m at a point x_m. In other words $\lambda(x)$ has two monotonic pieces, $\lambda_-(x)$, $0 \leqslant x \leqslant x_m$, and $\lambda_+(x)$, $x_m \leqslant x < \infty$. Let $x_\pm(\xi)$, $0 \leqslant \xi \leqslant m$, denote the inverses of $\lambda_\pm(x)$, respectively. Then the q-FT ($1 < q < 2$) of f can be expressed in the form, see [6]

$$\tilde{f}_q(\xi) = \int_{-\infty}^{\infty} F(\xi')\exp_q(\mathrm{i}\xi\xi')\mathrm{d}\xi',$$

where

$$F(\xi) = \frac{q-2}{q-1}\xi^{\frac{1}{q-1}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[x_-^{\frac{q-1}{q-2}}(\xi) - x_+^{\frac{q-1}{q-2}}(\xi)\right],\ \xi \in [0, m]. \tag{37}$$

Then the invariance principle yields

$$F(\xi) = \frac{q-2}{q-1}\xi^{\frac{1}{q-1}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[X_-^{\frac{q-1}{q-2}}(\xi) - X_+^{\frac{q-1}{q-2}}(\xi)\right],\ \xi \in [0, m], \tag{38}$$

where

$$X_\pm^{\frac{q-1}{q-2}}(\xi) = x_\pm^{\frac{q-1}{q-2}}(\xi) + H(\xi), \tag{39}$$

with $H(\xi)$ being a function defined on $[0, m]$, and such that $X_\pm(\xi)$ are invertible. Denote by $\Lambda(x)$ the function defined by the two pieces of inverses of $X_\pm(\xi)$, namely

$$\Lambda_H(x) = \begin{cases} X_-^{-1}(x), & \text{if } 0 \leqslant x \leqslant x_{m,H}, \\ X_+^{-1}(x), & \text{if } x > x_{m,H}, \end{cases}$$

where $x_{m,H} = \left[(q-1)^{\frac{q-1}{2(2-q)}} + H(m)\right]^{-\frac{2-q}{q-1}}$. The function $\Lambda_H(x)$ is continuous, since $X_-^{-1}(x_{m,H}) = X_+^{-1}(x_{m,H})$. Then

$$f_H(x) = \left(\frac{\Lambda(x)}{x}\right)^{\frac{1}{q-1}} \tag{40}$$

defines a density function with the same q-FT as of f. The density f_H coincides with f if $H(\xi)$ is identically zero.

Now assume that $f(x)$ is a q-Gaussian

$$f(x) = G_q(x) = \frac{C_q}{[1 + (q-1)x^2]^{\frac{1}{q-1}}},\ 1 < q < 2, \tag{41}$$

where C_q is the normalization constant. Obviously, $G_q(x)$ is symmetric, and the function $\lambda_q(x) = x[G_q(x)]^{q-1}$ considered on the semiaxis $[0, \infty)$ has a unique maximum $m = \frac{C_q^{q-1}}{2\sqrt{q-1}}$ attained at the point

$$x_m = (q-1)^{-\frac{1}{2}}. \tag{42}$$

Moreover, the functions $x_{\pm}(\xi)$ in this case take the forms

$$x_{\pm}(\xi) = \frac{C_q^{q-1} \pm [C_q^{2(q-1)} - 4(q-1)\xi^2]^{\frac{1}{2}}}{2\xi(q-1)}, \; 0 < \xi \leqslant m. \tag{43}$$

We denote the density $f_H(x)$ and the function $\Lambda_H(x)$ corresponding to the q-Gaussian by $G_{q,H}(x)$ and $\Lambda_{q,H}(x)$, respectively. Hilhorst, selecting $\mathrm{H}(\xi) = \mathrm{A} \geqslant 0$ constant, constructed a family of densities

$$G_{q,A}(x) = \frac{C_q \left(x^{\frac{q-2}{q-1}} - A\right)^{\frac{1}{q-2}}}{x^{\frac{1}{q-1}} \left[1 + (q-1)\left(x^{\frac{q-2}{q-1}} - A\right)^{2\frac{q-1}{q-2}}\right]^{\frac{1}{q-1}}}, \tag{44}$$

which have the same q-FT as the q-Gaussian for all A. The following statement shows that none of the densities $G_{q,A}(x)$ can serve as the limit distribution in the q-CLT, except the one, corresponding to $A = 0$, which coincides with the q-Gaussian, $G_{q,0}(x) = G_q(x)$.

Proposition 4.1. *Let $H(0) > 0$. Then the support of $G_{q,H}(x)$ is compact, and*

$$\text{supp } G_{q,H} = \left[-(H(0))^{\frac{q-1}{q-2}}, (H(0))^{\frac{q-1}{q-2}}\right].$$

Proof. Since $\lim_{\xi \to 0} x_+(\xi) = +\infty$, the largest value of X_+ is equal to $\lim_{\xi \to 0} X_+(\xi) = [H(0)]^{\frac{q-1}{q-2}}$. Therefore, the inverse of X_+ is defined on the interval $\left[x_0, [H(0)]^{\frac{q-1}{q-2}}\right]$, where $x_0 > 0$ is some number obtained by **a** shifting of x_m depending on $H(m)$. On the other hand the smallest value of x_- is zero, taken at $\xi = 0$. Therefore, the inverse of X_- is defined on the interval $[0, x_0]$. Hence, by symmetry, $G_{q,H}$ has the support $\left[-[H(0)]^{\frac{q-1}{q-2}}, [H(0)]^{\frac{q-1}{q-2}}\right]$. □

Remark 4.2. Note that $H(0)$ can not be negative. In fact, if $H(0) < 0$, then either $X_{\pm}$ is not invertible or, if it is invertible, its inverse does not define a density function.

Proposition 4.1 implies that if $H(0) > 0$ then, due to corollary 2.4, $G_{q,H}(x)$ can not be the density function of the limit distribution in the q-CLT. Thus none of the densities in Hilhorst's counterexamples[4], except the q-Gaussian, can serve as an attractor in the q-CLT.

Only one possibility is left, namely $H(0) = 0$. The next proposition establishes that, in this case, $G_{q,H}(x)$ is asymptotically equivalent to $G_q(x) \equiv G_{q,0}(x)$.

Proposition 4.3. *Let $H(0) = 0$. Then*

$$\lim_{|x| \to \infty} \frac{G_{q,H}(x)}{G_q(x)} = 1.$$

Proof. Since $H(0) = 0$, then obviously

$$\lim_{\xi \to 0} \frac{X_+(\xi)}{x_+(\xi)} = \lim_{\xi \to 0} \left(1 + \frac{H(\xi)}{x_+(\xi)}\right) = 1.$$

[4] See examples 2 and 3 in [6]. Example 4 is not relevant to the q-CLT, since in this case, $(2q-1)$-variance of the 2-Gaussian does not exist, and consequently the q-CLT is not applicable.

Therefore, for inverses one has

$$\lim_{x\to+\infty}\frac{X_+^{-1}(x)}{x_+^{-1}(x)}=1.$$

This implies

$$\lim_{x\to+\infty}\frac{G_{q,H}(x)}{G_q(x)}=\lim_{x\to+\infty}\left(\frac{\frac{X_+^{-1}(x)}{x}}{\frac{x_+^{-1}(x)}{x}}\right)^{\frac{1}{q-1}}=1.$$

□

Remark 4.4. Propositions 4.1 and 4.3 establish that $G_{q,H}$ can identify a limiting distribution in the q-CLT only if $H(0)=0$. However, in this case, independently from other values of $H(\xi)$, the density $G_{q,H}(x)$ is asymptotically equivalent to the q-Gaussian, i.e. $G_{q,H}(x)\sim G_q(x)$ as $|x|\to\infty$.

The statement of the following proposition can be proved exactly as proposition 4.3, replacing $X_+(\xi)$, $x_+(\xi)$ by functions $X_-(\xi)$, $x_-(\xi)$, respectively.

Proposition 4.5. *Let $H(0)=0$. Then*

$$\lim_{x\to0}\frac{G_{q,H}(x)}{G_q(x)}=1.$$

5. On the uniqueness of the limit distribution

Let X be a random variable with a symmetric density function G and let G_H be the density function obtained from G by H-deformation, where $H(\xi)$ is a continuous function such that $H(0)=0$ and does not change its sign on the interval $(0, x_m)$. Denote by X_H the random variable corresponding to the density function G_H.

Lemma 5.1. *Let X and X_H be random variables with the respective densities G and G_H, and let $QV(X)=QV(X_H)$. Then $\sigma^2_{2q-1}(X)=\sigma^2_{2q-1}(X_H)$ if and only if $H(\xi)$ is identically zero.*

Proof. *Sufficiency.* Let $\sigma^2_{2q-1}(X)=\sigma^2_{2q-1}(X_H)$ and assume that $H(\xi)$ is not identically zero. This equality together with $QV(X)=QV(X_H)$ implies that $\nu_{2q-1}(X)=\nu_{2q-1}(X_H)$. Due to conditions on $H(\xi)$ both densities, G and G_H, are symmetric, decreasing on the positive semiaxis. Propositions 4.1 and 4.5 imply that $G(0)=G_H(0)$ since $H(0)=0$. Moreover, since both G and G_H are densities there is a point $a>0$ such that $G(a)=G_H(a)$. Depending on the sign of $H(\xi)$, we have either

$$G(x)>G_H(x)\quad\text{on the interval}\quad(0,a)\quad\text{and}\quad G(x)<G_H(x)\quad\text{on the interval}\quad(a,\infty),\tag{45}$$

or

$$G(x)<G_H(x)\quad\text{on}\quad(0,a)\quad\text{and}\quad G(x)>G_H(x)\quad\text{on}\quad(a,\infty).\tag{46}$$

If necessary, switching the order of G and G_H we can always assume that condition (45) holds. Notice, that the case $G(x) \equiv G_H(x)$ is obviously excluded, since $H(\xi)$ is not identically zero. Further, due to proposition 4.3, G and G_H share the same asymptotic behavior at infinity: $G(x) \sim G_H(x)$, $x \to \infty$. Since G and G_H are symmetric about the origin, it suffices to consider these functions only for $x \geqslant 0$. Furthermore, it follows from (45) that $G^{2q-1}(x) > G_H^{2q-1}(x)$ on the interval $[0, a)$, and $G^{2q-1}(x) < G_H^{2q-1}(x)$ on the interval (a, ∞).

Consider the function $\omega(x) = a[G^{2q-1}(ax) - G_H^{2q-1}(ax)]$. This function is continuous by construction. Moreover, $\omega(1) = 0$, $\omega(x) > 0$ if $x \in (0, 1)$, and $\omega(x) < 0$ if $x > 1$. The existence of finite $(2q-1)$-variances of X and X_H implies that $\int_1^\infty x^2|\omega(x)|\mathrm{d}x < \infty$. The calculations below, where the symmetry of densities are taken into account, show that $\omega(x)$ satisfies condition (c) of lemma 3.1 as well:

$$\begin{aligned} 2\int_0^1 \omega(x)\mathrm{d}x &= a\int_{-1}^1 (G^{2q-1}(ax) - G_H^{2q-1}(ax))\mathrm{d}x = \int_{-a}^a (G^{2q-1}(x) - G_H^{2q-1}(x))\mathrm{d}x \\ &= \int_{-a}^a G^{2q-1}(x)\mathrm{d}x - \int_{-a}^a G_H^{2q-1}(x)\mathrm{d}x \\ &= \nu_{2q-1}(X) - \int_{|x|\geqslant a} G^{2q-1}(x)\mathrm{d}x - \left[\nu_{2q-1}(X_H) - \int_{|x|\geqslant a} G_H^{2q-1}(x)\mathrm{d}x\right] \\ &= \int_{|x|\geqslant a} |G^{2q-1}(x) - G_H^{2q-1}(x)|\,\mathrm{d}x = 2\int_1^\infty |\omega(x)|\mathrm{d}x. \end{aligned}$$

Here we have taken into account the equality $\nu_{2q-1}(X) = \nu_{2q-1}(X_H)$. It follows from lemma 3.1 that

$$\int_0^1 x^2\omega(x)\mathrm{d}x - \int_1^\infty x^2|\omega(x)|\mathrm{d}x < 0, \tag{47}$$

which is equivalent to

$$\int_0^a x^2(G^{2q-1}(x) - G_H^{2q-1}(x))\mathrm{d}x - \int_a^\infty x^2(G_H^{2q-1}(x) - G^{2q-1}(x))\mathrm{d}x < 0. \tag{48}$$

Inequality (48) is the same as $QV(X) < QV(X_H)$. Switching the order of G and G_H in the above analysis one can see that (46) implies $QV(X) > QV(X_H)$. Both obtained relations contradict to equality $QV(X) = QV(X_H)$. Hence, our assumption on $H(\xi)$ is wrong. Thus, we conclude that $H(\xi) \equiv 0$.

The necessity is obvious, since $H(\xi) \equiv 0$ immediately implies $G_H = G$, which consequently yielding $\sigma^2_{2q-1}(X) = \sigma^2_{2q-1}(X_H)$. □

Theorem 5.2. *Let X_N be a q-i.i.d. random variables with zero q-mean and finite quasivariance. Then the sequence Z_N defined in (1) has a unique limit distribution.*

Proof. The existence of a limit distribution was proved in [1]. Suppose that there are two limit distributions Z_∞ and Z_H of the sequence Z_N with respective distinct densities $G(x)$ and $G_H(x)$. Due to (33), both distributions have the same quasivariance

$$QV(Z_\infty) = QV(Z_H) = QV(X_1). \tag{49}$$

Moreover, due to condition (30),

$$\sigma^2_{2q-1}(Z_N) = \sigma^2_{2q-1}\left(\frac{X_1+\cdots+X_N}{N^{\frac{1}{2(2-q)}}}\right) = \frac{1}{N^{\frac{1}{2-q}}}\sigma^2_{2q-1}(X_1+\cdots+X_N)$$
$$= \frac{1}{N^{\frac{1}{2-q}}}\frac{NQV(X_1)}{\nu_{2q-1}(X_1+\cdots+X_N)}$$
$$= \frac{1}{N^{\frac{q-1}{2-q}}}\frac{QV(X_1)}{\nu_{2q-1}(X_1+\cdots+X_N)} \;\to\; CQV(X_1), \quad \text{as } N\to\infty,$$

where C is a positive constant. This yields that $\sigma^2_{2q-1}(Z_\infty) = \sigma^2_{2q-1}(Z_H) = C\nu_{2q-1}(X_1)\sigma_{2q-1}(X_1)$. Hence, all the conditions of lemma 5.1 are satisfied. Thus, $H(\xi)\equiv 0$, which implies $Z_\infty = Z_H$, that is the uniqueness of the limit distribution. □

6. Conclusion

Concluding, we note that with the present results, the gap detected by Hilhorst [6, 7] in the q-CLT [1], has been adequately filled. Naturally, this does not imply that other, more general, theorems can not be thought of. For example, the requirement of strict q-independence for all N can obviously be released, by only requiring asymptotic q-independence in the $N\to\infty$ limit. It might also be possible theorems similar to Lyapunov–Lindeberg type theorems [32], or q-versions of CLT for weakly dependent random variables with various mixing conditions [32–34]. The q-CLT assumes the finiteness of the quasivariance $QV(X)<\infty$. The uniqueness of the limiting q-Lévy processes studied in [2] which corresponds to the case $QV(X)=\infty$, is also a challenging problem.Moreover, at the present stage, we can not strictly refute existence of dependencies between the N random variables other than q-independence, that could also exhibit q-Gaussians as attractors in the space of probability distributions. Further efforts along these lines are of course welcome.

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