Can the black hole entropy be reconciled with classical thermodynamics?

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As early as 1902, Gibbs pointed out that systems with long-range interactions, like gravitation, lie outside the validity of standard statistical mechanics. Nevertheless, the entropy of a black hole has been repeatedly calculated within Boltzmann-Gibbs concepts. Since the pioneering Bekenstein-Hawking results, it has become common to state that the black-hole "entropy" is proportional to its area. Similarly it exists the *area law*, so named because the "entropy" of a wide class of d-dimensional quantum systems is proportional to L^{d-1} ($d > 1$; L is a characteristic length), instead of L^d. These results violate the extensivity of the thermodynamical entropy. This inconsistency disappears if we realize that the entropies of such nonstandard systems must *not* be associated with the additive expression $S_{BG} = k_B \ln W$ but with appropriate nonadditive generalizations. Here we introduce a generalized form of entropy which solves the puzzle for the black hole and the area law.

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In his 1902 book *Elementary Principles in Statistical Mechanics* [1], Gibbs emphatically points that systems involving long-range interactions are intractable within the Boltzmann-Gibbs (BG) theory, due to the divergence of the partition function. As an illustration of his remark he refers specifically to the case of gravitation. This serious difficulty emerges in fact for any d-dimensional classical system including two-body interactions whose potential energy asymptotically decays with distance like $1/r^\alpha$ ($r \to \infty$), with $0 \le \alpha/d \le 1$. Indeed, under such conditions the potential is not integrable, i.e., the integral $\int_{\text{constant}}^{\infty} dr\, r^{d-1}\, r^{-\alpha}$ diverges. ¿From the microscopic (classical) dynamical point of view, this is directly related to the fact that the entire Lyapunov spectrum vanishes in the $N \to \infty$ limit, which typically impeaches ergodicity (see [2, 3] and references therein). This type of difficulty is also present, sometimes in an even more subtle manner, in various quantum systems (the *free* hydrogen atom constitutes, among many others, an elementary such example; indeed its BG partition function diverges due to the accumulation of electronic energy levels just below the ionization energy).

Along closely related lines, since the pioneering works of Bekenstein [4, 5] and Hawking [6, 7], it has become frequent in the literature the (explicit or tacit) acceptance that the black-hole entropy is anomalous in the sense that it violates thermodynamical extensivity. Indeed we read all the time claims that the entropy of the black hole is proportional to the area of its boundary instead of being proportional to the black-hole volume [8–15]. More generally speaking we have the so called *area law* [16], which states that the entropy of a class of quantum d-dimensional systems is proportional to L^{d-1} instead of being $\propto L^d$, i.e., that it is $\propto N^{(d-1)/d}$, instead of being $\propto N \propto L^d$, where N is the number of elements of the system and L is a characteristic length ($d = 3$ precisely corresponds to the black hole).

Strangely enough, Gibbs's crucial remark and the dramatic theoretical features to which it is related are often overlooked in textbooks. Similarly, the thermodynamical violation related to the area law frequently is, somehow, not taken that seriously. The usual inclination is to consider that, for such complex systems, the entropy is not expected to satisfy thermodynamics. Physically speaking, we consider such standpoint a quite bizarre one. The goal of the present paper is to overcome this difficulty. We shall argue here along two inter-related lines: (i) the physically appropriate entropy of a given system should in all cases satisfy classical thermodynamics, and be therefore extensive; (ii) the fact (repeatedly illustrated in various manners for strongly entangled systems, black holes and, generically speaking, for systems satisfying the above mentioned area law) that the Boltzmann-Gibbs-von Neumann entropy is *not* proportional to N precisely shows that, for such strongly correlated systems, *the entropy is not the usual one but a substantially different one*; a possible such entropy is introduced here.

Extensivity of the entropy

Let now address point (i). We consider the classical Hamiltonian mentioned above, with (attractive) potential $V(r)$ which diverges for any dimensionless distance $r < 1$, and precisely equals $-A/r^\alpha$ for $r \ge 1$ with $\alpha \ge 0$ and $A > 0$. As mentioned, we verify that the potential energy per particle $U(N)/N \propto \int_1^\infty dr r^{d-1} r^{-\alpha}$ diverges for $0 \le \alpha/d \le 1$ (long-range interactions), and converges for $\alpha/d > 1$ (short-range interactions). We also verify straightforwardly that $\int_1^{N^{1/d}} dr\, r^{d-1} r^{-\alpha} = \tilde{N}/d$, where

$\tilde{N} \equiv \frac{N^{1-\alpha/d}-1}{1-\alpha/d}$ [17, 18]. This expression implies that $U(N)$ is extensive (i.e., $\propto N$) for $\alpha/d > 1$, and nonextensive otherwise. More precisely, $U(N)$ is $\propto N \ln N$ for $\alpha/d = 1$, and $\propto N^{2-\alpha/d}$ for $0 \leq \alpha/d < 1$. Let us now focus on a generic thermodynamical potential

$$\begin{aligned} &G(N,T,p,\mu,H,...) = U(N,T,p,\mu,H,...) \\ &-TS(N,T,p,\mu,H,...) + pV(N,T,p,\mu,H,...) \\ &-\mu N - HM(N,T,p,\mu,H,...) - \cdots , \end{aligned} \tag{1}$$

where T, p, μ, H are the temperature, pressure, chemical potential, external magnetic field, and U, S, V, N, M are the internal energy, entropy, volume, number of particles, magnetization. For a classical model in the above short/long range class, the thermodynamic limit corresponds to taking $N \to \infty$ in the following expression:

$$\begin{aligned} &\frac{G(N,T,p,\mu,H,...)}{N\tilde{N}} = \frac{U(N,T,p,\mu,H,...)}{N\tilde{N}} \\ &-\frac{T}{\tilde{N}}\frac{S(N,T,p,\mu,H,...)}{N} + \frac{p}{\tilde{N}}\frac{V(N,T,p,\mu,H,...)}{N} \\ &-\frac{\mu}{\tilde{N}}\frac{N}{N} - \frac{H}{\tilde{N}}\frac{M(N,T,p,\mu,H,...)}{N} - \cdots , \end{aligned} \tag{2}$$

hence

$$\begin{aligned} &g(\tilde{T},\tilde{p},\tilde{\mu},\tilde{H},...) = u(\tilde{T},\tilde{p},\tilde{\mu},\tilde{H},...) \\ &-\tilde{T}s(\tilde{T},\tilde{p},\tilde{\mu},\tilde{H},...) + \tilde{p}v(\tilde{T},\tilde{p},\tilde{\mu},\tilde{H},...) \\ &-\tilde{\mu} - \tilde{H}m(\tilde{T},\tilde{p},\tilde{\mu},\tilde{H},...) - \cdots , \end{aligned} \tag{3}$$

with $(\tilde{T},\tilde{p},\tilde{\mu},\tilde{H},...) \equiv \left(\frac{T}{\tilde{N}}, \frac{p}{\tilde{N}}, \frac{\mu}{\tilde{N}}, \frac{H}{\tilde{N}},...\right)$. The correctness of these (conjectural) scalings has been profusely verified in the literature (in ferrofluid [17], fluid [19], magnetic [20–22], diffusive [23], percolation [24] systems, among others; see [18] for an overview). In all these cases, it has been verified that *finite* equations of states are obtained, for both short- and long-range interactions, in the $N \to \infty$ limit when using these rescaled thermodynamic variables, whereas the use of the standard (i.e., non rescaled, or equivalently, rescaled with $\tilde{N}$ for $\alpha/d > 1$) variables works (naturally) correctly for short-range interactions, but fails for long-range ones. What Eq. (3) implies is that $S, V, N, M, ...$ play totally analogous thermodynamical roles, in particular that they are *extensive in all cases* (which, for N, is verified by mere construction). In contrast, U, G and all thermodynamic potentials are extensive for short-range interactions, but are superextensive for long-range interactions. Analogously, $T, p, \mu, H, ...$ are intensive for short-range interactions, but scale with $\tilde{N}$ for long-range interactions. We see therefore that the traditional thermodynamical variables which are extensive for short-range interactions split into two classes for long-range ones. The first class corresponds to energies (which become super-extensive in the long-range case). The second class corresponds to those variables which appear, within the usual Legendre transformations, in thermodynamically conjugated pairs. This class remains extensive even in the long-range case. The entropy belongs to this class.

A second argument pointing towards the correctness of using nonadditive entropic forms in order to re-establish the entropic extensivity of the system can be found in the mutually consistent results achieved by Hanel and Thurner [25, 26] by focusing on the Khinchine axioms and on complex systems with surface-dominant statistics. A third argument might arise from the possible large-deviation theory for an ubiquitous class of strongly correlated systems (more precisely, involving q-independence [27, 28]), as numerically illustrated in [29]. As a fourth indication we can refer to the analogy with the time t dependence of the entropy of simple nonlinear dynamical systems, e.g., the logistic map. Indeed, for the parameter values for which the system has positive Lyapunov exponent (i.e., strong chaos and ergodicity), we verify $S_{BG} \propto t$ (under appropriate mathematical limits), but for parameter values where the Lyapunov exponent vanishes (i.e., weak chaos and breakdown of ergodicity), it is a nonadditive entropy (S_q, discussed below) the one which grows linearly with t (see [30] and references therein). If we take into account that, in many such dynamical systems, t plays a role analogous to N in thermodynamical systems, we have here one more indication which aligns with the extensivity of the entropy for complex systems.

Nonadditive entropies

Let us now turn onto the above mentioned point (ii), namely the fact that entropies generalizing that of BG become necessary in order to recover thermodynamic extensivity for nonstandard systems.

As a possibility for addressing complexities such as those illustrated above it was proposed in 1988 [31] (see also [18, 32, 33] a generalization of the BG theory, currently referred to as nonextensive statistical mechanics. It is based on the nonadditive entropy

$$\begin{aligned} S_q &= k_B \frac{1-\sum_{i=1}^W p_i^q}{q-1} \\ &= k_B \sum_{i=1}^W p_i \ln_q \frac{1}{p_i} \quad \Big(q \in \mathcal{R};\ \sum_{i=1}^W p_i = 1\Big), \end{aligned} \tag{4}$$

with $\ln_q z \equiv \frac{z^{1-q}-1}{1-q}$ ($\ln_1 z = \ln z$). S_q recovers $S_{BG} = -k_B \sum_{i=1}^W p_i \ln p_i$ for $q \to 1$. If A and B are two probabilistically independent systems (i.e., $p_{ij}^{A+B} = p_i^A p_j^B$, $\forall (i,j)$), definition (4) implies

$$\begin{aligned} \frac{S_q(A+B)}{k_B} &= \frac{S_q(A)}{k_B} + \frac{S_q(B)}{k_B} \\ &+(1-q)\frac{S_q(A)}{k_B}\frac{S_q(B)}{k_B} . \end{aligned} \tag{5}$$

In other words, according to the definition of entropic additivity in [34], S_q is additive if $q=1$, and nonadditive otherwise.

If probabilities are all equal, we straightforwardly obtain

$$S_q = k_B \ln_q W \,. \tag{6}$$

If we extremize (4) with a (finite) constraint on the width of the probability distribution $\{p_i\}$ (in addition to its normalization), we obtain

$$p_i = \frac{e_q^{-\beta_q E_i}}{\sum_{j=1}^{W} e_q^{-\beta_q E_j}} \,, \tag{7}$$

e_q^z being the inverse function of the q-logarithmic function, i.e., $e_q^z \equiv [1+(1-q)z]^{1/(1-q)}$ $(e_1^z = e^z)$; $\{E_i\}$ are the energy levels; β_q is an effective inverse temperature.

Complexity frequently emerges in natural, artificial and social systems. It may be caused by various geometrical-dynamical ingredients, which include nonergodicity, long-term memory, multifractality, and other spatial-temporal long-range correlations between the elements of the system. During the last two decades, many such phenomena have been successfully approached in the frame of nonextensive statistical mechanics. Predictions, verifications and various applications have been performed in high-energy physics [35–39], spin-glasses [40], cold atoms in optical lattices [41], trapped ions [42], anomalous diffusion [43], dusty plasmas [44], solar physics [45, 46], relativistic and nonrelativistic nonlinear quantum mechanics [47], among many others.

If the N elements of the physical system are independent (or quasi-independent in some sense), we have that

$$W(N) \sim A\mu^N \quad (A>0;\, \mu>1;\, N\to\infty) \,. \tag{8}$$

Therefore, by illustrating the present point for the particular case of equal probabilities, we immediately verify that

$$S_{BG}(N) = k_B \ln W(N) \sim k_B (\ln \mu) N \quad (N\to\infty) \,, \tag{9}$$

hence thermodynamical extensivity is satisfied. This reconfirms that, for such systems, the thermodynamically admissible entropy is precisely given by the additive one, S_{BG}, as well known. If, however, strong correlations are present (of the type assumed in the q-generalization of the Central Limit and Lévy-Gnedenko Theorems [27, 28]), we can have

$$W(N) \sim B N^{\rho} \quad (B>0;\, \rho>0;\, N\to\infty) \,. \tag{10}$$

In this case, we straightforwardly verify that, for $q = 1-\frac{1}{\rho}$,

$$S_q(N) = k_B \ln_q W(N) \propto N \quad (N\to\infty) \,, \tag{11}$$

which satisfies thermodynamical extensivity, in contrast with $S_{BG}(N) \propto \ln N$, which violates it. Probabilistic and physical models which belong to this class are respectively available in [33] and [48].

It is clear that, for $N >> 1$, expression (10) becomes increasingly smaller than (8). A similar situation occurs for

$$W(N) \sim C\nu^{N^{\gamma}} \quad (C>0;\, ;\nu>1;\, 0<\gamma<1) \,, \tag{12}$$

which also becomes increasingly smaller that (8) (though larger than (10)). The entropy associated with $\gamma\to 1$ is of course S_{BG}. What about $0<\gamma<1$? The answer is in fact already available in the literature (footnote of page 69 in [18]), namely,

$$S_\delta = k_B \sum_{i=1}^{W} p_i \Big(\ln \frac{1}{p_i}\Big)^{\delta} \quad (\delta>0) \,. \tag{13}$$

The case $\delta=1$ recovers S_{BG}. This entropy is, like S_q for $q>0$, concave for $0<\delta\le(1+\ln W)$. And, also like S_q for $q\neq 1$, it is nonadditive for $\delta\neq 1$. Indeed, for probabilistically independent systems A and B, we verify

$$\begin{aligned} S_\delta(A+B) &= k_B \sum_{i=1}^{W_A}\sum_{j=1}^{W_B} p_i^A p_j^B \Big(\ln\frac{1}{p_i^A} + \ln\frac{1}{p_j^B}\Big)^{\delta} \\ &\neq k_B \sum_{i=1}^{W_A} p_i \Big(\ln\frac{1}{p_i^A}\Big)^{\delta} + k_B \sum_{J=1}^{W_B} p_i \Big(\ln\frac{1}{p_J^B}\Big)^{\delta} \\ &= S_\delta(A) + S_\delta(B) \qquad (\delta\neq 1) \,. \end{aligned} \tag{14}$$

For equal probabilities we have

$$S_\delta = k_B \ln^{\delta} W \,, \tag{15}$$

hence, for $\delta>0$,

$$\frac{S_\delta(A+B)}{k_B} = \Big\{ \Big[\frac{S_\delta(A)}{k_B}\Big]^{1/\delta} + \Big[\frac{S_\delta(B)}{k_B}\Big]^{1/\delta} \Big\}^{\delta} \,. \tag{16}$$

It is easily verified that, if $W(N)$ satisfies (12), $S_\delta(N)$ is extensive for $\delta = 1/\gamma$. This is in fact true even if

$$W(N) \sim \phi(N)\nu^{N^{\gamma}} \quad (\nu>1;\, 0<\gamma<1) \,, \tag{17}$$

$\phi(N)$ being any function satisfying $\lim_{N\to\infty}\frac{\ln\phi(N)}{N^{\gamma}} = 0$. Let us now unify S_q (Eq. (4)) and S_δ (Eq. (13)) as follows:

$$S_{q,\delta} = k_B \sum_{i=1}^{W} p_i \Big(\ln_q \frac{1}{p_i}\Big)^{\delta} \quad \Big(q\in\mathcal{R};\, \delta>0; \sum_{i=1}^{W} p_i = 1\Big) \tag{18}$$

$S_{q,1}$ and $S_{1,\delta}$ respectively recover S_q and S_δ; $S_{1,1}$ recovers S_{BG}. Obviously this entropy is nonadditive unless $(q,\delta)=(1,1)$, and it is expansible, $\forall(q,\delta)$. It is concave for all $q>0$ and $0<\delta\le(qW^{q-1}-1)/(q-1)$. In the limit $W\to\infty$, this condition becomes $0<\delta\le 1/(1-q)$, $\forall q\in(0,1)$, and any $\delta>0$ for $q\ge 1$; see Figs. 1 and 2. By the way, several two-parameter entropic functionals different from $S_{q,\delta}$ are in fact available in the literature (see, for instance, [25, 26, 49], and also [50]). Clearly, close connections among them exist.

Discussion and conclusion

We can address now the area law. It has been verified for those anomalous d-dimensional systems that essentially yield $\ln W(N) \propto L^{d-1} \propto N^{(d-1)/d}$, which implies that $W(N)$ is of the type indicated in (17). Therefore, $S_\delta = S_{1,\delta}$ for $\delta = d/(d-1)$ is extensive, thus satisfying thermodynamics. At the present state of knowledge we cannot exclude the possibility of extensivity of $S_{q,\delta}$ for other special values of (q, δ), particularly in the limit $\delta \to \infty$. Indeed, assume for instance that we have $\phi(N) \propto N^\rho$ in (17), and take the limit $\gamma \to 0$, hence $\delta \to \infty$. The condition $\lim_{N\to\infty} \frac{\ln \phi(N)}{N^\gamma} = 0$ is satisfied for any $\gamma > 0$, but it is violated for $\gamma = 0$, which opens the door for S_q, or some other nonadditive entropic functional, being the thermodynamically appropriate entropy.

For example, for the $d = 1$ gapless fermionic system in [48], we have analytically proved the extensivity of S_q for a specific value of $q < 1$ which depends on the central charge of the universality class that we are focusing on. For the $d = 2$ gapless bosonic system in [48], we have numerically found that, once again, it is S_q with a value of $q < 1$ the entropy which is extensive and consequently satisfies thermodynamics. This kind of scenario might be present in many d-dimensional physical systems for which $\ln W(N) \propto \ln_{2-d} N$ (i.e., $\propto \ln L$ for $d = 1$, and $\propto L^{d-1}$ for $d > 1$).

Summarizing, classical thermodynamics, and the thermostatistics of a wide class of systems whose elements are strongly correlated (for instance, through long-range interactions, or through strong quantum entanglement, or both, such as black holes, quantum gravitational dense systems, and others) can be reconciled (along lines similar to those illustrated in [32, 33, 48] for simple cases). It is enough to use for these complex systems the nonadditive entropies such as $S_{q,\delta}$, instead of the usual Boltzmann-Gibbs-von Neumann one.

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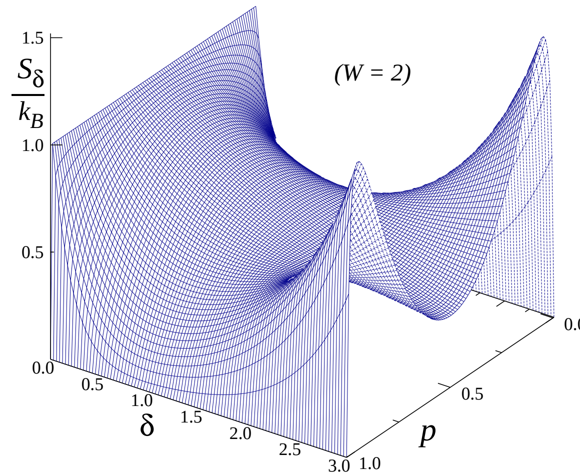

FIG. 1: Entropy S_δ as a function of the index δ and the probability p of a binary variable ($W = 2$). Concavity is lost for $\delta > 1 + \ln 2$.

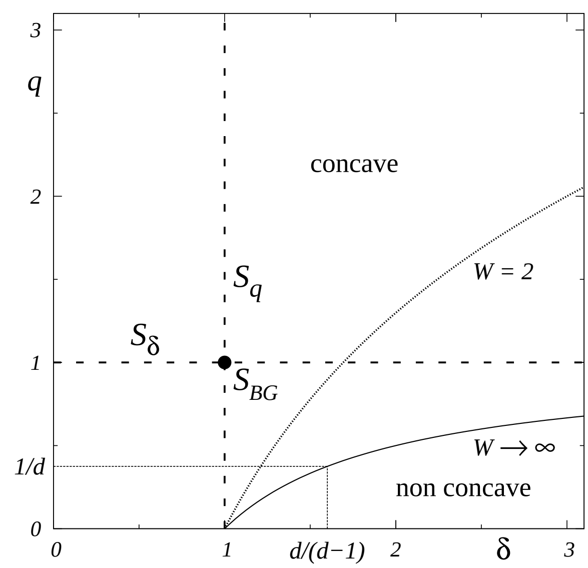

FIG. 2: Parameter space (q, δ) of the entropy $S_{q,\delta}$. At the point $(1, 1)$ we recover the BG entropy S_{BG}. At $\delta = 1$ ($q = 1$) we recover the nonadditive entropy S_q (S_δ). For any fixed W there is a frontier $q(\delta)$ such that, for δ values at its left, the entropy $S_{q,\delta}$ is concave, and, at its right, it neither concave nor convex. The $W = 2$ and $W \to \infty$ frontiers are indicated in the plot. The entropy S_δ is concave for $0 < \delta \leq 1 + \ln W$. If we impose the extensivity of $S_{q,\delta}$ for the class of systems represented by Eq. (17), it must be $\delta = 1/\gamma \geq 1$. If $S_{q,\delta}$ is used for other purposes, the region $0 < \delta < 1$ is accessible as well.

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Author contributions

C.T. conceived and wrote the paper. L.J.L.C. performed the computational calculations in order to establish the parameter-space regions of concavity of the (q,δ)-entropy. Both authors discussed all the results.