Nonadditive entropy S_q and nonextensive statistical mechanics – Aplications in geophysics and elsewhere*

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August 27, 2011

Abstract

The celebrated Boltzmann-Gibbs (BG) entropy $S_{BG} = -k\sum_i p_i \ln p_i$ and associated statistical mechanics are essentially based on hypothesis such as ergodicity, in other words, when ensemble averages (i.e., averages on the initial conditions of the entire system) coincide with time averages. This dynamical simplification occurs in classical systems (and in their quantum counterparts) whose microscopic evolution is governed by a *positive* largest Lyapunov exponent (LLE). Under such circumstances, relevant microscopic variables behave as (nearly) independent from the probabilistic viewpoint. Many phenomena exist, however, in natural, artificial and social systems that violate ergodicity, typically because of important space-time correlations between the elements of the system. Such is the case when the LLE approaches *zero*. This appears to be, for relevant dynamical degrees of freedom, the case of many geophysical, astrophysical, biophysical, economical systems, among several others. To cover a (possibly) wide class of such systems that, in one way or another, exhibit asymptotic scale-invariance, a generalization of the BG theory was proposed in 1988. This generalization is currently referred to as nonextensive statistical mechanics, and is based on the entropy $S_q = k\frac{1-\sum_i p_i^q}{q-1}$ (the index q is a real number, and $S_1 = S_{BG}$). This entropy is nonadditive for $q \neq 1$. In the present paper we comment some central aspects of this theory, and briefly review typical predictions, verifications and applications in geophysics and elsewhere, as illustrated through theoretical, experimental, observational and computational results.

*Invited review to appear in *Statistical Mechanics in Earth Physics and Natural Hazards*, Special Issue of *Acta Geophysica*, edited by P. Vallianatos and L. Telesca (2011).

1 Introduction

Statistical mechanics constitutes, together with electromagnetism — basically Maxwell equations —, and classical — basically Newton's law —, quantum — basically Schroedinger equation — and relativistic mechanics — basically Einstein special and general relativity —, one the pillars of contemporary physics. As its name indicates, *statistical mechanics* is essentially constructed from two ingredients, namely *mechanics* (including the electro-magnetic forces) and *theory of probabilities*: see Fig. 1. From this level, which is to be considered as from first principles, we may construct concepts such as energy and entropy, including their operational mathematical expressions. How to find the expressions for the energy is indicated in all good textbooks of mechanics (classical, quantum, relativistic). In what concerns a similar task for the entropy, it has proved along 140 years (since the 1860-1870's, when Clausius and Boltzmann first focused on this issue) to be a particularly delicate effort. In principle it goes like this. Guided by thermodynamics, we would like the entropy S of a macroscopic system to be *extensive* , i.e., $S(N) \propto N$ $(N \to \infty)$, N being the number of elements of the system. We expect this entropic extensivity to be valid for both short- and long-range interactions. For short-range interactions, it is well known that it should be so, as explained in any good textbook of thermodynamics. The question is more subtle for long-range interactions, for which the total energy U becomes nonextensive (more precisely, $U(N)$ is expected to increase faster than N for large N). But even in this case, the thermodynamic entropy should remain extensive, as lengthily argued and verified in the literature (see, for instance, Section 3.3.1 of [1]). Accepting this general thermodynamic demand, the next relevant question is what mathematical connection between S and its probabilistic expression (in terms of the admissible microscopic configurations) adequately takes into account the correlations existing between the N elements of the system in such a way that $S(N)$ is extensive. Let us illustrate this crucial point in what follows.

Assume that we have a system whose total number $W(N)$ of admissible microscopic configurations are equally probable, and satisfies

$$W(N) \propto \mu^N \quad (N \to \infty,\, \mu > 1). \tag{1}$$

Such hypothesis corresponds to probabilistic independence (or quasi-independence) of the N elements of the system, since $W(N+1) \sim \mu\, W(N)$.

We know of course that the Boltzmann-Gibbs (BG) entropy is given (for discrete variables) by

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \quad (\sum_{i=1}^{W} p_i = 1)\,, \tag{2}$$

hence, for equal probabilities, we have

$$S_{BG} = k \ln W\,. \tag{3}$$

If we introduce expression (1) in Eq. (3), we obtain

$$S_{BG}(N) = k \ln W(N) \propto N. \tag{4}$$

In other words, the BG formula precisely yields the desired extensivity for the entropy.

Let us assume now a system whose elements are strongly correlated in such a way that the number of equally probable admissible configurations (i.e., configurations whose probability is nonzero) satisfies

$$W(N) \propto N^{\rho} \quad (N \to \infty,\, \rho > 0). \tag{5}$$

We cannot use S_{BG} in this case, since it implies $S_{BG}(N) \propto \ln N$, which violates thermodynamics. But we may consider instead the following generalized expression [8]:

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} = -k\sum_{i=1}^{W} p_i^q \ln_q p_i = k\sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} \quad \Big(\sum_{i=1}^{W} p_i = 1\Big), \tag{6}$$

where $\ln_q z \equiv \frac{z^{1-q}-1}{1-q}$ ($\ln_1 z = \ln z$). We straightforwardly verify the $S_1 = S_{BG}$, and also that, for equal probabilities,

$$S_q = k \ln_q W\,. \tag{7}$$

If we introduce, within this entropy, expression (5), we obtain

$$S_q(N) = k\frac{[W(N)]^{1-q}-1}{1-q} \propto N^{\rho(1-q)}\,. \tag{8}$$

Consequently, if we choose

$$q = 1 - \frac{1}{\rho}\,, \tag{9}$$

we obtain $S_{1-\frac{1}{\rho}}(N) \propto N$, in agreement with thermodynamics. This is the basic reason why the BG entropy must be adequately replaced in cases where strong correlations exist in the system. It is appropriate to mention here that all kinds of asymptotic mathematical behaviors can in principle exist for $W(N)$. For those, other entropic functionals become necessary in order to have extensivity. In the present work, however, we focus on (5), which represents in fact a quite wide class of natural, artificial and social systems, as we shall see later on. Verifications on other probabilistic and physical models do exist in the literature which illustrate the fact that S_q is extensive for special values of q, which characterize the universality class of the system: see [2, 3, 4]. In Fig. 2 we exhibit that special value of q for fully quantum-entangled pure and random magnetic systems.

Let us close this section by emphasizing that *nonextensivity* must be well distinguished from *nonadditivity* [5]. Indeed, an entropy is said additive if, for

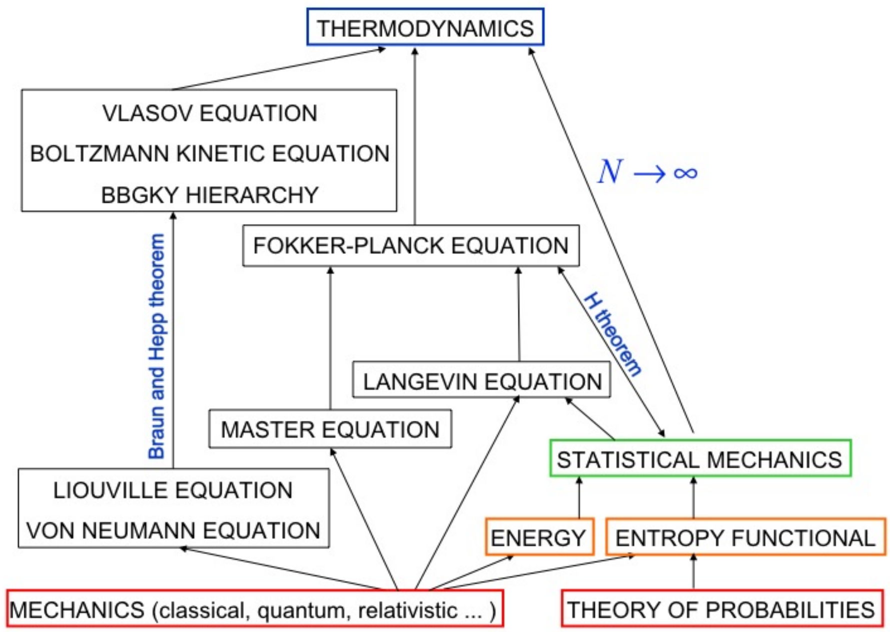

Figure 1: Schematic and non exhaustive connections that exist related with statistical mechanics. In *red* we have what we may call the first-principles or *microscopic* level, from where we can, in one or another, derive the concepts of energy and entropy (in *orange*). These concepts lead to statistical mechanics (in *green*), which, in turn, connects to thermodynamics or *macroscopic* level (in *blue*) for large systems. Many *mesoscopic* levels exist as well, of which we have indicated here only the most common ones.

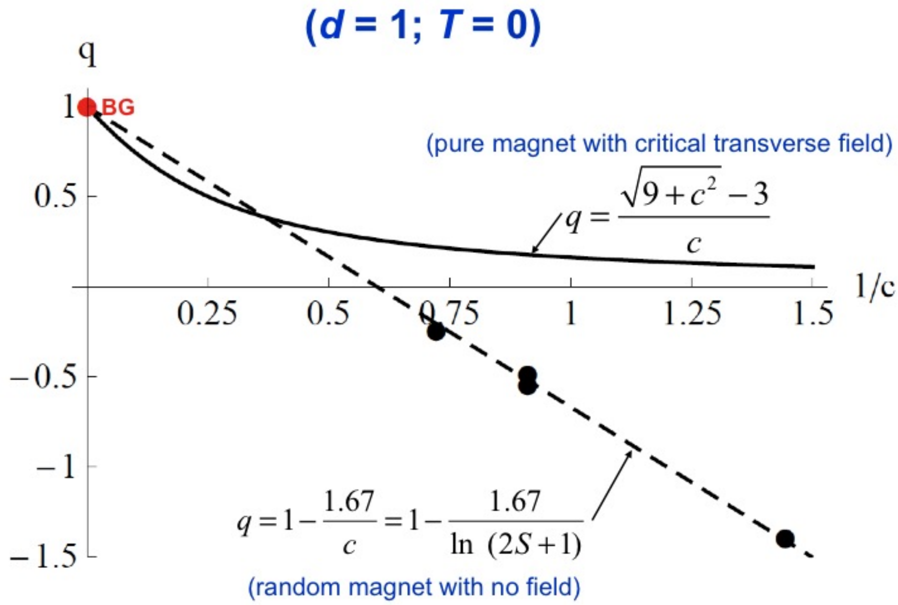

Figure 2: Dependence of q_{ent} on the central charge c of pure [4] and random [**?**] one-dimensional magnets undergoing quantum phase transitions at zero temperature, where the entire strongly entangled N-system is in its ground state (hence corresponding to a vanishing entropy since the ground state is a *pure* state), in contrast with the L-subsystem which is in a *mixed* state (hence corresponding to a nonvanishing entropy). For this value of q, the block *nonadditive* entropy S_q is *extensive*, whereas its *additive* BG entropy is *nonextensive.* Notice that, for the pure magnet, we have that $q_{ent} \in [0, 1]$, whereas, for the random magnet, we have that $q_{ent} \in (-\infty, 1]$. Both cases recover, in the $c \to \infty$ limit, the BG value $q_{ent} = 1$. These examples definitively clarify that additivity and extensivity are different properties. The only reason for which they have been confused (and still are confused in the mind of not few scientists!) is the fact that, during 140 years, the systems that have been addressed are simple, and not complex, thermodynamically speaking. For such non-pathological systems, the additive BG entropy happens to be extensive, and is naturally the one that should be used.

two probabilistically independent systems A and B, we verify that $S(A+B) = S(A)+S(B)$. We can easily establish that

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k} \,. \tag{10}$$

Therefore S_{BG} is additive, whereas, for $q \neq 1$, S_q is nonadditive.

2 Extremization of S_q and q-generalized central limit theorems

Let us now focus on the case where the random variable $\mathbf{x}$ is a continuous dimensionless D-dimensional one. The q-entropy is then given by [1]

$$\begin{aligned} S_q &= k\frac{1-\int d\mathbf{x}\,[p(\mathbf{x})]^q}{q-1} = -k\int d\mathbf{x}\,[p(\mathbf{x})]^q \ln_q p(\mathbf{x}) \\ &= k\int d\mathbf{x}\,p(\mathbf{x})\ln_q \frac{1}{p(\mathbf{x})} \qquad \Big(\int d\mathbf{x}\,p(\mathbf{x}) = 1\Big) \,. \end{aligned} \tag{11}$$

We further assume that we have a *cost function* $E(\mathbf{x})$ (e.g., the Hamiltonian of the total system in mechanical systems), and that we know the q-mean value (which characterizes the *width* of the distribution $p(\mathbf{x})$ even when its standard mean value $\langle E(\mathbf{x})\rangle_1$ diverges [6, 7]) of this cost function, i.e.,

$$\langle E(\mathbf{x})\rangle_q \equiv \frac{\int d\mathbf{x}\,E(\mathbf{x})[p(\mathbf{x})]^q}{\int d\mathbf{x}[p(\mathbf{x})]^q} \,. \tag{12}$$

Notice that this quantity is finite up to the same value of q for which the norm $\int d\mathbf{x}\,p(\mathbf{x})$ itself is finite.

If our system is a dynamical one with a physically relevant stationary state (a frequent case), this state is the one which, under the constraint (12), extremizes S_q. It is quite straightforward to verify that the maximizing distribution is given by [8, 9, 10]

$$p_q(\mathbf{x}) = \frac{e_q^{-\beta E(\mathbf{x})}}{\int dx\, e_q^{-\beta E(\mathbf{x})}} \qquad (\beta > 0)\,, \tag{13}$$

where e_q^z ($e_1^z = e^z$) is the inverse function of $\ln_q z$, i.e., $e_q^z \equiv [1+(1-q)z]_+^{\frac{1}{1-q}}$, with $[z]_+ = z$ if $z \geq 0$, and zero otherwise.

If $\mathbf{x}$ is a $D=1$ continuous variable x, and $E(x) \propto x$, the constraint (12) becomes the value of $\langle x\rangle_q$, hence

$$p_q(x) = \frac{e_q^{-\beta_1 x}}{\int dx\, e_q^{-\beta_1 x}} \qquad (\beta_1 > 0)\,. \tag{14}$$

[1] We should naturally have in mind that this type of expression can not be used in thermostatistics for extremely low temperatures, where the quantum nature of natural systems must be taken into account. In other words, if $p(\mathbf{x})$ is too thin, i.e., too close to a Dirac delta $\delta(\mathbf{x}-\mathbf{x_0})$, expression (11) will become negative ($\forall q$), which is inadmissible for an entropy.

This distribution is normalizable for $q < 2$, and has a finite mean value $\langle x \rangle_1$ for $q < 3/2$. For $q \geq 1$ it has an infinite support, whereas it is finite for $q < 1$.

If $\langle x \rangle_q = 0$, and we happen to know $\langle x^2 \rangle_q$, this quantity becomes the constraint (12), and can be used to characterize the width. The extremization of S_q then yields

$$p_q(x) = \frac{e_q^{-\beta_2 x^2}}{\int dx\, e_q^{-\beta_2 x^2}} \quad (\beta_2 > 0)\,, \tag{15}$$

which from now on will be referred to as q-Gaussian distribution [2]. It is normalizable for $q < 3$, and has a finite variance $\langle x^2 \rangle_1$ for $q < 5/3$ [3].

For $q \geq 1$ it has an infinite support, whereas it is finite for $q < 1$. For $q > 1$, q-Gaussians asymptotically decay as power-laws (more precisely like $x^{-2/(q-1)}$). However, they are quite different from the Lévy distributions, which also decay like power-laws (the only case in which q-Gaussians and Lévy distributions coincide is for $q = 2$, which corresponds to the Cauchy-Lorentz distribution). Quite frequently in the literature, any distribution decaying like a power-law is referred to as a "Lévy distribution", which constitutes a rather regrettable mistake.

Let us mention at this stage an interesting mesoscopic property of q-exponentials and q-Gaussians. We consider the following nonlinear diffusion-relaxation equation:

$$\frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2 [p(x,t)]^{2-q}}{\partial x^2} - R\,[p(x,t)]^q \quad \left(\int dx\, p(x,0) = 1 \right), \tag{16}$$

where D and R are constant phenomenological coefficients. If $D = 0$ we have

$$p(x,t) = p(x,0)\, e_q^{-R\,t} \;\; (\forall x)\,, \tag{17}$$

which has the form (14) with $\beta_1 = R$. If we have instead $R = 0$ we obtain [14, 15] the form (15):

$$p(x,t) \propto \; e_q^{-\beta_2(t) x^2} \tag{18}$$

where β_2 is related with D. In other words, q-exponentials and q-Gaussians are exact solutions of basic nonlinear diffusion-relaxation equations. They can be shown to also provide exact stationary states of similar though inhomogenous equations [16].

We have briefly reminded above that these distributions appear simultaneously as those which optimize (maximize for $q > 0$, and minimize for $q < 0$),

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[2] Since long known in plasma physics under the name *suprathermal* or κ *distributions* [11] if $q > 1$, and equal to the Student's t-distributions [12] for special rational values of $q > 1$; for special rational values of $q < 1$ coincides with the so-called r-distributions [12]. They are also occasionally referred to as *generalized Lorentzians* [13].

[3] If x is a D-dimensional real vector, normalizability mandates that $\int_0^\infty dx\, x^{D-1}\, e_q^{-\beta x^2}$ converges, hence $q < \frac{D+2}{D}$. If, in addition to that, the system has a density of states $\phi(x)$ which diverges like x^δ for $x \to \infty$ (a quite frequent case), then normalizability mandates $\int_0^\infty dx\, x^{D-1}\, \phi(x)\, e_q^{-\beta x^2}$ converges, hence $q < \frac{D+\delta+2}{D+\delta}$. Similarly, the finiteness of the second moment mandates that $\int_0^\infty dx\, x^{D+1}\, \phi(x)\, e_q^{-\beta x}$ converges, hence $q < \frac{D+\delta+4}{D+\delta+2}$.

under appropriate constraints, the nonadditive entropy S_q, and as those which exactly solve nonlinear/inhomogeneous diffusion-relaxation equations. Let us conclude by mentioning another remarkable property, namely that q-Gaussians constitute attractors in the sense of the classical central limit theorem (CLT). This theorem basically states that, if we consider the sum $S_N = \sum_{i=1}^{N} X_i$ of N *independent* (or nearly independent in some sense) random variables $\{X_i\}$, each of them having a *finite* variance, this sum converges for $N \to \infty$, after appropriate centering and rescaling, to a Gaussian. This most important theorem can be proved in a variety of manners and under slightly different hypothesis. One of those standard proofs uses the Fourier transform, which has been q-generalized [17, 18].

Around 2000 [19], q-Gaussians have been conjectured (see details in [20]) to be attractors in the CLT sense whenever the N random variables that are being summed are strongly correlated in a specific manner. The conjecture was recently proved in the presence of q-independent variables [17, 18]. The proof presented in [17] is based on a q-generalization of the Fourier transform, denoted as q-Fourier transform, and the theorem is currently referred to as the q-CLT. The validity of this proof has been recently challenged by Hilhorst [21]. His criticism is constructed on the inexistence of inverse q-Fourier transform for $q > 1$, which he illustrates with counterexamples. The inverse, as used in the [17] paper, indeed does not exist in general, which essentially makes the proof in [17] a proof of existence, but not of uniqueness. The q-generalization of the inverse Fourier transform appears then to be a quite subtle mathematical problem if $q \neq 1$. It has nevertheless been solved recently [22], and further considerations are coming [**?**] related to q-moments [6, 7, 23]. Work is in progress attempting to transform the existence proof in [17] (and also in [18]) into a uniqueness one. It is fair to say that, at the present moment, a gap exists in the complete proof (not necessarily in the thesis) of the q-CLT as it stands in [17]. In the meanwhile, several other forms [24, 25] of closely related q-generalized CLT's have already been published which do *not* use the inverse q-Fourier transform. See Fig. 3.

Probabilistic models have been formulated [26, 27] which, in the $N \to \infty$ limit, yield q-Gaussians. These models are scale-invariant, which might suggest that q-independence implies (either strict or asymptotic) scale-invariance, but this is an open problem at the present time. However, definitively, scale-invariance does *not* imply q-independence. Indeed, (strictly or asymptotically) scale-invariant probabilistic models are known [28] which do *not* yield q-Gaussians, but other limiting distributions instead, some of which have been proved to be amazingly close to q-Gaussians [29].

Some of the predictions, verifications and applications of this q-generalized theory are briefly reviewed in the rest of the present paper, which is based in fact on various previous books and reviews [1, 30, 31, 32, 33], parts of which are here followed/reproduced for simplicity and self-completeness.

	$q=1$ *[independent]*	$q \neq 1$ *(i.e.,* $Q \equiv 2q-1 \neq 1$*)* *[globally correlated]*
$\sigma_Q < \infty$ $(\alpha = 2)$	$F(x) =$ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$F(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \lvert x \rvert \ll x_c(q,2) \\ f(x) \sim C_q / \lvert x \rvert^{2/(q-1)} & \text{if } \lvert x \rvert \gg x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$F(x) =$ *Lévy distribution* $L_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *behavior* $L_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } \lvert x \rvert \ll x_c(1,\alpha) \\ f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \\ \quad \text{if } \lvert x \rvert \gg x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Lévy-Gnedenko CLT	$F(x) = L_{q,\alpha}$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\, \alpha}(x) \sim C^*_{q,\alpha} / \lvert x \rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \text{(intermediate regime)} \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\, 2}(x) \sim C^L_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \text{(distant regime)} \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg J Math Phys 51, 033502 (2010)

Figure 3: Central limit theorems (CLT) for $q \geq 1$: $N^{1/[\alpha(2-q)]}$-scaled attractor $F(x)$ when summing $N \to \infty$ identical q-independent random variables with symmetric distribution $f(x)$ with $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q \,/\, \int dx\, [f(x)]^Q$ ($Q \equiv 2q-1$). All the attractors of these theorems asymptotically decay as power-laws, excepting for the the classical CLT. See details in [17, 18].

3 Applications

In this Section we briefly, and non exhaustively, review various predictions, verifications and applications of q-exponentials and q-Gaussians through analytical, experimental, observational and computational methods in natural, artificial and social systems that are available in the literature (see [34] for full bibliography). The present list enriches the one recently presented in [33]. Several of these applications concern, as it can be checked in what follows, geophysical phenomena, very especially earthquakes and similar ones.

(i) The velocity distribution of (cells of) *Hydra viridissima* follows a q-Gaussian probability distribution function (PDF) with $q \simeq 3/2$ [35]. Anomalous diffusion has been independently measured as well [35], and an exponent $\gamma \simeq 4/3$ has been observed (where the squared space x^2 scales with time t like t^γ). Therefore, within the error bars, the prediction $\gamma = \frac{2}{3-q}$ [15] is verified in this system.

(ii) The velocity distribution of the point defects in in defect turbulence, as well as its corresponding anomalous diffusion, have been measured [36]. The results suggest a q-Gaussian PDF with $q \simeq 3/2$, and $\gamma \simeq 4/3$, which constitutes another verification of the prediction $\gamma = \frac{2}{3-q}$ [15].

(iii) The velocity distribution of cold atoms in a dissipative optical lattice was predicted [37] to be a q-Gaussian with $q = 1 + 44\frac{E_r}{U_0}$, where E_r and U_0 are parameters related to the optical lattice potential. This prediction was verified three years later, both in the laboratory and with quantum Monte Carlo techniques [38].

(iv) Computational simulations of the velocity distribution, and of the associated anomalous diffusion, during silo drainage suggest $q \simeq 3/2$ and $\gamma \simeq 4/3$ [39, **?**], once again satisfying the prediction $\gamma = \frac{2}{3-q}$ [15].

(v) The velocity distribution in a driven-dissipative 2D dusty plasma was found to be of the q-Gaussian form, with $q = 1.08 \pm 0.01$ and $q = 1.05 \pm 0.01$ at temperatures of $30000\,K$ and $61000\,K$ respectively [40].

(vi) The spatial (Monte Carlo) distributions of a trapped $^{136}Ba^+$ ion cooled by various classical buffer gases at $300\,K$ was verified to be of the q-Gaussian form, with q increasing from close to unity to about 1.9 when the mass of the molecules of the buffer increases from that of He to about 200 [41].

(vii) The distributions of price returns and stock volumes at the New York and NASDAQ stock exchanges are well fitted by q-Gaussians and q-exponentials respectively [42, 43]. The volatilities predicted within this approach fit well the real data. Various other economical and financial applications are available [44, 45, 46, 47, 48, 49, 50], including those associated with extreme values and risk[51].

(viii) The Bak-Sneppen model of biological evolution exhibits a time-dependence of the spread of damage which is well approached by a q-exponential with $q < 1$ [52].

(ix) The distributions of returns in the Ehrenfest's dog-flea model exhibit a q-Gaussian form [53].

(x) The distributions of returns in the coherent noise model are well fitted with q-Gaussians where q is analytically obtained through $q = \frac{2+\tau}{\tau}$, τ being the exponent associated with the distribution of sizes of the events [54].

(xi) The distributions of returns of the avalanche sizes in the self-organized critical Olami-Feder-Christensen model, as well as in real earthquakes exhibit a q-Gaussian form [55].

(xii) The distributions of angles in the HMF model approaches as time evolves towards a q-Gaussian form with $q \simeq 1.5$ [56].

(xiii) Experimental measurements of the turbulence in pure electron plasma are analytically reproduced with $q = 1/2$ [57].

(xiv) The relaxation in various paradigmatic spin-glass substances through neutron spin echo experiments is well reproduced by q-exponential forms with $q > 1$ [58].

(xv) The fluctuating time dependence of the width of the ozone layer over Buenos Aires (and, presumably, around the Earth) yields a q-triplet with $q_{sen} < 1 < q_{stat\,state} < q_{rel}$ [59].

(xvi) Diverse properties for conservative and dissipative nonlinear dynamical systems are well described within q-statistics [60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72].

(xvii) The degree distribution of (asymptotically) scale-free networks is numerically calculated and is well approached by a q-exponential distribution [73].

(xviii) The tissue radiation response follows a q-exponential form [74].

(xix) The overdamped motion of interacting particles in type II superconductors is analytically shown to follow, at vanishing temperature, a q-Gaussian with $q = 0$. Moreover, the entropy is the nonadditive one associated with this value of q [75].

(xx) Experimental and simulated molecular spectra due to the rotational population in plasmas are frequently interpreted as two Boltzmann distributions corresponding to two different temperatures. These fittings involve *three* fitting parameters, namely the two temperatures and the relative proportion of each of the Boltzmann weights. It has been shown [76] that equally good fittings can be obtained with a single q-exponential weight, which has only *two* fitting parameters, namely q and a single temperature.

(xxi) High energy physics has been since more than one decade handled with q-statistics [77]. During the last decade various phenomena, such as the flux of cosmic rays and others, have been shown to exhibit relevant nonextensive aspects [78, 79]. The distributions of transverse momenta of hadronic jets outcoming from proton-proton collisions (as well as others) have been shown to exhibit q-exponentials with $q \simeq 1.1$. These results have been obtained at the LHC detectors CMS, ATLAS and ALICE [80, 81, 82], as well as at SPS and RHIC in Brookhaven [83]. Predictions for the rapidities in such experiments have been advanced as well. These results stimulate an interesting possible dialog between nonextensive statistics and quantum chromodynamics (QCD).

(xxii) Various astrophysical systems exhibit nonextensive effects [84, 85, 86, 87].

(xxiii) Analysis of the magnetic field in the solar wind plasma using data from Voyager 1 and Voyager 2 strongly suggests nonextensive effects [88, 89].

(xxiv) Various geophysical applications exhibit nonextensive effects [90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101].

(xxv) Nonlinear generalizations of the Schroedinger, the Klein-Gordon and the Dirac equations have been implemented which admit q-plane wave solutions as free particles, i.e., solutions of the type $e_q^{i(kx-\omega t)}$ [102], with the energy given by $E = \hbar\omega$ and the momentum given by $\vec{p} = \hbar\vec{k}$, $\forall q$. The nonlinear Schroedinger equation yields $E = p^2/2m$ $(\forall q)$, and the nonlinear Klein-Gordon and Dirac equations yield the Einstein relation $E^2 = m^2c^4 + p^2c^2$ $(\forall q)$.

(xxvi) Phenomena in linguistics such as Zipf law and the frequency of words in various languages and literary styles [103].

(xxvii) Statistics of citations of scientific and technological papers [104, 105].

(xxviii) Processing of medical signals such as those emerging in epileptic crisis [106, 107].

(xxix) Processing of medical and other images [108, 109, 110].

(xxx) Global optimization algorithms generalizing Simulated Annealing and others [111, 112, 113].

(xxxi) Diversified applications in theoretical chemistry [114, 115, 116, 117, 118, 119, 120, 121, 122].

(xxxii) Cognitive psychology in relation with learning and remembering [123, 124, 125].

(xxxiii) Astronomical systems [126].

The systematic study of metastable or long-living states in long-range versions of magnetic models such as the finite-spin Ising [127] and Heisenberg [128] ones, or in hydrogen-like atoms [129, 130], might provide further hints and applications.

4 Final remarks

The Boltzmann-Gibbs entropy and exponential weight have been generalized, during the last two decades, in various manners [131, 132, 133, 134] (see further details in [1]). These various manners follow essentially from the 1988 proposal that we have focused on in this brief review. The corresponding entropy is noted S_q and it is nonadditive; it should be extensive for a special value of the index q, which reflects the class of strong correlations that the elements of the system have. The corresponding thermostatistics is currently referred to as nonextensive statistical mechanics (the word *nonextensive* stands here to reflect the fact that those systems typically have an internal energy which grows faster than the number of elements N).

The applicability of these concepts has been illustrated in the previous Section through analytical, numerical, experimental and observational results. Nevertheless, very many interesting questions still remain as opened issues. For example: Under what conditions q-independence and scale-invariance co-exist? Under what conditions the present generalized thermostatistics is compatible with classical thermodynamics? How well can be described with a q-exponential density matrix the mixed state of the block whose entropy has been discussed in [4])? Further analysis of these and other points would be very welcome. Finally, since the present special issue is also dedicated to natural hazards, let us suggest that the (all important) evaluation of their risks could perhaps benefit from analysis done along the lines that have proved useful for financial systems, as illustrated in [51].

Acknowledgments

I acknowledge useful recent remarks by L.J.L. Cirto, M. Jauregui, F.D. Nobre, G. Ruiz, P. Tempesta, as well as partial financial support by FAPERJ, CNPq and CAPES (Brazilian agencies).

References

[1] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World* (Springer, New York, 2009).

[2] C. Tsallis, *Nonextensive statistical mechanics: Construction and physical interpretation*, in *Nonextensive Entropy - Interdisciplinary Applications*, eds. M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004).

[3] C. Tsallis, M. Gell-Mann and Y. Sato, *Asymptotically scale-invariant occupancy of phase space makes the entropy S_q extensive*, Proc. Natl. Acad. Sc. USA **102**, 15377 (2005).

[4] F. Caruso and C. Tsallis, *Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics*, Phys. Rev. E **78**, 021101 (2008) .

[5] O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), p. 167.

[6] C. Budde, D. Prato and M. Re, *Superdiffusion in decoupled continuous time random walks*, Phys. Lett. A **283**, 309 (2001).

[7] C. Tsallis, A.R. Plastino and R.F. Alvarez-Estrada, *Escort mean values and the characterization of power-law-decaying probability densities*, J. Math. Phys **50**, 043303 (2009).

[8] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, J. Stat. Phys. **52**, 479-487 (1988).

[9] E.M.F. Curado and C. Tsallis, *Generalized statistical mechanics: connection with thermodynamics*, J. Phys. A **24**, L69-72 (1991); Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992).

[10] C. Tsallis, R.S. Mendes and A.R. Plastino, *The role of constraints within generalized nonextensive statistics*, Physica A **261**, 534-554 (1998).

[11] V.M. Vasyliunas, J. Geophys. Res., **73**, 2839 (1968); L.A. Rios and R.M.O. Galvao, *Modulation of whistler waves in nonthermal plasmas*, Phys. Plasmas **18**, 022311 (2011).

[12] A.M.C. Souza and C. Tsallis, *Student's t- and r-distributions: Unified derivation from an entropic variational principle*, Physica A, **236**, 52 (1997).

[13] R.A. Treumann, *Generalized-Lorentziann path integrals*, Phys. Rev. E **57**, 5150 (1998).

[14] A.R. Plastino and A. Plastino, *Non-extensive statistical mechanics and generalized Fokker-Planck equation*, Physica A **222**, 347 (1995).

[15] C. Tsallis and D.J. Bukman, *Anomalous diffusion in the presence of external forces: exact time-dependent solutions and their thermostatistical basis*, Phys. Rev. E **54**, R2197 (1996).

[16] B.C.C. dos Santos and C. Tsallis, *Time evolution towards q-Gaussian stationary states through unified Itô-Stratonovich stochastic equation*, Phys. Rev. E **82**, 061119 (2010).

[17] S. Umarov, C. Tsallis and S. Steinberg [2008], *On a q-central limit theorem consistent with nonextensive statistical mechanics*, Milan J. Math. **76**, 307-328.

[18] S. Umarov, C. Tsallis, M. Gell-Mann and S. Steinberg, *Generalization of symmetric α-stable Lévy distributions for $q > 1$*, J. Math. Phys. **51**, 033502 (2010).

[19] M. Bologna, C. Tsallis and P. Grigolini, *Anomalous diffusion associated with nonlinear fractional derivative Fokker-Planck-like equation: Exact time-dependent solutions*, Phys. Rev. E **62**, 2213 (2000).

[20] C. Tsallis, *Nonextensive statistical mechanics, anomalous diffusion and central limit theorems*, Milan J. Math. **73**, 145-176 (2005).

[21] H.J. Hilhorst, *Central limit theorems for correlated variables: some critical remarks*, Braz. J. Phys. **39**, 371-379 (2009), and *Note on a q-modified central limit theorem*, J. Stat. Mech., P10023 (2010).

[22] M. Jauregui and C. Tsallis, *q-generalization of the inverse Fourier transform*, Phys. Lett. A **375**, 2085 (2011). See also M. Jauregui and C. Tsallis, *New representations of π and Dirac delta using the nonextensive-statistical-mechanics q-exponential function*, J. Math. Phys. **51**, 063304 (2010).

[23] A. Rodriguez and C. Tsallis, *A generalization of the cumulant expansion. Application to a scale-invariant probabilistic model*, J. Math. Phys. **51**, 073301 (2010).

[24] C. Vignat and A. Plastino, *Central limit theorem and deformed exponentials*, J. Phys. A **40**, F969 (2007). See also C. Vignat and A. Plastino, *Geometry of the central limit theorem in the nonextensive case*, Phys. Lett. A **373**, 1713 (2009).

[25] M.G. Hahn, X.X. Jiang and S. Umarov, *On q-Gaussians and exchangeability*, J. Phys. A **43** (16), 165208 (2010).

[26] A. Rodriguez, V. Schwammle and C. Tsallis, *Strictly and asymptotically scale-invariant probabilistic models of N correlated binary random variables having* q–*Gaussians as $N \to \infty$ limiting distributions*, JSTAT P09006 (2008).

[27] R. Hanel, S. Thurner and C. Tsallis, *Limit distributions of scale-invariant probabilistic models of correlated random variables with the q-Gaussian as an explicit example*, Eur. Phys. J. B **72**, 263 (2009).

[28] L.G. Moyano, C. Tsallis and M. Gell-Mann, *Numerical indications of a q-generalised central limit theorem*, Europhys. Lett. **73**, 813 (2006); W.J. Thistleton, J.A. Marsh, K.P. Nelson and C. Tsallis, *q-Gaussian approximants mimic non-extensive statistical-mechanical expectation for many-body probabilistic model with long-range correlations*, Cent. Eur. J. Phys. **7**, 387 (2009).

[29] H.J. Hilhorst and G. Schehr, *A note on q-Gaussians and non-Gaussians in statistical mechanics*, J. Stat. Mech. (2007) P06003.

[30] M. Gell-Mann and C. Tsallis, eds., *Nonextensive Entropy - Interdisciplinary Applications*, eds. M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004).

[31] J.P. Boon and C. Tsallis, eds., *Nonextensive Statistical Mechanics: New Trends, New Perspectives*, Europhysics News **36** (6) (2005).

[32] C. Tsallis, *Entropy*, in *Encyclopedia of Complexity and Systems Science*, ed. R.A. Meyers (Springer, Berlin, 2009), 11 volumes [ISBN: 978-0-387-75888-6].

[33] C. Tsallis, *The nonadditive entropy S_q and its applications in physics and elsewhere: Some remarks*, invited review to appear in the Special Issue *Tsallis Entropy*, ed. A. Anastasiadis, *Entropy*, (2011).

[34] http://tsallis.cat.cbpf.br/biblio.htm

[35] A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada, *Anomalous diffusion and non-Gaussian velocity distribution of Hydra cells in cellular aggregates*, Physica A **293**, 549 (2001).

[36] K.E. Daniels, C. Beck and E. Bodenschatz, *Defect turbulence and generalized statistical mechanics*, Physica D **193**, 208 (2004).

[37] E. Lutz, *Anomalous diffusion and Tsallis statistics in an optical lattice*, Phys. Rev. A **67**, 051402(R) (2003).

[38] P. Douglas, S. Bergamini and F. Renzoni, *Tunable Tsallis distributions in dissipative optical lattices*, Phys. Rev. Lett. **96**, 110601 (2006); G.B. Bagci and U. Tirnakli, *Self-organization in dissipative optical lattices*, Chaos **19**, 033113 (2009).

[39] R. Arevalo, A. Garcimartin and D. Maza, *Anomalous diffusion in silo drainage*, Eur. Phys. J. E **23**, 191 (2007).

[40] B. Liu and J. Goree, *Superdiffusion and non-Gaussian statistics in a driven-dissipative 2D dusty plasma*, Phys. Rev. Lett. **100**, 055003 (2008).

[41] R.G. DeVoe, *Power-law distributions for a trapped ion interacting with a classical buffer gas*, Phys. Rev. Lett. **102**, 063001 (2009).

[42] L. Borland, *Closed form option pricing formulas based on a non-Gaussian stock price model with statistical feedback*, Phys. Rev. Lett. **89**, 098701 (2002).

[43] S.M.D. Queiros, *On non-Gaussianity and dependence in financial in time series: A nonextensive approach*, Quant. Finance **5**, 475 (2005).

[44] E.P. Borges, *Empirical nonextensive laws for the county distribution of total personal income and gross domestic product*, Physica A **334**, 255 (2004).

[45] A.P. Mattedi, F.M. Ramos, R.R. Rosa and R.N. Mantegna, *Value-at-risk and Tsallis statistics: Risk analysis of the aerospace sector*, Physica A **344**, 554 (2004).

[46] L. Borland and J.-P. Bouchaud, *A non-Gaussian option pricing model with skew*, Quant. Finance **4**, 499 (2004).

[47] L. Borland, *Long-range memory and nonextensivity in financial markets*, Europhysics News **36**, 228 (2005) [Europhysics News Special Issue *Nonextensive Statistical Mechanics: New Trends, new perspectives*, eds. J.P. Boon and C. Tsallis (November/December 2005].

[48] S.M.D. Queiros, *On the emergence of a generalised Gamma distribution. Application to traded volume in financial markets*, Europhys. Lett. **71**, 339 (2005).

[49] T. Takahashi, H. Ono and M.H.B. Radford, *Empirical estimation of consistency parameter in intertemporal choice based on Tsallis' statistics*, Physica A **387** (2008).

[50] J.C. Ferrero, *A statistical analysis of stratification and inequity in the income distribution*, Eur. Phys. J. B **80**, 255 (2011).

[51] J. Ludescher, C. Tsallis and A. Bunde, *Universal behaviour of interoccurrence times between losses in financial markets: An analytical description*, Europhys. Lett. (2011), in press.

[52] F.A. Tamarit, S.A. Cannas and C. Tsallis, *Sensitivity to initial conditions in the Bak-Sneppen model of biological evolution*, Eur. Phys. J. B **1**, 545 (1998).

[53] B. Bakar and U. Tirnakli, *Return distributions in dog-flea model revisited*, Physica A **389**, 3382 (2010).

[54] A. Celikoglu, U. Tirnakli and S.M.D. Queiros, *Analysis of return distributions in the coherent noise model*, Phys. Rev. E **82**, 021124.

[55] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra and A. Rapisarda, *Analysis of self-organized criticality in the Olami-Feder-Christensen model and in real earthquakes*, Phys. Rev. E **75**, 055101(R) (2007).

[56] L.G. Moyano and C. Anteneodo, *Diffusive anomalies in a long-range Hamiltonian system*, Phys. Rev. E **74**, 021118 (2006).

[57] B.M. Boghosian, *Thermodynamic description of the relaxation of two-dimensional turbulence using Tsallis statistics*, Phys. Rev. E **53**, 4754 (1996).

[58] R.M. Pickup, R. Cywinski, C. Pappas, B. Farago and P. Fouquet, *Generalized spin glass relaxation*, Phys. Rev. Lett. **102**, 097202 (2009).

[59] G.L. Ferri, M.F. Reynoso Savio and A. Plastino , *Tsallis q-triplet and the ozone layer*, Physica A **389**, 1829 (2010).

[60] C. Anteneodo and C. Tsallis, *Breakdown of the exponential sensitivity to the initial conditions: Role of the range of the interaction*, Phys. Rev. Lett. **80**, 5313 (1998).

[61] M.L. Lyra and C. Tsallis, *Nonextensivity and multifractality in low-dimensional dissipative systems*, Phys. Rev. Lett. **80**, 53 (1998).

[62] E.P. Borges, C. Tsallis, G.F.J. Ananos and P.M.C. Oliveira, *Nonequilibrium probabilistic dynamics at the logistic map edge of chaos*, Phys. Rev. Lett. **89**, 254103 (2002).

[63] G.F.J. Ananos and C. Tsallis, *Ensemble averages and nonextensivity at the edge of chaos of one-dimensional maps*, Phys. Rev. Lett. **93**, 020601 (2004).

[64] F. Baldovin and A. Robledo, *Nonextensive Pesin identity. Exact renormalization group analytical results for the dynamics at the edge of chaos of the logistic map*, Phys. Rev. E **69**, 045202(R) (2004).

[65] G.F.J. Ananos, F. Baldovin and C. Tsallis, *Anomalous sensitivity to initial conditions and entropy production in standard maps: Nonextensive approach*, Eur. Phys. J. B **46**, 409 (2005).

[66] U. Tirnakli, C. Beck and C. Tsallis, *Central limit behavior of deterministic dynamical systems*, Phys. Rev. E **75**, 040106(R) (2007).

[67] A. Pluchino, A. Rapisarda and C. Tsallis, *A closer look at the indications of q-generalized Central Limit Theorem behavior in quasi-stationary states of the HMF model*, Physica A **387**, 3121 (2008).

[68] U. Tirnakli, C. Tsallis and C. Beck, *A closer look on the time-average attractor at the edge of chaos of the logistic map*, Phys. Rev. E **79**, 056209 (2009).

[69] O. Afsar and U. Tirnakli, *Probability densities for the sums of iterates of the sine-circle map in the vicinity of the quasi-periodic edge of chaos*, Phys. Rev. E **82**, 046210 (2010).

[70] M. Leo, R.A. Leo and P. Tempesta, *Thermostatistics in the neighborhood of the π-mode solution for the Fermi-Pasta-Ulam β system: From weak to strong chaos*, J. Stat. Mech. P04021 (2010).

[71] M.A. Fuentes, Y. Sato and C. Tsallis, *Sensitivity to initial conditions, entropy production, and escape rate at the onset of chaos*, Phys. Lett. A **375**, 2988 (2011).

[72] T.D. Frank, *Stochastic processes and mean field systems defined by nonlinear Markov chains: An illustration for a model of evolutionary population dynamics*, Braz. J. Phys. **41** (2-3), 129 (2011).

[73] S. Thurner and C. Tsallis, *Nonextensive aspects of self-organized scale-free gas-like networks*, Europhys. Lett. **72**, 197 (2005).

[74] O. Sotolongo-Grau, D. Rodriguez-Perez, J.C. Antoranz and O. Sotolongo-Costa, *Tissue radiation response with maximum Tsallis entropy*, Phys. Rev. Lett. **105**, 158105 (2010).

[75] J. S. Andrade Jr., G.F.T. da Silva, A.A. Moreira, F.D. Nobre and E.M.F. Curado, *Thermostatistics of overdamped motion of interacting particles*, Phys. Rev. Lett. **105**, 260601 (2010).

[76] J.L. Reis Jr., J. Amorim and A. Dal Pino Jr., *Occupancy of rotational population in molecular spectra based on nonextensive statistics*, Phys. Rev. E **83**, 017401 (2011).

[77] I. Bediaga, E.M.F. Curado and J. Miranda, *A nonextensive thermodynamical equilibrium approach in $e^+e^- \to$ hadrons*, Physica A **286**, 156 (2000).

[78] G. Wilk and Z. Wlodarczyk, *Power laws in elementary and heavy-ion collisions - A story of fluctuations and nonextensivity?*, Eur. Phys. J. A **40**, 299 (2009).

[79] T.S. Biro, G. Purcsel and K. Urmossy, *Non-extensive approach to quark matter*, in *Statistical Power-Law Tails in High Energy Phenomena*, Eur. Phys. J. A **40**, 325 (2009).

[80] CMS Collaboration, *Transverse-momentum and pseudorapidity distributions of charged hadrons in pp collisions at $\sqrt{s} = 7\,TeV$*, Phys. Rev. Lett. **105**, 022002 (2010).

[81] ALICE Collaboration, *Production of pions, kaons and protons in pp collisions at $\sqrt{s} = 900\,GeV$ with ALICE at the LHC*, Eur. Phys. J. C **71**, 1655 (2011).

[82] ATLAS Collaboration, *Charged-particle multiplicities in pp interactions measured with the ATLAS detector at the LHC*, New J. Physics **13**, 053033 (2011).

[83] M. Shao, L. Yi, Z.B. Tang, H.F. Chen, C. Li and Z.B. Xu, *Examination of the species and beam energy dependence of particle spectra using Tsallis statistics*, J. Phys. G **37** (8), 085104 (2010).

[84] J.C. Carvalho, R. Silva, J.D. do Nascimento and J.R. de Medeiros, *Power law statistics and stellar rotational velocities in the Pleiades*, Europhys. Lett. **84**, 59001 (2008).

[85] A. Esquivel and A. Lazarian, *Tsallis statistics as a tool for studying interstellar turbulence*, Astrophys. J. **710**, 125 (2010).

[86] G. Livadiotis and D.J. McComas, *Beyond kappa distributions: Exploiting Tsallis Statistical Mechanics in space plasmas*, J. Geophys. Res. - Space Physics **114**, A11105 (2009).

Phys. Scripta **82**, 035003 (2010).

[87] G. Livadiotis, D.J. McComas, M.A. Dayeh, H.O. Funsten and N.A. Schwadron, *First sky map of the inner heliosheath temperature using IBEX spectra*, Astrophys. J. **734** (1) (2011).

[88] L.F. Burlaga and A.F.-Vinas, *Triangle for the entropic index q of non-extensive statistical mechanics observed by Voyager 1 in the distant heliosphere*, Physica A **356**, 375 (2005).

[89] L.F. Burlaga and N.F. Ness, *Transition from the sector zone to the unipolar zone in the heliosheath: Voyager 2 magnetic field observations*, Astrophys. J. **737** (1), 35 (2011).

[90] U. Tirnakli and S. Abe, *Aging in coherent noise models and natural time*, Phys. Rev. E **70**, 056120 (2004).

[91] S. Abe, N. V. Sarlis, E. S. Skordas, H. Tanaka and P. A. Varotsos, *Optimality of natural time representation of complex time series*, Phys. Rev. Lett. **94**, 170601 (2005).

[92] C. Papadimitriou, M. Kalimeri and K. Eftaxias, *Nonextensivity and universality in the earthquake preparation process*, Phys. Rev. E **77**, 036101 (2008).

[93] F. Vallianatos, *A non-extensive approach to risk assessment*, Nat. Hazards Earth Syst. Sci. **9**, 211 (2009).

[94] G. Balasis, I.A. Daglis, C. Papadimitriou, M. Kalimeri, A. Anastasiadis and K. Eftaxias, *Investigating dynamical complexity in the magnetosphere using various entropy measures*, J. Geophys. Res.-Space Phys. **114**, A00D06 (2009).

[95] L. Telesca, *Nonextensive analysis of seismic sequences*, Physica A **389**, 1911 (2010).

[96] L. Telesca, *Analysis of Italian seismicity by using a nonextensive approach*, Tectonophysics **494**, 155 (2010).

[97] F. Vallianatos and P. Sammonds, *Is plate tectonics a case of non-extensive thermodynamics?*, Physica A **389**, 4989 (2010).

[98] K. Eftaxias, *Footprints of nonextensive Tsallis statistics, selfaffinity and universality in the preparation of the L'Aquila earthquake hidden in a pre-seismic EM emission*, Physica A **389**, 133 (2010).

[99] F. Vallianatos, *A non-extensive statistical physics approach to the polarity reversals of the geomagnetic field*, Physica A **390**, 1773 (2011).

[100] F. Vallianatos, E. Kokinou and P. Sammonds, *Non-extensive statistical physics approach to fault population distribution. A case study from the Southern Hellenic Arc (Central Crete)*, Acta Geophysica **59** (4), 770 (2011).

[101] F. Vallianatos, D. Triantis and P. Sammonds, *Non-extensivity of the isothermal depolarization relaxation currents in uniaxial compressed rocks*, Europhys. Lett. **94**, 68008 (2011).

[102] F.D. Nobre, M.A. Rego-Monteiro and C. Tsallis, *Nonlinear generalizations of relativistic and quantum equations with a common type of solution*, Phys. Rev. Lett. **106**, 140601 (2011).

[103] M.A. Montemurro, *A generalization of the Zipf-Mandelbrot law in linguistics*, in *Nonextensive Entropy - Interdisciplinary Applications*, eds. M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004).

[104] C. Tsallis and M.P. de Albuquerque, *Are citations of scientific papers a case of nonextensivity?*, Eur. Phys. J. B **13**, 777 (2000).

[105] A.D. Anastasiadis, Marcelo P. de Albuquerque, M.P. de Albuquerque and D.B. Mussi, *Tsallis q-exponential describes the distribution of scientific citations - A new characterization of the impact*, Scientometrics **83**, 205 (2010).

[106] S. Tong, A. Bezerianos, J. Paul, Y. Zhu and N. Thakor, *Nonextensive entropy measure of EEG following brain injury from cardiac arrest*, Physica A **305**, 619 (2002).

[107] O.A. Rosso, M.T. Martin and A. Plastino, *Brain electrical activity analysis using wavelet-based informational tools (II): Tsallis non-extensivity and complexity measures*, Physica A **320**, 497 (2003).

[108] M.P. de Albuquerque, I.A. Esquef, A.R.G. Mello and M.P. de Albuquerque, *Image thresholding using Tsallis entropy*, Pattern Recognition Letters **25**, 1059 (2004).

[109] W.L. Shi, Y. Miao, Z.F. Chen and H.B. Zhang, *Research of automatic medical image segmentation algorithm based on Tsallis entropy and improved PCNN*, IEEE International Conference on Mechatronics and Automation, Proceedings **1-7**, 1004 (2009).

[110] J. Mohanalin, Beenamol, P.K. Kalra and N. Kumar, *A novel automatic microcalcification detection technique using Tsallis entropy and a type II fuzzy index*, Computers and Mathematics with Applications **60** (8), 2426 (2010).

[111] C. Tsallis and D.A. Stariolo, *Generalized simulated annealing*, Physica A **233**, 395 (1996); a preliminary version appeared (in English) as Notas de Fisica/CBPF 026 (June 1994).

[112] P. Serra, A.F. Stanton and S. Kais, *Pivot method for global optimization*, Phys. Rev. E **55**, 1162 (1997).

[113] A.D. Anastasiadis and G.D. Magoulas, *Nonextensive statistical mechanics for hybrid learning of neural networks*, Physica A **344**, 372 (2004).

[114] K.C. Mundim and C. Tsallis, *Geometry optimization and conformational analysis through generalized simulated annealing*, Int. J. Quantum Chem. **58**, 373 (1996).

[115] I. Andricioaei and J.E. Straub, *Generalized simulated annealing algorithms using Tsallis statistics: Application to conformational optimization of a tetrapeptide*, Phys. Rev. E **53**, R3055 (1996).

[116] U.H.E. Hansmann , M. Masuya and Y. Okamoto, *Characteristic temperatures of folding of a small peptide*, Proc. Natl. Acad. Sci. USA **94**, 10652 (1997).

[117] R.F. Gutterres, M. A. de Menezes, C.E. Fellows and O. Dulieu, *Generalized simulated annealing method in the analysis of atom-atom interaction*, Chem. Phys. Lett. **300**, 131 (1999).

[118] J.E. Straub and I. Andricioaei, *Computational methods inspired by Tsallis statistics: Monte Carlo and molecular dynamics algorithms for the simulation of classical and quantum systems*, in *Nonextensive Statistical Mechanics and Thermodynamics*, eds. S.R.A. Salinas and C. Tsallis, Braz. J. Phys. **29**, 179 (1999).

[119] D. Fuks, S. Dorfman, K.C. Mundim and D.E. Ellis, *Stochastic molecular dynamics in simulations of metalloid impurities in metals*, Int. J. Quant. Chem. **85**, 354 (2001).

[120] Y. Okamoto and U.H.E. Hansmann, *Protein folding simulations by a generalized-ensemble algorithm based on Tsallis statistics*, in *Nonextensive Statistical Mechanics and Its Applications*, eds. S. Abe and Y. Okamoto, Series *Lecture Notes in Physics* (Springer-Verlag, Heidelberg, 2001).

[121] M.A. Moret, P.G. Pascutti, P.M. Bisch, M.S.P. Mundim and K.C. Mundim, *Classical and quantum conformational analysis using Generalized Genetic Algorithm*, Physica A **363**, 260 (2006).

[122] J. Kim and J.E. Straub, *Optimal replica exchange method combined with Tsallis weight sampling*, J. Chem. Phys. **130**, 144114 (2009).

[123] A.C. Tsallis, C. Tsallis, A.C.N. Magalhaes and F.A. Tamarit, *Human and computer learning: An experimental study*, Complexus **1**, 181 (2003).

[124] T. Takahashi, T. Hadzibeganovic, S.A. Cannas, T. Makino, H. Fukui and S. Kitayama, *Cultural neuroeconomics of intertemporal choice*, Neuroendocrinology Lett. **30**, 185-191 (2009).

[125] R.S. Wedemann, R. Donangelo and L.A.V. de Carvalho, *Generalized memory associativity in a network model for the neuroses*, Chaos **19**, 015116 (2009).

[126] J.C. Carvalho, J.D. Jr. do Nascimento, R. Silva and J.R. De Medeiros, *Non-gaussian statistics and stellar rotational velocities of main-sequence field stars*, Astrophys. J. Lett. **696**, L48 (2009).

[127] F.D. Nobre and C. Tsallis, *Infinite-range Ising ferromagnet: thermodynamic limit within generalized statistical mechanics*, Physica A **213**, 337 (1995); Erratum: **216**, 369 (1995).

[128] A.O. Caride, C. Tsallis and S. I. Zanette, *Criticality of the anisotropic quantum Heisenberg model on a self-dual hierarchical lattice*, Phys. Rev. Lett. **51**, 145 (1983); **51**, 616 (1983).

[129] L.S. Lucena, L.R. da Silva and C. Tsallis, *Departure from Boltzmann-Gibbs statistics makes the hydrogen-atom specific heat a computable quantity*, Phys. Rev. E **51**, 6247 (1995).

[130] N.M. Oliveira-Neto, E.M.F. Curado, F.D. Nobre and M.A. Rego-Monteiro, *A simple model to describe the low-temperature behaviour of some atoms and molecules: An application to the hydrogen atom*, J. Phys. B - Atomic, Molecular and Optical Physics **40**, 1975 (2007).

[131] C. Beck and E.G.D. Cohen, *Superstatistics*, Physica A **322**, 267 (2003).

[132] R. Hanel and S. Thurner, *A comprehensive classification of complex statistical systems and an axiomatic derivation of their entropy and distribution functions*, Europhys. Lett. **93**, 20006 (2011).

[133] P. Tempesta, *Group entropies, correlation laws, and zeta functions*, Phys. Rev. E **84**, 021121 (2011).

[134] R. Hanel, S. Thurner and M. Gell-Mann, *Generalized entropies and the transformation group of superstatistics*, Proc. Nat. Acad. Sci. (USA) 1103539108 (2011).