JUAN GUILLERMO DUEÑAS LUNA

ASPECTS OF QUANTUM FIELD THEORY AND NUMBER THEORY

ASPECTS OF QUANTUM FIELD THEORY AND NUMBER THEORY

JUAN GUILLERMO DUEÑAS LUNA

Quantum Field Theory and Riemann Zeros

Theoretical Physics Department.
Brazilian Center For Physics Research.
July 02, 2015.

SUPERVISORS:
Nami Fux Svaiter.
LOCATION:
Rio de Janeiro.
TIME FRAME:
July 02, 2015.

ABSTRACT

The Riemann hypothesis states that all nontrivial zeros of the zeta function lie in the critical line $\mathrm{Re}(s) = 1/2$. Motivated by the Hilbert-Pólya conjecture which states that one possible way to prove the Riemann hypothesis is to interpret the nontrivial zeros in the light of spectral theory, a lot of activity has been devoted to establish a bridge between number theory and physics. Using the techniques of the spectral zeta function we show that prime numbers and the Riemann zeros have a different behaviour as the spectrum of a linear operator associated to a system with countable infinite number of degrees of freedom. To explore more connections between quantum field theory and number theory we studied three systems involving these two sequences of numbers.

First, we discuss the renormalized zero-point energy for a massive scalar field such that the Riemann zeros appear in the spectrum of the vacuum modes. This scalar field is defined in a $(d+1)$-dimensional flat space-time, assuming that one of the coordinates lies in an finite interval $[0, a]$. For even dimensional space-time we found a finite regularized energy density, while for odd dimensional space-time we are forced to introduce mass counterterms to define a renormalized vacuum energy.

Second, we established a remarkable connection between the asymptotic distribution of the Riemann zeros with the asymptotic behaviour of the two-point correlation function in the momentum space of the non-linear sigma model. This is done in the leading order in the $1/N$ expansion in a two-dimensional Euclidean space.

Finally, we studied the consequences of introducing disorder on an arithmetic gas. We study the thermodynamic variables as the average free energy density and average mean energy density in the complex β-plane. We define a Hagedorn temperature above which the system can not be heated up. At this temperature the divergence of the partition function is related to the prime number theorem. For an ensemble made by an enumerable infinite set of copies we show that the mean energy density depends strongly on the distribution of the Riemann zeros. Whereas considering an ensemble made by a non enumerable set of copies, the singular behavior of the average free energy density disappears. We obtain a non singular average energy density of the system using an analytic regularization procedure.

RESUMO

A hipótese de Riemann afirma que todos os zeros não triviais da função zeta estão na linha crítica $\text{Re}(s) = 1/2$. Motivado pela conjectura Hilbert-Polya a qual afirma que uma maneira possível de provar a hipótese de Riemann é interpretar os zeros não triviais à luz da teoria espectral, muita atividade tem se dedicado a estabelecer uma ponte entre a teoria dos números e a física. Utilizando as técnicas da função zeta espectral nós mostramos que os números primos e os zeros de Riemann tem um comportamento diferente como o espectro de um operador linear associado a um sistema com um número infinito contável de graus de liberdade. Para explorar mais conexões entre a teoria quântica de campos e a teoria dos números, estudamos três sistemas envolvendo essas duas sequências de números.

Primeiro, discutimos a energia do ponto zero renormalizada para um campo escalar massivo de tal forma que os zeros Riemann aparecem no espectro dos modos de vácuo. Este campo escalar é definido em um espaço-tempo plano $(d+1)$-dimensional, assumindo que uma das coordenadas encontra-se em um intervalo finito $[0, a]$. Para um espaço-tempo de dimensão par, encontramos uma densidade de energia regularizada finita, enquanto que para um espaço-tempo de dimensão ímpar é necesseario introduzir contratermos de massa para podermos definir uma energia do vácuo renormalizada.

Em segundo lugar, nós estabelecemos uma conexão notável entre a distribuição assintótica dos zeros de Riemann com o comportamento assintótico da função de correlação de dois pontos no espaço dos momentos do modelo sigma não-linear. Isto é feito na ordem dominante na expansão $1/N$ em um espaço euclidiano bidimensional.

Finalmente, estudamos as consequências da introdução de aleatoriedade em um gás aritmético. Nós estudamos as variáveis termodinâmicas como a densidade de energia livre média e a densidade de energia média no plano complexo β. Definimos uma temperatura de Hagedorn acima da qual o sistema não pode ser aquecido. Nessa temperatura a divergência da função de partição está relacionada com o teorema dos número primos. Para um ensemble formado por um conjunto infinito enumerável de cópias mostramos que a densidade de energia média depende fortemente da distribuição dos zeros de Riemann. Enquanto que, considerando um ensemble feito por um conjunto não enumerável de cópias, o comportamento singular da densidade de energia livre média desaparece. Nós obtemos uma densidade de energia média não singular do sistema utilizando um procedimento de regularização analítica.

PUBLICATIONS

Some ideas and figures have appeared previously in the following publications:

- C. H. G.Bessa, J. G. Dueñas, N. F. Svaiter. Accelerated detectors in Dirac vacuum: the effects of horizon fluctuations. *Classical and Quantum Gravity*, Vol 29, (2012) 215011.
- E. Arias, J. G. Dueñas, N. F. Svaiter, C. H. G. Bessa, G. Menezes. Casimir energy corrections by light-cone fluctuations. *International Journal of Modern Physics A*, vol 29, (2014) 1450024.
- J. G. Dueñas, N. F. Svaiter. Riemann zeta zeros and zero-point energy. *International Journal of Modern Physics A*, vol 29, (2014) 1450051.
- J. G. Dueñas, N. F. Svaiter, G. Menezes. One-loop effective action and the Riemann zeros. *International Journal of Modern Physics A*, vol 29, (2014) 1450182.
- J. G. Dueñas, N. F. Svaiter. Thermodynamics of the randomized Riemann gas. *Journal of Physics A: Mathematical and Theoretical*, vol 48, (2015) 315201.

″... la amistad era para mi
la prueba de que existe algo más fuerte
que la ideología, que la religión, que la nación...″

— *La identidad - Milan Kundera.*

ACKNOWLEDGMENTS

Although the connections between number theory and physics presented in this modest work are in very specific aspects, many persons have directly or indirectly contributed to its elaboration. I want to express my gratitude .

I should begin thanking to my advisor Prof. Nami Fux Svaiter. I would like to express my gratitude for the many hours in which discussing about some problem he encouraged me to research in physics, for the advice and guidance in the development of this work, for the friendship and a lot of patience needed in the difficult moments, and finally for bringing back to me the amazement of chess. I hope one day to win him!

I wish to thank to the teaching staff of CBPF, specifically to professor Jose Heläyel, for the efforts in organizing formative courses.

I would like to thanks to professors B. F. Svaiter, G. Menezes, T. Micklitz, for the discussions and explanations which keep me motivated and encouraged me to give a next step into the Riemann zeta function and physics. I really enjoy to learn from them in each shared coffee.

Thanks are also due to Prof. Sebastião Alves Dias for his courses and comments about the material of chapter 5. To Prof. E. F. M. Curado and F. D. Nobre for discussions on the material of chapter 6

To my relatives Dianita and Richy, and in the last year to Pedro for the fellowship and friendship, for the warm discussions about politics, ethics, linguistics and so on. Always we found an excuse to drink a cup of coffee.

To Miguel Angel for the daily discussions in the last year about physics and "related problems". Interesting ideas emerged!

To Enrique for clarifying many details of the calculations and his friendship, and in general for each one of whom this time was more pleasant and contributed me with his/her friendship.

Finally, I woud like to express my appreciation to the National Council for Scientific and Technological Development of Brazil (CNPq) for the financial assistance, in these four years staying in the Ph. D. program of Brazilian Center For Physics Research (CBPF).

CONTENTS

LIST OF FIGURES

LIST OF TABLES

LISTINGS

ACRONYMS

1 INTRODUCTION

There is no apparent reason why one number is prime and another not. To the contrary, upon looking at these numbers one has the feeling of being in the presence of one of the inexplicable secrets of creation.

– D. Zagier - The first 50 million prime numbers.

Connections between mathematics and physics have been fruitful many times for the advance of each other. Recently, a lot of activity has been devoted to establish a bridge between number theory and physics. Number theory [1, 2, 3, 4, 5] is being applied by physicists to solve physical problems and, perhaps even more unexpectedly, techniques developed by physicists applied to solve problems in number theory. Specifically, physicists are trying to shed some light in which is considered the most important problem of mathematics: *The Riemann hypothesis* [6, 7, 8, 9, 10]. It has been motivated by the Hilbert-Pólya conjecture and the apparent inaccessibility to a solution of the Riemann hypothesis from a full rigorous mathematical point of view for more than 150 years.

Let us give a context of ideas in which this thesis emerges: Random matrix theory has found a wider spectrum of applications in many branches of physics and mathematics. The success of this theory can be better described by the words of Dyson: "random matrix theory is a new kind of statistical mechanics where the realization of the systems is not relevant". It emerges as a formalism to describe complex systems in which the information about separate energy levels is neglected and only averaged quantities are studied. Examples of these quantities are the density of states, energy level correlations and wave function correlations.

The study of the statistical description of these quantities in complex nuclei opened the subject. In this proposal large number of particles interact in an unknown way, therefore a statistical description in terms of a probability distribution is required. In this case the matrix elements of the hamiltonian are assumed random with a Gaussian distribution. The density of states for this ensemble forms a semicircle (Wigner semicircle law). The level-level correlation function (level statistics), in the limit of spacing between levels going to infinity, tends to 1, which means that the correlations are lost. However, where the spacing between levels tends to 0, the level-level correlation function decays quadratically. This effect is known as level repulsion. Of course the observable quantities are different for different distributions. Other kinds of distribution can be implemented depending on the symmetries of the system. A classical review is [11], and the main developments in random matrix can be found in [12].

The connection between Riemann zeros and random matrices is given by the Montgomery-Odlyzko law which states that the distribution of spacing between nontrivial zeros of the Riemann zeta function $\zeta(s)$ is statistically identical to the distribution of eigenvalue spacings in a Gaussian unitary ensemble [13, 14]. This is equivalent to the result that the statistical asymptotic distribution of the Riemann zeros on the critical line computed numerically coincides with the statistical distribution of the eigenvalues of very large size random unitary matrices. This law is more empirical than analytic, because it is motivated in a theorem that Montgomery proved for a certain class of test functions $f(x)$ whose Fourier transform have support in the range $[-1, 1]$. The conjecture corresponds to the particular case in which $f(x)$ is taken to be the indicator function that does not fall in the above mentioned class of functions. Such a conjecture prompted many authors to consider the Riemann hypothesis in the light of random matrix theory and quantum mechanics of classically chaotic systems [15, 16].

The connection between quantum chaos and the Riemann zeros is based on two facts. First, there is a resemblance between the Gutzwiller trace formula [17, 18] for chaotic hamiltonians systems and the Weyl explicit formula [6] in number theory relating the Riemann zeros to prime numbers (See Eq. (42)).

The Gutzwiller trace formula expresses the density of states of a quantum chaotic system in terms of sums over classical periodic orbits. Specifically, these isolated periodic trajectories specify the fluctuations in the energy levels density. This formula is a kind of response function of the system in the complex energy plane such that the singularities along the real energy axis equals to the density of states. In other words, If the periodic orbit sum in the trace formula converges, it will have its poles at the approximated location of quantum levels. Therefore similarities with the zeta function has been established [17, 18].

Second, Bohigas, et al. [19, 20] and also Berry [21], conjectured that the quantum energy spectra of classically chaotic systems exhibit universal spectral correlations, i.e., the energy levels of classically chaotic systems are distributed as eigenvalues of random matrix ensembles [11]. This conjecture was demonstrated to be true by Simons, et al. in [22] where they showed that the properties of an integrable quantum hamiltonian and the spectra of nonintegrable systems can be deduced from a continuous matrix model.

Following these findings between trace formulas, quantum chaos and random matrix models, Berry and Keating [23, 24] established an analogy between the Riemann zeros and the eigenvalues E_n of an hypothetical quantum hamiltonian comparing the fluctuations of the density of Riemann zeros Eq. (42) and the energy level density for this hypothetical dynamical system. It implies that the density of the Riemann zeros can be written as a trace formula where the role of periodic orbits is played by the sequence of the prime numbers. For a recent discussion see [16]. They also suggested that the quantized version of the classical hamiltonian $H = xp$ could realize the Hilbert-Pólya conjecture [23, 24].

The main problem of such model is the fact that the hamiltonian is not compact. Therefore the quantized version does not have discrete spectrum. It is because the hamiltonian $H = xp$ breaks the time reversal symmetry, so the particle can not come back to its initial position along the time reversed path. However, in order to obtain a discrete spectrum Berry and Keating impose certain conditions over the phase space that allow them to calculate the semiclassical number of states obtaining the average term of the Riemann - Mangoldt formula, interpreting the height on the critical line T as $E/\hbar$.

Possible modifications and extensions to this interesting scenario were proposed where extra terms are added to guarantee closed periodic orbits and hence allowing a consistent quantization scheme. [25, 26, 27]. A discussion about the possible extension of this hamiltonian to quantum field theory is given in the section 3.4.

Although the disorder systems born about at the same time as the random matrices, it was only until the work by Efetov [28] on the supersymmetric method that both subjects converge. This method is based on showing how the Green function of a disordered metal can be expressed in terms of commuting and anticommuting variables. Efetov showed how the partition function of a disordered system is given by a supersymmetric nonlinear σ-model. In this case the two-point correlation functions coincides with the results obtained by Dyson for a level-level correlation function and therefore establishes a connection with the Montgomery conjecture. The connection between Riemann zeros, chaotic systems, disordered systems and random matrices has been widely discussed in the literature [29, 30, 31].

Arithmetic quantum theory has been developed to establish connections between number theory and statistical and quantum field theory. Under this approach, the concept of arithmetic gas is discussed in [32, 33, 34] and its disordered version is recently analyzed in [35]. In a similar way, it was considered a Fermi gas with a fully chaotic classical dynamics in [36, 37]. A discussion about the Möbius function as a random walk is given in [38]. Also, relations between the Möbius function and important inverse problems in statistical mechanics can be found in [39, 40].

The approach followed in this work to discuss relations between number theory and physics is based on the result of [41, 42]. Menezes, Svaiter and Svaiter showed that the prime numbers and the nontrivial zeros of the zeta function behave in different ways with respect to being the spectrum of a linear operator associated to a system with countable infinite number of degrees of freedom. In the modern functional approach to free quantum field theory, Gaussian path integrals yield expressions which depend on the differential operators determinant. Although these determinants diverge, because their spectra are unbounded, finite regularized values must be obtained using the spectral zeta functions. These finite values define the generating function of correlation functions of a physical theory and therefore depend strongly on the analytic properties of the spectral zeta functions.

Since nonseparability in wave mechanics leads to chaotic systems in the limit of short wavelengths, from the above discussion one cannot disregard the possibility that a quantum field model described by a nonseparable wave equation for a given boundary condition is able to reproduce the statistical properties of the nontrivial zeta zeros. Another possibility that could shed some light in the spectral interpretation for the zeros is the study of field theory in disordered media, such as wave equations in random fluids.

Let us mentioned two works in which we studied how disorder can influence the correlation functions. In [43] we consider an Unruh-DeWitt detector interacting with a massless Dirac field subject to fluctuations of an event horizon. We assume that the detector moves along a hyperbolic trajectory. We modelled the effects of the fluctuations of the event horizon through a Dirac equation with random coefficients. This approach is motivated by the way in which disorder is modelled in mesoscopic physics. We develop the perturbation theory for the fermionic field in the presence of randomness and evaluate corrections in the response function of the detector.

Another disordered system in which disorder can be relevant was studied in [44]. Here, we calculate the effects of light-cone fluctuations on the renormalized zero-point energy associated with a free massless scalar field in the presence of boundaries. In this case we introduced a space-time dependent random coefficient in the Klein-Gordon equation and assume that the field is defined in a domain with one confined direction. We choose the symmetric case of parallel plates separated by a distance a. We found a correction to the renormalized vacuum energy density between the plates that goes as $1/a^8$ and that the light-cone fluctuations break down the vacuum pressure homogeneity between the plates.

A good place to be introduced to the subject and to find the recent progress in research about the relationships between number theory and physics is the repository website by Martin Watkins [45].

This thesis is organized as follows. Firstly, we briefly shall show in chapter 2 the important results in number theory for this thesis. We shall discuss the prime number theorem, some arithmetic functions relevant for us, the amazing Riemann zeta function and its relative the prime zeta function.

Then we discuss in chapter 3 some examples in which it is possible to establish connections between number theory and physical systems. Afterwards, we discuss the spectral zeta function of a compact self-adjoint operator and its use in quantum field theory. The consequences of the analytic extension of this spectral zeta function at the origin are pointed out assuming two different sequence of numbers as spectra: the prime numbers and Riemann zeros.

Chapter 4 is devoted to analyse the consequences of the existence of an hypothetical differential operator conjecture by Hilbert and Pólya in a quantum field theory framework. We calculate the renormalized vacuum energy associated with a hypothetical system where the imaginary part of the complex zeros of the zeta function appear in the spectrum of the vacuum modes.

In chapter 5, we consider a large class of two dimensional field theory models in Euclidean space. We show specifically for the non-linear sigma model in the leading order in $1/N$ expansion that the asymptotic behavior of the Fourier transform of the two-point correlation function fits the asymptotic distribution of the non-trivial zeros of the Riemann zeta function.

In chapter 6 we study the consequences of introducing randomness in an arithmetic gas. We study the thermodynamic variables as the average free energy density and average mean energy density in the complex β-plane.

Finally, we presented some conclusions in chapter 7 and discuss some future perspectives to this work. Appendices A, B and C present analytic continuations of the Euler's gamma function, the Riemann zeta function and the superzeta function, respectively.

2 NUMBER THEORY PRELIMINARIES

The mystery that clings to numbers, the magic of numbers, may spring from this very fact, that the intellect, in the form of the number series, creates an infinite manifold of well distinguishable individuals. Even we enlightened scientists can still feel it e.g. in the impenetrable law of the distribution of prime numbers.

–H. Weyl - Philosophy of Mathematics and Natural Science.

The purpose of this chapter is to present briefly the number theoretical background required to visualize the connection between quantum field theory and number theory presented in this work. The presentation of this chapter pretends to give arguments that may guide the reader through the reasoning and provides the references to technical details.

2.1 PRIME NUMBERS AND FUNDAMENTAL THEOREMS

Every natural number n is called prime if $n > 1$ and if the only positive divisors of n are 1 and n. If $n > 1$ and if n is not prime, then n is called composite.

Fundamental theorem of arithmetic

Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors as

$$n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} = \prod_{i=1}^{r} p_i^{m_r} \qquad r_i \in \mathcal{N},\ r \geqslant 1. \tag{1}$$

The prime numbers derive their importance from this theorem, and therefore one of the most importance challenges in mathematics is to understand how they are distributed among the integers. They can be seen as the building blocks of the natural numbers. The first result about the prime numbers is known as the *Euclid theorem*. Here its simple proof:

There are infinitely many prime numbers

Let be $P = \prod_{i=1}^{r} p_i^{m_i}$, $r_i \in \mathcal{N}$ product of any finite set of primes. Let be $Q = P + 1$. the integers P and Q can have no prime factors in common, since such a factor would divide $Q - P = 1$, however 1 is not divisible by any number. Then must Q have at least one prime as factor. Hence this establishes the existence of at least one prime greater than these occurring in P. Here, it is natural to ask how many prime numbers there are up to any given number.

A natural question at this point to ask is how many prime numbers there are up to any given number? This is one of the most important questions made by mathematicians because it is related to the distribution of prime numbers in the real line. The first impression seeing the sequence of prime numbers is that they exhibit great irregularities, this happens when we see them "*locally*", i.e, when we take a consecutive finite number of them.

$$2, 3, 5, 7, 11, 13, \cdots, 101, 103, 109, \cdots, 163, 167, 169, ...$$

However the general distribution of the prime numbers presents some certain regularity when it is viewed asymptotically in the real line. If we look at the distribution of prime numbers, we realize that they do not appear to be closed between them. Moreover, they appear to be on average more widely spaced for a large prime number. To find an expression predicting the next prime number from one given has been a difficult problem. Therefore to find an approximate expression to the distribution of prime numbers has been a much more fruitful path for mathematicians. An introduction to this problem can be found in [1, 2, 3, 4, 5].

Prime counting function

We define the *prime counting function* $\pi(x)$ by the number of primes up to a given threshold x, that is

$$\pi(x) = \sum_{p \leqslant x} 1_p, \qquad (x > 0), \tag{2}$$

where 1_p is the indicator function adding one for each prime. By the Euclid theorem, we have that $\pi(x)$ tends to infinity with x. Gauss conjectured in the eighteen century that $\pi(x)$ can be approximated by the logarithmic integral $\text{Li}(x)$

$$\pi(x) \sim \text{Li}(x) \equiv \int_2^x \frac{du}{\ln(u)}. \tag{3}$$

The symbol $\sim$ means the limit $\pi(x)/\text{Li}(x) \to 1$ when $x \to \infty$. We can integrate by parts and obtain the asymptotic expansion for $\text{Li}(x)$ [46, 47]

$$\text{Li}(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \frac{2!x}{\ln^3(x)} + \frac{3!x}{\ln^4(x)} + \cdots + . \tag{4}$$

In his celebrated paper [48], Riemann outlined a way to prove the conjecture made by Gauss[1]. To obtain an exact expression for $\pi(x)$ Riemann defined the function $J(x)$ as

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots, \tag{5}$$

what means that each term of $J(x)$ increases by $1/n$ when the argument is equal to n-th power of some prime p^n. The first term contributing when x is a prime, the second term contributing one half when x is the square of a prime, and so on. Because of these jumps $J(x)$ is a discontinuous function at $x = p^n$, so at these points, $J(x)$ is defined as a halfway between its old value and its new value. Using the fact that $J(x) = 0$ for $0 \leqslant x < 2$, we can write $\pi(x)$ in terms of the Möbius function (using the Möbius inversion formula Eq. (12)) as

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{\frac{1}{n}}), \tag{6}$$

where the sum vanishes at such N that $x^{1/(N+1)} < 2 < x^{1/N}$. However, an explicit formula for the prime counting function was obtained by Von Mangoldt in 1895.He found that $\pi(x)$ is given by the Eq. (6) with

$$J(x) = \mathrm{Li}(x) - \lim_{T\to\infty}\left(\sum_{|\rho|\leqslant T} \mathrm{Ei}(\rho\log(x))\right) + \int_x^{\infty} \frac{dt}{(t^2-1)t\log(t)} - \log(2). \tag{7}$$

Here, ρ denotes the nontrivial zeros of the Riemann zeta function $\zeta(s)$, and $\mathrm{Li}(x)$ and $\mathrm{Ei}(x)$ are the logarithmic and exponential integrals, respectively [46, 47]. We see that each term in the summation Eq. (6) can be seen as smooth part coming from the $\mathrm{Li}(x)$ function and an oscillatory contribution related to the nontrivial zeros of $\zeta(s)$. Let us stress that the left hand side of the above equation is a step function, whereas the right hand side is a smooth function. It is remarkable how the powers of all zeta zeros contribute to deform smoothly the plot of the logarithm integral function into the graph with jumps of $\pi(x)$. The important point of this expression is that knowing the distribution of the nontrivial zeros of the Riemann zeta function allow us to understand the distribution of the prime numbers.

Prime number theorem

The prime number theorem is one of the most important results in mathematics [49]. This theorem can be expressed by

$$\lim_{x\to\infty} \pi(x) \Big/ \frac{x}{\log x} = 1. \tag{8}$$

1 The original version of the Riemann paper and an English translation is available at http://www.claymath.org/publications/riemanns-1859-manuscript

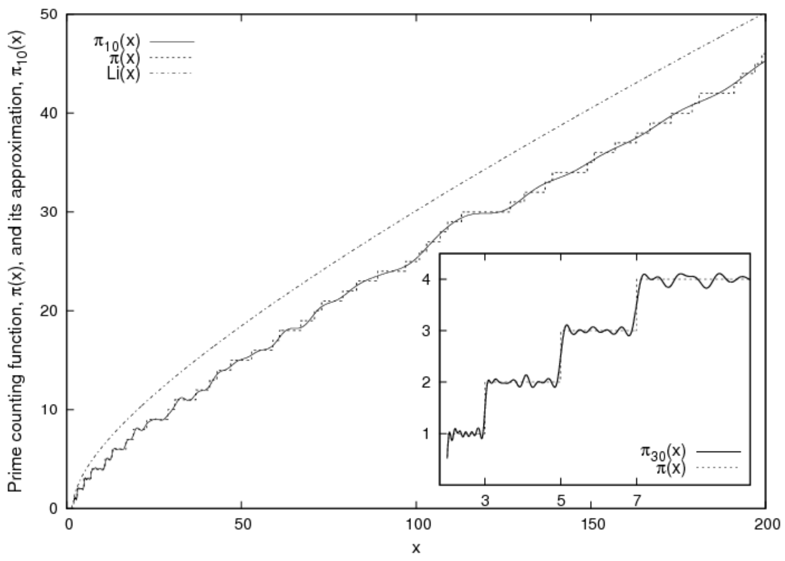

Figure 1: Prime counting function $\pi(x)$ (dash line): The dash-dotted line gives the first term $Li(x)$ of Eq. (6). The first ten nontrivial pairs of zeros of $\zeta(s)$ are used to plot the solid line. The inset shows how improve the fit taking the first 30 nontrivial zeros in the range $[2, 10]$. Picture taken from [50].

This gives the asymptotic behaviour of the prime numbers. This result can be reproduced from the Eq. (6) taking only the first terms of the series

$$\pi(x) \approx Li(x) \approx \frac{x}{\log x}, \qquad \text{for } (x \to \infty). \tag{9}$$

It implies that $Li(x)$ gives the main contribution to $\pi(x)$ and the other terms related to the nontrivial zeros represent corrections as shown in Fig. (1). Both, Hadamard and de La Vallée Poussin (1896) based on methods on complex analysis and extending the ideas of Riemann proved independently that the $\zeta(s)$ is free of zeros on the $Re(s) = 1$ boundary of the critical strip [51, 52, 53, 6]. The statement $\pi(x) \sim Li(x)$ is proved indirectly. We can obtain another equivalent statement for the prime number theorem related to the nth prime number p_n. The number of primes less than p_n is given by $\pi(p_n) = n$ therefore using Eq. (9) $p_n \approx n \log p_n \approx n \log n + n \log(\log p_n)$ and hence p_n satisfies

$$p_n \approx n \log n, \qquad \text{for } (n \to \infty). \tag{10}$$

Many other simple proofs have been given in the last century. Selberg and Erdös (1949) gave a elementary proof based on Tauberian Theorems. Donald Newman (1980) gave what is considered the simplest known proof based on the Cauchy integral theorem. See e.g [49].

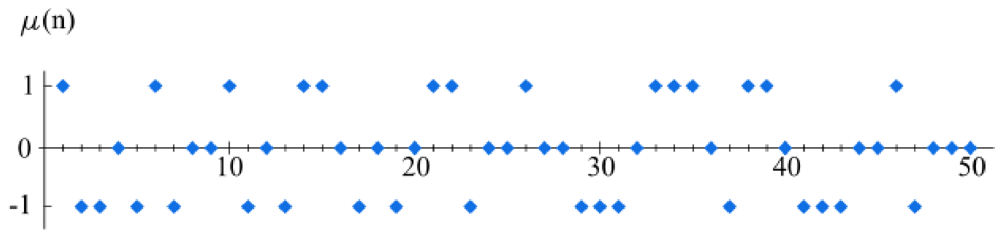

Figure 2: Möbius function $\mu(n)$: This function encodes the information about the prime number structure of a natural number and it is related with important results as the prime number and the Riemann hypothesis.

2.2 DIRICHLET SERIES AND ARITHMETICAL FUNCTIONS

A multiplicative arithmetical function is a real or complex valued function $f(n)$ defined on the positive integers satisfying that $f(1) = 1$ and whenever a and b are relatively prime, then $f(ab) = f(a)f(b)$. $f(n)$ is said to be completely multiplicative if it is multiplicative without the constraint over a and b to be relative primes.[2]

Möbius function $\mu(n)$

The multiplicative arithmetical Möbius function [1] encodes the multiplicative structure of an integer. It is defined by

$$\mu(n) = \begin{cases} 1, & \text{if n=1,} \\ (-1)^r, & \text{if n is the product of r } (\geqslant 1) \text{ distinct primes,} \\ 0, & \text{otherwise, i.e., if the square of at least one prime} \\ & \text{divides } n. \end{cases} \tag{11}$$

For example, $\mu(1) = 1, \mu(2) = -1, \mu(3) = 1, \mu(4) = 0, \mu(5) = -1, \mu(6) = 1, \mu(7) = -1, \mu(8) = 0, \mu(9) = 0$ and so on. A plot of this function is shown in Fig. (2). An important property of this function is that the sum over all possible divisors of n (including n and 1) of the Möbius function is zero except when $n = 1$

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if n = 1,} \\ 0 & \text{if n > 1.} \end{cases}$$

2 Two integers a and b are said to be relatively prime, mutually prime, or coprime if the only positive integer that evenly divides both of them is 1. That is, the only common positive factor of the two numbers is 1. Ex: (2,3), (3,4) are relative primes but (2,4) are not.

This last expression leads to the important Möbius inversion formula. If $f(n)$ is any number arithmetic function such that $F(n) = \sum_{d|n} f(d)$, then the Möbius inversion formula for $f(d)$ is given by

$$f(d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d). \quad (12)$$

In the chapter 7 we discuss different physical interpretations of the Möbius function in quantum field theory. Other applications in physics of the Möbius function and the inversion formula are discussed in [39, 40].

Liouville function $\lambda(n)$

The Liouville function $\lambda(n)$ is a completely multiplicative function defined by

$$\lambda(n) = (-1)^{r(n)}, \quad (13)$$

where $r(n)$ is the number of, not necessarily distinct, prime factors of n, multiple factors being counted according to their multiplicity.

Mangoldt function $\Lambda(n)$

For every integer $n \geqslant 1$ we define the Mangoldt function as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r, \ p \text{ prime and } r \text{ integer} \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

2.3 RIEMANN ZETA FUNCTION

The great advance to understand the prime numbers made by Riemann (1859) consisted to exploit the idea of continuing the zeta function to a meromorphic function on the complex s-plane. This allowed to extend the definition of $\zeta(s)$ function to $Re(s) < 1$, i. e., to give a meaning to $\zeta(s)$ even outside the domain of convergence of the Dirichlet series[3].

Let s be a complex variable, i. e., $s = \sigma + i\tau$ with $\sigma, \tau \in \mathcal{R}$. The Riemann zeta function [48] is defined as the Dirichlet series by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (15)$$

3 The difference between a Dirichlet series and a power series is that for the former the abscissa of convergence σ_0 may be different than the abscissa of absolute convergence $\bar{\sigma}$, while for the latter the radius of convergence and absolute convergence are the same. Series of the type $\sum_{n \geqslant 1} a_n e^{-\lambda_n s}$ converge for $Real(s) > \sigma_0$, with λ_n an increasing sequence of positive numbers. For $\lambda = n$ we have a power series whereas for $\lambda = \log(n)$ we have a Dirichlet series.

this series converges uniformly and absolutely for $\mathrm{Re}(s) = \sigma > 1$. Classical textbooks about this function are [6, 7, 8, 9, 2, 10]. A more recent reference including the original papers is [49]. By using the Euler product and the fundamental theorem of arithmetic Eq. (1) we can write the $\zeta(s)$ function as

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \qquad \mathrm{Re}(s) \geqslant 1, \tag{16}$$

where the infinite product is over all the primes. To show this, note that each factor in the product can be seen as an infinite geometric progression. Let $E_p(s)$ be defined by

$$E_p(s) = (1 - p^{-s})^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots = \sum_{m=0}^{\infty} \frac{1}{p^{ms}}, \tag{17}$$

for $\mathrm{Re}(s) > 1$ each of these factors is absolutely convergent and for the first N primes they can be multiplied term- by-term, such that

$$\prod_{i=1}^{N} E_{p_i}(s) = \prod_{i=1}^{N} \sum_{m=0}^{\infty} \frac{1}{p^{ms}} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \frac{1}{(p_1^{m_1} \cdots p_N^{m_N})^s}. \tag{18}$$

In the limit $N \to \infty$ because $\mathrm{Re}(s) > 1$ then we obtain the Eq. (16). This equation shows the connection that exists between the Riemann zeta function and the set of prime numbers. A direct implication of this representation is that the $\zeta(s)$ function does not vanish for $\mathrm{Re}(s) \geqslant 1$, because it is a product of positive numbers.

The Riemann zeta function can be expressed as a Dirichlet series for different number theoretical functions. For example, for a complex variable s such that $\mathrm{Re}(s) > 1$ the Möbius function can be generated by the Dirichlet series

$$\zeta^{-1}(s) = \sum_{n=1}^{\infty} \mu(n)/n^s, \tag{19}$$

which can be seen as expressing the $\zeta(s)$ function by its Euler product and expanding the product as sums. Also, the Dirichlet series for the Liouville function is related to the Riemann zeta function by

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n \geqslant 1} \frac{\lambda(n)}{n^s}. \tag{20}$$

The prime number theorem can be formulated as equivalent statements involving these arithmetic functions [49].

$$\sum_{n \geqslant 1} \frac{\lambda(n)}{n} = 0, \qquad \sum_{n \geqslant 1} \frac{\mu(n)}{n} = 0. \tag{21}$$

These expressions can be written also in terms of the zeta function Eq. (19) and Eq. (20), where the prime number theorem would be the limit where $s \to 1$, i.e, we recover the above expressions as a consequence of the divergence of the zeta function at its pole, which is related to the non existence of zeros of $\zeta(s)$ in the line $\text{Re}(s) = 1$ [51, 52, 53, 6].

In a similar way, the Euler product can be recast into another more convenient form, taking the logarithm of both sides in Eq. (16) and differentiating to obtain

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\log\zeta(s) = -\sum_{p}\sum_{r=1}^{\infty}(\log p)p^{-rs} = -\sum_{n=2}^{\infty}\Lambda(n)n^{-s}, \quad (22)$$

for $\text{Re}(s) > 1$ and using the Mangoldt function Eq. (14).

Analytical extension of $\zeta(s)$

The analytic extension of the Riemann zeta function may be in terms of different integral representations, each one being useful for different purposes [6]. By using the gamma function, we can write the zeta function as (see appendices (A) and (B))

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_{1}^{\infty} dx\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\Psi(x), \quad (23)$$

where $\Psi(x)$ is defined in appendix (B). Due to the exponential decay of $\Psi(x)$ the integral in the right side is convergent for all values of s and hence defines an entire function in $\mathbb{C}$. Therefore this expression gives the analytic extension to the whole complex plane of the zeta function with the exception of $s = 1$. The analytic structure of $\zeta(s)$ function is related to the $\Gamma(s)$ function. Then, to discuss this structure we derive a functional equation replacing s by $1-s$ in Eq. (23). We see that the Riemann zeta function $\zeta(s)$ satisfies the *functional equation*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \quad (24)$$

functional equation of $\zeta(s)$

for $s \in \mathbb{C}\setminus\{0,1\}$. From the discussion of the poles of the gamma function located at $s = -n$, we see that the $\zeta(s)$ has an infinite number of values on the negative real axis for which it vanishes. These zeros are situated at $s = -2n$ and are as the trivial zeros. The only pole singularities are the poles of $\Gamma(1-s)$, i. e., $s = 1, 2, \cdots$ so $\Gamma(0)$ is a singularity which is offset by the pole at $1/s$. Therefore we obtain a well defined value for $\zeta(0) = -1/2$. Hence the only possible singularity is a simple pole at $s = 1$.

Nontrivial zeros

Let us define the entire function $\xi(s)$ to study the zeros of the $\zeta(s)$ function as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \quad (25)$$

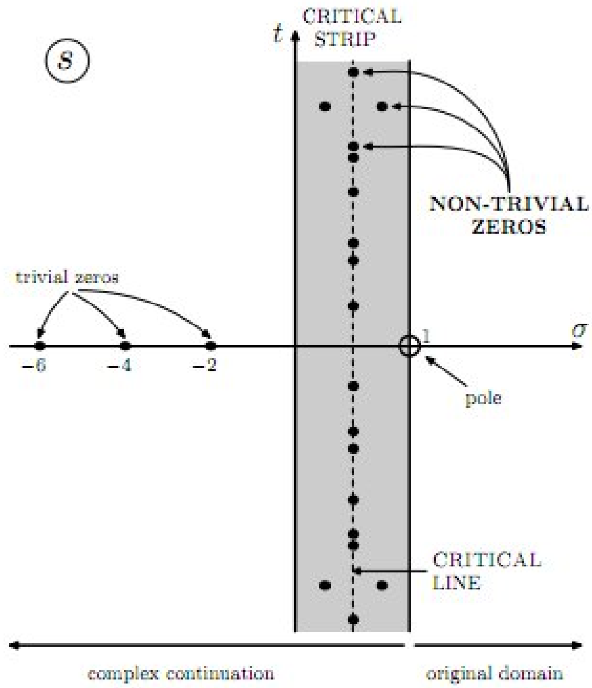

Figure 3: Analytic structure of the Riemann zeta function on the complex s plane:All the black dots represent the zeros of $\zeta(s)$. Picture taken from [50].

From Eq. (23), $\xi(s)$ is an entire function and so the functional equation Eq. (24) takes the form $\xi(s) = \xi(1-s)$. From the Euler product Eq. (16) and the analysis of trivial zeros, we conclude that the nontrivial zeros lie in the region $0 \leqslant \mathrm{Re}(s) \leqslant 1$ of the complex plane known as the critical strip. From the functional equation for $\xi(s)$, we see that the nontrivial zeros of $\zeta(s)$ are precisely the zeros of $\xi(s)$, and hence these zeros are symmetric around the vertical line $\mathrm{Re}(s) = 1/2$. Also from the analytic extension of $\zeta(s)$, Eq. (23), these zeros are symmetric about the real axis $\mathrm{Im}(s) = t = 0$. Therefore, if ρ is a zero of $\xi(s)$, then so is $1-\rho$. Since $\bar{\xi}(\rho) = \xi(\bar{\rho})$ we have that $\bar{\rho}$ and $1-\bar{\rho}$ are also zeros. The line $\mathrm{Re}(s) = 1/2$ is called the critical line (see Fig. (3)). The Hadamard product for $\xi(s)$ in terms of the nontrivial zeros takes the form

$$\xi(s) = e^{b_0 + b_1 s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \tag{26}$$

where b_0 and b_1 are given constants. By setting $s = 0$ in the last expression we obtain that $\xi(0) = e^{b_0}$. Since $\xi(0) = \xi(1)$ and the residue of $\zeta(s)|_{s=1} = 1$ we have $\xi(1) = 1/2$, therefore $b_0 = -\ln(2)$. For the constant b_1 we proceed by taking the logarithmic derivative of the Eq. (26), so we get

$$\frac{d}{ds} \ln \xi(s) = \frac{\xi'}{\xi}(s) = b_1 + \sum_{\rho} \frac{1}{(s-\rho)} + \sum_{\rho} \frac{1}{\rho}, \tag{27}$$

which combined with Eq. (25) gives

$$\frac{\zeta'}{\zeta}(s) = b_1 - \frac{1}{s-1} + \frac{1}{2}\ln\pi - \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \sum_{\rho}\frac{1}{(s-\rho)} + \sum_{\rho}\frac{1}{\rho}, \quad (28)$$

where the summation is over all nontrivial zeros of $\zeta(s)$, and we have used Eq. (194) and Eq. (201) for the logarithmic differentiation of the $\Gamma(s)$ function. Letting $s \to 0$ in the last expression with the values of $\Gamma'(1) = -\gamma$, $\Gamma(1) = 1$, and performing a logarithmic differentiation for $\zeta(s)$ in the functional equation it is possible to show that $b_1 = \ln 2 + \frac{1}{2}\ln\pi - 1 - \frac{1}{2}\gamma$ [8, 10]. Now we can go back to the Eq. (28) and instead of using the Eq. (194) evaluate the logarithmic derivative of $\Gamma(s/2)$ through Eq. (201) to obtain

$$\frac{\zeta'}{\zeta}(s) = C_1 - \frac{1}{s-1} + \sum_{\rho}\frac{1}{(s-\rho)} + \sum_{\rho}\frac{1}{\rho} + \sum_{n=1}^{\infty}\frac{1}{(s+2n)} - \sum_{n=1}^{\infty}\frac{1}{2n}, \quad (29)$$

where $C_1 = -1 - \ln(2\pi)$. This expression is very important for the calculation of the spectral density of the nontrivial zeros and we will provide a clear physical interpretation in the chapter 6.

Riemann Hypothesis

The Riemann hypothesis can be formulated in many and at first unrelated ways. The most common way states that *all nontrivial zeros of the Riemann zeta function lie on the critical line* $\mathrm{Real}(s) = 1/2$. Classical monographs are [6, 7, 8, 9, 10] and [49] which is a collection of original papers with a good introduction to the subject. We will assume throughout the thesis the validity of this conjecture. Let us assume that these zeros are simple and let us write the complex zeros of the zeta function as $\rho = \frac{1}{2} + i\gamma$, $\gamma \in \mathbb{C}$. The Riemann hypothesis implies that all γ are real. Because of the symmetries discussed above about these zeros, we can take $\gamma > 0$ such that $\gamma_{-k} = -\gamma_k$ and arrange this sequence $\rho_k = \frac{1}{2} + i\gamma_k$ so that $\gamma_{k+1} > \gamma_k$.

There is a relation between the prime number theorem and Riemann hypothesis. The connection comes for trying to expand the zero-free region to include the vertical line $\mathrm{Re}(s) = 1$. Similar to the proof of the prime number theorem by Hadamard [51, 49], the Riemann hypothesis implies that $\zeta(1+it) \neq 0$ for $t \in \mathcal{R}$, accordingly the truth of the Riemann hypothesis implies the prime number theorem.

There are about one hundred equivalent statements from which the truth of Riemann hypothesis follows. Many of these statements can be found in [49]. Different theorems have been proved establising necessary conditions for the truth of the Riemann hypothesis. One of the first theorems in favor of the Riemann hypothesis was given by Hardy (1928). He demostrated that there are infinitely many zeros of $\zeta(s)$ on the critical line. Following this approach, Selberg (1945) proved that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line. This result was improved first by Levinson (1970), who showed that at least $1/3$ of the zeros lie on the critical line, and then by Conrey (1980) who proved that at least $2/5$ of the zeros lie on the critical line. In 90' s Oldyzko veryfied the Riemann hypothesis computationally counting the number of zeros of $\zeta(s)$ in the critical strip up to any finite desired height[4].

Spectral density of zeros

In a similar way to the definition of the counting function for primes Eq. (2), the counting function for the nontrivial zeros of the $\zeta(s)$ function is defined as

$$N(T) = \sum_{\gamma_n \leqslant T} 1_{\gamma_n}, \qquad (T > 0). \tag{30}$$

It is possible to obtain an asymptotic expression for large T similar to the prime number theorem. Following Ivic [8], let us consider the rectangle R with vertices $2 \pm iT$, $-1 \pm iT$. Then by the argument principle theorem in complex analysis[5], since $\xi(s)$ does not vanish on the boundary of R and $N(T)$ is real we have that

$$N(T) = \frac{1}{2\pi} \mathrm{Im}\left(\int_R \frac{\xi'(s)}{\xi(s)} ds \right). \tag{31}$$

Let us write $\xi(s) = s(s-1)\eta(s)$ with $\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. By the above analysis for $\xi(s)$ we may write

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} + \frac{\eta'(s)}{\eta(s)}. \tag{32}$$

We notice that

$$\mathrm{Im}\left[\int_R \left(\frac{1}{s} + \frac{1}{s-1} \right) ds \right] = 4\pi, \tag{33}$$

then, due to $\eta(s)$ having the same symmetries as $\xi(s)$, i.e, $\eta(s) = \eta(1-s)$ and $\eta(s) = \eta(\bar{s})$, we have that

$$\mathrm{Im}\left(\int_R \frac{\eta'(s)}{\eta(s)} ds \right) = 4 \mathrm{Im}\left(\int_L \frac{\eta'(s)}{\eta(s)} ds \right), \tag{34}$$

4 The distribution of the nontrivial zeros has maintained as one of the most intriguing and challenging mathematical problems for about 150 years.

5 The argument principle theorem establishes: Let f(z) be a meromorphic function inside and on some closed contour R without poles or zeros on R, then $\oint_R \frac{f'(z)}{f(z)} dz = 2\pi\left(\sum_a n(R,a) - \sum_b n(R,b) \right)$, where the first summation is over the zeros a of $f(z)$ counted with their multiplicities, and the second summation is over the poles b of $f(z)$ counted with their orders, and $n(R,z)$ is the winding number of R around z.

where L consists of the segments $[2, 2+iT]$ and $[2+iT, \frac{1}{2}+iT]$. Therefore

$$\begin{aligned}\mathrm{Im}\left(\int_L \frac{\eta'(s)}{\eta(s)}ds\right) &= \mathrm{Im}\left[\int_L\left(-\frac{1}{2}\ln(\pi)+\frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)}+\frac{\zeta'(s)}{\zeta(s)}\right)ds\right]\\ &= -\frac{T}{2}\ln(\pi)+\mathrm{Im}\left[\int_L \frac{\Gamma'(s/2)}{\Gamma(s/2)}ds+\int_L\frac{\zeta'(s)}{\zeta(s)}ds\right]\end{aligned} \quad (35)$$

The integral in L for the Γ-term is

$$\mathrm{Im}\left(\int_L \frac{\Gamma'(s/2)}{\Gamma(s/2)}ds\right) = \mathrm{Im}\left[\ln\Gamma\left(\frac{s}{2}\right)\right]_{s=2}^{s=\frac{1}{2}+iT} = \mathrm{Im}\left[\ln\Gamma\left(\frac{1}{4}+\frac{T}{2}\right)\right]. \quad (36)$$

Using Stirling's formula Eq.(202) and the identity $\arctan(\alpha) = \frac{\pi}{2} - \arctan(1/\alpha)$ we obtain that

$$\mathrm{Im}\left(\int_L \frac{\Gamma'(s/2)}{\Gamma(s/2)}ds\right) = \frac{T}{2}\ln\left(\frac{T}{2}\right) - \frac{T}{2} - \frac{\pi}{8} + O\left(\frac{1}{T}\right), \quad (37)$$

thus the counting function $N(T)$ yields

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi}\mathrm{Im}\left(\int_L \frac{\zeta'(s)}{\zeta(s)}ds\right) + O\left(\frac{1}{T}\right). \quad (38)$$

It is possible to prove that a bound for the logarithmic derivative of $\zeta(s)$ is [8, 49]

$$\mathrm{Im}\left(\int_{\frac{1}{2}+iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)}ds\right) = O(\ln(T)). \quad (39)$$

Hence, the number $N(T)$ of Riemann zeros $\sigma + i\gamma$, with $0 < \gamma < T$ is asymptotically given by the Riemann-von Mangoldt formula:

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + O(\ln T), \qquad T \to \infty. \quad (40)$$

From this expression we see that $N(\gamma_n - 1) < n \leqslant N(\gamma_n + 1)$, from which we define the density of nontrivial zeros as $D(T) = \frac{d}{dT}N(T) \approx N(T+1) - N(T)$. For $T \to \infty$ or large T we find that

$$D(T) = \frac{1}{2\pi}\ln\left(\frac{T}{2\pi}\right) + \frac{1}{T} + O\left(\frac{1}{T}\right), \qquad T \to \infty. \quad (41)$$

For $T = \gamma_n$ we have a strong result about the asymptotic distribution of the $n-th$ nontrivial zero. We have that $|\rho_n| \sim \gamma_n \sim 2\pi n/\ln(\gamma_n/2\pi)$ as $n \to \infty$[6]. Therefore for the zeros of the zeta function we get $\gamma_n \sim n/\ln(n)$ [2]. It implies that the mean spacing between the nontrivial zeros at height T is $1/\ln(T)$. Using the prime number theorem, we get the asymptotic regimes $p_n \sim n\ln(n)$. Then we have that

6 By taking this limit we see that $\frac{n}{\ln(\gamma_n/2\pi)} \approx \gamma_n > 1$ for $n \to \infty$.

asymptotically the product of the prime numbers and Riemann zeros behaves as $p_n \gamma_n \approx n^2$ establishing a connection between these three different sets. This fact should in some way be expected because the primes are the building blocks of naturals Eq.(1) and the Riemann zeros are in some sense conjugated to the primes Eq. (6).

As it was seen, we can think of the nontrivial zeros of the zeta function and the prime numbers as Fourier-conjugated sets of numbers. For the prime numbers $\pi(x)$ was expressed as a harmonic sum over the Riemann zeros, and for these zeros $N(T)$ can be expressed as a harmonic sum over primes. Specifically, we can split the counting function $D(T)$ in a smooth and oscillatory parts $D(T) = D_{sm}(T) + D_{osc}(T)$. The smooth part can be interpreted as a local average of the number of Riemann zeros over a large range compared with the mean spacing of the zeros but small compared with T and is given by Eq. (41). The oscillatory part can be obtained substituting into Eq. (38) the Euler product representation of $\zeta(s)$ (See Eq. (22)), it yields

$$D_{osc}(T) = \frac{1}{\pi}\frac{d}{dT}(\mathrm{Im}\ln\zeta(s)) = -\frac{1}{\pi}\sum_{p}\sum_{r=1}^{\infty}\frac{\ln(p)}{\sqrt{p^r}}\cos\left[rT\ln(p)\right]. \quad (42)$$

This oscillatory part gives the fluctuactions as individual contributions from each prime number p, which is the analogue to Eq. (6), which is the explicit formula for $\pi(x)$. Although the Eq. (22) is valid only for $\mathrm{Re}(s) > 1$ (free-zeros region) the last formula contains information about the zeros on the critical line. For $\mathrm{Re}(s) < 1$ we can use the analytic extension of $\zeta(s)$ in Eq. (42). This kind of decomposition in a smooth and an oscillatory part can be obtained also using the Mellin transform of a test function $f(s)$ with the requirement to be analytic on $\mathrm{Im}(s) < 1/2 + \delta$, bounded, and decreasing as $f(s) = O(|s|^{-2-\delta})$ for some $\delta > 0$. For a discussion see [16][7]. This expression is known as the trace formula for the Riemann zeta function and has been connected with chaotic systems when it is compared with the Gutzwiller trace formula [17, 18, 21, 24].

Prime zeta function

Let us conclude this chapter discussing *the prime zeta function* $\zeta_p(s)$ [54, 55], which is defined for $s = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$ as

$$\zeta_p(s) = \sum_{k=1}^{\infty} p_k^{-s}, \ \mathrm{Re}(s) > 1, \quad (43)$$

where $\{p_k\}_{k\in\mathbb{N}}$ is the sequence of prime numbers. The series defined by Eq. (43) converges absolutely for $\mathrm{Re}(s) > 1$. The analytic structure of the prime zeta function can be obtained from the Euler product for

7 A similar relation has been obtained for the Laplace-Beltrami operator in constant negative curvature (hyperbolic) space corresponding to the Selberg trace formula.

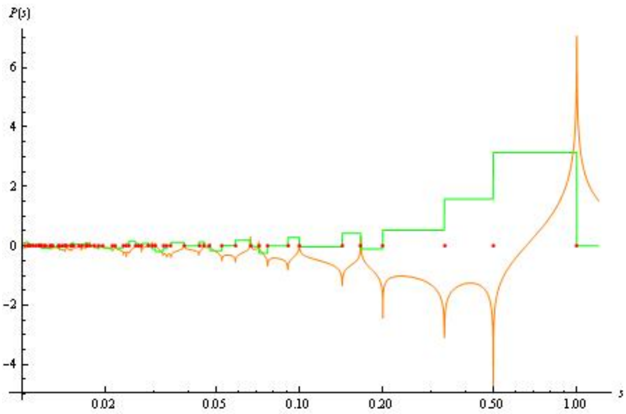

Figure 4: Prime zeta function. (green line) Real part of $\zeta_p(s)$. (yellow line) imaginary part of $\zeta_p(s)$.

the ζ-function Eq. (16). By taking the logarithm of this equation we have

$$\log \zeta(s) = \sum_p \sum_{r=1}^{\infty} \frac{p^{-rs}}{r}, \qquad \text{Re}(s) > 1, \tag{44}$$

using the Eq. (43) we can write the last equation as

$$\log \zeta(s) = \sum_{r=1}^{\infty} \frac{1}{r} \zeta_p(rs), \qquad \text{Re}(s) > 1. \tag{45}$$

This expression can be inverted using the Möbius function Eq. (12). Hence, the prime zeta function $\zeta_p(s)$ can be written as

$$\zeta_p(s) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \ln \zeta(rs), \quad \text{Re}(s) > 1, \tag{46}$$

The analytic continuation of the prime zeta function now can be obtained from the analytic continuation of the $\zeta(s)$ Eq. (23). From the pole of $\zeta(s)$ we see that $s = 1/r$ is a singular point for all square free positive integers r. It establishes a natural boundary of $\zeta_p(s)$. Together with this set of singularities coming from the poles of the $\zeta(rs)$ functions, we have another source of singularities coming from the Riemann zeros. In this case we have a clustering of singular points along the imaginary axis on the critical line $\text{Re}(s) = \frac{1}{2}$. This set of logarithmic singularities accumulates at $s = 0$ and therefore all points on the line $\text{Re}(s) = 0$ are limit points. The analytic structure of the $\zeta_p(s)$ is shown in the Figure (4). The prime zeta function can be analytically extended only in the strip $0 < \text{Re}(s) \leqslant 1$. A detailed demostration of this result was obtained by Landau and Walfisz and discussed by Fröberg [54, 55].

3 NUMBER THEORY AND PHYSICS

The shortest path between two truths
in the real domain passes
through the complex domain

– J. Hadamard.

3.1 HILBERT-PÓLYA CONJECTURE

Given the nontrivial zeros of Riemann zeta function Landau asked to Pólya about a possible physical reason for their the existence. This question was formulated as a strategy to prove the Riemann Hypothesis by Hilbert and Pólya, who proposed to find an hermitian operator whose eigenvalues are precisely the Riemann zeros. This conjecture implies a natural connection between the Riemann hypothesis and quantum mechanics allowing a spectral interpretation of the Riemann zeros. A review discussing the physics of the Riemann hypothesis is [50].

3.2 QUANTUM FIELD THEORY AND ZETA REGULARIZATION

Arithmetic quantum theory has been developed to establish connections between number theory and quantum field theory [32, 33, 34, 56, 57, 58, 38, 50]. Quantum field theory is the formalism where the probabilistic interpretation of quantum mechanics and the special theory of relativity was gathered to describe a plethora of phenomena not described by classical physics nor quantum mechanics of systems with finitely many degrees of freedom. Being an extension of quantum mechanics, a natural question arises. Can this formalism shed some light over the Riemann zeta zeros problem? We would like to emphasize that in the formalism of quantum field theory, there is no obstruction for the existence of this elusive operator conjectured by Hilbert and Pólya.

In the approach of quantum field theory using functional methods [59, 60], Gaussian path integrals yield expressions which depend on the differential operator's determinant. Although these determinants diverge, finite regularized values can be obtained using the spectral zeta function regularization [61, 62, 63, 64, 65]. Recent discussions about spectral methods in quantum field theory can be found in [66, 67]. For a more mathematical treatment of spectral zeta functions see [68, 69, 70]. The zeta regularization is a standard procedure to obtain a well defined one-loop effective action for different models in quantum field theory.

In the functional approach to Euclidean scalar field theory there are three fundamental objects to be considered. The generating functional of all Schwinger functions $Z[J]$, the generating functional of connected Schwinger functions $W[J]$, and finally, since in perturbation theory the proper vertices are given by the sum of one-particle irreducible diagrams, $\Gamma[\varphi]$ is defined as the generating functional of the sum of one-particle irreducible Green's functions. If a loop expansion is performed, we can define the one-loop effective action [71].

Let us consider a massive neutral free scalar field defined in a d-dimensional Minkowski spacetime. The Euclidean version of this theory can be obtained by analytic continuation (Wick rotation) of the generating function to imaginary time. Let us assume a compact Euclidean space with a suitable boundary where we impose over the fields certain conditions to satisfy. We can define the Euclidean generating functional $Z[h]$ by the following functional integral

$$Z[h] = \int [d\,\varphi]\, \exp\left(- S_0 + \int d^d h(x)\varphi(x) \right), \tag{47}$$

where $[d\varphi] = \prod_x d\varphi(x)$ is a translational invariant measure and $h(x)$ is a smooth function introduced to generate the Schwinger functions of the theory. The action that usually describes a free scalar field is

$$S_0 = \int d^d x d^d y\, \varphi(x) K(m_0; x-y)\varphi(y). \tag{48}$$

and the kernel $K(m_0; x-y)$ is defined by

$$K(m_0; x-y) = D\delta^d(x-y), \tag{49}$$

where D is an elliptic, self-adjoint differential operator acting on scalar functions on the Euclidean space. The usual case is to take $D = (-\Delta + m_0^2)$, where Δ is the d-dimensional Laplacian. We define the generating functional of connected Scwinger functions by $W[h] = \ln Z[h]$. To obtain a well defined object, it is required to regularize the determinant associated with the operator D, because $W[0] = -1/2 \ln \det D$. A similar regularization is required when one calculates the one-loop effective action for self-interacting scalar fields. In both cases, the use of the spectral zeta function $\zeta(s)_D$ (defined below) associated to the corresponding elliptic operator is used. In the following we use the Minakshisundaram-Pleijel zeta function [72].

Let be D a self-adjoint second order elliptic differential operator. For simplicity we suppose that the eigenvalues λ_n of D are real and positive such that $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_k \to \infty$ when $k \to \infty$, where the zero eigenvalue must be omitted. It implies that D is a positive definite operator. Let be $\{f_k\}_{k=1}^{\infty}$ a complete orthonormal set of eigenfunctions associated to D, hence D satisfies $Df_n(x) = \lambda_n f_n(x)$.

In this basis D is represented by an infinite diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots)$ and its determinant is given by $\det D = \prod_n \lambda_n$. This is a product over an infinite number of positive values, hence this is divergent. To regularize this product and give sense to the determinant we use the spectral zeta function.

The spectral zeta function $\zeta(s)_D$ associated to the operator D is defined as

$$\zeta_D(s) = \sum_n \frac{1}{\lambda_n^s}, \qquad \mathrm{Re}(s) > s_0, \tag{50}$$

for some s_0. Formally, we have

$$-\frac{d}{ds}\zeta_D(s)\bigg|_{s=0} = \ln \det D. \tag{51}$$

We see that the zeta regularization product can be defined as

$$\prod_{n\in\mathcal{N}} \lambda_n \equiv \exp\left(-\frac{d}{ds}\zeta_D(s)\bigg|_{s=0}\right), \tag{52}$$

providing that the spectral zeta function $\zeta_D(s)$ has an analytic extension and is holomorphic at $s = 0$. [1]

In the case of the functional integral $Z[h]$, we need to regularize the determinant in the connected generating functional $W[0]$. To do this, it is necessary to perform an analytic continuation of the spectral zeta function from $\mathrm{Re}(s) > s_0 > 0$ into a region containing a complex neighborhood of the origin, i.e. $s = 0$. This method can also be used in a perturbative regime with self-interacting fields. In this case the one-loop effective action can be computed by evaluating a determinant similarly as was mentioned above [67].

Frequently, in the calculation of the generating functional a constant μ of dimension of mass is introduced to keep proper dimensionality of the Eq. (51). Hence the zeta-function regularized determinant is given by

$$\frac{d}{ds}\zeta_{\mu^2 D}(s)\bigg|_{s=0} = \ln(\mu^2)\zeta_D(s)\bigg|_{s=0} + \frac{d}{ds}\zeta_D(s)\bigg|_{s=0}. \tag{53}$$

3.3 PRIME NUMBERS AND RIEMANN ZEROS AS SPECTRUM

Following Menezes and Svaiter [41], it is not possible to built a field theory with the prime numbers as spectrum. Using that the prime zeta function Eq. (46) can be analytically extended only in the strip $0 < \mathrm{Re}(s) \leqslant 1$ [54, 55], Menezes and Svaiter [41] concluded that the

1 In general, for a sequence of complex numbers this definition of the regularized product depends strongly on the choice of the argument of λ_n. Therefore, one must be careful to deal with this problem.

sequence of prime numbers cannot be associated with some hypothetical linear operator of a physical system with infinitely many degrees of freedom. This result was generalized by Andrade to other sequences of numbers motivated by number theory [73]. Later, Menezes, Svaiter, and Svaiter also considered the same situation with numerical sequences whose asymptotic distributions are not "far away" from the asymptotic distribution of prime numbers [42].

Next, using the construction of the so-called superzetas or secondary zeta functions built over the Riemann zeros, i.e., the nontrivial zeros of the Riemann zeta function [74, 75, 76, 77, 78, 79, 80], and the regularity properties of one of these secondary zeta functions at the origin (see Appendix (C)), it was shown that the sequence of the nontrivial zeros of the Riemann zeta function can in principle be interpreted as being the spectrum of a self-adjoint operator acting on scalar fields of some hypothetical system.

Although there is a relationship between the set of prime numbers and nontrivial zeros of the zeta function, these two sequences of numbers have totally distinct behavior with respect to being the spectrum of a linear operator associated to a system with countable infinite number of degrees of freedom.

3.4 RIEMANN ZEROS IN QUANTUM MECHANICS

Berry and Keating proposed that the hamiltonian

$$H = \frac{1}{2}\left(qp + pq\right) \tag{54}$$

could have as spectrum the Riemann zeros. However, it turns out that a simple generalization to quantum field theory leads one to rather suspicious consequences. A straightforward extension of the supersymmetric Wess-Zumino model [81] to spaces of dimension $d > 4$ leads one to equations of motion of higher order such as

$$\left(\Box^{\kappa/2} - m^{\kappa}\right)\varphi = 0, \tag{55}$$

where $\kappa = 2^{d/2-1}$ and φ is a scalar quantum field. As discussed in [82], these equations for any $d > 4$ have a tachyonic component. Being more specific, for the case of six dimensions, the above equation can be derived from the second-order lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\Box\varphi\Box\varphi - m^4\varphi^2\right), \tag{56}$$

whose associated hamiltonian has a tachyonic contribution. Such a term can be understood as representing a superposition of systems having for each momentum degree of freedom **k** the following hamiltonian

$$H_{\mathbf{k}} = \frac{1}{2}\left(q_{\mathbf{k}}p_{\mathbf{k}} + p_{\mathbf{k}}q_{\mathbf{k}}\right). \tag{57}$$

For more details see Ref. [82]. It is easy to see that equation (57) is the straightforward generalization of the hamiltonian (54) to the case of infinite number of degrees of freedom. This leads us to a model for tachyon quantization. As well known, such a model is plagued with many difficulties such as imaginary mass, lack of unitarity, causality issues, etc. Therefore, as long as quantum field theory is concerned, models based on Eq. (57) should be discarded on physical grounds.

4 RIEMANN ZETA ZEROS AND ZERO POINT ENERGY

We have all this evidence that the Riemann zeros are vibrations, but we don't know what's doing the vibrating.

— M. du Sautoy - The Music of the Primes.

The Casimir effect is the manifestation of the zero-point vacuum fluctuations. It was proposed more than 60 years ago in a seminal paper [83]. In its simplest form, it consists of the attraction between two infinitely large, parallel, conducting planes that are electrically neutral placed into a electromagnetic field in the ground state. The force per unit area, i.e the pressure between the planes is given by $P(a) \approx -\hbar c/a^4$, where $\hbar$ is the Planck constant, c is the speed of light and a is the separation distance between the planes. It is remarkable that a macroscopic effect is caused by the quantum vacuum fluctuations. For an extensive review of the Casimir effect see [84, 85, 86].

According to quantum mechanics, the energy levels of the harmonic oscillator are $E_n = \hbar\omega(n + 1/2)$, where ω is the angular frequency, and $n = 0, 1, 2, \cdots$ is the number of energy quanta. The energy of the ground state (vacuum) containing $n = 0$ energy quanta is $E_0 = \hbar\omega/2$ which is not null. Therefore this is the energy of a zero-point fluctuation with energy $\hbar\omega/2$. In the framework of quantum field theory a quantized field is considered as a collection of harmonic oscillators with all frequencies. Hence, the energy of the ground state of the field is the sum of the energies of zero-point fluctuations $E_0 = (\hbar/2)\sum_i \omega_i$, with i labeling the quantum numbers of the field modes. This sum diverges and therefore a suitable renormalization procedure must be employed to obtain a finite energy per unit area. Different schemes to deal with this kind of divergences have been built based on analytic zeta-regularization, the heat kernel, addition of counterterms, substracting of configurations with and without boundaries, and so on [87, 88, 89, 90].

A crucial question addressed here, is whether the Hilbert and Pólya conjecture is realized in a quantum mechanical system with countably infinite number of degrees of freedom. We are interested to investigate the consequences of the existence of an hypothetical differential operator conjecture by Hilbert and Pólya in the quantum field theory framework. Since there is a relationship between the Casimir energy and the one-loop effective action [83, 91, 92, 84, 90, 93], it is possible to calculate the renormalized vacuum energy associated with a hypothetical system where the imaginary part of the complex zeros of the zeta function appear in the spectrum of the vacuum modes. The assumptions are: the operator must be defined in a bounded region of space since the spectrum is discrete. This operator must be self-adjoint in some Hilbert space and acts on scalar functions defined in flat space-time.

The aim of this chapter is to discuss the renormalization of the vacuum expectation value of the energy operator, i.e., the renormalized zero-point energy associated to a massive scalar field defined in a $(d+1)$-dimensional flat space-time, assuming that only one of the coordinates lies in a finite interval $[0, a]$ and the others are unbounded.

4.1 THE ZETA ZEROS AND THE VACUUM ENERGY

The Weyl theorem and its generalization by Pleijel relate the asymptotic distribution of eigenvalues of an elliptic differential operator with geometric parameters of the surface where the fields satisfy some boundary condition. [94, 95, 96, 97, 98, 99]. For example, the asymptotic series for the density of eigenvalues of the Laplacian operator in a four dimensional space-time $N(\omega)$ is given by

$$N(\omega) = V\omega^2/2\pi^2 \mp S\omega/8\pi + Sq/6\pi^2 + O(\omega^{-2}), \tag{58}$$

where V is the volume of the three-space, S the area of the boundary and q is mean curvature of the boundary averaged over the surface, for Dirichlet (Neumann) boundary conditions. Since the asymptotic behavior of the non-trivial zeros has a regime quite different from the asymptotic behavior of the spectrum of the Laplacian, we consider a linear operator $-\Delta_{d-1} - L$, where Δ_{d-1} is the usual Laplacian defined in a $(d-1)$-dimensional space, and impose that the eigenvalues associated with the linear operator L are the imaginary part of the complex zeros of the Riemann zeta function.

We are using the same idea of the papers [100, 101, 102], where some confining potential acts as a pair of effective plates. We use Cartesian coordinates $x^\mu = (t, \mathbf{x}_\perp, z)$. One way to implement our model is to assume that the scalar field is coupled to a confined background field $\sigma(z)$ with an interaction Lagrangian $\mathcal{L}_{int} = \sigma(z)\,\varphi^2(t, \mathbf{x}_\perp, z)$ such that the system is confined in the interval $[0, a]$ in one dimension and is unrestricted in the other spatial directions. We call such configuration a slab-bag [103]. Here we are interested in studying finite size systems [104, 105, 106, 107], where the translational invariance is broken. We refer as finite size system to any system that has finite size in at least on space dimension. The idea of substituting the hard boundary conditions on some surface by potentials was used by many authors. For instance, the zero-point energy of a quantum field as the limit of quantum field theory coupled to a background was used in Ref. [108, 109].

Let us suppose a flat $(d+1)$-dimensional manifold $\mathcal{M}$, i.e, a Minkowski space-time with $(d+1)$ dimensions and consider a quantum field theory of a single scalar field $\varphi : \mathcal{M} \to \mathbb{R}$. The action functional of the theory is

$$S(\varphi) = \int d^{d+1}x\, \mathcal{L}(\varphi), \tag{59}$$

where $\mathcal{L}(\varphi)$ is the Lagrangian density of the system. We should assume that the scalar field is coupled to a background field $\sigma(z)$ such that the system is confined in the interval $[0, a]$ in one dimension and is unrestricted in the other spatial directions. The Lagrangian density of the system is given by

$$\begin{aligned} \mathcal{L} &= \varphi(t, \mathbf{x}_\perp, z)\big(\partial_t^2 - \Delta_{d-1} - L\big)\varphi(t, \mathbf{x}_\perp, z) \\ &- m^2\varphi^2(t, \mathbf{x}_\perp, z) - \delta m^2 \varphi^2(t, \mathbf{x}_\perp, z), \end{aligned} \tag{60}$$

where $m_B^2 = m^2 + \delta m^2$, and m^2 is the squared bare mass and δm^2 is the mass counterterm [109]. The counterterms have to be included in odd-dimensional space-time as will be shown below. The eigenfrequencies of the vacuum modes can be found from the equation

$$\big(-\Delta_{d-1} - L\big)\varphi(\mathbf{x}_\perp, z) = \omega^2 \varphi(\mathbf{x}_\perp, z). \tag{61}$$

Since we are assuming that the linear operator L has a differential and a background contribution we can write

$$-L = \big(-\mathcal{O}_z + \sigma(z)\big), \tag{62}$$

where $\mathcal{O}_z$ is an unknown differential operator. As we discussed before, since we are assuming that the eigenvalues of the L operator are the imaginary part of the Riemann zeta zeros we have that the linear operator $-L$ satisfies

$$\big(-\mathcal{O}_z + \sigma(z)\big)u_n(z) = \frac{\gamma_n}{a^2} u_n(z), \tag{63}$$

where $u_n(z)$ is a countable infinite set of eigenfunctions, and a is the size of the compact spatial dimension. The eigenfunctions $u_n(z)$ and the respectively complex conjugates $u_n^*(z)$ satisfy the completeness and orthonormality relations, i.e,

$$\sum_n u_n(z) u_n^*(z') = \delta(z - z') \tag{64}$$

and

$$\int_0^a dz\, u_n(z) u_{n'}^*(z) = \delta_{n,n'}, \tag{65}$$

where $\delta(z - z')$ is the Dirac delta and $\delta_{n,n'}$ is the Kronecker delta. The zero-point energy of a massive scalar field defined in a d-dimensional box of lengths $A_1, A_2, ..., A_d$ and volume $\Omega = \prod_{i=1}^{d} (A_i)$ in a $(d+1)$-dimensional flat space-time is given by

$$\langle 0|H|0\rangle = \frac{1}{2}\sum_{\mathbf{k}}^{\infty} \omega_{\mathbf{k}}. \tag{66}$$

The Cartesian components of $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ take the values $-\infty < k_i < \infty$, with $i = 1, 2, ..., d$. For example, if we assume that the field is defined in a spacelike d-torus of side A, then

$$k_i = \frac{2\pi j_i}{A}, \quad j_i = 0 \pm 1, \pm 2, \cdots, \quad i = 1, \cdots, d. \tag{67}$$

The sum in Eq. (66) means that we have to perform d summations. This sum is divergent because all the vacuum modes give contribution to the zero-point energy [87]. The eigenfrequencies of the vacuum modes are given by

$$\omega_{\mathbf{k}} = \sqrt{k_1^2 + k_2^2 + \cdots + k_{d-1}^2 + k_d^2 + m^2}. \tag{68}$$

Since the eigenfunctions associated to the L operator are not known, we cannot find the Green functions of the system. Therefore the local renormalized energy density can not be defined. In other words, local methods cannot be implemented here.

As we discussed the scalar field is trapped by a potential in the interval $[0, a]$ in one of its dimensions and free in the other directions. We assume from now that the lengths associated to the other dimensions satisfy $A_i \gg a$ $(i = 1, ..., d-1)$ and $A_d = a$ i.e., we are assuming the slab-bag configuration [103]. Using this fact, we can change the d summations that appear in Eq. (66) by $(d-1)$ integrations and one summation. As we have discussed, let us assume that

$$k_d = \frac{\sqrt{\gamma_n}}{a} \quad n = 1, 2, \cdots, \tag{69}$$

the quantities γ_n being the imaginary part of the nontrivial zeros of the Riemann zeta function. Therefore the zero-point energy of the scalar field in the slab-bag configuration, taking into account the above considerations, is given by

$$\langle 0|H|0\rangle = \frac{1}{2}\prod_{i=1}^{d-1}\left(\frac{A_i}{2\pi}\right)\int_0^\infty d^{d-1}k \sum_{n=1}^{\infty}\left[k_1^2 + \cdots + k_d^2 + m^2\right]^{1/2}. \tag{70}$$

Let us define the total zero-point energy per unit area of the hyperplates as:

$$\varepsilon_{d+1}(a) = \frac{\langle 0|H|0\rangle}{\prod_i A_i}. \tag{71}$$

Substituting Eq. (70) in Eq. (71) we get

$$\varepsilon_{d+1}(a, m) = \frac{1}{2}\sum_{n=1}^{\infty}\int_0^\infty \frac{d^{d-1}k}{(2\pi)^{d-1}}\left[k_1^2 + \cdots + k_d^2 + m^2\right]^{1/2}. \tag{72}$$

To proceed, we can use dimensional [110, 111, 112, 113, 114] and analytic regularization combined. Using the well known formula [114]

$$\int_0^\infty \frac{d^d u}{(u^2 + a^2)^s} = \frac{\pi^{d/2}}{\Gamma(s)}\Gamma\left(s - \frac{d}{2}\right)\frac{1}{(a^2)^{s-d/2}}, \tag{73}$$

and defining the function $f(d)$ as

$$f(d) = \frac{1}{2(2\sqrt{\pi})^d}, \tag{74}$$

the vacuum-energy per unit area $\varepsilon_{d+1}(a)$ can be written as

$$\varepsilon_{d+1}(a, m) = \frac{f(d)}{a^d}\Gamma(-d/2)\sum_{n=1}^{\infty}\left(\gamma_n + a^2m^2\right)^{\frac{d}{2}}. \tag{75}$$

For even dimensional space-time, due to the unboundedness of the eigenvalues of L, the vacuum-energy per unit area is divergent. In the odd dimensional case, the divergence of the vacuum-energy per unit area comes also from the singular structure of the gamma function. In the next section we will discuss the renormalization procedure to obtain a finite vacuum energy associated to the system.

4.2 THE RENORMALIZED ZERO-POINT ENERGY

In order to obtain the renormalized vacuum energy, let us use the analytic extension technique. Before continuing, we would like to point out that a very common procedure to study the divergent contribution of the vacuum energy given by Eq. (75) is to use the heat-kernel expansion in some asymptotic limit [90]. This is a good procedure for usual operators, such as the Laplacian, where there appears a logarithmic divergent mass term $\ln(4\mu^2/m^2)$. The μ is an parameter with mass dimension that we have to introduce in order to perform analytic regularizations. Since in our problem we are using an unknown operator, this procedure cannot be implemented.

In the next section we show that the renormalized vacuum energy is identified with the constant term in the asymptotic expansion of the regularized vacuum energy. To proceed, let us assume that $a^2m^2 < \gamma_1$. In other words, the square of the ratio between the length of the slab-bag in the z direction to the Compton length $l_c = m^{-1}$ is smaller than the first non-trivial Riemann zero. It is important to stress that in general it is not necessary to impose such a condition. Here we need this assumption to make use of the generalization of the binomial expansion. For any complex coefficient α and $|x| < 1$ we may write

$$(1+x)^\alpha = \sum_{n=0}^{N} C_\alpha^n x^n \quad |x| < 1, \tag{76}$$

where C_α^n are the generalizations of the binomial coefficients given by $C_\alpha^n = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}$. Let us define the function $g(q, d)$ as

$$g(d, q) = \frac{f(d)\Gamma(\frac{d}{2}+1)}{\Gamma(q+1)\Gamma\left(\frac{d}{2}-q+1\right)}. \tag{77}$$

The zero-point energy per unit area can then be written as

$$\varepsilon_{d+1}(a,m) = \frac{1}{a^d}\sum_{q=0}^{N(d)} g(d,q)\Gamma(-d/2)(ma)^{2q}\sum_{n=1}^{\infty}(\gamma_n)^{\frac{d}{2}-q}. \quad (78)$$

In the above equation, for odd dimensional space-times we get that the sum over q has a finite number of terms, i.e., $N = d/2$ and for even dimensional space-time we have $N \to \infty$.

Let us define the superzeta or secondary zeta function built over the Riemann zeros, i.e., the nontrivial zeros of the Riemann zeta function. Let s be a complex variable i.e. $s = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$. The superzeta $G_\gamma(s)$ is defined as

$$G_\gamma(s) = \sum_{n=1}^{\infty}\frac{1}{\gamma_n^s}, \quad \Re(s) > 1, \quad (79)$$

where we are assuming that $\gamma_n > 0$. The analytic continuation of different families of the superzeta has been discussed in the literature [74, 75, 76, 77, 78, 79, 80]. Here we are using the analytic extension for $G_\gamma(s)$ assuming the Riemann hypothesis and following the Ref. [75], see appendix (C). A more detailed study of different families of zeta functions for the Riemann zeros can be found in Ref. [80].

Let us study first the even dimensional space-time case. By using the definition of $g(d,q)$ of Eq. (77) we define in this case the regular function of d, $h(d,q) = g(d,q)\Gamma(-d/2)$. The zero-point energy per unit area can be written as

$$\varepsilon_{d+1}(a,m) = \frac{1}{a^d}\sum_{q=0}^{\infty} h(d,q)(ma)^{2q} G_\gamma\big(q - d/2\big). \quad (80)$$

Note that the factor $\frac{1}{\Gamma(q+1)}$ that appears in the function $h(d,q)$ is a strong convergence factor in the summation. Since the analytic extension of the superzeta $G_\gamma(s)$ (presented in appendix (C)) has no poles for even dimensional space-time we obtain that the regularized vacuum energy of the system is finite in any even dimensional space-time.

The zero-point energy per unit area $\varepsilon_{d+1}(a)$ in the odd dimensional space-time case can be written as

$$\varepsilon_{d+1}(a,m) = \frac{1}{a^d}\sum_{q=0}^{d/2} g(d,q)(ma)^{2q}\Gamma(-d/2)\, G_\gamma\big(q - d/2\big). \quad (81)$$

The $G_\gamma(s)$ is a meromorphic function of s in the whole complex plane with double pole at $s = 1$ and simple poles at $s = -1, -3, .., -(2l+1), \ldots$ The gamma function $\Gamma(z)$ is a meromorphic function of a complex variable z with simple poles at the points $z = 0, -1, -2..$ (see appendix (A)). In the neighborhood of any of its poles $z = -n$ for $n = 0, 1, 2..$ the gamma function has a representation given by

$$\Gamma(z) = \frac{(-1)^n}{n!(z+n)} + \Omega(z+n), \tag{82}$$

where $\Omega(z+n)$ stands for the regular part of the analytic extension. The same construction can be used for the superzeta function. The double pole for the superzeta function at $s = 1$ does not give contribution for the sum since we must have $q = \frac{d}{2} + 1$. In the neighborhood of any of its first order poles $z = -(2l+1)$ for $l = 0, 1, 2...$ the superzeta function has a representation given by

$$G_\gamma(z) = \frac{\alpha_1}{z + (2l+1)} + \Upsilon\big(z + (2l+1)\big), \tag{83}$$

where $\Upsilon(z + (2l+1))$ is the regular part of the analytic extension and α_1 is a constant. The regularized vacuum energy has second order and first order poles.

To analyze the analytic structure of $\varepsilon_{d+1}(a, m)$, let us discuss the behavior of the sum given by Eq. (81) in a generic odd dimensional space-time. Since the gamma function gives a first order pole in any odd dimensional space-time, let us study only the contribution coming from the superzeta. Let us write the zero-point energy per unit area $\varepsilon_{d+1}(a)$ as

$$\varepsilon_{d+1}(a, m) = \frac{1}{a^d}\Gamma(-d/2)\,\rho_{d+1}(a, m), \tag{84}$$

where $\rho_{d+1}(a, m)$ is given by

$$\rho_{d+1}(a, m) = \sum_{q=0}^{d/2} g(d, q)(ma)^{2q} G_\gamma\big(q - d/2\big). \tag{85}$$

Using the fact that the first order poles of the superzeta function are located at $z = -(2l+1)$ for $l = 0, 1, 2...$, we have to study the singularity structure of $\rho_{d+1}(a, m)$ coming from the equation $q - \frac{d}{2} = -(2l+1)$ for $l = 0, 1, 2....$ Note that we have that $l \leqslant \frac{d-2}{4}$. For $d = 2$ the only polar contribution to the sum comes from the term $q = 0$. This term is related to the first order pole at $s = -1$ of the superzeta. The last term of the sum, i.e., $q = 1$ is analytic. For $d = 4$ the contribution coming from the $q = 0$ and $q = 2$ terms are analytic and the contribution of the $q = 1$ term is polar, with a first order pole coming from the $s = -1$ pole of the superzeta. For $d = 6$ the contributions coming from $q = 1$ and $q = 3$ are analytic, and the contributions coming from $q = 0$ and $q = 2$ are singular with a first order pole coming from $s = -3$ and $s = -1$ poles of the superzeta. See the table 1 where the polar and analytic contributions in the sum which defines $\rho_{d+1}(a, m)$ are shown.

Table 1. Polar and analytic contributions in $\rho_{d+1}(a, m)$

d \ q	0	1	2	3	4
2	$l = 0$	-			
4	-	$l = 0$	-		
6	$l = 1$	-	$l = 0$	-	
8	-	$l = 1$	-	$l = 0$	-
10	$l = 2$	-	$l = 1$	-	$l = 0$

Therefore in order to make the energy density per unit area finite in odd dimensional space-time we have to introduce mass counterterms proportional to second order and first order poles. This problem has been discussed by Kay [115]. This author shows that the analytic regularization procedure does not yield a finite result automatically in the case of massive fields.

As an instructive example, let us discuss this renormalization procedure in a three-dimensional space-time. In this case we have

$$\varepsilon_3(a, m) = \frac{1}{a^2} \sum_{q=0}^{1} g(2, q)(ma)^{2q} \Gamma(-1+\epsilon) G_\gamma\big(q - 1 + \epsilon\big). \qquad (86)$$

The mass counterterm δm^2 in a generic odd-dimensional space-time has dimension $[\delta m^2] = \frac{1}{a^{d+1}} \mu^{1-d}$, where we introduced a mass parameter μ. Therefore we get

$$\delta m^2 = -\frac{1}{8\pi\, a^3 \mu} \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(\Omega_1 + \Upsilon_1) \right) - \frac{1}{8\pi\, a\mu} \frac{1}{\epsilon} m^2 G_\gamma(0), \qquad (87)$$

where Ω_1 is the regular part of the analytic extension for the gamma function around the pole in $s = -1$ and Υ_1 is the regular part of the analytic extension for the superzeta function around the pole in $s = -1$. The mass correction Δm^2 is given by

$$\Delta m^2(a, \mu) = \frac{1}{8\pi} \left(\frac{1}{a^3 \mu} \Omega_1 \Upsilon_1 + \frac{1}{a\mu} m^2 \Omega_1 G_\gamma(0) \right), \qquad (88)$$

where μ is a mass parameter that is introduced to perform the analytic regularization. We can define a renormalized square mass, the Riemann mass, given by

$$m_R^2 = m^2 + \Delta m^2(a, \mu). \qquad (89)$$

It is interesting to stress that the mass generation occurs only in odd dimensional space-time, and that the effect vanishes when the translational invariance is recovered. This mechanism is quite similar to the topological generation of mass [116, 117]. Nevertheless, in this last case the translational invariance is maintained.

Finally two comments are in order. First, the renormalized vacuum energy is identified with the constant term in the asymptotic expansion of the regularized vacuum energy. For the massless case it is possible to show, using an exponential ultraviolet cut-off [118, 88, 89, 119, 120], that the finite part of the regularized vacuum energy agrees with the renormalized energy density obtained by the analytic regularization procedure. Second, the Casimir energy for the case of fermionic fields was investigated in this configuration, based on the MIT bag model [121, 122]. The generalization for massive fermions was presented by Elizalde et al [65]. See also the Ref. [123].

5 ONE-LOOP EFFECTIVE ACTION AND RIEMANN ZEROS

The Riemann Hypothesis is a precise statement, and in one sense what it means is clear, but what it's connected with, what it implies, where it comes from, can be very unobvious.

— M. Huxley - Dr. Riemann's Zeros.

Recently there have been investigations whether a large class of systems with countable infinite number of degrees of freedom described by self-adjoint operators can be used to give a spectral interpretation for the Riemann zeros [45]. The main result we emphasize is that in the formalism of quantum field theory there is no obstruction for the existence of the above mentioned elusive operators, in the sense that it is possible to define a correlation generating function for a system with the Riemann zeros as its spectrum.

In this chapter we delve a different point of view aiming to shed some light on this problem. We consider a large class of two dimensional field theory models in Euclidean space. We show that the asymptotic behavior of the Fourier transform of the two-point correlation function in such models fits the asymptotic distribution of the non-trivial zeros of the Riemann zeta function. As an example we show that the non-linear sigma model in the leading order in $1/N$ expansion presents such a behavior. Although there is in the literature intense activity connecting the Riemann zeros, classical and quantum chaos, random matrices and disordered systems, as far as we know the result presented here is discussed for the first time.

We briefly mention that non-linear sigma models can be brought to the context of random matrices and systems with disorder. This is achieved by considering the fact that a gas of fermions in the presence of a random potential can be described by an effective low-energy field theory, which is precisely a non-linear sigma model [28, 29]. It can be shown that random matrix theory appears in an appropriate limit of such a model. The non-linear sigma model also emerges if we consider the problem of wave propagation in a medium with a correlated spatially varying index of refraction [124]. Since disorder in wave problems leads us to chaotic systems in the limit of short wavelengths, one has an example in which a link between the Riemann zeta zeros and the non-linear sigma model must be investigated.

5.1 THE ONE-LOOP EFFECTIVE ACTION

In this Section we digress on the fundamental object through which correlation functions in many models in two-dimensional Euclidean scalar field theory relate to the asymptotic distribution of Riemann zeros. We consider massive interacting fields ϕ with (bare) mass m. Here the basic quantity we are interested in is

$$\mathcal{S}_{\text{eff}}[M] = \ln\left[-\partial^2 + M^2(m,x)\right], \tag{90}$$

where $M(m,x)$ is some scalar function with square mass dimensions. Whenever a formal perturbative expansion in powers of a suitable parameter is available, the usual Feynman diagrams of field theory can be displayed. In particular, (90) generates one-loop diagrams, in particular bubble diagrams. The quantity (90) naturally appears in the semi-classical or loopwise expansion of field theory. As well known, this method amounts to introduce a formal expansion of the functional integral in powers of $\hbar$:

$$Z[J] = \int [d\phi] \exp -\frac{1}{\hbar}\left\{S[\phi] - \int d^2x\, J(x)\phi(x)\right\} \tag{91}$$

where $[d\phi]$ is a formal product of Lebesgue measures at every point of $\mathbb{R}^2$, $J(x)$ is an external source and $S[\phi]$ is the action given by

$$S[\phi] = \int d^2x \left[\frac{1}{2}\left(\partial_\mu\phi\partial_\mu\phi + m^2\phi^2\right) + V[\phi]\right]. \tag{92}$$

Here we regard $V[\phi]$ as a simple polynomial in ϕ. We are not going to present the standard derivation since it can be found in many textbooks, see for instance [125]. The important point is that, at order $\hbar$ one obtains the following expression for the generating functional Γ of proper vertices:

$$\Gamma[\varphi] = S[\varphi] + \frac{1}{2}\left\{\mathcal{S}_{\text{eff}}[M] - \ln\left[-\partial^2 + m^2\right]\right\}, \tag{93}$$

where $M^2 = m^2 + V''[\varphi]$, φ being the classical field, solution of the classical field equations $\delta S[\varphi]/\delta\varphi = 0$. The n-th functional derivative of Γ with respect to φ for $n \geqslant 3$ yields the one-particle irreducible correlation functions of the theory.

Let us present one model where the basic quantity (90) shows up. Consider a simple model of two Euclidean self-interacting scalar fields $\phi_1(x)$ and $\phi_2(x)$ with masses m_1 and m_2, respectively. Assume that $m_2 \gg m_1$. After the construction of an effective field theory and further imposition of the infinite mass limit for the heavy field, there is a decoupling between the light and heavy modes as stated by the Appelquist-Carrazzone theorem [126]. This is essentially the model studied in Ref [105] in two dimensions. Following such a reference, we consider the theory described by the following action with two real scalar fields

$$S[\phi_1,\phi_2] = \int d^2x \left[\frac{1}{2}\left(\partial_\mu\phi_1\partial_\mu\phi_1 + m_1^2\phi_1^2 + \partial_\mu\phi_2\partial_\mu\phi_2 + m_2^2\phi_2^2\right) + V[\phi_1] + \frac{\lambda}{2}(\phi_1\phi_2)^2\right]. \quad (94)$$

The precise form of $V[\phi_1]$ is not important for the construction of the effective theory, although it is essential in order to consistently implement the decoupling theorem [105]. The partition function is given by

$$Z[j_1,j_2] = N\int[d\phi_1][d\phi_2]\, e^{-S[\phi_1,\phi_2]}e^{-\int d^2x\,(j_1\phi_1+j_2\phi_2)}, \quad (95)$$

where $j_1(x), j_2(x)$ are external sources and N is a normalization. The above action can be rewritten as

$$S[\phi_1,\phi_2] = S_1[\phi_1] + S_2[\phi_1,\phi_2], \quad (96)$$

where $S_1[\phi_1]$ is the ϕ_2-independent part of the total action. In order to obtain the effective action for the light modes one must assume that $m_2 \gg m_1$. In this way we integrate out the heavy field ϕ_2 in the functional integral and define the effective action of the light modes $\Gamma_{\text{eff}}[\phi_1]$ by

$$e^{-\Gamma_{\text{eff}}[\phi_1]} = e^{-S_1[\phi_1]}\int[d\phi_2]\, e^{-S_2[\phi_1,\phi_2]} \quad (97)$$

By means of a simple Gaussian integration, we obtain the following expression for the effective action of the light field $\phi_1(x)$

$$\Gamma_{\text{eff}}[\phi_1] = S_1[\phi_1] + \frac{1}{2}\mathcal{S}_{\text{eff}}[M], \quad (98)$$

where $M^2 = m_2^2 + \lambda\phi_1^2$. So we see that $\mathcal{S}_{\text{eff}}$ here appears as the one-loop contribution to the effective action for the light fields.

Another example in which Eq. (90) appears is provided by an $O(N)$ symmetry Euclidean field theory with a N-component scalar field $\Phi(x)$. In a two-dimensional Euclidean space, the action is given by

$$S[\Phi] = \int d^2x \left\{\frac{1}{2}\partial_\mu\Phi^T\partial_\mu\Phi + N\,U\left[\frac{\Phi^2}{N}\right]\right\}, \quad (99)$$

where U is a general polynomial in Φ^2 and

$$\Phi^2 = \sum_{i=1}^{N}\varphi_i^2. \quad (100)$$

The correlation functions of the model are generated by the partition function

$$Z[h] = \int\prod_x d\Phi(x)\, \exp\left[-S[\Phi] + \int d^2x\, h^T(x)\Phi(x)\right], \quad (101)$$

where the last term in the argument of the exponential is the contribution for an auxiliary N-component external field $h(x)$. Since we are interested in finite theories, a cut-off consistent with the symmetries of the model is implicit. Introduce the field $\rho(x)$ defined by

$$\rho(x) = \frac{\Phi^2}{N}, \tag{102}$$

and the field $\lambda(x)$ which acts as a Lagrange multiplier, i.e.

$$1 = \frac{N}{4\pi i} \int d\rho \, d\lambda \, \exp\left[\frac{\lambda}{2}\left(\Phi^2 - N\rho\right)\right], \tag{103}$$

where the λ integration runs parallel to the imaginary axis. Using the fields $\lambda(x)$ and $\rho(x)$, and defining the respective functional measures $[\rho(x)]$ and $[\lambda(x)]$ we can write the partition function $Z[0]$ as

$$Z[0] = \int [d\Phi]\,[d\rho]\,[d\lambda] \exp\{-S[\Phi, \rho, \lambda]\}, \tag{104}$$

where $S[\Phi, \rho, \lambda]$ is given by

$$S[\Phi, \rho, \lambda] = \int d^2x \left\{\frac{1}{2}\left(\partial_\mu \Phi\right)^2 + N\, U\left[\rho(x)\right] + \frac{1}{2}\lambda(x)\left[\Phi^2(x) - N\rho(x)\right]\right\}. \tag{105}$$

Let us write the N-component field $\Phi(x)$ as

$$\Phi^T = (\pi^1, \pi^2, ..., \pi^{N-1}, \sigma). \tag{106}$$

Integrating out the $N-1$ components $\pi(x)$ and introducing a one-component source $h(x)$ we have

$$Z[h] = \int [d\sigma][d\rho][d\lambda] \exp\left[-S_N(\sigma, \rho, \lambda) + \int d^2x\, h(x)\sigma(x)\right], \tag{107}$$

where

$$\begin{aligned} S_N(\sigma, \rho, \lambda) \;&=\; \int d^2x \left\{\frac{1}{2}\left(\partial_\mu \sigma\right)^2 + N\, U\left[\rho(x)\right]\right. \\ &+\; \left.\frac{1}{2}\lambda(x)\left[\sigma^2(x) - N\rho(x)\right]\right\} + \frac{1}{2}(N-1)\mathcal{S}_{\text{eff}}[M], \end{aligned} \tag{108}$$

where now $M^2(x) = \lambda(x)$. In the large N expansion we can obtain the saddle point equations. We are looking for a uniform saddle-point: $\sigma(x) = \sigma$, $\rho(x) = \rho$ and finally $\lambda(x) = m^2$. We are particularly interested in studying fluctuations around the uniform saddle-point solution. One can show that the asymptotic behavior of the Fourier transform of the two-point correlation function of the linear sigma model in the leading order in $1/N$ expansion in a two-dimensional Euclidean space fits the asymptotic distribution of the Riemann zeros. In the next section we will present explicit calculations that prove our statement.

5.2 NON-LINEAR SIGMA MODEL AND RIEMANN ZEROS

The sigma model was first introduced in [127] as an example realizing chiral symmetry and partial conservation of the axial current. The model is constructed with a fermionic isodoublet field, a triplet of pseudoscalar fields and also a scalar field. The sigma model without a fermionic isodoublet field with only two scalar fields σ and π has been widely discussed in the literature, as an example of spontaneous symmetry breaking. This model can be generalized using a N-component field, in such a way that the Lagrangian density is invariant under the orthogonal group in N-dimension, $O(N)$ [128]. The local action of the model in a four-dimensional Euclidean space is

$$S[\Phi] = \frac{1}{2}\int d^4x \left[\partial_\mu \Phi^T \partial_\mu \Phi + V(\Phi^2)\right] \tag{109}$$

with $\Phi^T = (\pi^1, \pi^2, ..., \pi^{N-1}, \sigma)$ and $\Phi^2 = \Phi^T\Phi$. The potential $V(\Phi^2)$ is chosen to produce a minimum whenever $\Phi^2 = v^2 > 0$. The system exhibits spontaneous symmetry breaking. There is a non-trivial subgroup which leaves the vacuum invariant, the $O(N-1)$. For each generator of the $O(N)$ group which does not leave the vacuum invariant, there corresponds a massless Goldstone boson. An alternative description to produce spontaneous symmetry breaking is given by the non-linear sigma model [129]. The non-linear sigma model is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial_\mu \Phi \tag{110}$$

with the constraint $\Phi^T\Phi = 1$. The generating functional of the Schwinger functions can be written as

$$Z = \int [d\Phi]\delta[\Phi^T\Phi - 1] \exp\left(\left(\int d^4x\, \mathcal{L} + \text{source terms}\right)\right). \tag{111}$$

The quantization of non-linear sigma models was initiated in Refs [130, 131, 132]. Although this model is not renormalizable in a four dimensional Euclidean space, it makes sense as a low-energy effective theory. The two dimensional case is quite different. In this situation the model is perturbatively renormalizable. One can show that the beta function of the model is negative, therefore the theory is also asymptotically free for $N > 2$. The renormalizability of the two-dimensional non-linear sigma model was proved in Refs. [133, 134].

Now we present some evidence that the large N expansion of the two-dimensional $O(N)$ non-linear sigma model described by the Euclidean action

$$S[\Phi] = \frac{1}{2g_0^2}\int d^2x\, \partial_\mu \Phi^T \partial_\mu \Phi, \quad \Phi^T\Phi = 1 \tag{112}$$

might have a deep connection with the asymptotic distribution of the Riemann zeros. For a complete review of the large N expansion in quantum field theory, see the Ref. [135]. In equation (112) g_0 is a coupling constant and the field Φ is a N-component vector. We are following the discussion presented in Refs. [136, 137]. The partition function which describes such modes in this model reads

$$\begin{aligned} Z &= \int [d\Phi]\delta[\Phi^T\Phi - 1]\, e^{-\frac{1}{2g_0^2}\int d^2x\, \partial_\mu\Phi^T\partial_\mu\Phi} \\ &= \int_{C-i\infty}^{C+i\infty} [d\lambda(x)] \int [d\Phi]\, e^{-\frac{1}{2g_0^2}\int d^2x\, [\partial_\mu\Phi^T\partial_\mu\Phi + \lambda(x)(\Phi^T\Phi - 1)]}, \end{aligned} \tag{113}$$

where a Lagrange multiplier $\lambda(x)$ was introduced to substitute the functional Dirac delta function. The integral over the the field Φ can be performed, since all of them are Gaussian integrals. We can write the partition function as

$$Z = \int_{C-i\infty}^{C+i\infty} [d\lambda(x)]\, e^{-S[\lambda]}, \tag{114}$$

where the action $S[\lambda]$ is given by

$$S[\lambda] = -\frac{1}{2g_0^2}\int d^2x\lambda(x) + \frac{N}{2}\ln\det\left[-\partial^2 + \lambda(x)\right]. \tag{115}$$

As discussed in ([136]), the expansion above is formal since we have infrared and also ultraviolet divergences. To deal with the ultraviolet divergences we can introduce a cut-off or define the theory in a lattice [138]. Another approach is to use analytic regularization procedure. For an instructive example where an analytic regularization procedure is used to evaluate the Fredholm determinant, see the Ref. [105]. To cure the infrared divergences one can consider that the theory is defined in a finite volume. Introducing the Green function $G(x, x; \lambda)$ defined as

$$G(x, x') = \langle\, x|(-\partial^2 + \lambda)^{-1}|\, x'\,\rangle, \tag{116}$$

and computing the variation of Eq. (114) with respect to λ we get

$$\frac{1}{2g_0^2} = \frac{N}{2} G(x, x; \lambda). \tag{117}$$

In the saddle point approximation, using a Fourier representation for $G(x, x'; \lambda)$ we can solve the equation above. The cases $d = 2$ and $d > 2$ must be studied separately. We are interested only in the case $d = 2$, where continuous symmetries cannot be broken. The $O(N)$ symmetry is unbroken, since the expectation value for the components of Φ are zero, but the field $\lambda(x)$ has a nonzero vacuum expectation value, i.e. $\langle\, \lambda(x)\,\rangle = m^2$. We identify this quantity as a squared mass. The excitations associated with the components for the Φ field are massive particles. We are interested in studying fluctuations around this solution. With the identification

$$\lambda(x) = m^2 + i\alpha(x), \tag{118}$$

we get the effective action S_{eff} [see Eq. (90)]

$$S_{\text{eff}}[\alpha] = \frac{N}{2}\ln\left[-\partial^2 + m^2 + i\alpha(x)\right]. \tag{119}$$

Introduce the Fourier expansion

$$\alpha(x) = \int \frac{d^2p}{(2\pi)^2}\,\alpha(\mathbf{p})\,e^{i\mathbf{p}\mathbf{x}} \tag{120}$$

and let us consider the effective action for the modes $\mathbf{p} \neq 0$. Using a functional Taylor series one gets

$$\begin{aligned} S_{\text{eff}}[\alpha] &= \frac{N}{2}\ln\left[\Delta^{-1}\right] + \frac{1}{2}\int \frac{d^2p}{(2\pi)^2} u(-p)\Pi(p)u(p) \\ &+ \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{N}{2}\right)^{-n/2+1}\left[(iu(x)\Delta)^n\right], \end{aligned} \tag{121}$$

where we have defined the field

$$u(x) = \left(\frac{N}{2}\right)^{\frac{1}{2}}\alpha(x) \tag{122}$$

and the propagator

$$\Delta(x,x') = \langle\, x|(-\partial^2 + m^2)^{-1}|\,x'\,\rangle. \tag{123}$$

Also, in equation (121) we have the quantity

$$\Pi(p) = \int \frac{d^2q}{(2\pi)^2}\Delta(p/2+q)\Delta(p/2-q), \tag{124}$$

where

$$\Delta(p) = \frac{1}{p^2+m^2}. \tag{125}$$

The above series expansion furnishes a perturbative expansion for the u-field in the parameter $1/\sqrt{N}$. The zeroth-order effective action for the fluctuations is given by the quadratic terms in the mentioned expansion. Hence in the leading order, the Fourier transform of the two-point correlation function for the u-fields is given by

$$\langle u(-p)u(p)\rangle = [\Pi(p)]^{-1}. \tag{126}$$

Let us present an expression for this correlation function at large $|p| \gg m$. Consider the integral

$$I(\alpha,\beta,d,p) = \int \frac{d^dq}{(2\pi)^d}\frac{1}{[(q-p)^2+m^2]^{\alpha}\,[q^2+m^2]^{\beta}}. \tag{127}$$

It is easy to see that $I(1,1,2,p) = \Pi(p)$. After standard manipulations we get

$$I(\alpha,\beta,d,p) = \frac{1}{(4\pi)^{d/2}(p^2)^{\alpha+\beta-d/2}} \frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)} \times \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1}\left[x(1-x)+\frac{m^2}{p^2}\right]^{d/2-\alpha-\beta} \tag{128}$$

where $\Gamma(x)$ is the usual gamma function. Therefore, after we integrate we get

$$\Pi(p) = \frac{1}{4\pi f(m/p)p^2}\ln\left[\frac{1+f(m/p)}{-1+f(m/p)}\right], \tag{129}$$

where

$$f(x) = \left(1+4x^2\right)^{1/2}. \tag{130}$$

Since we are interested in the asymptotic result, let us assume that $|p| \gg m$. Under this assumption we get

$$\langle u(-p)u(p)\rangle|_{|p|\gg m} \approx \frac{2\pi t}{\ln(t/m^2)}\left[1-\frac{2}{(t/m^2)\ln(t/m^2)}+\frac{2}{(t/m^2)}\right], \tag{131}$$

where $t = t(p) = p^2 = p_0^2 + p_1^2$ is the equation of a circular paraboloid. We stress that the theory is finite after a renormalization procedure. Note that this result obtained in Eq. (131) has a remarkable resemblance with the asymptotic distribution of Riemann zeros. Remember that the number $N(T)$ of Riemann zeros $\sigma + i\gamma$, with $0 < \gamma < T$ is asymptotically given by the Riemann-von Mangoldt formula Eq. (40):

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\ln T), \tag{132}$$

as $T \to \infty$. The density $D(T) = N(T+1) - N(T)$ of zeros at height T given by Eq.(41) is

$$D(T) \sim \frac{1}{2\pi}\left[\ln\left(\frac{T}{2\pi}\right) + \frac{1}{T}\right]. \tag{133}$$

for high T. As we discussed in chapter 2 an important consequence of such a result is that the imaginary parts of consecutive zeta zeros in the upper half-plane $0 < \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \cdots$ satisfy $\gamma_n \sim n/D(n)$ as $n \to \infty$, i.e.

$$\gamma_n \sim \frac{2\pi n}{\ln(n/2\pi) + 1/n} \approx \frac{2\pi n}{\ln(n/2\pi)}\left[1 - \frac{1}{n\ln(n/2\pi)}\right]. \tag{134}$$

The similarity between the Eq. (131) and Eq. (134) is manifest. We have established the asymptotic equivalence between the distribution of the non-trivial zeros of the Riemann zeta function and the

Fourier transform of the two-point correlation function of the non-linear sigma model in the leading order in a large N expansion defined in a two-dimensional Euclidean space. Here the identification $n \leftrightarrow t$ leads one to the conclusion that n obeys a circular paraboloid equation in the momentum space of a field theory.

The non-linear sigma model in a two-dimensional Euclidean space was used as a toy model for the study of asymptotic freedom and dynamical mass generation [139]. As far as we know, although the literature emphasizes connections between the Riemann zeros, chaos, random matrices and disordered systems, the result presented above is original and new. Observe that in the semiclassical periodic orbit theory, the oscillating density of states is related through a trace formula to classical periodic orbits [17, 18]. It is possible to obtain trace formulas for the Riemann zeta function. Assuming the Riemannn hypothesis there is a remarkable parallel between the trace formulas of chaotic systems and trace formulas for the Riemann zeta function. This analysis is of order $\hbar$ as the large N expansion in the scalar field theory.

Of particular interest is to find other models where the behavior is similar to the linear and non-linear sigma model. One model that calls for attention is the $CP(N-1)$ model, which can be solved in the large N limit [140]. Also, since two-dimensional quantum chromodynamics (QCD) reduces to the non-linear sigma model in some limit [141], the result presented above indicates that a connection between the Riemann zeros and QCD must be investigated. On the other hand a pivotal matter that confronts us is the nature of $t(p)$. The Riemann zeros form a discrete set of points whereas for the quantum field theory considered here t is a continuous variable. This difficulty is already encountered within the Berry-Keating approach [23]. At the present moment we do not know how to circumvent this obstacle.

6 RANDOMIZED RIEMANN GAS

"[It has been] said that the zeros [of the Riemann zeta function] weren't real, nobody measured them. They are as real as anything you will measure in a laboratory - this has to be the way we look at the world."

Peter Sarnak [142]

In equilibrium statistical mechanics, macroscopic systems can develop phase transitions when external parameters change. To give a mathematical description of these phenomena it is usual to define the free energy, $F_N(\beta)$, where N is the number of particles of the system. For finite systems, the free energy and the partition function are analytic in the entire complex β-plane. In the thermodynamic limit, at points where analyticity is not preserved, one says that a phase transition occurs. The singularities in the free energy density correspond to the zeros of the partition function in the complex β-plane.

In two papers, Lee and Yang [143, 144] studied the zeros of the partition function of the Ising model in the complex magnetic field[1]. They obtained the following theorem: in the Ising model in a complex magnetic field h, the complex zeros of the partition function are located on the unit circle in the complex plane. Discussing the zeros in the complex temperature plane, Fisher studied an Ising model on a square two-dimensional lattice and showed that there is an accumulation of these complex zeros close to the critical point [145]. Systems with disorder and randomness also have been considered [146, 147].

There are many non-trivial features that are observed in disordered systems which are absent in their non-disordered counterparts. In disordered systems the dynamics of the disorder or impurities affect radically the way in which the thermodynamical physical quantities are computed. We can establish regimes of disorder comparing the dynamical time scales of the disorder and the field. If the time scale of the disorder is much longer than the time scale of the field we can assume that the disorder variable admits a stationary value. It means that each realization of the disorder (random noise) corresponds to a specific sample of the system[148, 149, 150, 151]

For all these systems the disorder is a random variable in the Hamiltonian where the probability distribution is known. In the spin-glass the disorder appears in the coupling between neighbor spins, and in

1 Early in 1915, the influence in statistical mechanics of the Pólya work on the Riemann zeta function was an example of the fruitful relationships between number theory and physics. Lee and Yang adapted the reasoning of Pólya, which dealed with an integral representation of the zeta function, to prove their celebrated Lee-Yang circle theorem.

the random field Ising model the random variable is a quenched magnetic field. One way to investigate such models is to replace the original Hamiltonian of each model by an effective Hamiltonian of the Landau-Ginzburg model where the order parameter is a continuous d-dimensional field. In such a case the disorder appears as a random temperature or a random external field. For the simplest case $d = 1$ we get the anharmonic oscillator with a random frequency ω. The quenched free energy for this system without the anharmonic contribution can be calculated in terms of the derivative of a particular spectral zeta function $\zeta(s, \omega)$ at $s = 0$. This derivative has to be determined by the analytic continuation from the domain where the series actually converges.

A very simple example of a disordered system which exhibits a phase transition is the random energy model. The zeros of the partition function in the complex temperature plane in this model has been studied numerically and also analytically [152]. In disordered systems, Matsuda, Nishimori and Hukushima studied the distribution of zeros of the partition function in Ising spin glasses on the complex field plane [153]. More recently Takahashi and collaborators studied the zeros of the partition function in the mean-field spin-glass models [154, 155]. Recently, the first experimental observation of Lee-Tang zeros has been achieved based on [156]. These measures have been performed through the quantum coherence of a probe spin coupled to an Ising-type spin bath [157]. A complex phase factor is generated by the evolution of the probe spin and therefore realizes an imaginary magnetic field. This experimental realization opens up the opportunity to study thermodynamics on the complex plane.

With respect to prime numbers, the models that have shown to be more accurate in practice to describe them are random models involving one or more random variables[158, 159, 4]. Prime numbers appear among the integers seemingly at random, and it is due to this local irregularity of its distribution that they can be thought to behave as an ideal gas. Despite of these are deterministic quantities, we can not expect a complete description of a deterministic set through a probabilistic model, we can gain some deeper understanding of its behavior and of the correlations between them.

Besides the Hilbert-Pólya conjecture, the interpretation of prime numbers related to energy eigenvalues of particles appears also in statistical mechanics. This can be seen studying the Riemann gas[2] and its disordered version. The purpose of this chapter is to study the consequences of introducing randomness on this arithmetic gas. We study the thermodynamic variables as the average free energy density and average mean energy density in the complex β-plane.

Due the particular spectrum of the arithmetic gas, the introduction of randomness in the Hamiltonian leads us to a quite inter-

2 The Riemann gas is also called an arithmetic gas.

esting situation where the argument of the Riemann zeta function is a random variable with some probability distribution. Therefore, this perspective provides a connection between the theory of the Riemann zeta function and the physics of disordered systems. Although a probabilistic approach in number theory is not new in the literature [158, 159, 4], as far as we know the approach presented here is discussed for the first time.

6.1 RIEMANN GAS

A bosonic Riemann gas is a second quantized mechanical system at temperature β^{-1}, with partition function given by the Riemann zeta function [32, 33, 34, 56, 57, 58, 50]. As was discussed by Weiss and collaborators, the hamiltonian for the Riemann gas can in principle be produced in a Bose-Einstein condensate [160].

Let us consider a non-interacting bosonic field theory with Hamiltonian

$$H_B = \omega \sum_{k=1}^{\infty} \ln(p_k) b_k^\dagger b_k, \tag{135}$$

where $b_k^\dagger$ and b_k are respectively the creation and annihilation operators of quanta associated to the bosonic field and the p_k 's are the prime numbers. The energy of each mode is given by $\nu_k = \omega \ln p_k$. As the particles are noninteracting, a many-body state in the second quantized formalism, can be represented by the occupation number n. By the fundamental theorem of arithmetic Eq. (1) this state is interpreted as m_1 particles are in the state $|p_1\rangle$, m_2 particles are in the state $|p_2\rangle$, and so on, hence

$$|n\rangle = |m_{1\,p_1}, m_{2\,p_2}, \cdots, m_{k\,p_k}\rangle, \tag{136}$$

because this unique factorization of $|n\rangle$, each many-body state is enumerated only once. The total energy of the system in the state $|n\rangle$ is

$$\begin{aligned} E_n &= \omega(m_1 \log p_1 + m_2 \log p_2 + \cdots + m_k \log p_k) \\ &= \omega \log(p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}) = \omega \log n. \end{aligned} \tag{137}$$

The arithmetic gas partition function therefore is

$$Z = \sum_{n=1}^{\infty} e^{-\beta E_n} = \sum_{n=1}^{\infty} \frac{1}{n^{\beta\omega}}, \tag{138}$$

which is exactly the Riemann zeta function. The parameter ω has dimension of frequency and β is the inverse of the temperature of the system. For $\beta\omega > 1$ or low temperatures we have a well defined partition function, whereas that for $\beta\omega \leqslant 1$ or high temperatures we do not have. This happens due to the simple pole of the Riemann zeta function at $s = 1$. This is a Hagedorn temperature [161] above which the system can not be heated up, since its energy becomes infinite $E \approx \frac{\omega}{\beta\omega - 1}$. In spite of this divergence of the partition function Julia claims that the continuation across or around this critical temperature can shed some understanding on the quark confinement [56]. At this stage we can notice that if we consider the partition function of the Riemann gas in the complex-β plane, the non-trivial zeros of the zeta function are the Fisher zeros.

Let us show that an alternative bosonic Hamiltonian similar to H_B can not be constructed with the sequence of prime numbers or the sequence of prime powers in the spectrum. Only the $\omega \ln p_n$ spectra is allowed. To proceed let us use the the following representation for $\ln x = \lim_{s\to 0} \frac{x^s - 1}{s}$, which is used in the replica formalism to study disordered systems [148, 149, 150, 151].

The Hamiltonian of this non-interacting bosonic field theory can be written as

$$H = \lim_{s\to 0} \omega \left(\sum_{k=1}^{\infty} \frac{p_k^s}{s} b_k^\dagger b_k - \sum_{k=1}^{\infty} \frac{b_k^\dagger b_k}{s} \right). \tag{139}$$

Although the Hamiltonian of the Riemann gas is obtained only when $s \to 0$, we can ask if there is a Hamiltonian for generic value of s. The case with $s = 1$, is exactly a system with the spectrum proportional to the prime numbers. In the following we use the results obtained by Menezes and Svaiter [41] to prove that only the Riemann gas has a well defined free energy. Using the definition of prime zeta function $P(s)$ it was shown that the free energy of a system with the sequence of prime numbers as the spectrum does not exist. As was discussed before, this is related to the fact that there is no analytic extension for $P(s)$ to the half-plane $\sigma \leqslant 0$[3].

Let us now discuss some features of this gas for fermions. For a free arithmetic fermionic gas the Hamiltonian is given by

$$H_F = \omega \sum_{k=1}^{\infty} \ln(p_k) c_k^\dagger c_k, \tag{140}$$

where $c_k^\dagger$ and c_k are respectively the creation and annihilation operators of quanta associated to the fermionic field and the p_k are as

3 The same argument can be used to prove that there is no fermionic system with infinitely number of degrees of freedom whose spectrum is composed by the sequence of prime numbers. Let us consider Dirac fermions interacting with an external field $A_\mu(x)$ in Euclidean space. The fermionic path integral is not well defined since the determinant of the Dirac operator diverges because it is an unbound product of increasing eigenvalues λ_n. It is possible to define a regularized determinant using the spectral zeta function associated with the Dirac operator and the principle of analytic continuation. The analytic continuation of the spectral zeta function must be regular at the origin, i.e., $s = 0$. The eigenvalues of the Dirac operator cannot be the sequence of prime numbers by the same reason we discussed for the bosonic case.

above prime numbers. The difference respect to the bosonic case is that we need to take into account the Pauli exclusion principle, therefore m_i just can be 0 or 1. This implies that the natural numbers are restricted to those that are free-square $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ with $m_k = (0, 1)$. Using the Möebius function and following the same steps as the bosonic case, the fermionic partition function can be written as

$$Z_F(\beta) = \sum_{\text{freesquare}} e^{-\beta\omega \log n} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\beta\omega}}. \tag{141}$$

Following Spector [58, 34] this partition function can be written as $Z_F(\beta) = \zeta(\beta\omega)/\zeta(2\beta\omega)$. It implies that the thermal partition functions associated to the Hamiltonians H_B and H_F are related by

$$Z_F(\beta)\, Z_B(2\beta) = Z_B(\beta). \tag{142}$$

The noninteracting mixture of two systems, each of its own temperature β^{-1} and $(2\beta)^{-1}$ is equivalent to another bosonic system with temperature β^{-1}.

For the fermionic arithmetic gas, the divergence of the partition function at the Hagedorn temperature implies two number theoretical results: the first one is that the zero-free region of the Riemann zeta function includes the vertical line $\mathrm{Re}(s) = 1$. The second one is that this divergence of $Z_F(\beta\omega)$ at $\beta\omega = 1$ implies the prime number theorem.

The first one is related to the effort to expand the zero-free region of the Riemann zeta function to include the vertical line $\mathrm{Re}(s) = 1$. In the classical proofs of Hadamard and de la Vallee-Poussin where $\zeta(1 + it) \neq 0$ for $\mathrm{Re}(s) = 1$, it was used that the Riemann zeta function has a pole at $s = 1$ with residue 1[51, 52, 53]. We conclude using the principle of analytic extension that the divergence of the partition function for the fermionic arithmetic gas at the Hagedorn temperature $\beta\omega = 1$ is also equivalent to the absence of zeros of the Riemann zeta function in the vertical line $\mathrm{Re}(s) = 1$. Also, as we know, the fact that the zero-free region of the Riemann zeta function includes the vertical line $\mathrm{Re}(s) = 1$ is equivalent at the prime number theorem [162, 163, 49]. We conclude that that the divergence of the partition function for the fermionic arithmetic gas at the Hagedorn temperature $\beta\omega = 1$ is equivalent to the prime number theorem.

The second way is using the Liouville function Eq. (13) and the equivalent sentence of the prime number theorem in terms of it Eq. (21). From the Eq. (20) and since the partition function for the fermionic arithmetic gas is given by $\zeta(\beta\omega)/\zeta(2\beta\omega)$ we have that the inverse of the partition function of this arithmetic gas can be written as

$$\Big(Z_F(\beta\omega)\Big)^{-1} = \sum_{n \geqslant 1} \frac{\lambda(n)}{n^{\beta\omega}}. \tag{143}$$

Using the principle of analytic extension we can see that the divergence of the partition function for the fermionic arithmetic gas at the Hagedorn temperature $\beta\omega = 1$ is equivalent to the prime number theorem.

6.2 RANDOMIZED RIEMANN GAS

Although the probabilistic approach in number theory is not new in the literature [158, 159, 4, 164], let us discuss the thermodynamics of an arithmetic gas introducing randomness in the system. We assume that the hamiltonian of the bosonic gas at a given temperature β^{-1} has a random variable ω_k. We take $\{\omega_k\}_{k\in\mathbb{N}}$ a set of uncorrelated random variables with some probability distribution $P(\omega_k)$ over an ensemble of hamiltonians. The thermodynamic quantities must be calculated averaging over an ensemble of realizations of the random quantity. Since the random variable is a parameter in an extensive quantity, we have to perform an ensemble average of some extensive quantity of interest, in our case the free energy [149, 150].

Assuming that the ensemble is made by a enumerable infinite set of copies, characterized by the parameter ω_k defined in the interval $\{\omega : \omega_1 \leqslant \omega_k < \infty\}_{k\in\mathbb{N}}$, the first copy the ensemble with parameter ω_1 provides a logarithmic singular contribution to the free energy density due to the pole of the Riemann zeta function. This temperature, where the average free energy density diverges is the Hagedorn temperature [161] of the random system. On the other hand, considering an ensemble of non-enumerable set of copies, the singular behaviour of the average free energy density disappears. Meanwhile, due to the behaviour of the Riemann zeta function in the critical region, the average free energy density acquires complex values. Finally, we show that the non-trivial zeros of the Riemann zeta function, which are the Fisher zeros of the system, contribute to the average energy density of the system. To proceed we have to perform ensemble averages of some extensive quantity of interest. Let us define the average free energy of the Riemann gas as

$$\langle F(\beta)\rangle = -\frac{1}{\beta}\langle \ln\zeta(\beta\omega)\rangle, \tag{144}$$

where $\langle(...)\rangle$ denotes the averaging over an ensemble of realizations of random variable with a given discrete probability distribution function. The average free energy density, for the case where the ensemble consists of an enumerable infinite set of copies of the system is

$$\langle f(\beta)\rangle = -\frac{1}{\beta V}\sum_{k=1}^{\infty} P(\omega_k)\ln\zeta(\omega_k\beta), \tag{145}$$

where V is the volume of the system and $P(\omega_k)$ is a given one-dimensional discrete distribution function defined in the interval $\{\omega :$

$\omega_1 \leqslant \omega_k < \infty\}_{k\in\mathbb{N}}$. Note that for low temperatures all of the copies of the ensemble contribute to the average free energy density of the system. Nevertheless, due to the pole of the Riemann zeta function there is a critical temperature where the first copy of the ensemble gives a singular contribution to the average free energy density of the system. This is the Hagedorn temperature of the random arithmetic gas.

In the next section, we first show that considering an ensemble made by a non-enumerable set of copies, the singular behavior of the average free energy density disappears. We are also interested in studying the average energy density of the system, which is related to the logarithmic derivative of the zeta function $\frac{\zeta'}{\zeta}(s)$. As we will see, the divergent contributions that appear in the average energy density of the system can be circumvented using an analytic regularization procedure.

6.3 AVERAGE FREE AND MEAN ENERGY DENSITY

In this section we will show that the thermodynamic quantities associated with the Riemann random gas are defined in terms of some number theoretical formulas. With this aim, our next task is to calculate relevant thermodynamic physical quantities as the average energy density $\langle\, \varepsilon \,\rangle$ and the average entropy density $\langle\, s \,\rangle$. These thermodynamic quantities are given respectively by

$$\langle\, \varepsilon \,\rangle = -\frac{1}{V}\frac{\partial}{\partial\beta}\sum_{k=1}^{\infty} P(\omega_k)\ln\zeta(\omega_k\beta) \tag{146}$$

and

$$\langle\, s \,\rangle = \frac{1}{V}\left(1-\beta\frac{\partial}{\partial\beta}\right)\sum_{k=1}^{\infty} P(\omega_k)\ln\zeta(\omega_k\beta). \tag{147}$$

The properties of the model depend strongly on the analytic structure of the Riemann zeta function. In the following, instead of considering an ensemble made by an enumerable infinite set of copies, we extend these definitions for a non-enumerable set of copies. This approach is analogous to the one that makes the classical Gibbs ensemble in phase space a continuous fluid [165]. Therefore the average over an ensemble of realizations can be represented by an integral with ω defined in the continuum i.e. $\{\omega : \omega \in \mathbb{R}^+\}$. The average free energy density and average energy density can be written as

$$\langle\, f(\beta,\lambda)\rangle = -\frac{1}{\beta V}\int_0^{\infty} d\omega\, P(\omega,\lambda)\ln\zeta(\omega\beta) \tag{148}$$

and

$$\langle\, \varepsilon(\beta,\lambda)\rangle = -\frac{1}{V}\int_0^{\infty} d\omega\, P(\omega,\lambda)\frac{\partial}{\partial\beta}\ln\zeta(\omega\beta), \tag{149}$$

where λ is a parameter with length dimension that we have to introduce to keep the right dimension in the expressions. For simplicity let us assume that the probability density distribution is given by $P(\omega,\lambda) = \lambda e^{-\lambda\omega}$. It is important to stress that the choice of the $P(\omega)$ does not affect the conclusions about the behaviour of the system. By changing the variable $s = \omega\beta$ we can write the average free energy density as

$$\langle f(\beta,\lambda)\rangle = -\frac{\lambda}{\beta^2 V}\int_0^\infty ds\, e^{-\frac{\lambda}{\beta}s}\ln\zeta(s). \tag{150}$$

Although there is a singularity in the integrand at $s = 1$, it is easy to show that the integral is bounded. On the neighborhood of $s = 1$ we can substitute the function $\log\zeta(s) \approx \log\frac{1}{s-1} = -\log(s-1)$. This is an integrable singularity, therefore the average free energy density is non-singular and continuous for all temperatures. Note that overcomes the Hagedorn temperature implies to go into the critical region of the Riemann zeta function, i.e, the region $0 < s < 1$, that may lead to complex values of the average free energy density.

We would like to point out that complex values for free energy are found in the study of open and dissipative systems [166][4]. Also, there is an alternative formulation of quantum mechanics and quantum field theory with non-Hermitian Hamiltonians [167]. The main problem to go into the critical region is the existence of a branch point of $\ln(z)$ at the origin. This fact generates an ambiguity in the free energy density. It is clear that this problem disappear if we deal with the logarithmic derivative of the zeta function $\frac{\zeta'}{\zeta}(s)$. The picture that emerges from this discussion is that the mean energy is a well behaved function of the temperature.

As we discussed above, the logarithmic derivative of the zeta function $\frac{\zeta'}{\zeta}(s)$, which is fundamental in the study of the density of non-trivial zeros of the zeta function, must be used in the definition of the average energy density. Substituting the Eq. (29) in Eq. (149) we can write the average energy density as

$$\langle \varepsilon(\beta,\lambda)\rangle = \varepsilon_1(\lambda) + \varepsilon_2(\beta,\lambda) + \varepsilon_3(\beta,\lambda) + \varepsilon_4(\lambda) + \varepsilon_5(\beta,\lambda) + \varepsilon_6(\lambda), \tag{151}$$

where each of these terms are given by

$$\varepsilon_1(\lambda) = -\frac{C_1}{V}\int_0^\infty d\omega\,\omega P(\omega,\lambda), \tag{152}$$

$$\varepsilon_2(\beta,\lambda) = \frac{1}{V}\int_0^\infty d\omega\,\omega P(\omega,\lambda)\frac{1}{\beta\omega - 1}, \tag{153}$$

4 The fact that the energy is complex implies that the perturbative ground-state is unstable. The life-time of the ground state is just the reciprocal of the imaginary part of the ground-state energy.

$$\varepsilon_3(\beta,\lambda) = -\frac{1}{V}\int_0^\infty d\omega\,\omega P(\omega,\lambda)\sum_\rho \frac{1}{\beta\omega-\rho}, \tag{154}$$

$$\varepsilon_4(\lambda) = -\frac{1}{V}\int_0^\infty d\omega\,\omega P(\omega,\lambda)\sum_\rho \frac{1}{\rho}, \tag{155}$$

$$\varepsilon_5(\beta,\lambda) = -\frac{1}{V}\int_0^\infty d\omega\,\omega P(\omega,\lambda)\sum_{n=1}^\infty \frac{1}{(\beta\omega+2n)} \tag{156}$$

and finally

$$\varepsilon_6(\lambda) = \frac{1}{2V}\int_0^\infty d\omega\,\omega P(\omega,\lambda)\sum_{n=1}^\infty \frac{1}{n}. \tag{157}$$

Usually, the information of the thermodynamics of the system is contained in derivatives of the mean free energy density. With the exception of the contribution coming from the pole of the zeta function, the above expressions for the mean free energy density are divergent series. We can use a standard regularization procedure to give meaning to these divergent terms [168]. Here, we choose to use an analytic regularization procedure introduced in quantum field theory in [110, 111] and used extensively since then.

6.4 superzeta functions and average energy density

In this section we will discuss each of the terms that contributes to the average energy density given by Eq. (151). The contribution of the first term given by Eq. (152) to the average energy density can be written as

$$\varepsilon_1(\lambda) = -\frac{C_1}{\lambda V}. \tag{158}$$

The second term that contributes to the average energy density $\varepsilon_2(\beta)$ can be written as

$$\varepsilon_2(\beta,\lambda) = \frac{1}{\beta V} - \frac{\lambda}{\beta^2 V}e^{-\frac{\lambda}{\beta}}\,\mathrm{Ei}(\lambda/\beta), \tag{159}$$

where $\mathrm{Ei}(x)$ is the exponential-integral function [46, 47] defined by

$$\mathrm{Ei}(x) = -\int_{-x}^\infty dt\,\frac{e^{-t}}{t} \quad x<0, \tag{160}$$

and

$$\mathrm{Ei}(x) = -\lim_{\varepsilon\to 0}\int_{-x}^{-\varepsilon} dt\,\frac{e^{-t}}{t} + \int_{-\varepsilon}^\infty dt\,\frac{e^{-t}}{t} \quad x>0. \tag{161}$$

For the third term that contributes to the average energy density $\varepsilon_3(\beta,\lambda)$ we have

$$\varepsilon_3(\beta,\lambda) = -\frac{\lambda}{V}\int_0^\infty d\omega\, \omega e^{-\lambda\omega} \sum_\rho \frac{1}{\beta\omega - \rho}. \tag{162}$$

As we will see, the contributions given by $\varepsilon_3(\beta,\lambda)$ and $\varepsilon_4(\lambda)$ can be written in terms of superzeta functions. A straightforward calculation gives us for $\varepsilon_4(\lambda)$

$$\varepsilon_4(\lambda) = -\frac{1}{\lambda V}\sum_\rho \frac{1}{\rho}. \tag{163}$$

Let us discuss the construction of the so-called superzeta or secondary zeta function built over the Riemann zeros, i.e., the nontrivial zeros of the Riemann zeta function [80]. As was discussed by Voros, in view of the central symmetry of the Riemann zeros $\rho \leftrightarrow 1-\rho$, leads us to generalized zeta functions of several kinds. Each one of these superzeta functions reflects our choice of a set of numbers to build zeta functions over the Riemann zeros. The first family that we call $G_1(s,t)$ is defined by

$$G_1(s,t) = \sum_\rho \frac{1}{(\frac{1}{2}+t-\rho)^s} \qquad \Re(s) > 1, \tag{164}$$

valid for $t \in \Omega_1 = \{t \in \mathbb{C}\,|\,(\frac{1}{2}+t-\rho) \notin \mathbb{R}_-(\forall\rho)\}$. This is the simplest generalized zeta-function over the Riemann zeros. The sum runs over all zeros symmetrically and t is just a shift parameter. In view of the fact that

$$\sum_\rho \frac{1}{\rho} = \sum_{k=1}^\infty \frac{1}{\frac{1}{2}+i\tau_k} + \frac{1}{\frac{1}{2}-i\tau_k} = \sum_{k=1}^\infty \frac{1}{\frac{1}{4}+\tau_k^2}, \tag{165}$$

the second generalized superzeta function is defined as

$$G_2(\sigma,t) = \sum_{k=1}^\infty (\tau_k^2+t^2)^{-\sigma} \qquad \Re(\sigma) > \frac{1}{2}, \tag{166}$$

valid for $t \in \Omega_2 = \{t \in \mathbb{C}\,|\,t \pm i\tau_k \notin \pm i\mathbb{R}_-\,(\forall k)\}$. The central symmetry $\tau_k \leftrightarrow -\tau_k$, is preserved in the family of superzeta functions $G_2(\sigma,t)$. Using the above definitions, the contributions to the average energy density given by $\varepsilon_3(\beta,\lambda)$ and $\varepsilon_4(\lambda)$ can be written respectively as

$$\varepsilon_3(\beta,\lambda) = -\frac{\lambda}{V}\int_0^\infty d\omega\, \omega e^{-\lambda\omega} G_1(1,\beta\omega-1/2) \tag{167}$$

and

$$\varepsilon_4(\lambda) = -\frac{1}{\lambda V} G_2(1,1/2). \tag{168}$$

Since we are interested in the region $\Re(\sigma) > \frac{1}{2}$ we are in the region of convergence of the series in Eq. (166). Therefore, assuming the Riemann hypothesis we can write $\varepsilon_4(\lambda)$ as

$$\varepsilon_4(\lambda) = -\frac{1}{\lambda V}\sum_{n=1}^{\infty}\frac{1}{\frac{1}{4}+\gamma_n^2}. \tag{169}$$

After this discussion of the terms that involve superzeta functions, let us proceed with the contribution to the average energy density given by terms that involve the Hurwitz zeta function and the Riemann zeta function. The Hurwitz zeta function $\zeta(z,q)$ is the analytic extension of the series

$$\zeta(z,q) = \sum_{n=0}^{\infty}\frac{1}{(q+n)^z} \qquad q \neq 0,-1,-2,..., \quad \mathrm{Re}(z) > 1, \tag{170}$$

which is a meromorphic function in the whole complex plane with a single pole at $z = 1$. We can write $\varepsilon_5(\beta,\lambda)$ in terms of the Hurwitz zeta function as

$$\varepsilon_5(\beta,\lambda) = \lim_{z\to 1}\left(\frac{1}{\beta V} - \frac{\lambda}{2V}\int_0^{\infty} d\omega\, \omega e^{-\lambda\omega}\zeta(z,\beta\omega/2)\right). \tag{171}$$

Finally, let us discuss the last term given by $\varepsilon_6(\lambda)$. Using the zeta function $\zeta(s)$ defined by Eq. (204) and its analytic extension Eq. (23) we can write $\varepsilon_6(\beta,\lambda)$ as

$$\varepsilon_6(\lambda) = \lim_{s\to 1}\frac{1}{2\lambda V}\zeta(s). \tag{172}$$

In the next section we will discuss the structure of the singularities for $\varepsilon_3(\beta,\lambda)$, $\varepsilon_5(\beta,\lambda)$ and $\varepsilon_6(\lambda)$ to the average energy density.

6.5 REGULARIZED AVERAGE ENERGY DENSITY

The aim of this section is to find the contributions given by $\varepsilon_3(\beta,\lambda)$, $\varepsilon_5(\beta,\lambda)$ and $\varepsilon_6(\lambda)$ to the average energy density. From the previous section we have that the contribution to the average energy density given by $\varepsilon_3(\beta,\lambda)$ can be written as

$$\varepsilon_3(\beta,\lambda) = -\frac{\lambda}{V}\int_0^{\infty} d\omega\, \omega e^{-\lambda\omega} G_1(1,\beta\omega - 1/2). \tag{173}$$

The superzeta function $G_1(s,t)$ admits an analytic extension to the half-complex plane given by

$$G_1(s,t) = -Z(s,t) + \frac{1}{(t-\frac{1}{2})^s} + \frac{\sin\pi s}{\pi}\mathcal{J}(s,t), \tag{174}$$

where $Z(s,t)$ is the superzeta function $G_1(s,t)$ evaluated in the trivial zeros of the Riemann zeta function [80]. It is possible to write $Z(s,t)$ in terms of the Hurwitz zeta function, defined in Eq. (170), as

$$Z(s,t) = \sum_{k=1}^{\infty}\left(\frac{1}{2}+t+2k\right)^{-s} = 2^{-s}\zeta\left(s,\frac{1}{4}+\frac{1}{2}t\right). \tag{175}$$

The function $\mathcal{J}(s,t)$, a Mellin transform of the logaritmic derivative of the Riemann zeta function, is defined as

$$\mathcal{J}(s,t) = \int_0^\infty \frac{\zeta'}{\zeta}\left(\frac{1}{2}+t+y\right)y^{-s}dy \quad \mathrm{Re}(s) < 1. \tag{176}$$

As discussed by Voros, all the discontinuities of $(t-1/2)^{-s}$ and $-Z(s,t)$ in the interval $(-\infty, \frac{1}{2}]$ cancel against jumps of $\mathcal{J}(s,t)$. Therefore, the analytic continuation of $G_1(s,t)$ is a regular function of $t \in \Omega_1$. Using the analytical properties of the Mellin transformation of $(\zeta'/\zeta)(x)$ the function $\mathcal{J}(s,t)$ has a known global meromorphic structure which implies the meromorphic continuation to the whole plane of the superzeta function $G_1(s,t)$. The function $\mathcal{J}(s,t)$ has simple poles at $s = +1, +2, ...$ with residues given by

$$\mathrm{Res}(\mathcal{J}(s,t))|_{s=n} = -\frac{1}{(n-1)!}\frac{d^n}{dt^n}\log|\zeta(\frac{1}{2}+t)|, \tag{177}$$

and this structure makes the product $\sin(\pi s)\,\mathcal{J}(s,t)$ free of singularities. Hence, the singularity structure of the function $G_1(s,t)$ is the same as the one of $Z(s,t)$, i.e, of the Hurwitz zeta function.

The analytic extension of the Hurwitz zeta function $\zeta(s,q)$ can be performed as given in [10]. The result is a meromorphic function for $\mathrm{Re}(s) > 0$, with a simple pole at $s = 1$ and of residue 1. It is possible to show that

$$\lim_{s\to 1}\left(\zeta(s,q) - \frac{1}{s-1}\right) = -\psi(q), \tag{178}$$

where $\psi(q)$ is the Euler's Psi-function defined by

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x). \tag{179}$$

Since $\sin(\pi s)$ vanishes at integers, the contribution to the average energy density given by $\epsilon_3(\beta,\lambda)$ can be written as

$$\epsilon_3(\beta,\lambda) = -\frac{\lambda}{V}\int_0^\infty d\omega\,\omega e^{-\lambda\omega}\left(\frac{1}{2}\psi\left(\frac{\beta\omega}{2}\right) + \frac{1}{\beta\omega - 1/4}\right). \tag{180}$$

To perform the integral of the first term in the parenthesis, we can use the functional relation for the psi function $\psi(x+1) = \psi(x) + 1/x$ and the following series representation for $\psi(x+1)$

$$\psi(x+1) = -C + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1}. \tag{181}$$

The integral of the second term is similar to the one calculated for $\epsilon_2(\beta,\lambda)$ which can be expressed in terms of the exponential-integral function $\mathrm{Ei}(x)$. Accordingly, the contribution to the average energy density given by$\epsilon_3(\beta,\lambda)$ can be written as

$$\epsilon_3(\beta,\lambda) = \frac{C}{2\lambda V} - \frac{1}{\beta V}\sum_{k=2}^{\infty} g(k)\left(\frac{\beta}{\lambda}\right)^k + \frac{\lambda}{4\beta^2 V}e^{-\frac{\lambda}{4\beta}}\mathrm{Ei}\left(\frac{\lambda}{4\beta}\right) \tag{182}$$

where $g(k) = \frac{(-1)^k}{2^k}\Gamma(k+1)\zeta(k)$. From the last section, the contribution to the average energy density given by $\varepsilon_5(\beta,\lambda)$ can be written as

$$\varepsilon_5(\beta,\lambda) = \lim_{s\to 1}\left(\frac{1}{\beta V} - \frac{\lambda}{2V}\int_0^\infty d\omega\, \omega e^{-\lambda\omega}\zeta(s,\beta\omega/2)\right).$$

A similar procedure as the mentioned above in the analysis of the Hurwitz zeta function can be performed with the aid of equations (178), (179) and (181). In this case we can see that $\varepsilon_5(\beta,\lambda)$ has three contributions, the first two are finite and the last one is singular. We have

$$\varepsilon_5(\beta,\lambda) = \frac{1}{\beta V} + \frac{\lambda}{2V}\int_0^\infty d\omega\, \omega\, e^{-\lambda\omega}\psi\left(\frac{\omega\beta}{2}\right) - h(\lambda,V). \tag{183}$$

We can write $\varepsilon_5(\beta,\lambda)$ as

$$\varepsilon_5(\beta,\lambda) = -\frac{C}{2\lambda V} + \frac{1}{\beta V}\sum_{k=2}^{\infty} g(k)\left(\frac{\beta}{\lambda}\right)^k - h(\lambda,V), \tag{184}$$

where $h(\lambda,V) = \lim_{s\to 1}\left(\frac{1}{2\lambda V}\frac{1}{s-1}\right)$ and $g(k)$ being the same coefficient as before. From the previous section we have that the contributions to the average energy density given by $\varepsilon_6(\lambda)$ is

$$\varepsilon_6(\lambda) = \lim_{s\to 1}\frac{1}{2\lambda V}\zeta(s).$$

The $\varepsilon_6(\lambda)$ contribution is proportional to the Riemann zeta function. Using the analytic extension of the Riemann zeta function and the fact that

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = -\psi(1), \tag{185}$$

where $\psi(1) = -0.577215$ is the Euler's constant, we can write $\varepsilon_6(\beta)$ as a finite contribution and again a singular part. We have

$$\varepsilon_6(\lambda) = -\frac{1}{2\lambda V}\psi(1) + h(\lambda,V). \tag{186}$$

Note that the singular contribution in $\varepsilon_5(\beta,\lambda)$ and $\varepsilon_6(\beta,\lambda)$ cancel each other.

We can split the contribution of each term to the average energy density into two categories. The first one coming from $\varepsilon_A(\lambda) = \varepsilon_1(\lambda) + \varepsilon_4(\lambda) + \varepsilon_6(\lambda)$. These terms are temperature independent and, therefore, we can interpret them as coming from the vacuum modes associated to the arithmetic gas. The other terms $\varepsilon_2(\beta,\lambda)$, $\varepsilon_3(\beta,\lambda)$ and $\varepsilon_5(\beta,\lambda)$ are temperature dependent contributions. We can write this thermal contribution $\epsilon_B(\beta,\lambda)$ as

$$\varepsilon_B(\beta,\lambda) = \frac{1}{\beta V} - \frac{\lambda}{\beta^2 V}e^{-\frac{\lambda}{\beta}}\mathrm{Ei}\left(\frac{\lambda}{\beta}\right) + \frac{\lambda}{4\beta^2 V}e^{-\frac{\lambda}{4\beta}}\mathrm{Ei}\left(\frac{\lambda}{4\beta}\right). \tag{187}$$

We have shown that the thermodynamic quantities associated to the arithmetic gas can be calculated. A similar calculation can be performed to find the average entropy density. Since we have an ambiguity in the mean free energy density the mean entropy also keeps such ambiguity.

7 CONCLUSIONS AND PERSPECTIVES

Probability is not a notion of pure mathematics,
but of physics or philosophy.

– G. Hardy and J. Littlewood.

7.1 CONCLUSIONS

We present a scenario where it is possible to establish some links between the Riemann zeta function theory and physics.

From Hadamard's theory it is possible to define analytic functions by its zeros and singularities. In the case of the Riemann zeta function, it is possible to represent it as an Euler product and a Hadamard's product. This shows a relationship between the set of prime numbers and nontrivial zeros of the zeta function. Nevertheless these two sequences of numbers have totally distinct behavior with respect to being the spectrum of a linear operator associated to a system with countable infinite number of degrees of freedom.

The renormalized zero-point energy of a massive scalar field with the Riemann zeros in the spectrum of the vacuum modes was presented. Using analytic and dimensional regularization, for even dimensional space-time, we show that the series that defines the regularized energy density is finite, therefore there is no necessity of introducing counterterms. For odd dimensional space-time the analytic regularization procedure does not produce finite results because the coefficients of the series are divergent. We concluded that in order to renormalise the vacuum energy we are forced to introduce mass counterterms in the interaction lagrangian only for odd dimensional space-time, it guarantees the cancellation of the polar contributions in the series.

We established a precise connection between the asymptotic distribution of the Riemann zeros with the asymptotic behavior of the Fourier transform of the two-point correlation function of the non-linear sigma model in the leading order in $1/N$ expansion in a two-dimensional Euclidean space. This result shows us that there is a deep connection between number theory and quantum field theory, that deserves a further investigation.

We show that using an arithmetic quantum field theory with randomness it is possible to connect strongly the non-trivial zeros of the Riemann zeta function with a measurable physical quantity defined in the system. For the Riemann gas, since the Riemann zeta function has a simple pole in $s = 1$, there is a Hagedorn temperature above which the system can not be heated up. The divergence of the partition function at the Hagedorn temperature is related to the prime number theorem. This can be seen expanding the zero-free region of the Riemann zeta function to include the vertical line $\mathrm{Re}(s) = 1$. For the randomized Riemann gas assuming that the ensemble is made by an enumerable infinite set of copies we show that the mean energy density depends on the distribution of the Riemann zeros. This is because the mean energy density is defined in terms of the quantity $\frac{\zeta'}{\zeta}(s)$, which is related to the non-trivial zeros of the Riemann zeta function.

On the other hand, considering an ensemble made by a non enumerable set of copies, the singular behavior of the average free energy density disappears, i.e, it is non-singular for all temperatures. Due to the behavior of the Riemann zeta function in the critical region, the average free energy density acquires complex values and appears an ambiguity in the free energy density. When we study the average energy density of the system, the above mentioned problem disappears, since it is related to the logarithmic derivative of the zeta function $\frac{\zeta'}{\zeta}(s)$. We showed that the divergent contributions that appear in the average energy density of the system can be circumvented using an analytic regularization procedure.

7.2 PERSPECTIVES

7.2.1 *Regularized zeta function*

Hilbert pointed out in his millenium problems article that one of the relevant functions to prove the Riemann hypothesis would be the function

$$\xi(s,x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}. \tag{188}$$

Basically what this function implies is a regularization method for the Riemann zeta function. The Hilbert proposal is a power law regularization scheme, where in the limit $x \to 1$ we recover the Riemann zeta function. It is clear that other possible regularization schemes can be implemented as the quantum field theory has taught us. One of the immediate possibilities is to explore an exponential cut-off λ. Many trends can be addressed at this stage. The first possibility is to figure out if there exists the analytic extension to the complex λ-plane of the new cut-off zeta function. This analytic extension could provide new insights on how to deal with the Riemann hypothesis and establish some kind of renormalization group. Based on this analytic approach, one can find out if the cut-off zeta function has zeros in the complex λ-plane, and if it does, one can ask how these complex zeros flow when the limit $\lambda \to 0$ is reached.

One of the most important teachings of the dimensional regularization of the quantum field theory was that a good regularization scheme must preserve the symmetries of the theory. Reminding that from the functional equation for the Riemann zeta function, it is possible to define the function $\xi(s)$ which presents the symmetry $\xi(s) = \xi(1-s)$, a possible question is: what kind of new symmetries could be inserted in these regularization schemes that can serve us to deal with the Riemann hypothesis?

7.2.2 *Physics of the Möbius function*

Some proposal showing connections between random walks and prime numbers [38] employed the Möbius function Eq. (11). In this approach the Mertens function defined as $M(n) = \sum_{k=1}^{n} \mu(k)$ denotes the cumulative values of $\mu(n)$ which are equiprobably equal to 1 and -1 and therefore, $M(n)$ reassembles a one dimensional random walk. Another physical interpretation to $\mu(n)$ was given in [33], where it was shown that $\mu(n)$ can be interpreted as the operator $(-1)^F$ giving the number of fermions in quantum field theory. The fact that $\mu(n) = 0$ when n is not squarefree is equivalent to the Pauli exclusion principle.

A more recent proposal is due to P. Sarnak [164]. It is based on the intuition that the Möbius function behaves like a random variable taking very erratic and unpredictable values. This proposal has been formalized as the Sarnak's conjecture with many implications in number theory if it results to be true.

Following these ideas, we can consider a physical disordered system of spin 1 like a spin glass that resemblance in some sense the Möbius function. One of the challenges is to choose an appropriate probability distribution function to fit in good approximation the behaviour of the Möbius function.

7.2.3 *Riemann zeros and non-linear σ-model*

The asymtotic behaviour of the Fourier transform of the two-point correlation function of the nonlinear sigma model fits remarkably the asymptotic behavior of the Riemann zeta zeros. Based on this fact, two questions emerge:

- Could the Fourier transform of higher order correlation functions of the nonlinear sigma model fit the statistical distribution of the Riemann zeta zeros?

- Since two-dimensional quantum chromodynamics (QCD) reduces to the non-linear sigma model in some limit [141], is there a connection between the Riemann zeros, QCD and confinement?.

7.2.4 *Parafermion arithmetic gas*

The parafermion arithmetic gas of order r is a quantum gas where the exclusion principle states that no more than $r-1$ parafermions can have the same quantum numbers [169]. The partition function for an arithmetic parafermion gas or order r is given by $Z_r(\beta, \omega) = \zeta(\beta\omega)/\zeta(r\beta\omega)$. A natural extension of the analysis made for the randomized Riemann gas is to assume that, as the previous case, ω is a random variable with a given probability distribution over an ensemble of hamiltonians.

7.2.5 *Probabilistic Riemann zeta function*

Similarly as the celebrated probabilistic model by Cramer of prime numbers [158, 159] gave deep insights on the understanding of prime numbers, the following question arises: Does exist a probabilistic model for the Riemann zeta function in the complex plane?. Although this issue is more speculative and deep work must be done to clarify the meaning of this question, there are some hints that point out that this is a possible issue to address. The quenched free energy for an anharmonic oscillator with a random frequency can be calculated in terms of the derivative of a particular spectral zeta function $\zeta(s, \omega)$ at $s = 0$. This derivative is determined by the analytic continuation of the spectral zeta function from the domain where the series actually converges. The spectral zeta function in this case is the Hurwitz zeta function $\zeta(s, a)$ where a is a random variable with some probability distribution function. The spectrum of the Riemann gas is a very specific spectrum for which the randomness in the frequency introduces some randomness in the argument of the partition function (Riemann zeta function). In this case we obtain a way to go further over the pole of the Riemann zeta function and regularize all the divergent expressions to get a finite average energy density. In this stage we could ask the following questions: Is it possible to assume the argument of the Riemann zeta function $s = \sigma + i\tau$ as a random complex variable with some complex measure?, What new information could this approach give us about the Riemann zeta function?. The randomized and parafermion Riemann gas are different physical realizations of a probabilistic model for the Riemann zeta function which deserves further investigation.

A

GAMMA FUNCTION

In this appendix we discuss briefly the gamma function. Many of its properties are frequently used to the analyses of the Riemann zeta function and different regularization procedures in quantum field theory.

The gamma function is defined in the complex plane by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad \mathrm{Re}(z) > 0. \tag{189}$$

It is possible to study the behaviour of $\Gamma(z)$ in the complex plane splitting the integral into two parts $\Gamma(z) = P(z) + Q(z)$ where

$$P(z) = \int_0^1 e^{-t} t^{z-1}\, dt, \qquad Q(z) = \int_1^\infty e^{-t} t^{z-1}\, dt, \tag{190}$$

the integrand $e^{-t}t^{z-1}$ is analytic in z and continuous in z and t for $\mathrm{Re}(z) > 0$ and $0 < t < \infty$. For t near to the origin the integrand diverges if $\mathrm{Re}(z) \leqslant 0$, hence the function $P(z)$ is analytic just in the half-plane $\mathrm{Re}(z) > 0$, while the function $Q(z)$ is analytic in the whole complex plane. It implies that $\Gamma(z)$ is analytic in the region $\mathrm{Re}(z) > 0$. We can perform an analytic continuation of the $\Gamma(z)$ to the rest of the complex plane using the series expansion of the exponential in the function $P(z)$ and then integrating term-by-term as follows

$$P(z) = \int_0^1 dt\, t^{z-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(k+z)}. \tag{191}$$

The exchange of the sum and the integral is possible to do because the integral converges absolutely for $\mathrm{Re}(z) > 0$. However, we can see that the Eq. (191) is analytic in the z-plane if $z \neq 0, -1, -2, \cdots$. Therefore, the series is a merophormic function with simple poles at the points $z = -n$. For $\mathrm{Re}(z) > 0$ this function coincides with the integral which defines $P(z)$ and hence the Eq. (191) is the analytic continuation of $P(z)$to the whole complex plane. An analytic extension for $\Gamma(z)$ is given by

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(k+z)} + \int_1^\infty e^{-t} t^{z-1}\, dt, \qquad z \neq -n,\ n \in N, \tag{192}$$

and we can represent it in a neighborhood of the pole $z = -n$, $(n = 0, 1, \cdots)$ in terms of the residues $(-1)^n/n!$ and a regular part $\Omega(z+n)$ as

$$\Gamma(z) = \frac{(-1)^n}{n!} \frac{1}{z+n} + \Omega(z+n). \tag{193}$$

Properties of $\Gamma(z)$.

Many different properties are satisfied by the $\Gamma(z)$ function. We list here the most relevant used in this work.

$$\Gamma(z+1) = z\Gamma(z), \tag{194}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad \textit{Euler's reflection formula} \tag{195}$$

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z), \quad \textit{duplication formula}. \tag{196}$$

We can use the Eq. (194) to calculate $\Gamma(z)$ for some specific values of z. We notice from Eq. (189) that $\Gamma(1) = 1$, then we find by induction that

$$\Gamma(n+1) = n!, \qquad n = 0, 1, 2, \cdots \tag{197}$$

$$\Gamma(1/2) = \sqrt{\pi}, \tag{198}$$

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}. \tag{199}$$

The $\Gamma(z)$ function has no zeros in the complex plane. To see this note that the points $z = n\,(n = 0, \pm 1, \pm 2, \cdots)$ can not be zeros, since for positive n we have $\Gamma(n) = (n-1)!$ and for negative n it diverges. From the Eq. (195) we see that any other value of z can be a zero of $\Gamma(z)$, because if there is one value of z for which $\Gamma(z) = 0$, then it would be a pole of $\Gamma(1-z)$, which is impossible. Then we have that $\Gamma(z)$ does not vanish and therefore $\Gamma(z)^{-1}$ is well defined in the whole complex plane with zeros at $z = -n$.

By the factorization Hadamard theorem we can represent the $\Gamma(z)^{-1}$ in terms of its zeros. This representation is the Weierstrass formula for the $\Gamma(z)$ function given by

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \tag{200}$$

where γ is Euler's constant. By logarithmic differentiation we obtain that

$$-\frac{d}{dz}\ln\Gamma(z) = -\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right). \tag{201}$$

The Stirling's formula is useful to obtain some important results of the $\zeta(z)$ function. It is express as

$$\ln\Gamma(z) = \left(1-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi), \qquad |z| \to \infty, \tag{202}$$

and let us give a useful integral representation of a^z derived from Eq. (189). By changing of variable $t = ax$ we have that

$$\frac{1}{a^z} = \frac{1}{\Gamma(z)}\int_0^{\infty} d\,t'\, e^{-a\,t'} t'^{z-1}. \tag{203}$$

B

ANALYTIC EXTENSION OF $\zeta(s)$−FUNCTION

This appendix is a briefly review of the analytic continuation of the Riemann function.

We start remembering the definition of the Riemann zeta function Eq. (204) For a complex variable s,i. e. $s = \sigma + i\tau$ with $\sigma, \tau \in \mathcal{R}$. The Riemann zeta function is defined as the Dirichlet series by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{204}$$

To obtain the analytic continuation to the whole complex plane of $\zeta(s)$ we use an integral representation obtained from the integral formula for the gamma function Eq. (203). If we take $a = n^2$, $t = \pi x$ and renaming $2s$ as s to maintain the notation we can write the zeta function as

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} dx\, x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-n^2\pi x} \tag{205}$$

by defining the function $\Psi(x)$ as

$$\Psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}, \tag{206}$$

and splitting the integral into the intervals $[0, 1]$ and $[1, \infty)$, we can write the $\zeta(s)$ function as

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} = \int_0^1 dx\, x^{\frac{s}{2}-1}\Psi(x) + \int_1^{\infty} dx\, x^{\frac{s}{2}-1}\Psi(x), \tag{207}$$

the Poisson's summation formula for $\Psi(x)$ is given by

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{\frac{-n^2\pi}{x}}, \tag{208}$$

which is equivalent to

$$2\Psi(x) + 1 = \frac{1}{\sqrt{x}}\left(2\Psi\left(\frac{1}{x}\right) + 1\right), \tag{209}$$

may be used to obtain an integral representation for $\zeta(s)$ as

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} = \frac{1}{s(s-1)} + \int_1^{\infty} dx\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\Psi(x). \tag{210}$$

The integral in the right side is convergent for all values of s and therefore this is the analytic extension to the whole complex plane of the zeta function.

C

THE ANALYTIC EXTENSION OF THE SUPER-ZETA FUNCTION $G_\gamma(s)$

In this appendix we present the analytic extension of one of the super-zeta functions. We are following the Ref. [75]. The superzeta function $G_\gamma(s)$ is defined by Eq. (79) as

$$G_\gamma(s) = \sum_{n=1}^{\infty} \frac{1}{\gamma_n^{\frac{s}{n}}}, \quad \Re(s) > 1, \tag{211}$$

where we are assuming that $\gamma_n > 0$. Similar as was done with the Riemann zeta function, we have that the superzeta function can be written as

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} G_\gamma(s) = \int_0^\infty dx\, x^{\frac{s}{2}-1} \sum_{\gamma>0} e^{-\pi\gamma^2 x}. \tag{212}$$

Let us split the integral that appears in Eq. (212) in the intervals $[0, 1]$ and $[1, \infty)$, and define the functions

$$A(s) = \int_0^1 dx\, x^{\frac{s}{2}-1} \sum_{\gamma>0} e^{-\pi\gamma^2 x} \tag{213}$$

and

$$B(s) = \int_1^\infty dx\, x^{\frac{s}{2}-1} \sum_{\gamma>0} e^{-\pi\gamma^2 x}. \tag{214}$$

Note that $B(s)$ is an entire function. To proceed let us use that

$$\sum_{\gamma>0} e^{-\pi\gamma^2 x} = -\frac{1}{2\pi\sqrt{x}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} e^{-\frac{(\ln n)^2}{4\pi x}} + e^{\frac{\pi x}{4}} - \frac{1}{2\pi}\int_0^\infty dt\, e^{-\pi x t^2}\Psi(t), \tag{215}$$

where the function $\Psi(t)$ is given by

$$\Psi(t) = \frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)} + \frac{\zeta'(\frac{1}{2}-it)}{\zeta(\frac{1}{2}-it)}. \tag{216}$$

Substituting Eq. (215) in (213) we get that A-function can be written as

$$A(s) = A_1(s) + A_2(s) + A_3(s), \tag{217}$$

where

$$A_1(s) = -\frac{1}{2\pi}\int_0^1 dx\, x^{\frac{s}{2}-\frac{3}{2}}\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\, e^{-\frac{(\ln n)^2}{4\pi x}}\right), \tag{218}$$

$$A_2(s) = \int_0^1 dx\, x^{\frac{s}{2}-1}\, e^{\frac{\pi x}{4}} \tag{219}$$

and finally

$$A_3(s) = -\frac{1}{2\pi}\int_0^1 dx\, x^{\frac{s}{2}-1}\left(\int_0^{\infty} e^{-\pi x t^2}\Psi(t)\right). \tag{220}$$

Changing variables in the $A_1(s)$, i.e., $x \to 1/x$ we get

$$A_1(s) = -\frac{1}{2\pi}\int_1^{\infty} dx\, x^{-\frac{s}{2}-\frac{1}{2}}\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\, e^{-\frac{x(\ln n)^2}{4\pi}}\right). \tag{221}$$

It is clear that $A_1(s)$ is an entire function of s. Let us define $\Phi(s)$ as

$$\Phi(s) = A_1(s) + B(s). \tag{222}$$

Using Eqs. (214), (217), (219), (220) and (222) we can write expression (212) as

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}G_\gamma(s) = \Phi(s) + A_2(s) + A_3(s). \tag{223}$$

Since $\Phi(s)$ is an entire function and we have the integrals that define $A_2(s)$ and $A_3(s)$, the above formula is the analytic extension of the secondary zeta function. The function $G_\gamma(s)$ is a meromorphic function of s in the whole complex plane with double pole at $s = 1$ and simple poles at $s = -1, -3, .., -(2n+1), \ldots$ Therefore $(s-1)^2 G_\gamma(s)(\Gamma(s))^{-1}$ is an entire function.

BIBLIOGRAPHY

[1] G. H. Hardy and E. M. Wright. *An introduction to the Theory of Numbers*. Oxford Clarenton Press, London, 1956. (Cited on pages 1, 8, and 11.)

[2] A. E. Ingham. *The Distribution of Prime Numbers*. Cambridge University Press, Cambridge, 1990. (Cited on pages 1, 8, 13, and 18.)

[3] E. Landau. *Elementary Number Theory*. AMS Chelsea Publishing, Providence, Rhode Island, 1991. (Cited on pages 1 and 8.)

[4] G. Tenenbaum and M. M. France. *The Prime Numbers and Their Distribution*. American Mathematical Society, New York, 2000. (Cited on pages 1, 8, 48, 49, and 52.)

[5] G. J. O. Jameson. *The Prime Number Theorem*. Cambridge UNiversity Press, New York, 2003. (Cited on pages 1 and 8.)

[6] E. C. Titchmarsh. *The Zeta-Function of Riemann*. Stechert-Hafner Service Agency, New York, 1964. (Cited on pages 1, 2, 10, 13, 14, and 16.)

[7] H. M. Edwards. *Riemann's Zeta Function*. Dover Publication, Inc., New York, 1974. (Cited on pages 1, 13, and 16.)

[8] A. Ivić. *The Riemann zeta-function: the theory of Riemann zeta-function with applications*. Wiley, New York, 1985. (Cited on pages 1, 13, 16, 17, and 18.)

[9] S. J. Patterson. *An introduction to the theory of the Riemann Zeta-Function*. Cambridge University Press, Cambridge, 1988. (Cited on pages 1, 13, and 16.)

[10] A. A. Karatsuba and S. M. Voronin. *The Riemann Zeta-Function*. Walter de Gruyter, New York, 1992. (Cited on pages 1, 13, 16, and 58.)

[11] M. L. Mehta. Random matrices. *Elsevier*, 2004. (Cited on page 2.)

[12] N. C. Snaith P. J. Forrester and J. J. M. Verbaarschot. Developments in random matrix theory. *J. Phys. A*, 36:R1, 2003. (Cited on page 2.)

[13] H. L. Montgomery. The pair correlation of zeros of the zeta function, in analytic number theory. *Proc. Symp. Pure Math.*, 24: 181, 1973. (Cited on page 2.)

[14] A. M. Odlyzko. On the distribution of spacings between zeros of the zeta function. *Math. Comp.*, 48:273, 1987. (Cited on page 2.)

[15] J. P. Keating and N. C. Snaith. Random matrix theory and $\zeta(1/2 + it)$. *Comm. Math. Phys.*, 214:57, 2000. (Cited on page 2.)

[16] P. Bourgade and J. P. Keating. Quantum chaos, random matrix theory, and the riemann $\zeta(s)$-function. *Séminaire Poincaré XIV*, 115, 2010. (Cited on pages 2, 3, and 19.)

[17] M. G. Gutzwiller. The energy spectrum according to classical mechanics. *J. Math. Phys.*, 11:1791, 1970. (Cited on pages 2, 19, and 45.)

[18] M. G. Gutzwiller. Periodic orbits and classical quantization conditions. *J. Math. Phys.*, 12:343, 1971. (Cited on pages 2, 19, and 45.)

[19] M. J. Giannoni O. Bohigas and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Phys. Rev. Lett.*, 52:1, 1984. (Cited on page 2.)

[20] O. Bohigas and M. J. Gianonni. *Chaotic motion and random matrix theory, Mathematical and Computational Methods in Nuclear Physics*. Springer Verlag, Berlin, 1984. (Cited on page 2.)

[21] M. V. Berry. *Chaotic Behavior of Deterministic Systems, Les Houches Section XXXVI*. North Holland, Amsterdam, 1981. (Cited on pages 2 and 19.)

[22] B. D. Simons, P. A. Lee, and B. L. Altshuler. Matrix models, one-dimensional fermions, and quantum chaos. *Phys. Rev. Lett.*, 72:1, 1994. (Cited on page 2.)

[23] M. V. Berry and J. P. Keating. H=xp and the riemann zeros. *Supersymmetry and Trace Formulae: Chaos and Disorder, eds. Lerner et al*, 1999. (Cited on pages 2, 3, and 45.)

[24] M. V. Berry and J. P. Keating. The riemann zeros and eigenvalue asymptotics. *Siam Review*, 41:236, 1999. (Cited on pages 2, 3, and 19.)

[25] G. Sierra. $h = xp$ with interaction and the riemann zeros. *Nucl. Phys. B*, 776:327, 2007. (Cited on page 3.)

[26] G. Sierra and J. R. Laguna. H=xp model revisited and the riemann zeros. *Phys. Rev. Lett.*, 106:200201, 2011. (Cited on page 3.)

[27] M. Srednicki. The berryâkeating hamiltonian and the local riemann hypothesis. *J. Phys. A*, 44:305202, 2011. (Cited on page 3.)

[28] K. Efetov. Supersymmetry and theory of disordered metals. *Adv. Phys.*, 32:53, 1983. (Cited on pages 3 and 37.)

[29] K. Efetov. *Supersymetry in Disorder and Chaos*. Cambridge University Press, Cambridge, 1997. (Cited on pages 3 and 37.)

[30] T. Guhr, A. M. Groelinga, and H. A. Weindermüler. Random-matrix theories in quantum physics: common concepts. *Phys. Rep.*, 299:199, 1998. (Cited on page 3.)

[31] E. Brezin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin. *Application of Random Matrices in Physics*. Springer, Netherlands, 2006. (Cited on page 3.)

[32] B. L. Julia. Statistical theory of numbers. *Number Theory and Physics, Proceedings in Physics*, 47:276–293, eds. J. M. Luck, P. Moussa, and M. Waldschmidt 1990. (Cited on pages 3, 21, and 49.)

[33] D. Spector. Supersymmetry and the möbius inversion function. *Communications in Mathematical Physics*, 127:238–252, 1990. (Cited on pages 3, 21, 49, and 63.)

[34] I. Bakas and M. J. Bowick. Curiosities of arithmetic gases. *J. Math. Phys.*, 32:1881, 1991. (Cited on pages 3, 21, 49, and 51.)

[35] J. G. Dueñas and N. F. Svaiter. Thermodynamics of the bosonic randomized riemann gas. *ArXiv:1401.8190*, 2014. (Cited on page 3.)

[36] P. Leboeuf, A. G. Monastra, and O. Bohigas. The riemannium. *Regular and Chaotic Dynamics*, 2:205, 2001. (Cited on page 3.)

[37] P. Leboeuf and A. G. Monastra. Quantum thermodynamic fluctuations of a chaotic fermi-gas model. *Nucl. Phys. A*, 724:69, 2003. (Cited on page 3.)

[38] M. Wolf. Applications of statistical mechanics in number theory. *Physica A*, 274:149, 1999. (Cited on pages 3, 21, and 63.)

[39] N. Cheng. Modified möbius inverse formula and its applications in physics. *Phys. Rev. Lett.*, 64(11):1193, 1990. (Cited on pages 3 and 12.)

[40] B. W. Ninham, B. D. Hughes, N. E. Frankel, and M. L. Glasser. Möbius, mellin, and mathematical physics. *Physica A*, 186:441–481, 1992. (Cited on pages 3 and 12.)

[41] G. Menezes and N. F. Svaiter. Quantum field theory and prime number spectrum. *ArXiv:math-ph/1211.5198*. (Cited on pages 3, 23, and 50.)

[42] G. Menezes, B. F. Svaiter, and N. F. Svaiter. Riemann zeta zeros and prime number spectra in quantum field theory. *Int. Jour. Mod. Phys. A*, 28:1350128, 2013. (Cited on pages 3 and 24.)

[43] C. H. G. Bessa, J. G. Dueñas, and N. F. Svaiter. Accelerated detectors in dirac vacuum: the effects of horizon fluctuations. *Class. Quantum Grav.*, 29:215011, 2012. (Cited on page 4.)

[44] E. Arias, J. G. Dueñas, N. F. Svaiter, C. H. G. Bessa, and G. Menezes. Casimir energy correcions by light-cone fluctuations. *Int. Journ. Mod. Phys A.*, 29:1450024, 2014. (Cited on page 4.)

[45] Matthew R. Watkins. Number theory and physics archive. 2015. URL http://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/physics.htm. (Cited on pages 4 and 37.)

[46] M. Abramowits and I. A. Stegun (Editors). *Handbook of Mathematical Functions*. Dover Publication, Inc., New York, 1965. (Cited on pages 8, 9, and 55.)

[47] I. M. Ryzhik. I. S. Gradshteyn. *Table of integrals, series and products*. Academic Press, Elsevier Inc, Oxford, a. jeffrey, d. zwillinger (editors) edition, 2007. (Cited on pages 8, 9, and 55.)

[48] B. Riemann. Uber die anzahl der primzahler unter einer gegebenen grosse. *Monats berichte der Berliner Akademie*, 671, 1859. (Cited on pages 9 and 12.)

[49] P. Borwein, S. Choi, B. Rooney, and A. (eds.) Weirathmueller. *The Riemann Hypothesis*. Springer Science+Bussines Media LLC, New York, 2008. (Cited on pages 9, 10, 13, 16, 18, and 51.)

[50] D. Schumayer and D. A. W. Hutchinson. Colloquium: Physics of the riemann hypothesis. *Rev. Mod. Phys.*, 83:307, 2011. (Cited on pages 10, 15, 21, and 49.)

[51] J. Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bull. Soc. Math. France*, 24:119, 1896. (Cited on pages 10, 14, 16, and 51.)

[52] C. de la Vallée-Poussin. Recherches analytiques la théorie des nombres premiers. *Ann. Soc. Sci. Bruxelles*, 202:183, 1896. (Cited on pages 10, 14, and 51.)

[53] C. de la Vallée-Poussin. Sur la fonction $\zeta(s)$ de riemann et le nombre des nombres premiers inférieurs à une limite donnée. *Ann. Soc. Sci. Bruxelles*, 202:281, 1896. (Cited on pages 10, 14, and 51.)

[54] Von E. Landau and A. Walfisz. Uber die nichtfortsetzbarkeit einiger durch dirichletsche reihen definierter funktionen. *Rendiconti del Circolo Matematico di Palermo*, 44:82, 1919. (Cited on pages 19, 20, and 23.)

[55] Carl-Erik Fröberg. On the prime zeta function. *BIT Numerical Mathematics*, 8(3):187–202, 1968. (Cited on pages 19, 20, and 23.)

[56] B. L. Julia. Thermodynamic limit in number theory: Riemann-beurling gases. *Physica A*, 203:425, 1994. (Cited on pages 21, 49, and 50.)

[57] D. Spector. Supersymmetry, vacuum statistics, and the fundamental theorem of algebra. *Communications in Mathematical Physics*, 177(1):13–25, 1996. (Cited on pages 21 and 49.)

[58] D. Spector. Duality, partial supersymmetry, and arithmetic number theory. *J. Math. Phys.*, 39:1919, 1998. (Cited on pages 21, 49, and 51.)

[59] R. J. Rivers. *Path Integral Methods in Quantum Field Theory*. Cambridge University Press, Cambridge, 1990. (Cited on page 21.)

[60] M. Masujima. *Path Integral Quantization and Stochastic Quantization*. Springer-Verlag, Berlin, 2009. (Cited on page 21.)

[61] R. T. Seeley. Complex powers of and elliptic operator. *Amer. Math. Soc. Proc. Symp. Pure Math*, 10:288, 1967. (Cited on page 21.)

[62] D. B. Ray and I. M. Singer. R-torsion and the laplacian on riemannian manifolds. *Advances in Mathematics*, 7(2):145 – 210, 1971. (Cited on page 21.)

[63] J. Dowker and R. Critchley. Effective lagrangian and energy-momentum tensor in de sitter space. *Phys. Rev. D*, 13. (Cited on page 21.)

[64] S. W. Hawking. Zeta function regularization of path integrals in curved spacetime. *Comm. Math. Phys.*, 55(2):133–148, 1977. (Cited on page 21.)

[65] E. Elizalde, F. C. Santos, and A. C. Tort. The casimir energy of a massive fermionic field confined in a $(d+1)$-dimensional slab-bag. *Int. Jour. Mod. Phys. A*, 18:1761, 2003. (Cited on pages 21 and 35.)

[66] G. V. Dunne. Functional determinants in quantum field theory. *Journal of Physics A: Mathematical and Theoretical*, 41(30):304006, 2008. (Cited on page 21.)

[67] D. Fursaev and D. Vassilevich. *Operators, Geometry and Quanta. Methods of Spectral Geometry in Quantum Field Theory*. Springer, New York, 2011. (Cited on pages 21 and 23.)

[68] A. Voros. Spectral functions, special functions and the selberg zeta function. *Comm. Math. Phys.*, 110(3):439–465, 1987. (Cited on page 21.)

[69] A. Voros. Spectral zeta functions. *Adv. Stud. Pure Math.*, 21:327, 1992. (Cited on page 21.)

[70] J. R. Quine, S. H. Heydary, and R. Y. Song. Zeta regularized products. *Trans. Amer. Math. Soc.*, 338:213, 1993. (Cited on page 21.)

[71] J. Iliopoulos, C. Itzykson, and A. Martin. Functional methods and perturbation theory. *Reviews of Modern Physics*, 47(1):165, 1975. (Cited on page 22.)

[72] Minakshisundaram Pleijel. Some properties of the eigenfunctions of the laplace-operator on riemanian manifolds. *Can. J. Math.*, 1:242, 1949. (Cited on page 22.)

[73] J. C. Andrade. Hilbert polya conjecture, zeta functions and bosonic quantum field theory. *Int. Jour. Mod. Phys. A*, 28: 1350072, 2013. (Cited on page 24.)

[74] A. P. Guinand. A summation formula in the theory of prime numbers. *Proc. Lond. Math. Soc*, 50:107, 1945. (Cited on pages 24 and 32.)

[75] J. Delsarte. Formules de poisson avec reste. *Jour. Anal. Math.*, 17:419, 1966. (Cited on pages 24, 32, and 69.)

[76] I. C. Chakravarty. The secondary zeta-functions. *Journal of Mathematical Analysis and Applications*, 30(2):280 – 294, 1970. (Cited on pages 24 and 32.)

[77] I. C. Chakravarty. Certain properties of a pair of secondary zeta-functions. *Journal of Mathematical Analysis and Applications*, 35(3):484 – 495, 1971. (Cited on pages 24 and 32.)

[78] A. Ivic. The distribution of zeros of the zeta function. *Bull. CXXI l'Acad. Scien. Arts*, 26:39, 2003. (Cited on pages 24 and 32.)

[79] A. Voros. Zeta functions for the riemann zeros. *Ann. Inst. Fourier, Grenoble*, 53:665, 2003. (Cited on pages 24 and 32.)

[80] A. Voros. *Zeta Functions over Zeros of Zeta Functions*. Springer-verlag, Heidelberg, 2010. (Cited on pages 24, 32, 56, and 57.)

[81] J. Wess and B. Zumino. Supergauge transformations in four dimensions. *Nucl. Phys. B*, 70:39, 1974. (Cited on page 24.)

[82] C. G. Bollini and J. J. Giambiagi. *Essays in Honour of Jayme Tiomno - Frontiers in Physics*. World Scientific, Singapore, 1991. (Cited on pages 24 and 25.)

[83] H. B. G. Casimir. On the attraction between two perfectly conducting plat. *Proc. Kon. Ned. Akad. Wet.*, 51:793, 1948. (Cited on page 27.)

[84] V. M. Mostepanenko and N. N. Trunov. *The Casimir Effect and Its Applications*. Oxford Science Publications, Great Britain, 1997. (Cited on page 27.)

[85] M. Bordag, G. L. Klimchitsa, U. Mohideen, and V. M. Mostepanenko. *Advances in the Casimir Effect*. Oxford University Press, New York, 2009. (Cited on page 27.)

[86] K. A. Milton and M. Bordag (eds.). *Quantum Field Theory Under the Influence of External Conditions (QFTEX09)*. World Scientific, Singapore, 2010. (Cited on page 27.)

[87] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge University Press, New York, 1982. (Cited on pages 27 and 30.)

[88] N. F. Svaiter and B. F. Svaiter. The analytic regularization zeta function method and the cut-off method in the casimir effect. *Journal of Physics A: Mathematical and General*, 25(4):979. (Cited on pages 27 and 35.)

[89] B. Svaiter and N. Svaiter. Zero point energy and analytic regularizations. *Phys. Rev. D*, 47:4581–4585, 1993. (Cited on pages 27 and 35.)

[90] M. Bordag, U. Mohideen, and Mostepanenko V. M. New developments in the casimir effect. *Physics Reports*, 353(1):1 – 205, 2001. (Cited on pages 27 and 31.)

[91] K. A. Milton. *The Casimir Effect: Physical Manifestation of Zero-Point Energy*. World Scientific, Singapore, 2001. (Cited on page 27.)

[92] G. Plunien, B. Muller, and W. Greiner. The casimir effect. *Phys. Rep.*, 134:87, 1986. (Cited on page 27.)

[93] S. K. Blau, M. Visser, and A. Wipf. "zeta functions and the casimir energy. *Nuclear Physics B*, 310(1):163 – 180, 1988. (Cited on page 27.)

[94] H. Weyl. Uber die asymptotische verteilung der eigenwert. *Nachr. Konigl. Ges. Wiss. Gottingen*, 110, 1911. (Cited on page 28.)

[95] A. Pleijel. A study of certain green's functions with applications in the theory of vibrating membranes. *Ark. Mat.*, 2:553, 1954. (Cited on page 28.)

[96] M. Kac. Can one hear the shape of a drum? *Am. Math. Mon.*, 73:1, 1966. (Cited on page 28.)

[97] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Interscience Publishers, Inc., New York, 1953. (Cited on page 28.)

[98] D. Deutsch and P. Candelas. Boundary effects in quantum field theory. *Phys. Rev. D*, 20:3063–3080, 1979. (Cited on page 28.)

[99] M. H. Protter. Can one hear the shape of a drum? revisited. *SIAM Review*, 29:185, 1987. (Cited on page 28.)

[100] A. Actor and I. Bender. Casimir effect for soft boundaries. *Phys. Rev. D*, 52:3581–3590, 1995. (Cited on page 28.)

[101] F. Caruso, R. De Paola, and N. F. Svaiter. Zero-point energy of massless scalar fields in the presence of soft and semihard boundaries in d dimensions. *International Journal of Modern Physics A*, 14(13):2077–2089, 1999. (Cited on page 28.)

[102] M. Beauregard, G. Fucci, K. Kirsten, and P. Morales. Casimir effect in the presence of external fields. *J. Phys. A*, 46:115401, 2013. (Cited on page 28.)

[103] R. D. M. De Paola, R. B. Rodrigues, and N. F. Svaiter. Casimir energy of massless fermions in the slab-bag. *Modern Physics Letters A*, 14(34):2353–2361, 1999. (Cited on pages 28 and 30.)

[104] C. D. Fosco and N. F. Svaiter. Finite size effects in the anisotropic $(\lambda/4!)(\varphi_1^4 + \varphi_2^4)_d$ model. *Journal of Mathematical Physics*, 42(11), 2001. (Cited on page 28.)

[105] M. I. Caicedo and N. F. Svaiter. Effective lagrangians for scalar fields and finite size effects in field theory. *Journal of Mathematical Physics*, 45(1), 2004. (Cited on pages 28, 38, 39, and 42.)

[106] N. F. Svaiter. Finite size effects in thermal field theory. *Journal of mathematical physics*, 45(12):4524–4538, 2004. (Cited on page 28.)

[107] M. A. Alcalde, G. F. Hidalgo, and N. F. Svaiter. The two-loop massless $(\lambda â4!)\phi^4$ model in nontranslational invariant domain. *Journal of Mathematical Physics*, 47(5), 2006. (Cited on page 28.)

[108] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel. Calculating vacuum energies in renormalizable quantum field theories: A new approach to the casimir problem. *Nucl. Phys. B*, 645:49, 2002. (Cited on page 28.)

[109] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel. Casimir energies in light of quantum field theory. *Phys. Lett. B*, 572:196, 2003. (Cited on pages 28 and 29.)

[110] J. J. Giambiagi C. G. Bollini and A. Goanzáles Domíngez. Regularization and renormalization of gauge fields. *Il Nuovo Cimento*, 31(3):550–561, 1964. (Cited on pages 30 and 55.)

[111] C. G. Bollini and J. J. Giambiagi. Dimensional renorinalization: The number of dimensions as a regularizing parameter. *Il Nuovo Cimento*, 12(1):20–26, 1972. (Cited on pages 30 and 55.)

[112] G. tH́ooft and M. Veltman. Regularization and renormalization of gauge fields. *Nuclear Physics B*, 44(1):189–213, 1972. (Cited on page 30.)

[113] J. F. Ashmore. A method of gauge-invariant regularization. *Lettere Al Nuovo Cimento (1971–1985)*, 4(8):289–290, 1972. (Cited on page 30.)

[114] T. Cheng and L. Li. *Gauge theory of elementary particle physics*. Clarenton Press, Oxford, 1984. (Cited on page 30.)

[115] B. Kay. Casimir effect in quantum field theory. *Phys. Rev. D*, 20. (Cited on page 34.)

[116] L. H. Ford and T. Yoshimura. Mass generation by self-interaction in non-minkowskian spacetimes. *Physics Letters A*, 70(2):89–91, 1979. (Cited on page 34.)

[117] N. Birrell and L. Ford. Renormalization of self-interacting scalar field theories in a nonsimply connected spacetime. *Phys. Rev. D*, 22:330–342, 1980. (Cited on page 34.)

[118] J. R. Ruggiero, A. Villani, and A. H. Zimerman. Some comments on the application of analytic regularisation to the casimir forces. *Journal of Physics A: Mathematical and General*, 13(2):761, 1980. (Cited on page 35.)

[119] B. F. Svaiter and N. F. Svaiter. The stress tensor conformal anomaly and analytic regularizations. *Journal of Mathematical Physics*, 35(4), 1994. (Cited on page 35.)

[120] C.G. Beneventano and E.M. Santangelo. Connection between ζ and cutoff regularizations of casimir energies. *International Journal of Modern Physics A*, 11(16):2871–2886, 1996. (Cited on page 35.)

[121] A. Chodos, R. L. Jaffe, K. Johnson, and V. F. Weisskopf. New extended model of hadrons. *Phys. Rev. D*, 9:3471, 1974. (Cited on page 35.)

[122] A. Chodos and C. B. Thorn. Electromagnetic self-energy of confined massless quarks. *Phys. Lett. B*, 53:359, 1974. (Cited on page 35.)

[123] V. K. Oikonomou and N. D. Tracas. Slab bag fermionic casimir effect, chiral boundaries and vector boson - majorana fermion pistons. *Int. Jour. Mod. Phys. A*, 25:5935, 2010. (Cited on page 35.)

[124] S. John and M. J. Stephen. Wave propagation and localization in a long-range correlated random potential. *Phys. Rev. B*, 28: 6358, 1983. (Cited on page 37.)

[125] J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. Claredon Press, Oxford, 2nd edition, 1994. (Cited on page 38.)

[126] T. Appelquist and J. Carrazzone. Infrared singularities and massive fields. *Phys. Rev. D*, 11:2850, 1975. (Cited on page 38.)

[127] M. Gell-Mann and M. Lévy. The axial vector current in beta decay. *Nuovo Cimento*, 16:705, 1960. (Cited on page 41.)

[128] L. Dolan and R. Jackiw. Symmetry behavior at finite temperature. *Phys. Rev. D*, 9:3320, 1974. (Cited on page 41.)

[129] J. Wess S. Coleman and B. Zumino. Structure of phenomenological lagrangians. i. *Phys. Rev.*, 177:2239, 1969. (Cited on page 41.)

[130] J. Honerkamp and K. Meetz. Chiral-invariant perturbation theory. *Phys. Rev. D*, 3:1996, 1971. (Cited on page 41.)

[131] B. W. I. S. Gerstein, R. Jackiw and S. Weinberg. Chiral loops. *Phys. Rev. D*, 3:2486, 1971. (Cited on page 41.)

[132] A. A. Slavnov and L. D. Faddeev. Invariant perturbation theory for nonlinear chiral lagrangians. *Theor. Math. Phys.*, 8:843, 1971. (Cited on page 41.)

[133] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin. Renormalization of the nonlinear σ model in $2 + \hat{I}\mu$ dimensions. *Phys. Rev. D*, 14: 2615, 1976. (Cited on page 41.)

[134] C. Becchi and O. Piguet. On the renormalization of two-dimensional chiral models. *Nucl. Phys. B*, 315:153, 1989. (Cited on page 41.)

[135] M. Moshe and J. Zinn-Justin. Quantum field theory in the large n limit: a review. *Phys. Rep.*, 385:69, 2003. (Cited on page 42.)

[136] A. M. Polyakov. *Gauge Fields and Strings*. Harood Academic Publishers, London, 1987. (Cited on page 42.)

[137] Alexei M. Tsvelik. *Quantum Field Theory in Condensed Matter Physics*. Cambridge University Press, Cambridge, 2003. (Cited on page 42.)

[138] K. Gawedzki and A. Kupiainen. Continuum limit of the hierarchical $o(n)$ non-linear σ-model. *Comm. Math. Phys.*, 106:533, 1986. (Cited on page 42.)

[139] A. M. Polyakov. Interaction of goldstone particles in two dimensions. applications to ferromagnets and massive yang-mills fields. *Phys. Lett. B*, 59:79, 1975. (Cited on page 45.)

[140] M. Lusher. The secret long range force in quantum field theories with instantons. *Phys. Lett. B*, 78:465, 1978. (Cited on page 45.)

[141] H. Falomir and E. M. Santangelo. Nonlinear sigma model from two-dimensional qcd. *Phys. Rev. Lett.*, 56:1659, 1986. (Cited on pages 45 and 64.)

[142] P. Sarnak. Random matrix theory and zeroes of zeta functions - a survey. *MSRI lecture*, 1999. (Cited on page 47.)

[143] T. D. Lee and C. N. Yang. Statistical theory of equations of state and phase transitions. i. theory of condensation. *Phys. Rev.*, 87: 404, 1952. (Cited on page 47.)

[144] T. D. Lee and C. N. Yang. Statistical theory of equations of state and phase transitions. ii. lattice gas and ising model. *Phys. Rev.*, 87:410, 1952. (Cited on page 47.)

[145] M. E. Fisher. The nature of critical points. *Lectures in Theoretical Physics*, 7C:edited by W. E. Britten, 1965. (Cited on page 47.)

[146] B. Derrida. Random-energy model: Limit of a family of disordered models. *Phys. Rev. Lett*, 45:79, 1980. (Cited on page 47.)

[147] B. Derrida. Random-energy model: An exactly solvable model of disordered systems. *Phys. Rev. B*, 24:2613, 1981. (Cited on page 47.)

[148] M. Mézard, G. Parisi, and M.A. Virasoro. *Spin Glass Theory and Beyond*. World Scientific, Singapore, 1987. (Cited on pages 47 and 50.)

[149] V. Dotsenko. *Introduction to the Replica Theory of Disordered Statistical Systems*. Cambridge University Press, Cambridge, 2001. (Cited on pages 47, 50, and 52.)

[150] H. Nishimori. *Statistical Physics of Spin Glasses and Information Processing*. Oxford Science Publication, Oxford, 2001. (Cited on pages 47, 50, and 52.)

[151] C. De Dominicis and I. Giardina. *Random Fields and Spin Glasses: A Field Theory Approach*. Cambridge University Press, New York, 2010. (Cited on pages 47 and 50.)

[152] B. Derrida. The zeroes of the partition function of the random energy model. *Physica A*, 177:31, 1991. (Cited on page 48.)

[153] Y. Matsuda, H. Nishimori, and K. Hukushima. The distribution of leeâyang zeros and griffiths singularities in the $\pm j$ model of spin glasses. *J. Phys. A*, 41:324012, 2008. (Cited on page 48.)

[154] K. Takahashi. Replica analysis of partition-function zeros in spin-glass models. *J. Phys. A*, 44:235001, 2011. (Cited on page 48.)

[155] K. Takahashi and T. Obuchi. Zeros of the partition function and dynamical singularities in spin-glass systems. *J. Phys. Conf. Series*, 473:012023, 2013. (Cited on page 48.)

[156] B. B. Wei and R. B. Liu. Lee-yang zeros and critical times in decoherence of a probe spin coupled to a bath. *Phys. Rev. Lett.*, 109:285701, 2012. (Cited on page 48.)

[157] X. Peng, H. Zhou, B. B. Wei, J. Cui, J. Du, and R. B. Liu. Experimental observation of lee-yang zeros. *Phys. Rev. Lett.*, 114: 010601, 2015. (Cited on page 48.)

[158] H. Cramer. Prime numbers and probability. *Skand. Mat.-Kongr*, 8:107, 1935. (Cited on pages 48, 49, 52, and 64.)

[159] A. Granville. Harald cramer and the distribution of prime numbers. *Scan. Actuarial J*, 1:12, 1995. (Cited on pages 48, 49, 52, and 64.)

[160] C. Weiss, S. Page, and M. Holthaus. Factorising numbers with a bose-einstein condensate. *Physics A*, 341:586, 2004. (Cited on page 49.)

[161] R. Hagedorn. Statistical thermodynamics of strong interactions at high energies. *Supp. Nuovo Cimento*, 3:147, 1965. (Cited on pages 50 and 52.)

[162] A. E. Ingham. On two conjectures in the theory of numbers. *Amm. Jour. Math*, 64:313, 1942. (Cited on page 51.)

[163] J. Korevaar. On newman's quick way to the prime number theorem. *Math. Intell*, 4:108, 1982. (Cited on page 51.)

[164] P. Sarnak. Three lectures on the mobius function randomness and dynamics. 2011. URL http://publications.ias.edu/node/506. (Cited on pages 52 and 63.)

[165] O. Penrose. *Foundations of Statistical Mechanics*. Dover Publications Inc., New York, 2005. (Cited on page 53.)

[166] U. Weiss. *Quantum Dissipative Systems*. World Scientific Publishing, Singapore, 4 edition, 2012. (Cited on page 54.)

[167] Carl M Bender. Making sense of non-hermitian hamiltonians. *Reports on Progress in Physics*, 70(6):947. (Cited on page 54.)

[168] G. H. Hardy. *Divergent Series*. Oxford University Press, London, 1973. (Cited on page 55.)

[169] H. S. Green. A generalized method of field quantization. *Phys. Rev.*, 90:270, 1953. (Cited on page 64.)

colophon

This document was typeset using the typographical look-and-feel `classicthesis` developed by André Miede. The style was inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*". `classicthesis` is available for both LaTeX and LyX:

http://code.google.com/p/classicthesis/

Happy users of `classicthesis` usually send a real postcard to the author, a collection of postcards received so far is featured here:

http://postcards.miede.de/

Final Version as of November 16, 2015 (`classicthesis`).