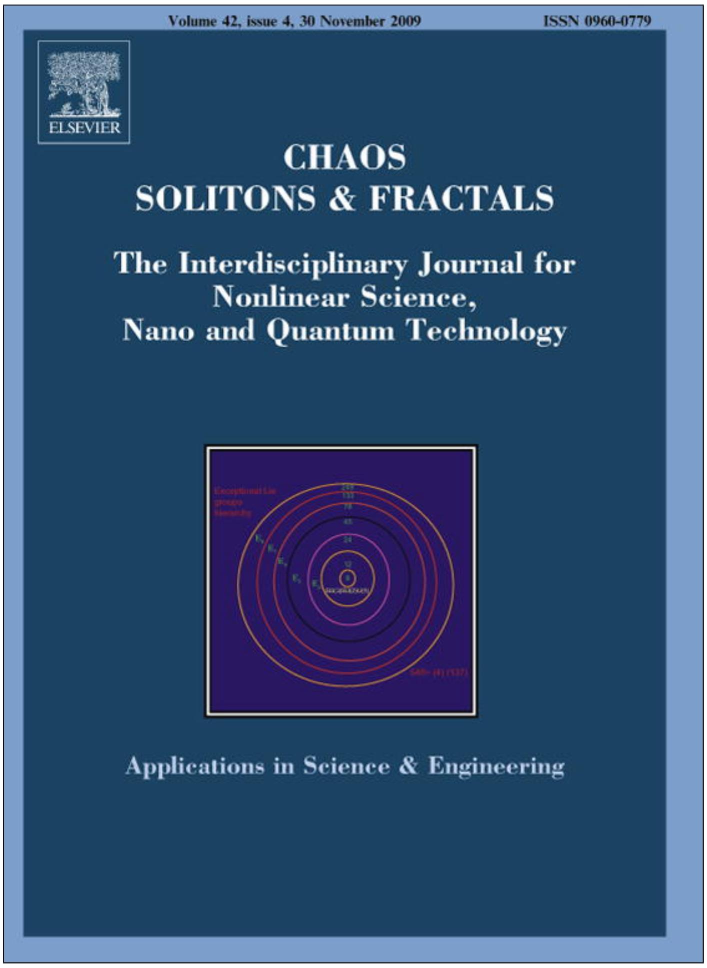
Volume 42, issue 4, 30 November 2009
ISSN 0960-0779
ELSEVIER
CHAOS
SOLITONS & FRACTALS
The Interdisciplinary Journal for
Nonlinear Science,
Nano and Quantum Technology
Applications in Science & Engineering

Chaos, Solitons and Fractals 42 (2009) 2512–2521

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

# Effective multifractal features of high-frequency price fluctuations time series and $\ell$-variability diagrams ☆

Jeferson de Souza [a,b], Sílvio M. Duarte Queirós [b,*,1]

[a] *Laboratório de Análise de Bacias e Petrofísica, Departamento de Geologia, Universidade Federal do Paraná, Centro Politécnico – Jardim das Américas, Caixa Postal 19001, 81531-990 Curitiba-PR, Brazil*
[b] *Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*

A R T I C L E I N F O

*Article history:*
Accepted 31 March 2009

Communicated by Prof. L. Marek-Crnjac

A B S T R A C T

In this manuscript we present a comprehensive study on the multifractal properties of high-frequency price fluctuations and instantaneous volatility of the equities that compose the Dow Jones Industrial Average. The analysis consists about the quantification of the influence of dependence and non-Gaussianity on the multifractal character of financial quantities. Our results point out an equivalent importance of dependence and non-Gaussianity on the multifractality of time series. Moreover, we analyse $\ell$-diagrams of price fluctuations. In the latter case, we show that the fractal dimension of these maps is basically independent of the lag between price fluctuations that we assume.



## 1. Introduction

Scale invariance and fractality, i.e., the absence of a characteristic scale, can be found in a widespread of natural and human created phenomena [1–8,10–12]. Mathematically, scale invariance of a certain function, $f$, of an observale $\mathcal{O}$ is written as

$$f(\lambda\mathcal{O}) = \lambda^{\alpha} f(\mathcal{O}), \tag{1}$$

and it has consensually been taken as a signature of complexity [9]. Concerning time series and fractality,[2] if many of them seem to be *monofractal* [15], i.e., they are characterised by a single scale exponent, just as in Eq. (1), several others, namely price fluctuations in financial markets have shown a spectrum of locally dependent $\alpha$ exponents [13,14]. Analytically, this is noted as

$$f(\{\lambda\mathcal{O}\}^{\upsilon}) = \lambda^{\alpha(\upsilon)} f(\mathcal{O}^{\upsilon}). \tag{2}$$

The previous equation (2), which corresponds to a signature of multiscaling and *multifractality* as well, has consistently been associated with the main statistical features of price fluctuations. Consequently, this close relation has been prominent in either developing dynamical models or validating previous approaches. In the former, pioneering works by B. Mandelbrot have opened the door to a new treatment of financial markets dynamics [16].

In sequel of this manuscript we perform an extensive analysis of the statistical features of high-frequency (log-) price fluctuations,

$$\tilde{r}_i'(t) = \ln S_i(t) - \ln S_i(t-1), \tag{3}$$

☆ The main contents of this manuscript is part of a CBPF document by JdS publicly presented at this institute on the 12th February 2007.

* Corresponding author.
*E-mail addresses:* jdesouza@ufpr.br (J. de Souza), sdqueiro@googlemail.com, sdqueiro@cbpf.br (S.M. Duarte Queirós).
[1] Present address: Unilever R&D Port Sunlight, Quarry Road East, Wirral CH63 3JW, United Kingdom.
[2] Time series can only be considered scale invariante in a self-affine context.

0960-0779/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2009.03.198

of the 30 equities that compose the Dow Jones Industrial Average (DJIA) from the 1st of July until the 31st December of 2004 in a total of around 50,000 data points for each equity, $i$, for which we removed the intra-day pattern [17] ($S$ represents the price). Previous studies on daily price fluctuations have shown the existence of multifractal behaviour [18]. Hence, with this high-frequency analysis, it is our aim to study the multiscaling of price fluctuations at a level that is closer to the transaction dynamics as it has been made for other financial observables [23]. Our study is driven on the evaluation of the multifractal spectra of both of time series and $(\ell = 1)$-diagrams, $(r_t, r_{t+1})$, describing the weight multi-scaling factors like heavy tails and memory. In addition, we enquire into the absolute value of price fluctuations, $| r_t |$, also called *instantaneous volatility*, $v_t \equiv | r_t |$, multifractal behaviour and analyse its weight on the multiscaling characteristics of price fluctuations.

To compute the multifractal features of our time series, we have chosen to apply MF-DFA [19] in lieu of the Wavelet Transform Modulus Maxima (WTMM) [25] taking into account a recent comparative study where it has been shown that, in the majority of situations, MF-DFA presents more reliable results [26]. For this procedure, it was proved that the $z$th order fluctuation function, $F_z(s)$, presents the following scale behavior, $F_z(s)/s \sim s^{h(z)}$. The correspondence between MF-DFA and the standard formalism of multifractals is obtained using Legendre transform

$$f(\alpha) = z\alpha - \tau(z). \tag{4}$$

We can relate exponent $\tau(z)$ with Hölder exponent, $\alpha$,

$$\alpha = h(z) + z\frac{dh(z)}{dz}, \tag{5}$$

and

$$f(\alpha) = z[\alpha - h(q)] + 1. \tag{6}$$

For $z = 2$, $h(2) \equiv H$, which corresponds to the Hurst exponent [27,28], and for $z = 0$, $f(\alpha)$ obtained from Eqs. (4)–(6) corresponds to the support dimension.

Multiscaling is introduced in a time series twifold: from memory and from asymptotic power-law probability density functions. If we aim to size up the weight of non-Gaussianity, we must destroy memory in the signal. And from it, by using the independence conjecture, we determine memory influence. Memory is basically destroyed if we shuffle time series elements. Doing that, we reorder the values of our original time series, but we keep the stationary probability density function unalted. On the other hand, we can destroy non-Gaussianity by implementing the procedure that we call *phase randomisation* corresponding to a introduction of random phase in half of the series and their conjugates in the remaining half (see [20] for details).

For both shuffled and phase randomised time series obtained from the original signal, we can also carry out a MF-DFA analysis. For each case, we appraise multiscaling using exponents $h_{shf}(z)$ for the shuffled time series, and $h_{rnd}(z)$ for the phase randomised case. Assuming independency between multifractal factors, we have measured the contribution of correlations, $h_{cor}(z)$, by, $h_{cor}(z) \equiv h(z) - h_{shf}(z)$. If only these two factors introduce multiscaling on the signal, then, when we perform the phase randomisation process on a shuffled signal, we should obtain a Gaussian and uncorrelated signal, i.e., $h_{shf\text{-}rnd}(z) = \frac{1}{2}$ for all $z$. Theoretically, we can evaluate the contribution of non-Gaussianity, $h'_{PDF}(z)$, from phase randomised time series with $h'_{PDF}(z) = h_{shf}(z) - h_{shf\text{-}rnd}(z)$. However, the probability density function of a finite time series is influenced by its size, particularly for small time series [21]. In this sense, comparing results obtained from times series with different probability density functions, such is the case of $\{x^{shf}(t)\}$ and $\{x^{shf\text{-}rnd}(t)\}$, introduces error factors that we are not able to quantify. Regarding this factor, we have opted to define an effective contribution of non-Gaussianity, $h_{PDF}(z)$,

$$h_{PDF}(z) = h_{shf}(z). \tag{7}$$

Moreover, in order to avoid, or at least minimise finite size spurious features, we have chosen to compute every quantity for $s$ between 8 and 11,585, and $z$ between $-3$ and 5. Within this range of values, we were able to obtain numerical curves which concur to the theoretical scaling curve of independent and Gaussian time series.

Multifractality can be effectually quantified through the difference between scale exponents of $z_{\min}$ and $z_{\max}$,

$$\Delta h \equiv h(z_{\min}) - h(z_{\max}). \tag{8}$$

Eq. (8) can be used for the original time series, $\Delta h$, and for the shuffled time series, $\Delta h_{shf}$. From these values, we finally compute the weight of non-Gaussianity, $\Delta h_{shf}/\Delta h$, and of correlations $1 - \Delta h_{shf}/\Delta h$.

## 2. Results for time series

### 2.1. Multifractality for price fluctuations time series

In Fig. 1(left), we present our results for the multifractal spectrum of price fluctuations, shuffled, phase randomised, and shuffled plus phase randomised time series. The values have been obtained by performing, for each moment $\tau$, an average over the 30 companies. Despite of the fact that it has been verified the influence of equities liquidity on the multifractal properties of financial time series, all companies of our data set have presented liquidity values within the same order of magnitude turning out our average over the companies perfectly plausible. As can be seen, the price fluctuations time series

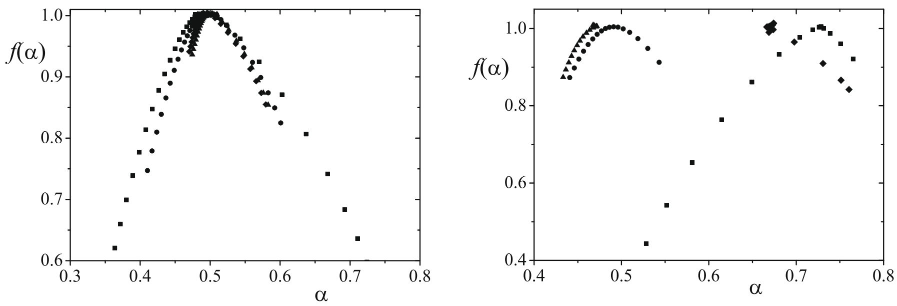


**Fig. 1.** Left panel: Multifractal spectra $f$ vs. $\alpha$ of the price fluctuations (■), shuffled time series (•), phase randomised (♦), and shuffled plus phase randomised (▲) time series of DJIA equities. As it is depicted, when elements that introduce multiscaling are removed the multifractal spectrum becomes narrower. Right panel: Multifractal spectra $f$ vs. $\alpha$ of the instantaneous volatility (■), shuffled time series (•), phase randomised (◇), and shuffled plus phase randomised (▲) time series of DJIA equities. As it is shown, when elements that introduce multiscaling are removed the multifractal spectrum becomes clearly narrower.

present a wide multifractal spectrum with $\alpha_{\min} = 0.364$ and $\alpha_{\max} = 0.724$. Furthermore, we verify a strong asymmetry between the part of the spectrum that goes from $\alpha_{\min}$ up to $\alpha\ (z = 0)$ and the remaining part of the spectrum. The asymmetry in Fig. 1 is contrary to the $f(\alpha)$ curve that has been measured in fully developed turbulent flows [31] often considered a price fluctuation analogue. Concerning the other time series, we observe that the multifractal spectrum of the shuffled time series is slightly narrow than the spectrum for the original time series. In addition, the shuffled signals have larger spectrum than the randomised and shuffled plus phase randomised signals. Analysing scaling exponent $h(2)$, that is the common Hurst exponent, $H$, we have obtained a value around $\frac{1}{2}$ concomitant with a white noise sequence, and in accordance with the Efficient Market Hypothesis (EMH) [32]. Furthermore, we have $\max(f(\alpha)) = 1$, i.e., price fluctuations time series are *fat-fractals* as it occurs for a large variety of other signals and non-linear phenomena [1][33]. For the $h$ difference defined in Eq. (8) we have obtained $\Delta h = 0.15$ and $\Delta h_{shf} = 0.08$. These values yield a weight of 54% for non-Gaussianity and 46% for correlations in the multifractal properties of our time series. In spite of this result appears to be at odds with the $H = \frac{1}{2}$, we must call attention to the fact that there is a more delicate relation for random variables, *the statistical dependence* [34], which cannot be described by the Hurst exponent. The statistical dependence of financial observables [13,35,36] has been verified by means of mutual information measures [37]. We attribute to this statistical feature the multiscaling of price fluctuations we have perceived. This assignment is also supported by the structure of $(\ell = 1)$-diagrams that we analyse in Section 3.

From an analysis of moments $\tau$ as a function of $z$, we observe that only the shuffled plus phase randomised signal are in compliance with the theoretical curve, $\tau = z/2 - 1$, of an Gaussian time series of independent elements [38].

### 2.2. *Multifractality for instantaneous volatility time series*

Albeit volatility is not directly measurable, it plays a central role in financial modelling [39], and it is usually related to the magnitude of price fluctuations. It is from this quantity that long-lasting covariances associated with asymptotic power-laws are measured. As a matter of fact, the appropriate mimicry of a long-lasting autocorrelation function of the volatility associated with a white noise character of the variable upon study is one of prime challenges in several areas of scientific research. Aiming to appraise its potential multiscaling nature, we have performed a MF-DFA analysis on instantaneous volatility time series. The main results are shown in Figs. 1 and 2. From our analysis, we have verified that there are clear differences between multifractal spectra for price fluctuations and absolute values.

In first place, and against our primary expectations, we have observed that price fluctuations have a wider multifractal spectrum. Specifically, we have computed $\Delta h = 0.15$ for price fluctuations, and $\Delta h = 0.10$ for volatilities. This corresponds to a ratio of 3 over 2. As it happens for price fluctuations, the multifractal spectrum is asymmetric. We have also obtained $h(2) = 0.71$, which indicates a strong persistency of volatility time series in accordance with previous empirical findings. We clarify that, we expected to obtain a wider spectrum for instantaneous volatility because of correlations and non-Gaussianity of this quantity. For shuffled instantaneous volatility time series, we have observed a shift of $f(\alpha)$, and a lessen of curve width. On the other hand, when we turn instantaneous volatility into a Gaussian variable the multifractality tends do be clearly diminished, though still present. This is in accordance to previous verifications about local fluctuations of the Hurst exponent for financial time series [53] which introduce multifractality. Bearing in mind the value $\Delta h = 0.10$, the difference between scaling exponents of the shuffled time series, $\Delta h_{shf} = 0.05$, points non-Gaussianity and dependence as equally responsible for the multiscaling of instantaneous volatility. From Fig. 2, it is visible that $\tau_{shf}$ almost coincides with the theoretical curve of an independent and Gaussian time series. Such a result indicates that the probability density function presents a nearly exponential decay. We corroborate this result with Fig. 3 in which we present absolute values probability density function, $p(v)$. In line with Fig. 3 we verify that $p(v)$ fits for a $F$-distribution,

$$F(v) \propto \left(\frac{v}{\theta}\right)^{\phi}\left[1-(1-q)\,\frac{v}{\theta}\right]^{\frac{1}{1-q}}, \tag{9}$$

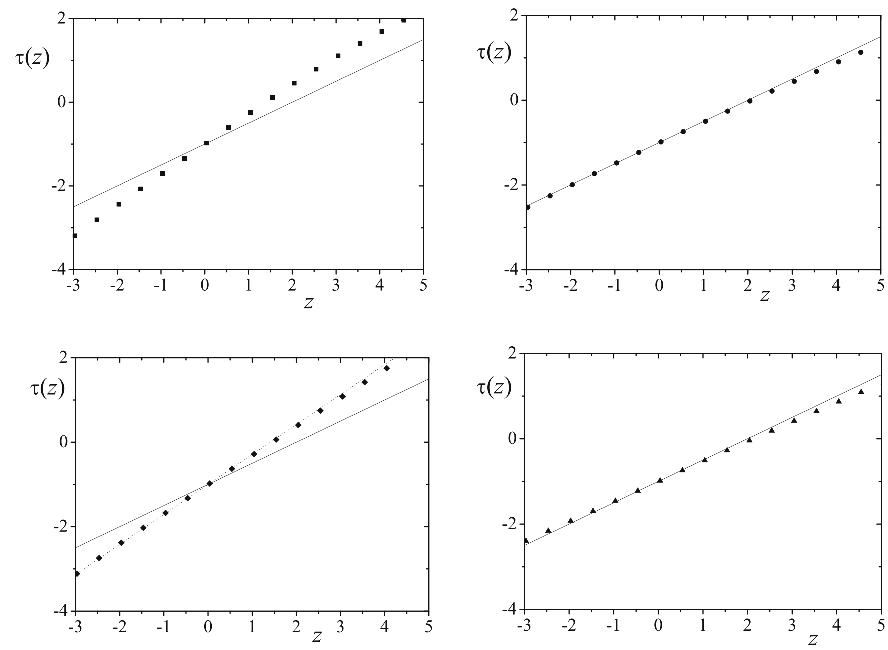


**Fig. 2.** Scaling exponent $\tau$ vs. $z$ of the instantaneous volatility (upper left), shuffled (upper right), phase randomised (lower left), and shuffled plus phase randomised (lower right) time series of DJIA equities. In all panels, the solid line $\tau = \frac{z}{2} - 1$ represents the theoretical curve of Gaussian time series of independent elements. Regarding instantaneous volatility results, it is visible the departure from Gaussian independent behaviour that persists when we destroy the Gaussianity. In the lower left panel the dotted line represents the monofractal curve $\tau = Hz - 1$ with $H = h(2) = 0.71 \pm 0.01$. If we considered the phase randomised time series as a pure monofractal set we would have the best fit for $H = 0.692 \pm 0.002$, a bit ouside error margin of $h(2)$.

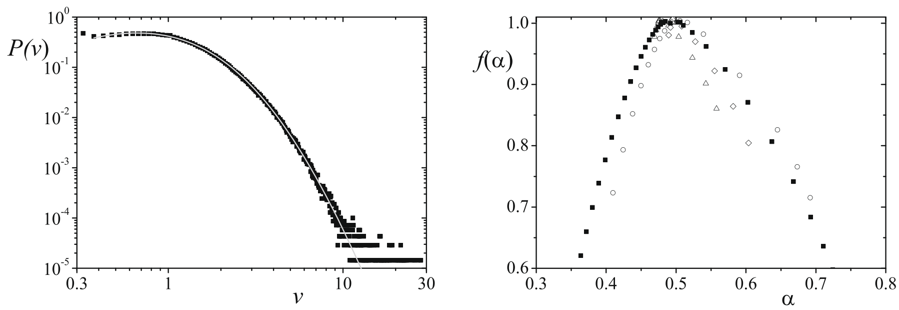


**Fig. 3.** Left panel: Instantaneous volatility probability density function $P(v)$ vs. $v$ averaged over DJIA equities. Symbols are the empirical PDF and the line the best fit using a $F$-distribution, Eq. (9) ($\chi^2/n = 4.4 \times 10^{-6}$ and $R^2 = 0.999$). Right panel: Multifractal spectra $f$ *vs.* $\alpha$ of the price fluctuations (■), and of time series $\{r(t)\}$ that use shuffled (∘), phase randomised (◇), and shuffled plus phase randomised (△) volatility time series of DJIA equities. As it is depicted, the multifractal character of volatility plays an essential role at the multifractal nature of price fluctuations. This role is clear for the non-Gaussianity of $v(t)$.

where $\theta = 0.32 \pm 0.02$, $\phi = 1.83 \pm 0.01$, and $q = 1.08 \pm 0.02$. Taking into account error margins, the small deviation from exponential decay given by numerical adjustment is in agreement with the slight deviation of $\tau_{shf}$ from the theoretical curve that we have measured.

### *2.3. Effects of the signal and (instantaneous) volatility multifractal behaviour on price flctuations multiscaling*

In this section, we assess the influence of the multifractal character of instantaneous volatility on the multifractal nature of price fluctuations. To that, we have proceeded the following way. We have separated price fluctuations, $r(t)$, considering each element as the product of elements of two other time series, i.e., one comprising the signal of the price fluctuation, $s(t) = \pm 1$, and the other which takes into account the magnitude or instantaneous volatility, $v(t) = | r(t) |$. Preserving the signal time series, we have multiplied $\{s(t)\}$ by time series that were obtained after shuffle, $v_{shf}(t)$, phase randomisation, $v_{rnd}(t)$, and shuffle plus phase randomisation, $v_{shf\text{-}rnd}(t)$, procedures. The results we have obtained are depicted in Figs. 3 and 4. From Fig. 3, we see that the statistical properties of volatility *do* influence the multifractal spectrum of price fluctuations. If we only shuffle $\{v(t)\}$ elements, the $u(t)$ time series,

$$u(t) \equiv s(t)\, v(t),$$

just has a paltry narrower $f(\alpha)$ curve than $\{r(t)\}$. Explicitly, it has $\Delta h = 0.13$ in opposition to $\Delta h = 0.15$ of $\{r(t)\}$. This is an unexpected result regarding the influence of $\{v(t)\}$ ordering on its multifractal spectrum. However, when we destroy the

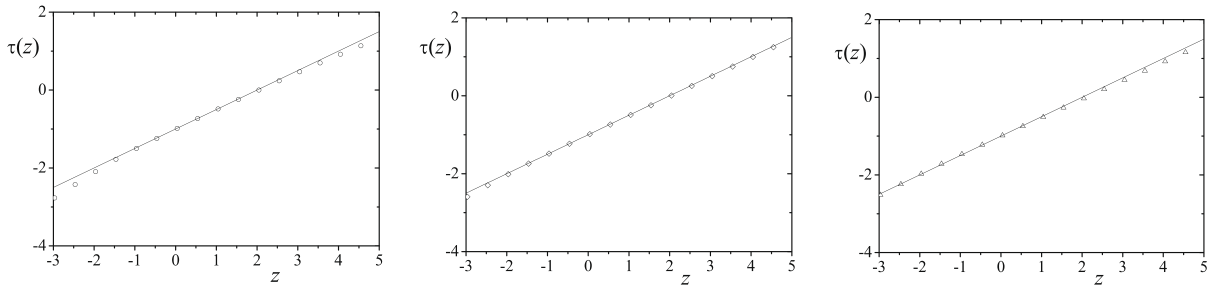


**Fig. 4.** Scaling exponent $\tau$ vs. $z$ of time series $\{r(t)\}$ that use shuffled (right), phase randomised (centre), and shuffled plus phase randomised (left) volatility time series of DJIA equities. In all panels, the line $\tau = \frac{z}{2} - 1$ represents the theoretical curve for an independent and Gaussian time series. The importance of the multifractal characteristics of volatility are demonstrated by the clear approach of these three results towards the theoretical curve of an independent Gaussian process.

non-Gaussianity of instantaneous volatility probability density function, we basically destroy the multifractal spectrum of price fluctuations, since $\Delta h = 0.04$, or $\Delta h = 0.03$ when we combine shuffling with phase randomisation procedures on $\{v(t)\}$. The latter result also sets the influence of the signal ordering on the price fluctuations multifractal character at the order of error in absolute accordance with previous analysis for other characteristics, namely the approach to the Gaussian when of cummulative price fluctuations probability density functions [48].

As it has been observed [13,36], many of the dynamical and statistical properties of price fluctuations depend on the volatility. Although it is a pivotal variable in finance, the truth is that the definition of volatility is still ambiguous [40]. If in many situations it is presented has we have been doing, volatility is often determined as the standard deviation of price fluctuations over windows of length $l$.[3] The latter definition is widely applied on stochastic volatility models. In that particular case, superstatistical models have been applied in problems of financial origin [13,41,49] to define such models. Concisely, *superstatistics* or "*statistics of statistics*" [42] is a compound method which has emerged within statistical mechanics. It is based on the assumption of a local statistics dependent on a parameter that fluctuates (smoothly) on a time scale that is very large when compared with the time needed for a system to reach a local equilibrium or stationarity. In a superstatistical context, it has been proved that, if we have a set of local Gaussian random variables,

$$\mathscr{G}_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right], \tag{10}$$

and the inverse variance, $\sigma^{-2}$, is associated with a $\Gamma$-distribution,

$$\Gamma(x) \propto \left(\frac{x}{\delta}\right)^\gamma \exp\left[-\frac{x}{\delta}\right], \tag{11}$$

then, the stationary distribution given by

$$p(x) = \int \mathscr{G}_\sigma(x)\Gamma(\sigma^{-2})d(\sigma^{-2}),$$

is equal to a Tsallis (or Student $t$-) distribution [43],

$$p(x) = \frac{1}{Z}\left[1 - (1-q)\frac{x^2}{\lambda}\right]^{\frac{1}{1-q}}, \tag{12}$$

where

$$q = 1 + \frac{2}{3 + 2\gamma}. \tag{13}$$

In this way, superstatistics has been considered has the first dynamical foundation for non-extensive framework [44] that has non-additive entropy, $S_q$ [45], as its cornerstone. Distribution (12) has regularly been used to fit price fluctuations of several financial markets including the data set we have been analysing for which it has been found a value of $q = 1.31 \pm 0.02$ [24]. If we assume a superstatistical approach for the data set upon analysis from Eq. (13) we obtain $\gamma = 1.82$.

In what follows, we analyse a discrete *ARCH*-like process [46] that can be catalogued as superstatistical. Explicitly, we have generated time series, $\{y(t)\}$, from the product of an uncorrelated Gaussian signal, $\{\omega(t)\}$, with $\langle\omega(t)\rangle = 0$, and $\langle\omega(t)^2\rangle = 1$ by an uncorrelated volatility signal, $\{\sigma(t)\}$,

$$y(t) \equiv \sigma(t)\omega(t),$$

such that $\sigma^{-2}$ follows a $\Gamma$-distribution with $\gamma = 1.82$ as we have obtained. In this case, because we have neglected memory on volatility, we compare the multifractal study of this time series with the results that we have presented at the beginning

[3] When $l = 1$ we obtain the instantaneous volatility definition.

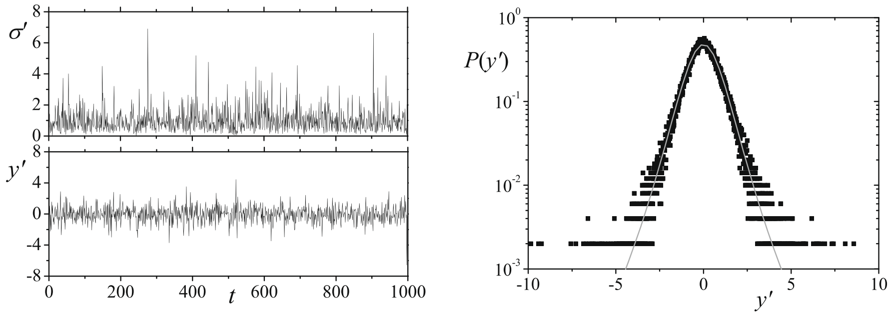

**Fig. 5.** Left: Excerpt of the *ARCH*-like signal (lower panel) where $y'$ represents $y$ divided by the average value of $\sigma$, $\sigma_m$. The elements of $\sigma$ signal (upper panel) follow a PDF such that $P(\sigma^{-2})$ is a $\Gamma$-distribution with $\gamma = 1.82$, and $\delta = 2$. Right: Stationary PDF of $y'$, $P(y')$ vs. $y'$. Symbols have been obtained from the time series and the line is the numerical adjustment for a $q$-Gaussian distribution with $q = 1.3$. Although this is not an exact approach, the adjustment is rather nice ($\chi^2/n = 1.2 \times 10^{-5}$ and $R^2 = 0.99$).

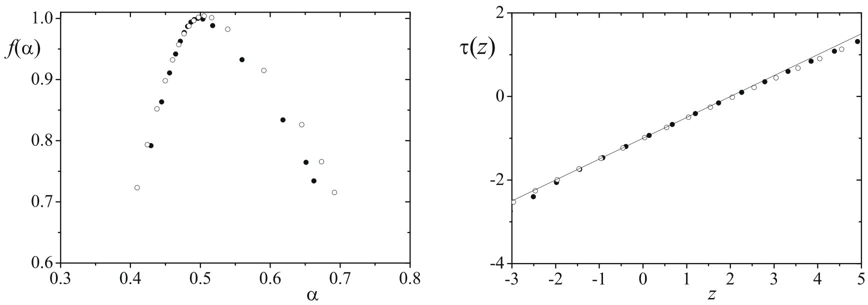

**Fig. 6.** Left panel: Multifractal spectra $f$ vs. $\alpha$ of the $\{y(t)\}$ ($\bullet$), and of time series $\{r(t)\}$ that use shuffled ($\circ$) volatility time series of DJIA equities. Right panel: Scaling exponent $\tau$ *vs.* $z$ of $\{y(t)\}$ ($\bullet$) and of time series $\{r(t)\}$ that use shuffled ($\circ$) volatility time series of DJIA equities. The line $\tau = \frac{z}{2} - 1$ represents the theoretical curve for an independent and Gaussian time series.

of this Section 2.3 for $\{u(t)\}$ with a shuffled instantaneous volatility. We have opted for this comparison because, just as $s(t)$, $\omega(t)$ does not contribute to the multifractal spectrum. The excerpt of the time series we have generated is presented in Fig. 5. In the same figure, we comprove that $\{y(t)\}$ follows PDF (12) with $q = 1.3$.

Afterwards, we have performed a multifractal analysis along the same lines as we made for the price fluctuations analysis. Even though both multifractal spectra are very similar, we can verify a noticeable difference. As a matter of fact, we have obtained $\Delta\alpha = 0.28$ for $\{y(t)\}$ with shuffled volatility time series, and $\Delta\alpha = 0.24$ for the generated series that we have priory analysed, i.e., an error of 17%, see in Fig. 6. This means that superstatistics can be considered as an acceptable first approach, although models that consider long term memory in variance [47] are certainly more appropriate. Since the only source of multifractality in this case is the asymptotic power-law behaviour of the stationary PDF $P(y)$, $\{y(t)\}$ should be in fact called a *bifractal* with $\tau(z) = 0$ for $z > 5.45$.

## 3. Results for $\ell$-diagrams

A rich and interesting way of representing time series is by considering a mapping of the time series onto a plane where each point signalled is obtained by pairing elements $x_t$ and $x_{t+\ell}$ of the time series as ordinate and abcissa, respectively. These $\ell$-diagrams and related methods [51] are frequently used on studies about biological [50], and dynamical systems [51]. Moreover, they have also been introduced to study daily fluctuations of some securities [52]. This type of representation, full named *$\ell$-diagram variability method* [50], is in fact quite illustrative since it is a simple way of capturing regular aspects of systems which are apparently irregular. Such regularities can be characterised by regions which are more visited in space $x_t \times x_{t+\ell}$. Specifically, taking into account price fluctuations time series and the value of $\ell = 1$ as an example, it allows one to verify how prices evolve in segments of two time intervals. Next, we analyse the first return map of the price fluctuations. In Fig. 7 we show the plot of $r_{t+1}$ *versus* $r_t$ for some of the companies of our set[4] [38]. Plots A and D present a very interesting structure. Over the four quadrants (anticlockwise) we have got stripes with high density of points and "forbidden" regions close to the axes. We assign to transaction costs the emergence of this banned regions. In the 3rd quadrant we can see a highly visited region close to the origin, pointing that small falls in price induce small other small falls. We have investigated the probabilities for each quadrant and we have found a very peculiar behaviour for DJ30.[5] The probabilities for each quadrant (anticlockwise)

[4] The plotted companies have been chosen in order to represent different sectors of activity and ways of trading (NYSE and NASDAQ).

[5] We have also calculated these probabilities for original data – intra-day trend mask the effects observed in first return maps – and, once again, we have found the same behaviour, but with different probabilities.

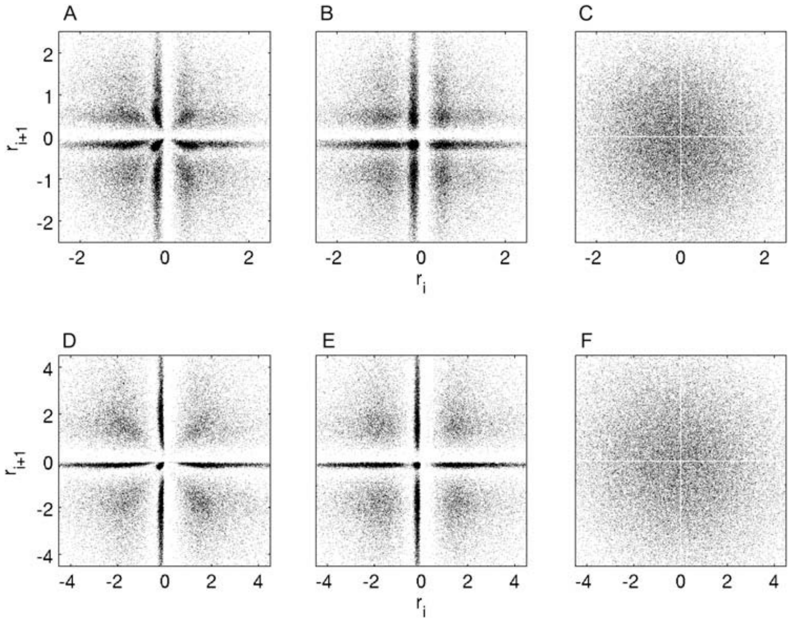

**Fig. 7.** Recurrence maps (step $s = 1$) for the companies Caterpilar (upper panels) and Intel (lower panels) for detrended data [(A) and (D)], shuffled data [(B) and (E)], and shuffled *plus* phase randomised data [(C) and (F)]. ($i \equiv t$).

**Table 1**
Probabilities for each quadrant (columns 2–4) of 30 companies of the DJ30. In column 6 is shown the difference between positives price fluctuations and negative price fluctuations, and in column 7 is shown the sum of all returns. The last two columns have been obtained from trended data. Even though most price fluctuations are negative (for most companies) though the sum is positive. Note also that there is a clear pattern for the quadrants for the 30 companies.

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $N(r_i > 0) - N(r_i < 0)$ | $\sum r_i$ |
|---|---|---|---|---|---|---|
| aa | 0.17 | 0.33 | 0.17 | 0.33 | −269 | 4.99 |
| aig | 0.18 | 0.33 | 0.17 | 0.33 | −55 | 4.16 |
| axp | 0.18 | 0.33 | 0.17 | 0.33 | −157 | 1.38 |
| ba | 0.18 | 0.33 | 0.17 | 0.33 | −262 | 4.01 |
| c | 0.18 | 0.33 | 0.16 | 0.33 | 132 | 4.80 |
| cat | 0.17 | 0.33 | 0.18 | 0.33 | −477 | 6.55 |
| dd | 0.18 | 0.33 | 0.17 | 0.33 | −130 | 5.09 |
| dis | 0.17 | 0.33 | 0.17 | 0.33 | −237 | 5.02 |
| ge | 0.17 | 0.33 | 0.17 | 0.33 | −280 | 4.71 |
| gm | 0.18 | 0.33 | 0.17 | 0.33 | −366 | 2.92 |
| hd | 0.18 | 0.33 | 0.17 | 0.33 | −113 | 0.76 |
| hon | 0.18 | 0.33 | 0.17 | 0.33 | −269 | 1.11 |
| hpq | 0.17 | 0.33 | 0.17 | 0.33 | 10 | 5.34 |
| ibm | 0.18 | 0.32 | 0.17 | 0.32 | −45 | 0.90 |
| intc | 0.19 | 0.32 | 0.18 | 0.32 | 172 | 2.23 |
| jnj | 0.17 | 0.33 | 0.17 | 0.33 | −96 | 0.57 |
| jpm | 0.17 | 0.33 | 0.17 | 0.33 | −276 | 0.76 |
| ko | 0.18 | 0.33 | 0.17 | 0.33 | −289 | 6.22 |
| mcd | 0.18 | 0.33 | 0.16 | 0.33 | 33 | 1.15 |
| mmm | 0.18 | 0.33 | 0.16 | 0.33 | −348 | 7.23 |
| mo | 0.18 | 0.33 | 0.16 | 0.33 | −82 | 6.17 |
| mrk | 0.17 | 0.33 | 0.17 | 0.33 | −298 | 1.16 |
| msft | 0.19 | 0.31 | 0.18 | 0.32 | −16 | 1.58 |
| pfe | 0.17 | 0.33 | 0.17 | 0.32 | −162 | 3.63 |
| pgn | 0.18 | 0.33 | 0.17 | 0.33 | −62 | 5.16 |
| sbc | 0.18 | 0.32 | 0.17 | 0.33 | −336 | 6.54 |
| utx | 0.18 | 0.32 | 0.17 | 0.33 | −478 | 5.85 |
| vz | 0.17 | 0.33 | 0.17 | 0.33 | −191 | 4.65 |
| wmt | 0.17 | 0.33 | 0.18 | 0.32 | −583 | 4.03 |
| xom | 0.17 | 0.33 | 0.17 | 0.33 | −452 | 6.02 |
| Average | 0.17 | 0.33 | 0.17 | 0.33 | −199 | 3.82 |

can be interpreted as follows: 1st quadrant – probability of two consecutive profits, $P_1$; 2nd quadrant – probability of a profit after a loss, $P_2$; 3rd quadrant – probability of two consecutive loss, $P_3$; 4th quadrant – probability of a loss after profit, $P_4$. These results are shown in Table 1. As it is easily observed, the dynamics of the system is basically a sequence os alternating ups and downs since the fourth and second quadrants together represent 2/3 of the points plotted on 1-diagrams, with the probabilities

**Table 2**
Fractal dimension of the $r_t \times r_{t+\ell}$ space ($\ell = 1, 2, 10, 50$) for the companies of the DJ30 estimated with Hou algorithm.

| | $d_f(\ell = 1)$ | $d_f(\ell = 2)$ | $d_f(\ell = 10)$ | $d_f(\ell = 50)$ | $d_f(\ell = 1)$ (shuf.) | $d_f(\ell = 1)$ (rand) |
|---|---|---|---|---|---|---|
| aa | 1.45 | 1.42 | 1.43 | 1.45 | 1.45 | 1.68 |
| aig | 1.28 | 1.25 | 1.26 | 1.29 | 1.40 | 1.67 |
| axp | 1.46 | 1.44 | 1.45 | 1.44 | 1.45 | 1.67 |
| ba | 1.44 | 1.45 | 1.43 | 1.44 | 1.45 | 1.67 |
| c | 1.33 | 1.31 | 1.31 | 1.31 | 1.33 | 1.67 |
| cat | 1.37 | 1.37 | 1.36 | 1.41 | 1.41 | 1.67 |
| dd | 1.45 | 1.47 | 1.47 | 1.48 | 1.47 | 1.66 |
| dis | 1.50 | 1.50 | 1.50 | 1.51 | 1.51 | 1.66 |
| ge | 1.52 | 1.53 | 1.54 | 1.51 | 1.52 | 1.66 |
| gm | 1.46 | 1.43 | 1.46 | 1.45 | 1.46 | 1.67 |
| hd | 1.37 | 1.39 | 1.37 | 1.39 | 1.36 | 1.68 |
| hon | 1.48 | 1.46 | 1.48 | 1.47 | 1.48 | 1.67 |
| hpq | 1.46 | 1.46 | 1.47 | 1.47 | 1.50 | 1.67 |
| ibm | 1.43 | 1.47 | 1.44 | 1.45 | 1.45 | 1.64 |
| intc | 1.35 | 1.40 | 1.40 | 1.43 | 1.41 | 1.69 |
| jnj | 1.36 | 1.36 | 1.37 | 1.37 | 1.41 | 1.65 |
| jpm | 1.44 | 1.44 | 1.41 | 1.44 | 1.43 | 1.67 |
| ko | 1.46 | 1.44 | 1.45 | 1.44 | 1.45 | 1.67 |
| mcd | 1.44 | 1.45 | 1.43 | 1.44 | 1.45 | 1.67 |
| mmm | 1.33 | 1.31 | 1.31 | 1.31 | 1.33 | 1.67 |
| mo | 1.37 | 1.37 | 1.36 | 1.41 | 1.41 | 1.67 |
| mrk | 1.45 | 1.47 | 1.47 | 1.48 | 1.47 | 1.66 |
| msft | 1.50 | 1.50 | 1.50 | 1.51 | 1.51 | 1.66 |
| pfe | 1.52 | 1.53 | 1.54 | 1.51 | 1.52 | 1.66 |
| pgn | 1.46 | 1.43 | 1.46 | 1.45 | 1.46 | 1.67 |
| sbc | 1.37 | 1.39 | 1.37 | 1.39 | 1.36 | 1.68 |
| utx | 1.48 | 1.46 | 1.48 | 1.47 | 1.48 | 1.67 |
| vz | 1.46 | 1.46 | 1.47 | 1.47 | 1.50 | 1.67 |
| wmt | 1.43 | 1.47 | 1.44 | 1.45 | 1.45 | 1.64 |
| xom | 1.35 | 1.40 | 1.40 | 1.43 | 1.41 | 1.69 |

of having either two consecutive profits or two consecutive losses equal to 17%, in average. Interestingly, we have verified that, although the number of negative price fluctuations surpasses the number of positive price fluctuations, $N(r_i > 0) - N(r_i < 0) = -199$ (related to the skewness of the distributions), the cumulative sum of price fluctuations yields a positive value for all equities, $\sum_t r_i(t) = 3.82$ (in average). In other words, although during the period upon analysis there was a larger number of negative price fluctuations than positive price fluctuations, the magnitude of the latter were greater so that a positive evolution arose. As a matter of fact, during this period the DJIA index increased its magnitude from 10334.16 to 10783.01, or a rise of 4.8%.

Be aware that, looking at Fig. 7, there exists a clear pattern for these probabilities. In order to further show that these characteristic patterns go beyond the uncorrelated essence of price fluctuations time series, we have performed immediate 1-diagrams for the shuffled signals. Some of the results are presented in Fig. 7 (B and E panels). Therein, it is visible that these diagrams are different from the diagrams A and C, namely the accumulation around lines $r_{t+1} = \pm r_t$ becomes less clear. Furthermore, analysing shuffled plus randomised times series which we exemplify in Fig. 7 with panels C and F. As it can be seen, these panels do not present of any pattern, forbidden stripes inclusive. Actually, both of the two latter representations are more homogeneous. In our opinion, this is a clear evidence about the importance of dependencies and non-Gaussianity on price fluctuations dynamics. At this point, it is absolutely necessary to stress that our profiles for 1-diagrams do not contradict the EMH. In other words, if one tried to make use of this property for immediate trading, transaction costs would surpass any possible (read likely) income.

Analysing $\ell$-diagrams for $\ell = 2, 4, 10$ we have verified an equal occupancy of all quadrants, $P_1 = P_2 = P_3 = P_4 = 25\%$, which indicates the loss of any predictability on the time series. To quantify the properties of $\ell$-diagrams we have carried out a fractal analysis using a modified box-counting algorithm [22,29,30]. Namely, we have mapped the space $r_t \times r_{t+1}$ onto interval $[2^0, 2^{16}]$, and we have estimated the fractal dimension of this space structure for the 30 companies. It has also been noticed that, for the majority of the companies the scale regime holds over a large range of scales. We have then used the interval $2^2$–$2^8$ to numerically obtain the fractal dimensions that are shown in Table 2. There, it is verifiable that the fractal dimension varies slight as the step ($s$) is changed, as well as after shuffling (keeping the order of the diagram) time series elements. However, it is strongly affected by phase randomisation, and, for this case, it presents values that are compatible with a two-dimensional Gaussian distribution.

## 4. Final remarks

To summarise, our results indicate that dependence and non-Gaussianity have similar weights on the multifractal features of high-frequency price fluctuations. Contrarily to some stylised facts, and especially for instantaneous volatility

[53], we have not verified a solid asymptotic power-law decay of the probability density functions, i.e., fair deviations from exponential decay. This result is substantiated by the clear approach of $\tau(z)$ curves to the theoretical curve of a independent gaussian signal when we perform a shuffling on time series elements. If we consider persistence as a major factor for multiscaling, it might be puzzling to verify that multifractality for price fluctuations is stronger than it is for magnitude price fluctuations. Such an apparent contradiction is cleared up if we take into consideration that price fluctuations PDF appears to be more fat tailed than instantaneous volatility PDF. Being like this, the former introduces a larger contribution to multiscaling. Besides, in respect of probability density functions, we have observed that a superstatistical approach to price fluctuations appears to be valid as a first approach. Still on multiscaling, we have tried to appraise the robustness of instantaneous volatility by means of measuring the effect of its possible multifractal nature on price fluctuations multifractal properties. Our results have indicated that the non-Gaussinity of instantaneous volatility (price fluctuation magnitudes) is the chief element of multifractal properties of price fluctuations. This occurs because the uncorrelated character of the signal annihilates the influence of dependencies of instantaneous volatility leading to the non-Gaussianity of latter quantity the chief role of introducing multifractality on price fluctuations time series. In this perspective, heteroskedastic (i.e., *ARCH*) approaches, within superstatistics is enclosed, to price fluctuations are validated.

Analysing $\ell$-diagrams obtained from price fluctuations time series, we have got sequences of immediate price fluctuations around Cartesian axes that are forbidden. We have attributed this fact to transaction costs. We have also observed that despite the number of negative price fluctuations is greater than the number of positive price fluctuations, the sum all returns is in fact positive, which is in accordance with both price fluctuations skewness and economical evolution. By means of a box-counting algorithm we have computed the fractal dimension of such diagrams. We have verified that, the fractal dimension varies slightly when time ordering is destroyed, and it is deeply affected by randomisation procedures. This provides an important clue on the fundamental role of non-Gaussinity of price fluctuations in several properties usually observed.

## Acknowledgements

We thank L.G. Moyano who has performed the intra-day pattern removal, as well as E.M.F. Curado and C. Tsallis for their several comments on the matters which are enclosed in this manuscript. One of us (JdS) acknowledges support of S.P. Rostirolla and F. Mancini. The code for estimating multifractal spectrum of time series was written during JdS visit to The Abdus Salam International Centre for Theoretical Physics, Trieste – Italy. The data used were provided by Olsen Data Services to whom we are also grateful. We appreciate the useful remarks from M.L. Lyra and R.L. Viana at the final stage of the work. This work has benefited from infrastructural support from PRONEX and PETROBRAS, and financial support from CNPq (Brazilian agency) and FCT/MCES (Portuguese agency).

## References

[1] Mandelbrot BB. The fractal geometry of nature. San Francisco: W.H. Freeman; 1983.
[2] Paladin G, Vulpiani A. Anomalous scaling laws in multifractal objects. Phys Rep 1987;156:147–225.
[3] Benzi R, Paladin G, Parisi G, Vulpiani A. On the multifractal nature of fully-developed turbulence and chaotic systems. J Phys A – Math General 1984;17:3521–31.
[4] Huckestein B. Scaling theory of the integer quantum hall-effect. Rev Mod Phys 1995;67:357–96.
[5] El Naschie MS. The concepts of e infinity: an elementary introduction to the Cantorian-fractal theory of quantum physics. Chaos, Solitons & Fractals 2004;22:495–511.
[6] El Naschie MS. On an eleven dimensional E-infinity fractal spacetime theory. Int J Nonlinear Sci Numer Simulat 2006;7:407–9.
[7] El Naschie MS. Feigenbaum scenario for turbulence and Cantorian E-infinity theory of high energy particle physics. Chaos, Solitons & Fractals 2007;32:911–5.
[8] El Naschie MS. E12 Lie symmetry group with 685 dimensions, KAC-Moody algebra and E-infinity Cantorian spacetime. Chaos, Solitons & Fractals 2008;38:990–2.
[9] Gell-Mann M. The quark and the jaguar. Adventures in the simple and the complex. San Francisco: W.H. Freeman; 1994;
Skjeltrop AT, Vicsek T. Complexity from the microscopic to macroscopic scales: coherence and large deviations. Dordrecht: Kluwer Academic Publishers; 2002.
[10] Peng C-K, Mietus J, Hausdorff JM, Havlin S, Stanley HE, Goldberger AL. Long-range anticorrelations and non-Gaussian behavior of the heartbeat. Phys Rev Lett 1993;70:1343–6.
[11] Hausdorff JM, Purdon PL, Peng C-K, Ladin Z, Wei JY, Goldberger AL. Fractal dynamics of human gait: stability of long-range correlations in stride interval fluctuations. J Appl Physiol 1996;80:1448–57.
[12] Burlaga LF, Vinas AF. Triangle for the entropic index $q$ of non-extensive statistical mechanics observed by Voyager 1 in the distant heliosphere. Physica A 2005;356:375–84.
[13] Bouchaud J-P, Potters M. Theory of financial risks: from statistical physics to risk management. Cambridge (MA): Cambridge University Press; 2000.
[14] Mantegna RN, Stanley HE. An introduction to econophysics: correlations and complexity in finance. Cambridge (MA): Cambridge University Press; 1999.
[15] Feder J. Fractals. New York: Plenum Press; 1988.
[16] Mandelbrot BB. The variation of certain speculative. J Business 1963;36:394–419;
Mandelbrot BB. Fractals and scaling in finance. New York: Springer; 1997.
[17] Admati AR, Pfleiderer P. A theory of intraday patterns: volume and price variability. Rev Finance Stud 1988;1:3–40.
[18] Andreadis I, Serletis A. Evidence of a random multifractal turbulent structure in the Dow Jones industrial average. Chaos, Solitons & Fractals 2002;13:1309–15;
Ivanova K, Ausloos M. Low $q$-moment multifractal analysis of Gold price, Dow Jones industrial average and BGL-USD exchange rate. Eur Phys J 1999;8:665–9;
Di Matteo T. Multi-scaling in finance. Quantitative Finance 2007;7:21–36.
[19] Kantelhardt JW, Zschiegner SA, Koscielny-Bunde E, Havlin S, Bunde A, Stanley HE. Physica A 2002;316:87–114.

[20] Ivanov PCh, Amaral LAN, Goldberger AL, Havlin S, Rosenblum MG, Stanley HG, et al. From 1/*f* noise to multifractal cascades in heartbeat dynamics. Chaos 2001;11:641–52;
Matia K, Ashkenazy Y, Stanley HE. Multifractal properties of price fluctuations of stocks and commodities. Europhys Lett 2003;61:422–8;
Ashkenazy Y, Baker DR, Gildor H, Havlin S. Nonlinearity and multifractality of climate change in the past 420,000 years. Geophys Res Lett 2003;30:2146;
Telesca L, Lapenna V, Macchiato M, Multifractal fluctuations in earthquake-related geoelectrical signals. New J Phys 2005;7:214.
[21] Kruger A. Implementation of a fast box-counting algorithm. Comput Phys Commun 1996;98:224–34.
[22] de Souza J, Rostirolla SP. A fast MATLAB® program to estimate the multifractal spectrum of multidimensional data: application to fractures (preprint, 2007).
[23] Moyano LG, de Souza J, Duarte Queirós SM. Multi-fractal structure of traded volume in financial markets. Physica A 2006;371:118–21.
[24] Duarte Queirós SM, Moyano LG, de Souza S, Tsallis C. A nonextensive approach to the dynamics of financial observables. Eur Phys J B 2007;55:161–7.
[25] Muzy JF, Bacry E, Arneodo A. Wavelets and multifractal formalism for singular signals: application to turbulence data. Phys Rev Lett 1991;67:3515–8.
[26] Oświęcimka P, Kwapień J, Drożdż S. Wavelet versus detrended fluctuation analysis of multifractal structures. Phys Rev E 2006;74:016103.
[27] Hurst HE. Long-term storage capacity of reservoirs. Trans Am Soc Civil Eng 1951;116:770–808.
[28] Peng C-K, Buldyrev SV, Havlin S, Simons M, Stanley HE, Goldberger AL. Mosaic organization of DNA nucleotides. Phys Rev E 1994;49:1685–9.
[29] Hou X-J, Gilmore R, Mindlin GB, Solari HG. An efficient algorithm for fast $O(N * ln(N))$ box counting. Phys Lett A 1990;151:43.
[30] Liebovitch LS, Toth T. A fast algorithm to determine fractal dimensions by box counting. Phys Lett A 1989;141:386–90.
[31] Meneveau C, Sreenivasan KR. Simple multifractal cascade model for fully developed turbulence. Phys Rev Lett 1987;59:1424–7.
[32] Fama EF. Efficient capital markets: a review of theory and empirical work. J Finance 1970;25:383–417.
[33] Umberger DK, Farmer JD. Fat fractals on the energy surface. Phys Rev Lett 1985;55:661–4.
[34] Feller W. Probability theory and its applications. New York: Wiley; 1950.
[35] Serletis A, Shintani M. No evidence of chaos but some evidence of dependence in the US stock market. Chaos, Solitons & Fractals 2003;17:449–54;
de Souza J, Moyano LG, Duarte Queirós SM. On statistical properties of traded volume in financial markets. Eur Phys J B 2006;50:165–8.
[36] Duarte Queirós SM. On non-Gaussianity and dependence in financial time series: a nonextensive approach. Quantitative Finance 2005;5:475–87.
[37] Granger C, Lin J-L. Using the mutual information coefficient to identify lags in nonlinear models. J Time Ser Anal 1994;15:371–84;
Borland L, Plastino AR, Tsallis C. Information gain within nonextensive thermostatistics. J Math Phys 1998;39:6490–501.
[38] Suplementary material can be found at the following URL address: <http://lanl.arxiv.org/abs/0711.2550/>.
[39] Engle RF, Patton AJ. What good is a volatility model? Quantitative Finance 2001;1:237–45;
Embrechts P, Kluppelberg C, Mikosch T. Modelling extremal events for insurance and finance (applications of mathematics). Berlin: Springer; 1997.
[40] Engle RF, Gallo GM. J. Econometrics 2006;131:3–27.
[41] Kozaki M, Sato A-H. Physica A 2007. doi:10.1016/j.physa.2007.10.023.
[42] Beck C, Cohen EGD. Superstatistics. Physica A 2003;322:267–75.
[43] Tsallis C. Nonextensive statistical mechanics, anomalous diffusion and central limit theorems. Milan J Math 2005;73:145–76.
[44] Cohen EGD. Boltzmann and Einstein: statistics and dynamics – an unsolved problem. Pramana – J Phys 2005;64:635–43.
[45] Tsallis C. Possible generalization of Boltzmann–Gibbs statistics. J Statist Phys 1988;52:479–87.
[46] Engle RF. Autoregressive conditional heteroskedasticity with estimates of the variance of UK inflation. Econometrica 1982;50:987–1008.
[47] Duarte Queirós SM. On discrete stochastic processes with long-lasting time dependence in the variance. Eur J Phys B 2008;66:161–7.
[48] Viswannathan GM, Fulco UL, Lyra ML, Serva M. The origin of fat-tailed distributions in financial time series. Physica A 2003;329:273–80.
[49] Stein EM, Stein JC. Stock price distributions with stochastic volatility: an analytic approach. Rev Finance Stud 1991;4:727–52.
[50] Babloyantz A, Maurer P. A graphical representation of local correlations in time series – assessment of cardiac dynamics. Phys Lett A 1996;221:43–55.
[51] Marwan N, Romano MC, Thiel M, Kurths J. Recurrence plots for the analysis of complex systems. Phys Rep 2007;438:237–329.
[52] Ivanova K, Ausloos M. Low-order variability diagrams for short-range correlation evidence in financial data: BGL-USD exchange rate, Dow Jones industrial average, gold ounce price. Physica A 1999;265:279–91.
[53] Liu Y, Gopikrishnan P, Cizeau P, Meyer M, Peng C-K, Stanley HE. Statistical properties of the volatility of price fluctuations. Phys Rev E 1999;60:1390–400.