

Bouncing models with a cosmological constant

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Bouncing models have been proposed by many authors as a completion, or even as an alternative to inflation for the description of the very early and dense Universe. However, most bouncing models contain a contracting phase from a very large and rarefied state, where dark energy might have had an important role as it has today in accelerating our large Universe. In that case, its presence can modify the initial conditions and evolution of cosmological perturbations, changing the known results already obtained in the literature concerning their amplitude and spectrum. In this paper, we assume the simplest and most appealing candidate for dark energy, the cosmological constant, and evaluate its influence on the evolution of cosmological perturbations during the contracting phase of a bouncing model, which also contains a scalar field with a potential allowing background solutions with pressure and energy density satisfying $p = w\epsilon$, w being a constant. An initial adiabatic vacuum state can be set at the end of domination by the cosmological constant, and an almost scale invariant spectrum of perturbations is obtained for $w \approx 0$, which is the usual result for bouncing models. However, the presence of the cosmological constant induces oscillations and a running towards a tiny red-tilted spectrum for long wavelength perturbations.

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I. INTRODUCTION

Bouncing models [1, 2] have been widely investigated as a solution of the singularity problem, and possibly as an alternative to inflation as long as it can also solve, by its own way, the horizon and flatness problems, and justify the power spectrum of primordial cosmological perturbations inferred by observations.

In the case where the contracting phase of a regular bouncing model is dominated by some matter content with a constant ratio between pressure and energy density, $p/\epsilon = w = \text{const.}$, it was shown by many authors, in different frameworks [3–8], that this matter content must be dust-like, perhaps connected to cold dark matter, in order to obtain a scale invariant spectrum of scalar and tensor cosmological perturbations.

On the other hand, since 1998 [9], cosmologists were confronted with a highly unexpected observation: the Universe is presently in a state of accelerated expansion. This may be caused by the existence of some field violating the strong energy condition, called dark energy, by a modification of general relativity at large scales, by the influence of some large scale inhomogeneities, or simply by a well suited cosmological constant. This last option is by far the simplest explanation to the present acceleration of the Universe, although it poses a problem to quantum field theory on how to accommodate its observed value with vacuum energy calculations. Anyway, the so called Λ CDM standard model assumes that there exists a cosmological constant term in Einstein's equations, which becomes dynamically important when the typical scale of the Universe has the size of the present Hubble radius.

In bouncing models without a cosmological constant, vacuum initial conditions for quantum cosmological perturbations are set in the far past of the contracting phase,

when the Universe was very big and almost flat, justifying the choice of an adiabatic Minkowski vacuum in that phase. However, if a cosmological constant is present, the asymptotic past of bouncing models will approach de Sitter rather than Minkowski spacetime. Furthermore, the large wavelengths today become comparable with the Hubble radius in the contracting phase when the Universe was still slightly influenced by the cosmological constant. Hence, the existence of a cosmological constant can modify the spectrum and amplitude of cosmological perturbations. Note that this is not a question for inflation because initial conditions for quantum perturbations and the moment of Hubble radius crossing in such models take place when the cosmological constant is completely irrelevant: the Universe is fully dominated by the inflaton field.

The aim of this paper is to investigate this issue in detail in the context of a Friedmann-Lemaître-Robertson-Walker geometry with a cosmological constant, and a scalar field with potential allowing a constant equation of state $p = w\epsilon$ for the background field, like the exponential potential in the scenario without cosmological constant. Hence, this paper can be considered as an extension of Ref. [4] through the introduction of a cosmological constant in the model. Here, as in Ref. [4], our background scenario is not intended to be a fully realistic description of the contracting phase of a bouncing model, but to yield a suitable framework to calculate the spectrum of linear cosmological perturbations in bouncing models, and to study how it depends on the presence of a cosmological constant and on the equation of state of the matter content. In our model, the bounce itself takes place at very short length scales, where the cosmological constant has no role. Hence, its presence does not modify the evolution of the background and perturbations in that period, and the descriptions provided in Refs. [3–7, 10–12]

can still be considered to be valid at the bounce. The main difference is originated from processes much before the bounce, when the initial conditions are set and the cosmological constant is not irrelevant. In that case, a Minkowski adiabatic vacuum can only be defined in a precise time domain, i.e., at the end of cosmological constant domination, but when the Universe was still very big and rarefied. However, even in this time domain, as the length scale associated with the cosmological constant, given by the present acceleration of the Universe, is not much bigger than the long wavelengths of physical interest today, the spectrum of these scales can still be slightly affected by the cosmological constant. And indeed we will show, analytically and numerically, that the usual result for bouncing models, namely, that the fluid should satisfy $w \approx 0$ in order to have an almost scale invariant spectrum of long wavelength perturbations, still holds, but the presence of the cosmological constant induces small oscillations and a small running towards a red-tilted spectrum for these scales.

In the next section, we will present the background model and obtain the equations for the evolution of cosmological perturbations on this background. In section III, we will discuss the choice of the initial state of the cosmological perturbations on this background. In section IV, we will obtain analytically and numerically the power spectrum of perturbations for the model presented in section III, and discuss its physical consequences. We end up with the conclusions.

II. THE BACKGROUND MODEL AND THE EQUATIONS FOR SCALAR PERTURBATIONS

The gravitational action we shall begin with is that of General Relativity with a cosmological constant, *i.e.*

$$\mathcal{S}_{\text{GR}} = -\frac{1}{6\ell_{\text{Pl}}^2} \int \sqrt{-g}(R + 2\Lambda)d^4x, \quad (1)$$

where $\ell_{\text{Pl}} = (8\pi G_{\text{N}}/3)^{1/2}$ is the Planck length in natural units ($\hbar = c = 1$), and Λ is the cosmological constant.

The geometry of the background is given by the spatially flat homogeneous and isotropic line element in conformal time:

$$ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j). \quad (2)$$

The matter content of the model is described by a canonical minimally coupled scalar field φ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - U(\varphi), \quad (3)$$

where the potential energy density of the scalar field is given by

$$U(\varphi) = U_0 \sinh^2\left(\frac{\varphi}{F}\right), \quad (4)$$

$$U_0 = \frac{3(1-w)H_0^2\Omega_\Lambda}{16\pi G}, \quad F = \sqrt{\frac{1}{6\pi G(1+w)}}, \quad (5)$$

and w is a constant. This potential was already studied in Refs. [13, 14] for the case $w = -1/3$.

In the case of a homogeneous and isotropic background, one can find the scalar field solution

$$\varphi(t) = \pm \sqrt{\frac{1}{6\pi G(1+w)}} \ln \left| \tanh \left[\frac{3(1+w)}{4} \sqrt{\Omega_\Lambda} H_0 t \right] \right|, \quad (6)$$

and under these conditions the energy density and pressure of the scalar field

$$\epsilon = \frac{1}{2a^2}\varphi'^2 + U(\varphi), \quad p = \frac{1}{2a^2}\varphi'^2 - U(\varphi), \quad (7)$$

satisfy $p = w\epsilon$ (a prime denotes derivative with respect to conformal time).

The Friedmann equations in conformal time read

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2\epsilon + a^2\Lambda, \quad (8)$$

$$\mathcal{H}' - \mathcal{H}^2 = -4\pi G a^2(\epsilon + p), \quad (9)$$

and ϵ satisfies the conservation equation

$$\epsilon' = -3\mathcal{H}(\epsilon + p), \quad (10)$$

where $\mathcal{H} \equiv a'/a$.

In the present situation, where the pressure and energy density of the matter content satisfy $p = w\epsilon$, with w constant, the solution for the scale factor in terms of cosmic time $dt = a d\eta$ reads,

$$a(t) = a_0 \left(\frac{\Omega_{0\omega}}{\Omega_\Lambda} \right)^{1/3(1+\omega)} \left[\sinh \left(-\frac{3\sqrt{\Omega_\Lambda}(1+\omega)H_0}{2} t \right) \right]^{2/3(1+\omega)}, \quad (11)$$

where $H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is the present Hubble parameter, $\Omega_{0\omega} \equiv \epsilon_0/\epsilon_{\text{crit}}$ with $\epsilon_{\text{crit}} \equiv 3H_0^2/(8\pi G)$, and $\Omega_\Lambda \equiv \Lambda/H_0^2$. The subscript 0 indicates the present values of the respective quantities.

Note from Eqs. (4,7) and $p = w\epsilon$ that $\dot{\varphi} = \sqrt{2(1+w)U_0/(1-w)} \sinh \varphi/F$, and the kinetic energy of the scalar field grows exponentially with φ , as usually expected for a scalar field in a contracting phase of a Friedmann model. However, the potential increases in the same way, and that is why $p = w\epsilon$ is maintained. Nevertheless, there is the question about the instability against initial conditions of this tracking between potential and kinetic energies of the scalar field in order to keep $p = w\epsilon$ in the contracting phase, which is also an issue for Ref. [4], but note that it is not necessary that this tracking must be valid all the way to the bounce: here, as in Ref. [4], we are interested in the spectrum of long wavelength perturbations, which become comparable to the Hubble radius in the contracting phase at scales not much smaller than the Hubble radius today. Hence, our calculations for the spectrum of long wavelength perturbations should hold even if the tracking $p = w\epsilon$ ceases to be valid at smaller scales. This may affect the spectrum of small wavelength perturbations, but it will not affect our forthcoming calculations for long wavelength perturbations if the tracking $p = w\epsilon$ holds at scales of the order of the Hubble radius today, when the contracting Universe is still very big. Hence, in the same way that the conclusions of Ref. [4] proved to be valid for more elaborate bouncing models, we expect that the results we will present later on will also be valid in more elaborate models containing a cosmological constant.

Note also that as $1/\sqrt{\Lambda}$ is of the order of the Hubble radius today, then all scales of physical interest had causal contact in the contracting phase of the model because the particle horizon in that phase is of the order of $1/\sqrt{\Lambda}$.

The evolution of linear scalar perturbations are described by the metric

$$g_{\mu\nu} = g_{\mu\nu}^{\text{back}} + h_{\mu\nu}, \quad (12)$$

where $g_{\mu\nu}^{\text{back}}$ represents the homogeneous and isotropic cosmological background given in Eq. (2), and the perturbations $h_{\mu\nu}$ can be decomposed into

$$\begin{aligned} h_{00} &= 2a^2\phi \\ h_{0i} &= -a^2B_{,i} \\ h_{ij} &= 2a^2(\psi\gamma_{ij} - E_{,ij}). \end{aligned} \quad (13)$$

Due to the constraint equations present in the Einstein's equations, the evolution of quantum perturbations in a classical background is described by a single quantum field, the gauge invariant Mukhanov-Sasaki variable defined by (see Ref. [15] for details)

$$v \equiv a \left(\delta\varphi + \frac{\varphi'_{\text{back}}}{\mathcal{H}} \psi \right), \quad (14)$$

where $\delta\varphi$ is the perturbed scalar field, and φ_{back} is its background solution.

The Mukhanov-Sasaki variable satisfies the equation,

$$v'' - \nabla^2 v - \frac{z''}{z} v = 0, \quad (15)$$

where

$$z \equiv \frac{a^2 \sqrt{4\pi G_N (\epsilon + p)}}{\mathcal{H}}. \quad (16)$$

The equations above are not altered by the presence of a cosmological constant, essentially because Eqs. (9,10) are not modified by its presence and because, of course, $\delta\Lambda = 0$.

In our choice of units a is dimensionless, hence we will define the dimensionless conformal time $\tilde{\eta} \equiv \eta/R_H$, where $R_H \equiv 1/(a_0 H_0)$ is the co-moving Hubble radius. From now on we will omit the tilda over η . We will also work with the dimensionless comoving wavenumber $k \equiv R_H/\lambda$, where λ is the comoving wavelength of the perturbation modes. The region corresponding to long wavelengths today is the interval $1 < k < 10^3$.

Taking the model with scalar field and scale factor given by Eqs. (6) and (11), respectively, one obtains that

$$z(t) = -\sqrt{\frac{3(1+w)}{2}} \frac{a(t)}{\cosh(\gamma t)}, \quad (17)$$

and

$$V(t) \equiv \frac{z''}{z} = \frac{\Omega_\Lambda a^2}{a_0^2} \left\{ \frac{(1-3w)}{2} \left[\frac{1}{\sinh^2(\gamma t)} - \frac{(1+3w)}{2} \right] - \frac{9(1+w)^2}{2 \cosh^2(\gamma t)} \right\}, \quad (18)$$

where

$$\gamma \equiv \frac{3\sqrt{\Omega_\Lambda}(1+\omega)H_0}{2}, \quad (19)$$

and a is given by Eq. (11).

Solution (11) is defined in two domains: $-\infty < t < 0$ and $0 < t < \infty$. The first one describes a universe contracting from an asymptotic de Sitter spacetime in the far past to a singularity at $t = 0$. The second one describes a universe expanding from a singularity at $t = 0$ to an asymptotic de Sitter expansion in the far future. Around $t = 0$, the field dominates the dynamics, and the cosmological constant is unimportant. These behaviours can be viewed by taking the limits e.g. , in the contracting solution, $t \rightarrow -\infty$ and $t \rightarrow 0^-$ in Eq. (11).

For $t \rightarrow -\infty$, Eq. (11) yields $a(t) \approx \exp(-\sqrt{\Lambda}t)$. In conformal time

$$\eta + \eta_\infty = \left(\frac{4}{\Omega_{0w}}\right)^{1/3(1+\omega)} \frac{\exp(\sqrt{\Lambda}t)}{\Omega_\Lambda^{(1+3w)/3(1+\omega)}}, \quad (20)$$

where $-\eta_\infty < \eta \ll 0$, and η_∞ is a positive constant, the scale factor behaves as

$$a(\eta) = \frac{a_0}{\sqrt{\Omega_\Lambda}(\eta + \eta_\infty)}. \quad (21)$$

This is the usual de Sitter behaviour. In this case, the potential (18) reads

$$V(\eta) \equiv \frac{z''}{z} \approx \frac{9w^2 - 1}{4(\eta + \eta_\infty)^2}, \quad (22)$$

yielding the equation

$$v_k'' + \left[k^2 - \frac{(9w^2 - 1)}{4(\eta + \eta_\infty)^2} \right] v_k = 0. \quad (23)$$

This equation is completely equivalent to an equation for a massive scalar field in a de Sitter spacetime, with mass given by

$$m = \frac{3\sqrt{\Lambda}}{2} \sqrt{1 - w^2}. \quad (24)$$

Its general solution reads

$$v_k = \sqrt{\eta} \left[b_1(k) H_\nu^{(1)}(k(\eta + \eta_\infty)) + b_2(k) H_\nu^{(2)}(k(\eta + \eta_\infty)) \right], \quad (25)$$

where the $H_\nu^{(1,2)}$ are Hankel functions of first and second kind, $\nu = 3w/2$. As $k(\eta + \eta_\infty) \ll 1$, we can write this solution as

$$v_k \approx c_1(k)(\eta + \eta_\infty)^{(1+3w)/2} + c_2(k)(\eta + \eta_\infty)^{(1-3w)/2}. \quad (26)$$

For $t \rightarrow 0^-$, or $\eta \rightarrow 0^-$, one obtains from Eq. (11) that $a(t) \propto t^{2/[3(1+w)]}$ or, in conformal time, $a(\eta) \propto \eta^{2/(1+3w)}$. This is the usual Friedmann evolution for $p = w\epsilon$ without a cosmological constant. In this regime, we obtain

$$V(\eta) \equiv \frac{z''}{z} \approx \frac{2(1-3w)}{(1+3w)^2\eta^2}. \quad (27)$$

In this situation, $z \propto a$ (see Eq. (17)) and $z''/z = a''/a$.

Note that the potential $V(t)$ diverges to $\pm\infty$ at the infinity past for $w > 1/3$ and $w < 1/3$, respectively, and diverges to $\pm\infty$ near the singularity at $t \propto \eta \approx 0$ for $w < 1/3$ and $w > 1/3$, respectively. Hence, it must cross zero in the middle of the line $-\infty < t < 0$. In Fig. 1 we present the behaviours of the potential $V(t)$ for $t < 0$ in the cases $w < 1/3$, $w = 1/3$ and $w > 1/3$.

Our idea is that the singularity at $t = 0$ separating the contracting and expanding solutions can be eliminated through some new physics which produces a regular bounce connecting these two phases. As in the region around $t = 0$ the cosmological constant is unimportant, one can evoke the bounce descriptions provided e.g. , in Refs. [3–7, 10–12]. For instance, the quantum cosmological bounces with a perfect fluid studied in Refs. [7, 10–12] present a regular scale factor given by

$$a(T) = a_b \left[1 + \left(\frac{T}{T_b}\right)^2 \right]^{\frac{1}{3(1-w)}}, \quad (28)$$

where $dT = a^{1-3w}d\eta$, and a_b and T_b are positive constants. Note that for $|T| \gg T_b$, this solution approaches the classical Friedmann solution for a perfect fluid given by the limit $t \rightarrow 0^-$ of Eq. (11): $a(\eta) \propto \eta^{2/(1+3w)}$. Hence, the scale factor (11) can be smoothly connected to the scale factor (28).

It was shown in Ref. [12] that the potential present in the equations for the perturbations around these quantum bounces reads

$$V(T) \equiv \frac{a''}{a} = \frac{2 a_b^{3(1-w)}}{3(1-w)T_b^2} \left[\frac{1}{a^{1+3w}} - \frac{2}{3} \left(\frac{T}{T_b}\right)^2 \frac{a_b^{3(1-w)}}{1-w} a^{-4} \right], \quad (29)$$

where Eq. (28) has been used. Their shapes are presented in Fig. 2 for the cases $w < 1/3$, $w = 1/3$ and $w > 1/3$, and they tend to the potential $V(t)$ of Eq. (27) for $\gamma t \ll 1$ in the limit $|T| \gg T_b$, but they do not diverge in $\eta = 0$. Hence, when these quantum effects become important near $\eta = 0$ inducing the bounce, the two disjoint parts of the classical potentials presented in Fig. 1 corresponding to the contracting and expanding classical universes separated by a singularity can now be softly connected with the potentials presented in Fig. 2. Then one can evolve smoothly the perturbations from the contracting phase to the expanding phase, and calculate their properties in the present era. For other regular bouncing models, the

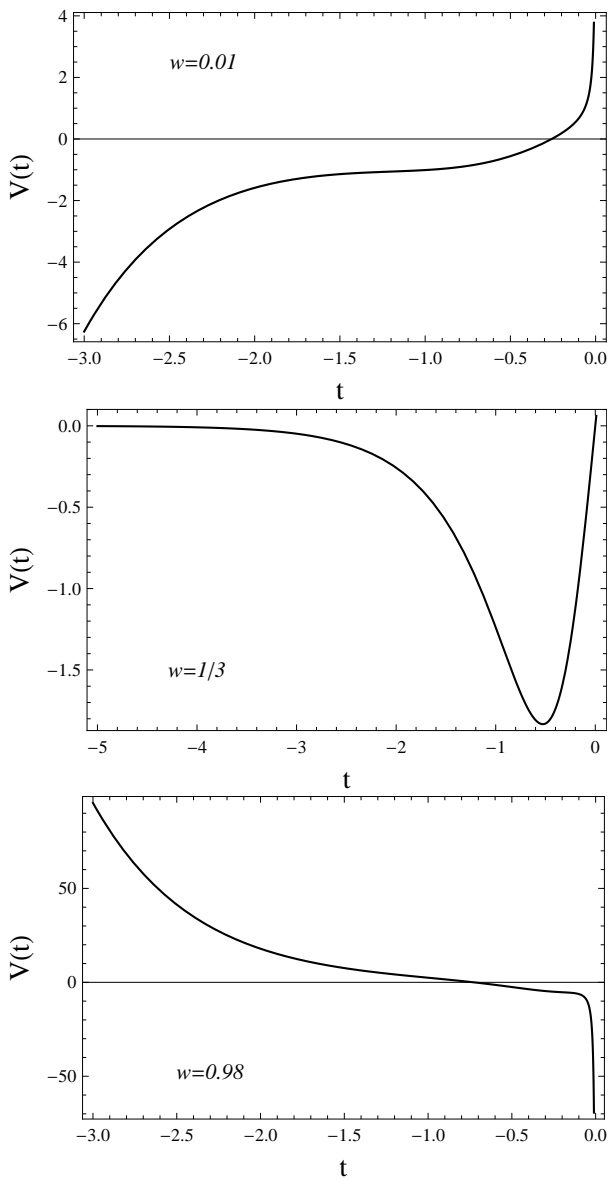


FIG. 1: Behaviour of the potential $V(t)$ given by Eq. (18) for three different values of w .

situation must be similar, although somewhat more intricate in the case there is an extra field which induces the bounce.

In order to accomplish this program, one must set the initial conditions for the perturbations in the past contracting phase. Without the cosmological constant, the Universe tends, in the far past, to Minkowski spacetime, where the potentials become null (see Fig. 2). Hence, an adiabatic Minkowski vacuum can be prescribed there. In the presence of the cosmological constant, neither the Universe tends to Minkowski spacetime in the far past (in fact, it tends to de Sitter spacetime), nor the potential becomes null there (see Fig. 1, except for $w = 1/3$, which is not physically interesting because it does not

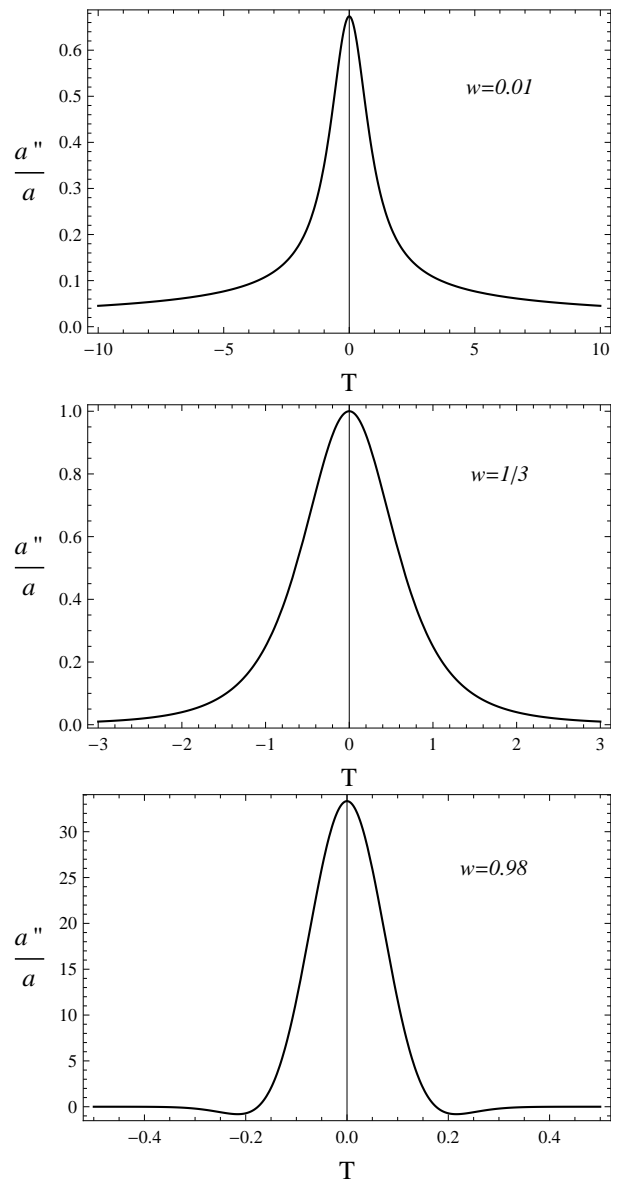


FIG. 2: Behaviour of the quantum bouncing potential given by Eq. (28) for three different values of w .

yield an almost scale invariant spectrum of perturbations). However, as we have shown above, the potential crosses zero somewhere in the middle of its evolution (which coincides with the moment when physically interesting long wavelengths perturbation modes become smaller than the Hubble radius), and perhaps one could define an adiabatic Minkowski vacuum there. We will show in the next section that this is indeed possible for the scalar field model presented above.

III. THE CHOICE OF INITIAL STATE FOR THE QUANTUM PERTURBATIONS

In this section we will check whether an adiabatic Minkowski vacuum can indeed be prescribed in the time interval when the potential (18) is negligible and the Universe is starting to be dominated by the field. In this regime the scale factor is approaching the form $a(\eta) \propto \eta^{2/(1+3w)}$. Taking $\Omega_{0\omega} \approx 0.3$ and $\Omega_\Lambda \approx 0.7$, the zero of this potential occurs when $y = y_V \equiv \gamma t_V$ (see Eq. (19) for the definition of γ). As y_V depends on the ω parameter, we will consider the following values for $\omega < 1/3$:

$$\begin{aligned} y_V(\omega \approx 0) &\approx -0.34, \\ y_V(\omega \approx 1/8) &\approx -0.23, \\ y_V(\omega \approx 1/4) &\approx -0.13. \end{aligned}$$

Around these points, one can numerically approximate Eq. (15), now expressed in terms of the modes $v_k(\eta)$, through

$$\frac{d^2 v_k}{dx^2} + (k^2 + \beta x)v_k = 0, \quad (30)$$

where

$$\beta \approx -\frac{3(1+w)\sqrt{\Omega_\Lambda}}{2} \left(\frac{a}{a_0} \frac{dV}{dy} \right) \Big|_{y_V}, \quad (31)$$

yielding

$$\begin{aligned} \beta(\omega \approx 0) &\approx -1.05, \\ \beta(\omega \approx 1/8) &\approx -1.65, \\ \beta(\omega \approx 1/4) &\approx -2.31. \end{aligned}$$

and $x \equiv \eta - \eta_V$, with η_V being the dimensionless conformal time corresponding to y_V defined above.

It is important to remark the dependence of β with Ω_Λ by looking at Eqs. (11) and (18). Noting that y_V is independent of Ω_Λ , and as $\Omega_{0\omega} + \Omega_\Lambda = 1$, one gets

$$\beta = \frac{\Omega_\Lambda^{(1+3w)/[2(1+w)]} (1 - \Omega_\Lambda)^{1/(1+w)}}{0.7^{(1+3w)/[2(1+w)]} 0.3^{1/(1+w)}} \beta_{0.7}, \quad (32)$$

where $\beta_{0.7}$ are the values of β for $\Omega_\Lambda = 0.7$. One can see that $\beta \rightarrow 0$ as $\Omega_\Lambda \rightarrow 0$.

Note also that, although y_V is independent of Ω_Λ , η_V depends on Ω_Λ as

$$\eta_V = \frac{0.7^{(1+3w)/[6(1+w)]} 0.3^{1/[3(1+w)]}}{\Omega_\Lambda^{(1+3w)/[6(1+w)]} (1 - \Omega_\Lambda)^{1/[3(1+w)]}} \eta_{V(0.7)}, \quad (33)$$

where, again, $\eta_{V(0.7)}$ are the values of η_V for $\Omega_\Lambda = 0.7$. One can see now that $\eta_V \rightarrow \infty$ as $\Omega_\Lambda \rightarrow 0$, as expected. In this last calculation, we have assumed that the field dominates at η_V .

The adiabatic vacuum is defined by the solution

$$v_k(x) = \frac{1}{2[\Omega_k(x)]^{1/2}} \exp \left[-i \int^x \Omega_k(x') dx' \right], \quad (34)$$

where $\Omega_k(x)$ must satisfy the equation

$$\Omega_k^2 = f_k^2 - \frac{1}{2\Omega_k} \frac{d^2 \Omega_k}{dx^2} + \frac{3}{4\Omega_k^2} \left(\frac{d\Omega_k}{dx} \right)^2, \quad (35)$$

and $f_k^2 \equiv k^2 + \beta x$.

Order by order, one has:

$$\begin{aligned} (\Omega_k^{(0)})^2 &= f_k^2; & (\Omega_k^{(2)})^2 &= f_k^2 \left(1 + \frac{5}{16} \frac{\beta^2}{f_k^6} \right) \\ (\Omega_k^{(4)})^2 &= f_k^2 \left[1 + \frac{5}{16} \frac{\beta^2}{f_k^6} - \frac{490}{256} \left(\frac{\beta^2}{f_k^6} \right)^2 \right], \end{aligned} \quad (36)$$

where the upper indices (n) denote the order of the approximation. Hence, an adiabatic Minkowski vacuum can be obtained if the parameter expansion β^2/f_k^6 satisfies $\beta^2/f_k^6 \ll 1$. In fact, as β is of order unity, $x \ll 1$, and the long wavelengths of physical interest satisfy $1 < k < 10^3$, the condition $\beta^2/f_k^6 \approx \beta^2/k^6 \ll 1$ is satisfied. Note that for the largest scales (k approaching 1), deviations from the Minkowski vacuum become more significant, and one should expect modifications against the standard results.

This problem can be presented under another point of view. A Minkowski vacuum can be defined for quantum perturbations with wavelengths much smaller than the Hubble radius, defined by $\mathcal{R}_H(t) \equiv 1/H(t)$. From Eq. (11), one obtains that

$$\mathcal{R}_H(t) = \sqrt{\frac{1}{\Lambda}} \tanh(-y). \quad (37)$$

One has to compare this quantity with the physical wavelength $\lambda_{\text{phys}} = a\lambda$ which, from Eq. (11), reads

$$\lambda_{\text{phys}} = \lambda_{\text{phys}}^{\text{now}} \left(\frac{\Omega_{0\omega}}{\Omega_\Lambda} \right)^{1/3(1+\omega)} \sinh^{2/3(1+\omega)}(-y). \quad (38)$$

The maximum value of \mathcal{R}_H , at $t \rightarrow -\infty$, is $\mathcal{R}_H^{\text{max}}(t) = \Lambda^{-1/2}$, while λ_{phys} diverges there. Eventually, they can be comparable at some time in the contracting phase. As in the case this is true one expects to obtain a similar spectrum as the one obtained in bouncing models without a cosmological constant, we will concentrate on the case $0 < w \ll 1$, which yields an almost scale invariant spectrum. The quantities defined in Eqs. (37) and (38) are comparable when

$$\Omega_{0\omega}^{1/3} \Omega_\Lambda^{1/6} \frac{\lambda_{\text{phys}}^{\text{now}}}{\mathcal{R}_H^{\text{now}}} \approx \frac{\sinh^{1/3}(-y)}{\cosh(-y)}, \quad (39)$$

where $\mathcal{R}_H^{\text{now}}$ is the Hubble radius today. As $\sinh^{1/3}(-y)/\cosh(-y) < 0.73$, $\Omega_{0m}^{1/3}\Omega_{0\Lambda}^{1/6} \approx 0.63$, and $10^{-3} < \lambda_{\text{phys}}^{\text{now}}/\mathcal{R}_H^{\text{now}} < 1$, this equality can be achieved for some finite domain of y .

Note also from Eq. (39) that this domain interval of y could be extended to large values of $|y|$ if Ω_Λ were much smaller than our prescribed values. This can also be seen from the analysis coming from the potential, where a smaller Ω_Λ would result in a smaller β in Eq. (36), and an adiabatic vacuum could also be achieved for smaller values of k .

IV. SPECTRUM OF QUANTUM COSMOLOGICAL PERTURBATIONS

Let us now calculate the spectrum of quantum cosmological perturbations for this scenario. In Section III, we have shown that an adiabatic Minkowski vacuum, for the case of a canonical scalar field, could be prescribed in the time interval where the potential becomes negligible and the Universe is starting to be dominated by the field.

Hence, substituting the zero order term ($\Omega_k^{(0)}$) given in Eq. (36) in the solution (34) of Eq. (30), we obtain

$$v_k(x) \approx \frac{1}{2(k^2 + \beta x)^{1/4}} \exp\left[-\frac{2ik^3}{3\beta}\left(1 + \frac{\beta x}{k^2}\right)^{3/2}\right], \quad (40)$$

and the initial conditions are given by

$$v_k(0) \approx \frac{1}{2\sqrt{k}} \exp\left(-\frac{2ik^3}{3\beta}\right), \quad (41)$$

$$\left.\frac{dv_k}{dx}\right|_{x=0} \approx -v(0)\left(\frac{\beta}{4k^2} + ik\right). \quad (42)$$

We have, therefore, to solve

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0, \quad (43)$$

with initial conditions given by Eqs. (41) and (42).

We calculated numerically the solution of Eq. (43) by changing the time variable from η to y , defining a new function $u_k \equiv a^{1/2}v_k$, and setting the above initial conditions at $y_{\text{ini}} = y_V$. Taking the potential (18), the transformed equation reads

$$\frac{d^2 u_k}{dy^2} + \left\{ \frac{4k^2}{9(1+w)^2[\Omega_\Lambda^{(1+3w)/2}\Omega_{0w}]^{2/[3(1+w)]}\sinh^{4/[3(1+w)]}(-y)} - \frac{w^2}{(1+w)^2} + \frac{w}{(1+w)^2\sinh^2(y)} + \frac{2}{\cosh^2(y)} \right\} u_k = 0. \quad (44)$$

The results are shown in Fig. 3.

The solutions of equation (43) can be expanded in powers of k^2 according to the formal solution [16]

$$\frac{v}{z} \simeq A_1(k) \left[1 - k^2 \int^\eta \frac{d\bar{\eta}}{z^2(\bar{\eta})} \int^{\bar{\eta}} z^2(\bar{\eta}) d\bar{\eta} + \dots \right] + A_2(k) \left[\int^\eta \frac{d\bar{\eta}}{z^2(\bar{\eta})} - k^2 \int^\eta \frac{d\bar{\eta}}{z^2(\bar{\eta})} \int^{\bar{\eta}} z^2(\bar{\eta}) d\bar{\eta} \int^{\bar{\eta}} \frac{d\bar{\eta}}{z^2(\bar{\eta})} + \dots \right], \quad (45)$$

In Eq. (45), the coefficients $A_1(k)$ and $A_2(k)$ are two constants depending only on the wavenumber k through the initial conditions.

Since expansion (45) is valid at all times during contraction, the $A_1(k)$ and $A_2(k)$ dependencies coming from the initial conditions hold when the Universe enters the field dominated phase just before performing the bounce. From then on, the evolution is guided by the particular physics of the bounce. For instance, in the quantum bounce with potentials shown in Fig. 2, everything goes as described in Ref. [7], with power spectrum in the expanding phase given by

$$\mathcal{P} \propto k^3 |A_2(k)|^2, \quad (46)$$

and spectral index

$$n_s = 1 + \frac{d \ln(\mathcal{P})}{d \ln(k)}. \quad (47)$$

This is a general feature of bouncing models: the spectrum $A_2(k)$ of the growing mode of the contracting phase, which will be the decaying mode of the expanding phase after the bounce, is transferred to the growing mode of the expanding phase and dominates over $A_1(k)$. Hence, to find the spectrum after the bounce one has to obtain $A_2(k)$, which, in the case without cosmological constant, reads $A_2(k) \propto k^{\frac{3(\omega-1)}{2(3\omega+1)}}$ (see e.g., Ref. [7]), yielding the spectral index

$$n_s = 1 + \frac{12\omega}{1+3\omega}, \quad (48)$$

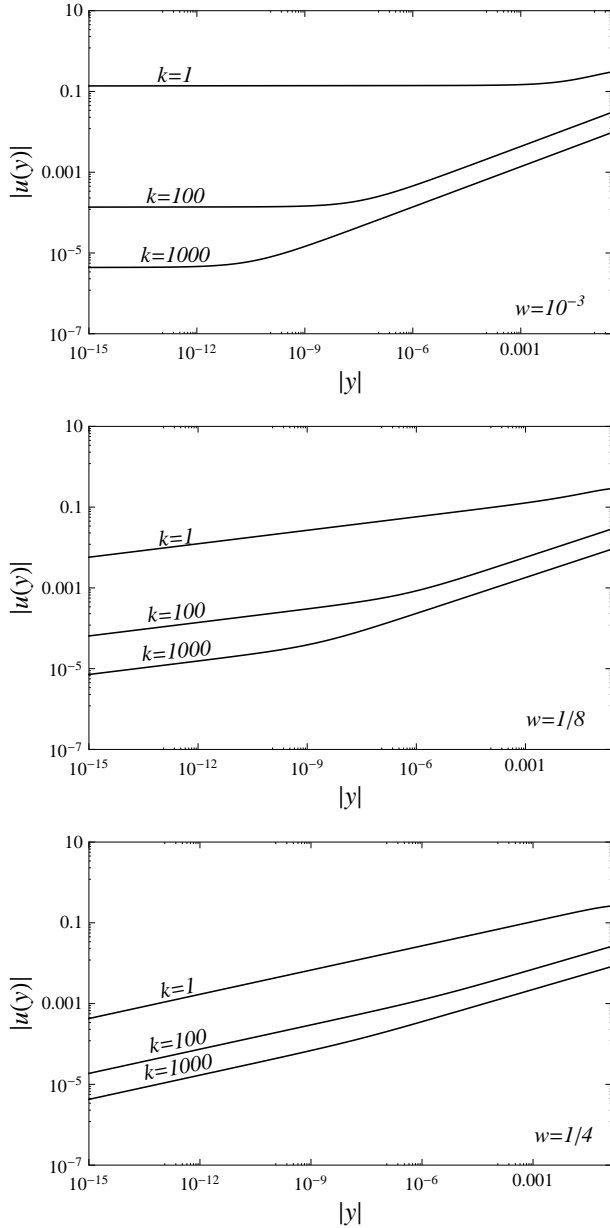


FIG. 3: Numerical results of Eq. (44) for $w = 10^{-3}, 1/8$ and $1/4$. Each curve shows $u_k(y)$ as a function of y for different values of k , as indicated in the graphs.

with $w \approx 0$ for an almost scale invariant spectrum.

In order to analytically predict the behaviour of the coefficient $A_2(k)$ for the case with a cosmological constant, we have taken two approximate solutions of Eq. (43) close to the point where the potential vanishes, where it has the form given in Eq. (30), and matched them at a point $x_* = (\eta_* - \eta_V)$, where $0 < x_* \ll 1$.

The first approximate solution comes from Eq. (40), and is given by

$$v_k(x_*) \approx \frac{1}{2\sqrt{k}} \left(1 - \frac{\beta x_*}{4k^2}\right) \exp \left[-i \left(\frac{2k^3}{3\beta} + kx_* \right) \right], \quad (49)$$

while its first derivative reads

$$\begin{aligned} \frac{dv_k}{dx} \Big|_{x=x_*} &\approx -\frac{1}{2\sqrt{k}} \left[\frac{\beta}{4k^2} + ik \left(1 + \frac{\beta x_*}{4k^2}\right) \right] \\ &\times \exp \left[-i \left(\frac{2k^3}{3\beta} + kx_* \right) \right]. \end{aligned} \quad (50)$$

The second approximate solution comes from the remark that at η_* and afterwards, up to the bounce phase, the evolution of the background is dominated by the scalar field, where the potential approaches the form given in Eq. (27), yielding the solution

$$v_k(\eta) = \sqrt{\eta} \left[C_1(k) H_\nu^1(k\eta) + C_2(k) H_\nu^2(k\eta) \right], \quad (51)$$

where $\nu \equiv 3(1-w)/2(1+3w)$. In the domain $|k\eta_*| \gg 1$ this solution reads

$$v_k(\eta_*) \approx \frac{B_1(k)}{2\sqrt{k}} \left[1 - \frac{\alpha_1}{2ik\eta_*} \right] + \frac{B_2(k)}{2\sqrt{k}} \left[1 + \frac{\alpha_1}{2ik\eta_*} \right], \quad (52)$$

and

$$\frac{dv_k}{dx} \Big|_{\eta=\eta_*} \approx \frac{1}{2\sqrt{k}} \left\{ B_1(k) \left[ik - \frac{\alpha_1}{2\eta_*} \right] - B_2(k) \left[ik + \frac{\alpha_1}{2\eta_*} \right] \right\}, \quad (53)$$

where

$$B_1(k) \equiv 2C_1(k) \exp \left[i \left(k\eta_* - \frac{\pi\nu}{2} - \frac{\pi}{4} \right) \right], \quad (54)$$

$$B_2(k) \equiv 2C_2(k) \exp \left[-i \left(k\eta_* - \frac{\pi\nu}{2} - \frac{\pi}{4} \right) \right], \quad (55)$$

$$\alpha_1 \equiv \frac{\Gamma(\nu + 3/2)}{\Gamma(\nu - 1/2)} = \frac{2(1-3w)}{(1+3w)^2}. \quad (56)$$

Performing the matching between Eqs. (49,50) and Eqs. (52,53) at η_* , one gets for $C_1(k)$ and $C_2(k)$,

$$C_1(k) \approx \left(\frac{i\beta}{16k^3} - \frac{\beta x_*}{8k^2} \right) \exp \left[-i \left(\frac{2k^3}{3\beta} + kx_* + \alpha_2 \right) \right] \quad (57)$$

and

$$\begin{aligned} C_2(k) &\approx \frac{1}{2} \left(1 + \frac{i\alpha_1}{2k\eta_*} - \frac{i\beta}{8k^3} \right) \\ &\times \exp \left[-i \left(\frac{2k^3}{3\beta} + kx_* - \alpha_2 \right) \right], \end{aligned} \quad (58)$$

where

$$\alpha_2 \equiv k\eta_* - \frac{\pi\nu}{2} - \frac{\pi}{4}. \quad (59)$$

In the limit $k\eta \rightarrow 0^-$, just before the new physics which generates the bounce, the solution $v_k(\eta)$ approximately

reads

$$\begin{aligned}
v_k(\eta) &\equiv A_1(k)\eta^{1/2+\nu} + A_2(k)\eta^{1/2-\nu} \\
&\approx \sqrt{\eta} \left\{ \left(\frac{k\eta}{2} \right)^\nu \frac{1}{\Gamma(\nu+1)} \left[C_1(k) + C_2(k) + \right. \right. \\
&\quad \left. \left. + i[C_1(k) - C_2(k)] \cot(\nu\pi) \right] + \right. \\
&\quad \left. + \left(\frac{k\eta}{2} \right)^{-\nu} \frac{i[C_2(k) - C_1(k)]}{\Gamma(1-\nu)\sin(\nu\pi)} \right\}, \quad (60)
\end{aligned}$$

and the coefficient $A_2(k)$ of the growing mode of the contracting phase is given by

$$\begin{aligned}
|A_2(k)|^2 &\approx \frac{1}{4} \left(\frac{k}{2} \right)^{-2\nu} \frac{1}{[\Gamma(1-\nu)\sin(\nu\pi)]^2} \\
&\times \left[1 + \frac{\beta x_*}{2k^2} \cos(2\alpha_2) + \frac{\alpha_1^2}{4k^2\eta_*^2} - \frac{\beta}{4k^3} \sin(2\alpha_2) \right]. \quad (61)
\end{aligned}$$

Calculating the spectral index as defined in Eq. (47), we find

$$\begin{aligned}
n_S &= 1 + \frac{12w}{1+3w} - \frac{\beta x_* \eta_*}{k} \sin(2\alpha_2) \\
&- \frac{2}{k^2} \left[\frac{\alpha_1^2}{4\eta_*^2} + \frac{\eta_* \beta}{4} (1+2x_*) \cos(2\alpha_2) + \frac{\eta_* \beta}{4} \cos(2\alpha_2) \right] \\
&+ \frac{3\beta}{4k^3} \sin(2\alpha_2). \quad (62)
\end{aligned}$$

Substituting the parameters $w \approx 0$, $\beta \approx -1.05$, $|\eta_*| \approx |\eta_V| \approx 2.19$, $x_* \ll 1$, $\alpha_1 \approx 2$, $\alpha_2 \approx k\eta_* - \pi$, and $1 < k < 10^3$ in Eq. (62), we obtain an almost scale invariant spectrum. Besides the usual $12w/(1+3w)$ result, there are additional terms in Eq. (62) inducing a running red-tilted spectrum and oscillations, both decreasing with k . Note that for a vanishing cosmological constant we have $\beta \approx 0$ and $|\eta_*| \rightarrow \infty$. In this case, the extra terms in Eq. (62) disappear and $n_S \rightarrow 1$ even for small values of k .

In order to check numerically this analytic calculation, we took the following steps: from the numerical solutions $u_k = a^{1/2}v_k$ presented in Fig. 3, we obtained v_k , evaluated it at very small y ($y \approx -10^{-15}$), expressed the result in conformal time (whose relation with y is trivial at field domination), multiplied the result by $\eta^{-\nu-1/2}$ (see Eq. (60)), and differentiated the final result with respect to η in order to isolate $A_2(k)$. The results are shown in Figs. 4 and 5.

It can be seen that $A_2(k)$ and n_S follow the predicted behaviour, $A_2(k) \propto k^{3(w-1)/2(1+3w)}$ and $n_S \approx 1 + \frac{12w}{1+3w}$, for $k \gtrsim 1$. We have also verified that small oscillations and a red-tilted running with amplitudes decreasing with k are superimposed to the power law overall behaviour, as predicted in Eqs. (61) and (62). Note from Fig. 5 that the oscillations do indeed become smaller for smaller Ω_Λ , showing that they are a consequence of the presence of the cosmological constant.

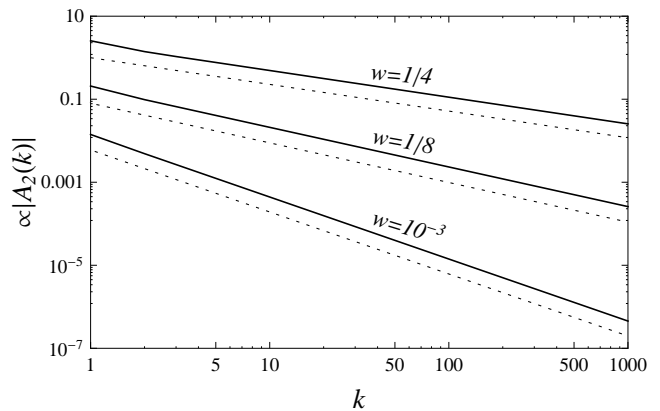


FIG. 4: Numerical results for the behaviour of $|A_2(k)|$ for $w = 10^{-3}, 1/8$ and $1/4$ evaluated at $y = -10^{-15}$. The solid lines show the numerical results and the dotted ones show, for comparison, a curve proportional to $k^{\frac{3(w-1)}{2(3w+1)}}$.

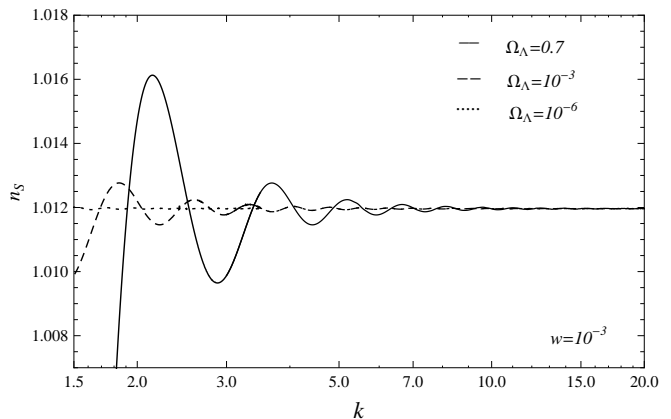


FIG. 5: Numerical results for $n_S(k)$ evaluated at $y = -10^{-15}$, obtained using $w = 10^{-3}$. The solid line indicates the result obtained using $\Omega_\Lambda = 0.7$, the dashed line for $\Omega_\Lambda = 10^{-3}$ and the dotted line for $\Omega_\Lambda = 10^{-6}$. Note that the oscillations become smaller for smaller Ω_Λ , showing that they are due to the presence of the cosmological constant. This result is in agreement to Eq. (62).

V. CONCLUSION

In this paper we have investigated the effects of the presence of a cosmological constant in the contracting phase of a bouncing model. It turns out that the initial vacuum state, usually determined in the contracting phase when the Universe was very large and rarefied, is affected by the presence of the cosmological constant. In order to get an almost scale invariant spectrum, one still must have some phase with dust-like contraction, but now the spectral index gets a red-tilted running and oscillations directly caused by the cosmological constant. It is interesting to realize that bouncing models allow such

an important role to the cosmological constant in the physics of primordial cosmological perturbations, which is not at all the case for always expanding models. This opens a new area of research, which is to investigate the influence of other models of dark energy on the primordial spectrum of bouncing models.. In other words, if the Universe had really bounced in the past, investigating its primordial spectrum can yield information about dark energy.

There is also the question about the possibility of an enormous growth of perturbation amplitudes in the deflationary contraction in the far past of the model, as discussed in other contexts [17]. Note, however, that the cosmological constant in our model is small and this almost de Sitter deflation will take place when the Universe was very large compared to the present Hubble radius, and for a fixed time interval. In conformal time, this time interval can be estimated using Eq. (20) by saying that it should be smaller than $\eta_* + \eta_\infty$ given by this equation when $t \approx -1/\sqrt{\Lambda}$. Taking the usual values $\Omega_\Lambda = 0.7$ and $\Omega_{0w} = 0.3$, one can see that $\eta_* + \eta_\infty < 1$. Note that for a cosmological constant dominated model ($\Omega_{0w} \approx 0$), one would obtain $\eta_* + \eta_\infty \gg 1$. Let us examine the behaviour of the Bardeen potential in this phase. From Eq. (26) we obtain that

$$\Phi \propto \frac{(\epsilon + p)a^2}{k^2 \mathcal{H}} \left(\frac{v}{z}\right)' \approx c_1(\eta + \eta_\infty)^3 + \frac{c_2}{k^2}(\eta + \eta_\infty) \quad (63)$$

for $w = 0$. Hence, as this almost de Sitter deflationary phase will not take long enough in conformal time, $\eta_* + \eta_\infty < 1$, because of the smallness of the cosmological constant, perturbations will not grow alarmingly in this epoch. Now, once the Universe leaves this deflationary contraction to a non-deflationary contraction when it is still very large, then it can be subjected to dissipation effects, as the ones discussed in Ref. [7], which could dissipate the existing inhomogeneities. Another approach to this problem should be to think in terms of the Anthropic Principle and state that the Universe is composed fundamentally by a small cosmological constant (dark energy) and some matter content as the one used in our model (dark matter?). In many regions it will expand to de

Sitter and it will freeze, in some it will contract inhomogeneously, and in a few homogeneous regions within one particle horizon size it may contract to make a bounce, where new particles (photons and baryons) will be created, and expand again to a Universe with some galaxies and stars where intelligent life can exist. The results of our paper can then be applied to this last possibility, the only one which can interest us. Of course these tentative answers to this basic question must be worked out more precisely, but we think that a final and complete answer to the issue on why the primordial Universe was so homogeneous, in any cosmological scenario, demands a theory of initial conditions, perhaps quantum cosmology. One interesting investigation should be a quantum cosmological analysis of eternal asymptotically (in time) de Sitter models.

Our approach here was concentrated on general features of the spectrum of cosmological perturbations in the presence of a cosmological constant in general bouncing models, and because of that we were not able to fix the amplitude of the perturbations. Our next step will be to take a specific model in which the physics at the bounce fixes the bounce scale, and hence the amplitude of the perturbations, in order to determine the influence of the new features of the spectrum of primordial perturbations we obtained in this paper in the anisotropies of the microwave background radiation, and to compare the results with observations. Another interesting problem should be to investigate the situation where the required dust-like contraction was not caused by a scalar field but by a hydrodynamical fluid with $c_s^2 = w$. In this case, the prescription of an adiabatic vacuum can be much more involved because of the smallness of the sound horizon.

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