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UCB Campus Universitário II
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SGAN 916 Módulo B – Asa Norte
70790-160 – Brasília/DF
Fone/Fax.: 55 (61) 3448-7127
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UNIVERSIDADE CATÓLICA DE BRASÍLIA
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SUMÁRIO

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Analysing and modelling finance within nonextensive statistical mechanics formalism

Abstract: In this article we revisit results about financial quantities, namely return and traded volume, which have been obtained within the non-extensive statistical mechanics based on entropy $S_q = k\left(1-\sum_{i=1}^{W} p_i^q\right)/(q-1) \quad (q \in \Re)$.
These results have been derived from both numerical (empirical) and analytical considerations. Concerning, the former, properties such as multi-fractality, self-correlation, and degree of dependence, have permitted not only the establishment of a basis for the application of non-extensive statistical mechanics on the treatment of financial variables, but also the achievement of relations between typical statistical properties and entropic indices q . Pertaining to the latter, dynamical scenarios of differential stochastic equations have taken into account. In these cases, dynamical interpretations for index q have been presented.

PACS: 05.45.Tp — Time series analysis; 89.65.Gh — Economics, econophysics, financial markets, business and management; 05.40.-a — Fluctuation phenomena, random processes, noise and Brownian motion.

Keywords: Financial markets, Nonextensivity statistical mechanics.

Sílvio M. Duarte Queirós*

* Centro Brasileiro de Pesquisas Físicas, 150, 22290-180, Rio de Janeiro-RJ, Brazil.
E-mail address sdqueiro@cbpf.br

1. Introduction

The word *physics* has its root in the Greek word physike which means *"of Nature"*. For that reason, if Nature is defined as the set of everything that forms Universe, interactions between its elements, and phenomena produced within it, then, we could study everything inside physical science. This comprises phenomena which occur at so different scales such as interactions between the most basic elements of matter, and dynamics of Galaxies. Between these two poles apart, we certainly find phenomena at human scale, specifically, interaction between humans, or organisations controlled by them. In this minute sub-set of Universe we can place financial markets, which are ruled, at least, by decisions of buying/selling given amounts of financial instruments at some price.

To a physicist, two ingredients are highly appealing in a financial market: the complexity of this type of system, and the time series profile of large part of financial quantities, very similar to Einstein's Brownian motion. Regarding that, simple questions can be asked about financial markets: What is the distribution for the price fluctuations of some financial instrument? Which dynamical mechanisms originate that distribution? How these mechanisms evolve in time? Questions like these have actually been raised up by physicists, particularly, by those working on Statistical Physics which is based on concept of probability, and it has as goal the description of natural complex behaviour [1–4]. This description is done from simple laws which depend on parameters that define systems.

Statistical Physics, specifically, statistical mechanics, is strongly attached to the concept of entropy originally introduced by Rudolf Julius Emmanuel Clausius in 1865 and its relation to the number of allowed microscopic states introduced by Ludwig Boltzmann around 1877 when he was studying the approach to equilibrium of an ideal gas [5]. This connection can be expressed as

$$S = k \ln W, \tag{1}$$

where k is a positive constant, and W is the number of microstates compatible with the macroscopic state of an isolated system. This equation, known as *Boltzmann principle*, is one of the cornerstones of standard statistical mechanics.

When the system is not isolated, but instead in contact to some large reservoir, it is possible to extend Eq. (1), under some assumptions, and obtain the *Boltzmann-Gibbs entropy*

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \,, \tag{2}$$

where p_i is the probability of the microscopic configuration i [6]. The Boltzmann principle should be derivable from microscopic dynamics, since it refers to microscopic states, but the implementation of such calculation has not been yet achieved. So, Boltzmann-Gibbs (BG) statistical mechanics is still based on hypothesis such as the molecular chaos [5] (*Stosszahlansatz*) and ergodicity [7]. In spite of the lack of an actual fundamental derivation, BG statistics has been undoubtedly successful in the treatment of systems in which *short* spatio/temporal interactions dominate[1]. For such cases, ergodicity and (quasi-) independence are favoured and Khinchin's approach to S_{BG} is valid [7]. Therefore, it is entirely feasible that other physical entropies, in addition to the BG one, can be defined in order to properly treat anomalous systems, for which the simplifying hypothesis of ergodicity and/or independence are not fulfilled. Examples are: metastable states in long-range interacting Hamiltonian dynamics, metaequilibrium states in small systems (*i.e.*, systems whose number of particles is much smaller than Avogrado's number), glassy systems, some types of dissipative dynamics, and other systems that, in some way, violate ergodicity. This includes systems with non-Markovian memory (i.e., long-range memory) like it seems to be the case of financial ones. Generically speaking, systems that might have a multi-fractal, scale-free or hierarchical structure in the occupancy of their phase space.

Inspired by this kind of systems, specifically multi-fractals, it was proposed in 1988 the entropy [9]

1 To give the reader an idea of the difficulty of the problem, it is worth to say that it took around one century so that the ergodic hypothesis could be verified for the simplest of the systems, the ideal gas [8].

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \qquad (q \in \Re), \tag{3}$$

which generalises S_{BG} ($\lim_{q\to 1} S_q = S_{BG}$), as the basis of a possible generalisation of BG statistical mechanics [10,11]. The fundamental ideas was to introduce a bias amid the set of probabilities by raising probability to some power q. In other words, since $0<p_i<1$, $q<1$ benefits less probable events, $p_i^q > p_i$ because while $q>1$ enhances most probable events inasmuch as $p_i^q > p$. The value of the *entropic index* q for a specific system is to be determined *a priori* from microscopic dynamics. Just like S_{BG} , S_q is *nonnegative*, *concave*, *experimentally robust* (or *Lesche-stable* [12]) ($\forall q>0$), and leads to a *finite entropy production per unit time* [13]. Moreover, it has been recently shown [14] that it is also *extensive*, i.e.,

$$S_q\left(A_1 + A_2 + ... + A_N\right) \simeq \sum_{i=1}^{N} S_q\left(A_i\right) , \tag{4}$$

for special kinds of *correlated* systems, more precisely when the phase-space is occupied in a scale-invariant form. By being *extensive*, for an appropriate value of q, S_q complies with Clausius' concept on macroscopic entropy, and with thermodynamics.

Since its proposal, entropy (3) has been the source of several results in both fundamental and applied physics, as well as in other scientific areas such as biology, chemistry, economics, geophysics and medicine [15]. Herein, we review some new results concerning applications to the dynamics and empirical analysis of financial markets observables, namely the price fluctuations and traded volumes. Specifically, we introduce stochastic dynamical mechanisms which are able to reproduce some features of quantities such as the probability density functions (PDFs) and the Kramer-Moyal moments. Moreover, we present some results concerning the return multi-fractal structure, and its relation to sensitivity to initial conditions.

Our dynamical proposals are faced against empirical analysis of 1 minute returns and traded volumes of the 30 companies that were used to compose the Dow Jones Industrial Average (DJ30) between the 1st July and the 31st December 2004. In order to eliminate specious behaviours we have removed the well-known intra-day pattern following a standard procedure [16]. After that, the return values have been subtracted from its average value and expressed in standard deviation units, whereas the traded volumes are expressed in mean traded volume units.

2. Generating probability density function from S_q

Before dealing with specific financial problems, let us analyse the probability density function which emerges when the variational principle is applied to S_q [11].

Let us consider its continuous version, i.e.,

$$S_q = k\frac{1-\int\left[p(x)\right]^q dx}{1-q}. \tag{5}$$

The natural constraints in the maximisation of (5) are

$$\int p(x)\,dx = 1, \tag{6}$$

corresponding to normalisation, and

$$\int x\, P_q(x)\frac{\left[p(x)\right]^q}{\int\left[p(x)\right]^q dx}\,dx \equiv \langle x\rangle_q = \bar{\mu}_q, \tag{7}$$

$$\int \left(x-\bar{\mu}_q\right)^2 P_q(x)\frac{\left[p(x)\right]^q}{\int\left[p(x)\right]^q dx}\,dx \equiv \left\langle\left(x-\bar{\mu}_q\right)^2\right\rangle_q = \bar{\sigma}_q^2, \tag{8}$$

corresponding to the *generalised* mean and variance of x, respectively [11]. The notation $P_q(x)$ equal $\frac{\left[p(x)\right]^q}{\int\left[p(x)\right]^q dx}$ and it is known in the literature as escort probability. It is worth mention that averages weighted with a function of the escort probability allow to mimic the way individuals behave when they face to risky choices. The prospect theory of D. Kahneman and A. Tversky to analysing decision-making under risk is founded on the concept of decision weights that can be modelled analogous functional forms [17].

From the variational problem using (5) under the above constraints, we obtain

$$p(x) = A_q\left[1+(q-1)B_q\left(x-\bar{\mu}_q\right)^2\right]^{\frac{1}{1-q}}, \quad (q<3), \tag{9}$$

where,

$$A_q = \begin{cases} \frac{\Gamma\left[\frac{5-3q}{2-2q}\right]}{\Gamma\left[\frac{2-q}{1-q}\right]} \sqrt{\frac{1-q}{\pi} B_q} & \Leftarrow \quad q<1 \\ \frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2q-2}\right]} \sqrt{\frac{q-1}{\pi} B_q} & \Leftarrow \quad q>1 \end{cases}, \tag{10}$$

and

$$B_q = \left[(3-q)\,\bar{\sigma}_q^2\right]^{-1}. \tag{11}$$

Standard and generalised variances, $\bar{\sigma}^2$ and $\bar{\sigma}_q^2$ respectively, are related by

$$\bar{\sigma}_q^2 = \bar{\sigma}^2 \frac{5-3q}{3-q}. \tag{12}$$

Defining the q *-exponential* function as

$$e_q^x \equiv \left[1+(1-q)x\right]^{\frac{1}{1-q}} \quad \left(e_1^x \equiv e^x\right), \tag{13}$$

($e_q^x = 0$ if $1+(1-q)\,x \leq 0$) we can rewrite PDF (9) as

$$p(x) = A_q\, e_q^{-B_q\left(x-\bar{\mu}_q\right)^2}, \tag{14}$$

hereafter referred to as q *-Gaussian*.

For $q = \frac{3+m}{1+m}$, the q -Gaussian form recovers the Student's t -distribution with m degrees of freedom (m = 1, 2, 3,...) with finite moment up to order m^{th}. So, for $q>1$, PDF (14) presents an asymptotic *power-law* behaviour. On the other hand, if $q = \frac{n-4}{n-2}$ with n = 3, 4, 5,... , $p(x)$ recovers the r -distribution with n degrees of freedom. Consistently, for $q<1$, $p(x)$, has a *compact support* which is defined by the condition $\left|x-\bar{\mu}_q\right| \leq \sqrt{\frac{3-q}{1-q}\bar{\sigma}_q^2}$.

Many other entropic forms have been introduced in the literature for various interesting purposes. One of the advantages of entropy (3) is that it yields power-law tails, which play a particularly relevant role, as well known.

Let us recall succinctly the two basic central limit theorems: (1) A convoluted distribution with a finite second moment approaches, in the limit of $N \to \infty$ convolutions, a Gaussian attractor; (2) A convoluted distribution with a divergent second moment, approaches, in the same limit, a Lévy distribution $L_\gamma(x)$ (with $0<\gamma<2$) [18]. However, through dynamics different from the convolution one, for instance with some sort of memory across successive steps (i.e., *non-independence* of the successive distributions), different ubiquitous distributions might arise (see also [14]). In this specific case the recent introduction of a correlation mechanism, by imposing Liebnitz rule to a set of probabilities based on the q-product, which has allowed the establishment of a new attractor in probability space different from Lévy and Gauss [19]. This work has turned out to be the first non-extensive generalisation of Central Limit Theorem which embraces a q generalisation of symmetric Lévy distributions [20]. The relation between Liebnitz rule and q -Gaussian distributions has also been verified in binomial processes for option pricing [21].

3. Defining consistent testing from non-extensive statistical mechanics

Discrimination between two hypothesis, *consistent testing*, is ubiquitous in science. Examples are the stationary/non-stationary character of time series or the dependence degree between its elements. Concerning the latter, the most widely applied measure of "dependence" between variables is the correlation function mathematically defined as,

$$C[x, y] = \frac{\langle x\,y\rangle - \langle x\rangle\langle y\rangle}{\sqrt{\langle x^2\rangle - \langle x\rangle^2}\sqrt{\langle y^2\rangle - \langle y\rangle^2}}.$$

Since the correlation function is basically a normalised covariance (or the second cumulant of the stochastic process), it will only be a suitable statistical procedure for linear correlations or correlations that can be written in a linear way. In other words, the correlation function is not able to determine conveniently non-linearities in a given group of data. Aiming to consistently test the dependence or independence of stochastic variables within non-extensive statistical mechanics framework, and using as starting point the Kullback-Leibler information measure [22],

$$KL = -\int p(y)\ln\frac{p'(y)}{p(y)}dy, \quad (15)$$

which is customarily applied in finance [23], it was defined in Ref. [24] a *non-extensive generalised mutual information measure*,

$$I_{q'} = -\int p(y)\ln_{q'}\frac{p'(y)}{p(y)}dy, \quad (16)$$

where $\ln_{q'}(y) = \frac{y^{q'-1}-1}{q'-1}$ ($\ln_{q'}(y) = \ln_1(y)$). For $q'=1$, Eq. (16) is equivalent to the Eq. (15), *i.e.*, $KL=I_1$.

Consider that y is a two-dimensional random variable $y=(x,z)$In this situation, the quantification of the degree dependence between x and z can be made by computing I_q for $p(x, z)$ and $p'(x,z) = p_1(x)p_2(z)$, where $p_{...}(...)$ represents the marginal probability. In the analysis of two-dimensional random variables, $I_{q'}$ presents both a lower bound and an upper bound. The former, $I_{q'}^{MIN} = 0$, corresponds to *total independence* between x and z, *i.e.*, $p(x,z)= p'(x,z)$. The latter, $I_{q'}^{MAX}$, represents a *one-to-one dependence* between variables and is given by,

$$I_{q'}^{MAX} = -\iint p(x,z)\Big[\ln_{q'} p_1(x) + (1-q)\ln_{q'} p_1(x)\ln_{q'} p_2(z)\Big]dx\,dz.$$

Existence of these two extreme values has allowed the definition of a normalised measure,

$$R_{q'} = \frac{I_{q'}}{I_{q'}^{MAX}} \in [0,1], \quad (17)$$

which has an *optimal* index, q^{op} (where the prime was suppressed for clarity).

This index is optimal in the sense that the gradient of the measure R is most sensitive and hence most capable of determine variations in the dependence among the variables. Moreover, it is optimal because its two extreme values are associated to full dependence and full independence between x and z. Analytically, q^{op} is determined as the inflection point of $R_{q'}$ *vs* q' curves. For one-to-one dependence we have $q^{op}=0$, and $q^{op}=\infty$ for total independence (see reference [25] for a detailed discussion).

4. Basic stochastic processes

The Gaussian distribution, recovered in the limit $q\rightarrow 1$ of expression (14), can be derived on a variety of grounds. For instance, it has been derived, through arguments based on the dynamics, by L. Bachelier in his 1900 work on price changes in Paris stock market, and also by A. Einstein in his 1905 article on Brownian motion. In particular, starting from a Langevin dynamics, we are able to write the corresponding Fokker-Planck equation and, from it, to obtain, as solution, the Gaussian distribution. Analogously, it is also possible, from certain classes of stochastic differential equations and their associated Fokker-Planck equations, to obtain the distribution given by (14).

In the remaining of this section it is our objective to discuss dynamical mechanisms which lead to probability functions with asymptotic power-law behaviour of q-Gaussian form.

A. *Stochastic differential processes*

Stochastic differential dynamics containing multiplicative noise might be encountered in many dynamical processes and, due to its significance, it has been subject of several studies over the last decades. A special and very peculiar kind of stochastic differential process with multiplicative noise was in introduced in [28] by L. Borland,

$$dr = \sqrt{2D}\left[p(r,t)\right]^{(1-q)/2}dW_t, \quad (18)$$

which, when q equal 1 becomes the traditional Einstein equation for Brownian motion (W_t is a regular Wiener process). Specifically, for this case, the diffusion coefficient, or *volatility* in financial jargon, is proportional to the probability of having a certain value, x, for the stochastic variable at time t.

Stochastic dynamics (18) has as associated Fokker-Planck equation,

$$\frac{\partial p(r;t)}{\partial t} = \frac{\partial^2}{\partial r^2}\left\{\theta\left[p(r;t)\right]^{(2-q)}\right\}, \quad (19)$$

which corresponds to a well-kown *Nonlinear Fokker-Planck Equation* whose solution is

$$p(x,t) = \frac{1}{Z_q(t)}\left[1+(q-1)\beta_q(t)x^2\right]^{\frac{1}{1-q}}, \quad (20)$$

where

$$Z_q(t)=\left\{\left[Z_q(0)\right]^{3-q}+2(2-q)(3-q)D\beta_q(0)\left[Z_q(0)\right]^2 t\right\}^{1/(3-q)} \quad (21)$$

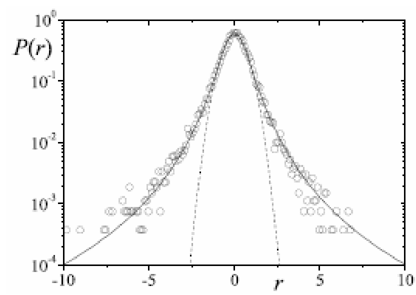
and

$$\beta_q(t)=\beta_q(0)\left[\frac{Z_q(0)}{Z_q(t)}\right]^2 \quad (22)$$

From eq. (20), we verify that the distribution presents always the same asymptotic power law exponent, $\frac{2}{1-q}$, and that variance (squared volatility), which is proportional to $\beta(t)^{-1}$, evolves as $t^{\frac{2}{3-q}}$, in the limit $t\to\infty$. This power-law dependence has recently been tested for price fluctuations in São Paulo financial market and has proved to be rather appropriate [29].

Taking into account that empirical returns where found to follow a q -Gaussian distribution (see fig. 1), Eqs. (18) and (19) provide a simple mechanism to model the dynamics of prices. Along similar lines it has been worked out an option-pricing model which is more realistic than the celebrated Black-Scholes one (recovered as the q=1 particular case) [30][31].

Figure 1 Probability density function of returns, P(r), versus return, r. The symbols represent P(r) for the Dow Jones Industrial daily return index from 1900 until 2003. The solid line represents the best q-Gaussian numerical adjust with q=1.54 and $\sigma_q^2 = 0338$ (as obtained in [32]) and the dashed line a Gaussian fit.



B. Stochastic processes with varying intensive parameters

Intricate dynamical behaviour is a common feature of many systems, which can be also characterized by power-law probability density functions. Inside this class we have systems whose dynamical behaviour shows spatio/temporal fluctuations of some intensive quantity. This quantity might be, in traditional physical systems, the inverse temperature, the energy dissipation, whereas in financial models, it could be the width of some white noise, as assumed in the famous Heston model [33]. The connection between this type of dynamics and non-extensive entropy was made for the first time by G. Wilk and Z. Włodarczyk [34] and later extended by C. Beck and E.G.D. Cohen[35], who called it superstatistics. In this "statistics of statistics", Beck and Cohen aimed to treat non-equilibrium systems from the point of view of long-living stationary states characterised by a temporally or spatially fluctuating intensive parameter. Such a condition can be mathematically expressed by

$$B[E(z)] = \int_0^\infty f(\beta)e^{-\beta E(z)}\, d\beta, \quad (23)$$

where $B[E(z)]$ is a kind of effective Boltzmann factor, $E(z)$ a function of some relevant variable z, and $f(\beta)$ the probability density function of the inverse temperature β. Superstatistics is intimately connected with non-extensive statistical mechanics. More precisely, it is possible to derive a generalised Boltzmann factor which is exactly $B[E]$, when $f(\beta)$ is the Gamma distribution, *i.e.*,

$$e_q^{-\beta' E(z)} = \int \frac{e^{-\beta/b}}{b\Gamma[c]}\left(\frac{\beta}{c}\right)^{c-1} e^{-\beta/b},$$

where the q-exponential functional form of the effective Boltzmann factor turns out clearly visible its asymptotic power-law behaviour. It is noteworthy that the above effective Boltzmann factor is also a good approximation for other $f(\beta)$ probability density functions [35].

5. Applications to financial observables

A. ARCH (1) and GARCH (1,1) processes from a non-extensive perspective

The fluctuating character of volatility in financial markets has been considered, since a few decades ago, as a key factor for price changes dynamics [36]. In fact, the intermittent character of return time series is usually associated to localised bursts in volatility and thus called volatility *clustering* [37]. The temporal evolution of the second-order moment, known as *heteroskedasticity* [38], has proven to be of extreme importance in order to define enhancing option-price models [33,39,40].

The first proposal aiming to modelise and analyse economical time series with time-varying volatility was made by R.F. Engle [38], who defined the autoregressive conditional heteroskedasticity (ARCH) process. In his seminal article, Engle defined a heteroskedastic observable z (e.g., the return) as

$$z_t = \sigma_t\,\omega_t, \tag{24}$$

where ω_t represents an independent and identically distributed stochastic process with null mean and unitary variance ($\langle\omega_t\rangle=0$, $\langle\omega_t^2\rangle=1$) associated to a probability density function $P_n(\omega)$, and σ_t the volatility. In the same work, Engle also suggested a simple dynamics for volatilities, a linear dependence of σ_t^2 on the n previous values of z_t^2,

$$\sigma_t^2 = a + \sum_{i=1}^{n} b_i\, z_{t-i}^2, \quad (a, b_i > 0), \tag{25}$$

later named as *ARCH*(n) linear process [41]. The *ARCH*(n) process is uncorrelated, and for n=1, it presents an exponentially decaying self-correlation function for volatility with a characteristic time of order $|\log b_1|^{-1}$ [42].

In order to give a more flexible structure to the functional form of σ_t^2, T. Bollerslev generalised Eq. (25) defining the *GARCH*(n,m) process[43]

$$\sigma_t^2 = a + \sum_{i=1}^{n} b_i\, z_{t-i}^2 + \sum_{i=1}^{s} c_i\,\sigma_{t-i}^2, \quad (c_i > 0), \tag{26}$$

which reduces to *ARCH*(n) process, when $c_i=0, \forall_i$.

For the *GARCH*(1,1) (b_1=b and $c_1 \equiv c$), we can straightforwardly determine the k^{th} moment of the *stationary* probability density function $P(z)$, particularly its second moment

$$\bar{\sigma}^2 \equiv \langle z_t^2\rangle = \langle\sigma_t^2\rangle = \frac{a}{1-b-c}, \qquad (b+c)<1,$$

and the fourth moment, which equal kurtosis ($k_x \equiv \frac{\langle x^4\rangle}{\langle x^2\rangle^2}$),

$$\langle z_t^4\rangle = k_z = k_\omega\left(1 + b^2\,\frac{k_\omega - 1}{1-c^2-2bc-b^2k_\omega}\right),$$

$$\left(c^2+2bc+b^2k_\omega<1\right),$$

for processes with unitary variance, i.e., $\bar{\sigma}^2 = 1$. The self-correlation function for volatilities generated from a *GARCH*(1,1) process is also of exponential kind with a decaying characteristic time of $|\log(b+c)|^{-1}$.

Despite, *ARCH*-like processes fail to reproduce the power-law-like volatility self-correlation function decay [44], they are still considered a cornerstone in econometrics due to their simplicity and satisfactory efficiency in financial time series mimicry.

Having a glance at Eq. (24), we can verify that the distribution $P(z)$ of the stochastic variable z has, at each time step t, the same functional form of the noise distribution, $P(\omega)$, but with a standard deviation (or volatility) σ_t. This property allows one to look at process $\{z\}$ as a process similar to those occurring in some non-equilibrium systems with a long-lasting stationary state. Specifically, this principle has allowed to establish, firstly for *ARCH*(1) [45] and then for *GARCH*(1,1) [46], a connection between b and c, $P(z)$ and, $P_n(\omega)$, the latter assumed to be of the following q_n-Gaussian form

$$P_n(\omega) = \frac{A_{q_n}}{\left[1+\frac{q_n-1}{5-3q_n}\omega^2\right]^{\frac{1}{q_n-1}}} =$$

$$A_{q_n}\, e_{q_n}^{-\omega^2/(5-3)\,q_n}, \qquad \left(q_n < \frac{5}{3}\right), \tag{27}$$

By making the ansatz $P(z) \cong p(z)$, where $p(z)$ is the q-Gaussian probability density function which maximises S_q (Eq. (3)), and by imposing the matching of second ($\bar{\sigma}^2 = 1$) and fourth order moments, it is possible to establish, for *GARCH*(1,1), a relation containing the dynamical parameters b and c and entropic indexes q and q_n:

$$b(5-3q_n)(2-q) = \sqrt{(q-q_n)-\left[(5-3q_n)(2-q)-c^2(5-3q)(2-q_n)\right]} - c(q-q_n). \tag{28}$$

For c = 0, Eq. (28) reduces to the one corresponding to *ARCH*(1),

$$q = \frac{q_n + 2b^2(5-3q_n)}{1+b^2(5-3q_n)} \tag{29}$$

which we depict in fig. 3. for $b = c = 0$, one has $q = q_n$. The validity of Eqs. (28) and (29) is depicted in fig. 2. The discrepancy between $p(z)$ and $P(z)$ can be evaluated by computing the sixth-order moment percentual difference, which is never greater than 3% [45,46]. In fig. 4 we illustrate eq. (28) for typical values of q_n.

Since $\omega_t = z_t/\sigma_t$ and $\langle\omega_t\,\sigma_t\rangle=0$, for q_n=1 we can write

$$p(z\,|\,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{z^2}{2\sigma^2}},$$

as the conditional probability density function of z given σ_2. Considering that,

$$p(z) = \int_0^\infty p(z\,|\,\sigma^2)\, P_\sigma(\sigma^2)\, d(\sigma^2),$$

and $P(z) \cong p(z)$, we obtain the stationary probability density function for squared volatility[46],

$$P_\sigma\left(\sigma^2\right)=\frac{\left(\sigma^2\right)^{-1-\lambda}}{\left(2\kappa\right)^{\lambda}\Gamma[\lambda]}\,e^{-\frac{1}{2\kappa\sigma^2}},$$

where

$$\lambda=\frac{1}{q-1}-\frac{1}{2},\qquad \kappa=\frac{1-q}{\bar{\sigma}^2\left(3q-5\right)}.$$

Figure 2 Typical runs and $GARCH(1,1)$ with $\bar{\sigma}^2=1$. For each plot the symbols represent the relative frequency, $F(z)$, and the line the corresponding probability function, $\int_{z-\delta}^{z+\delta} p\left(\tilde{z}\right)d\tilde{z}$. Upper panels ($q_n$=1): (0.1;0.88), q=1.287 (x^2=4.59×10^{-10}) (left);(0.4; 0.4), q = 1.38 (x^2 = 3.22 × 10^{-7}) (right). Lower panels (qn = 1.2): (0.1; 0.5), q = 1.221 (x^2 = 6.01 × 10^{-10}) (left); (0.3; 0.45), q = 1.35 (x^2 = 7.36 × 10^{-9}) (right); .

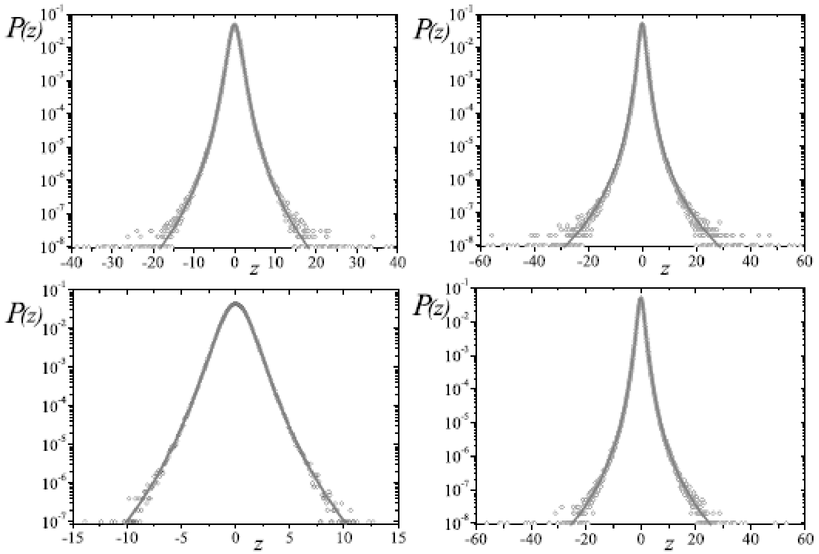


Figure 3 Diagram (q, qn, b) for ARCH (1) processes with $\langle z^2\rangle$ = 0. When b = 0 we obtain the straigth line q = q_n.

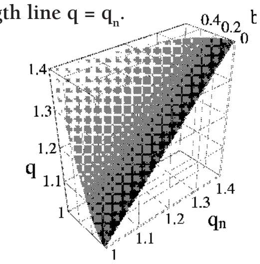


Figure 4 Diagrams (q, b, c) of GARCH (1, 1) processes with z2_= 1 and qn = 1 (Left panel), qn = 1.2 (Right panel). No ´ultimo caso o valor m´aximo poss´ıvel para b (com c = 0) ´e b = 1 p4.2' 0.488

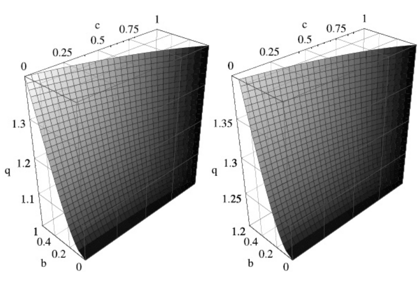
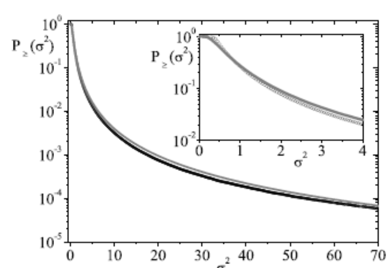


Figure 5 The symbols in black represent the inverse cumulative frequency, C (σ^2), numerically obtained for a Gaussian noise with b=c=0.4 and the gray line the respective inverse cumulative distribution, $\int_{\sigma^2}^{\infty} P_\sigma\left(\tilde{\sigma}^2\right)d\left(\tilde{\sigma}^2\right)$ with ($\kappa,\lambda,\bar{\sigma}^2$) = (0.444,2.125,1) for $P_\sigma\left(\tilde{\sigma}^2\right)$.



As one can observe in figure 5, the ansatz gives a quite satisfactory description for σ^2 probability density function, suggesting a connection between the *ARCH* class of processes and non-extensive entropy. These explicit formulas can be helpful in applications related, among others, to option prices, where volatility forecasting plays a particularly important role.

Albeit uncorrelated, stochastic variables $\{z_t\}$ are *not* independent. In other words, they present correlations which cannot be written in a purely linear fashion. Applying the q-generalised Kullback-Leibler relative entropy [24,25] to *stationary joint probability density function* $p_1(z_t,z_{t-1})$ and

$$p_2(z_t,z_{t-1}) \equiv p(z_t)p(z_{t-1}) = \left[p(z)\right]^2,$$

it is possible to quantify the degree of dependence between successive returns, through an optimal entropic index, q^{op}. In Ref. [46], it has been verified the existence of a direct relation between dependence, q^{op}, the non-Gaussianity, q, and the nature of the noise, q_n. An interesting property has emerged, namely that, whatever the pair (b,c), which results in a certain q for the stationary probability density function, we obtain the same q^{op} and, consequently, the time series will present the same degree of dependence [46]. Furthermore, the relation between q and q^{op} does not depend on the correlation function. Expressly, although pairs (b=0.2, c=0.2) and (b=0.4, c=0) present the same characteristic time for z_t^2 correlation function, they do not present the same dependence degree. In point of fact, we verify that pair (b=0.4, c=0), which presents a greater value for q, it also has a greater dependence degree (smaller value for q^{op}). The same comparison can be made with pairs (b=0.3; c=0) and (b=0.1, c=0.2) pointed at fig. 7. This shows that R_q is not a measure of linear correlations.

Figure 6 Plot of q^{op} *versus* q for typical (b, c, q_n) triplets. The arrow points two examples which were obtained from *different* triplets and nevertheless coincide in what concerns the resulting point (q,q^{op}).

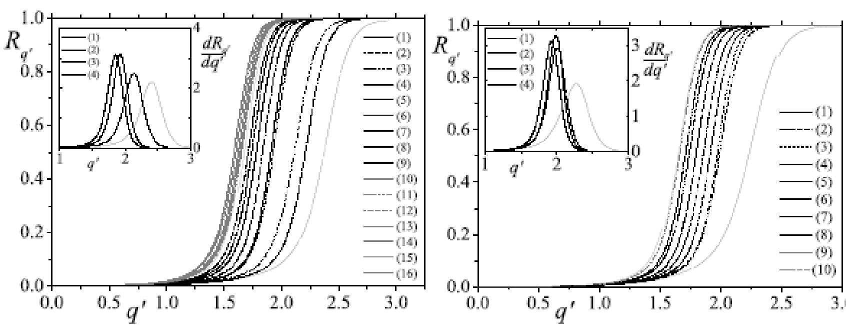


Figure 7 Representation of q^{op} vs. q for (q_n,b,c) values exhibited in fig. 2. One of the pairs indicates an example of different triplets, (1,0.4,0) e (1,0.2,0.6888) which, although they have different characteristic times for z^2 self-correlation function, present the same value for (q, q^{op}) (discrepancy beyond third decimal digit). The other two triplets (in addition to the pair cited in the text) present the same characteristic times for z^2 self-correlation function. The fact that (q, q^{op}) does not coincide it is a proof that covariance and q^{op} are different quantities.

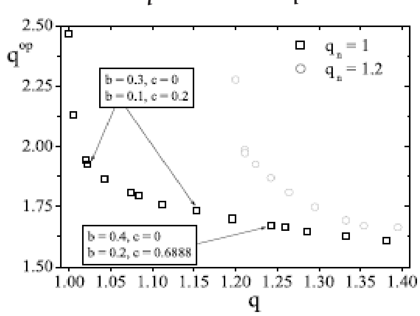
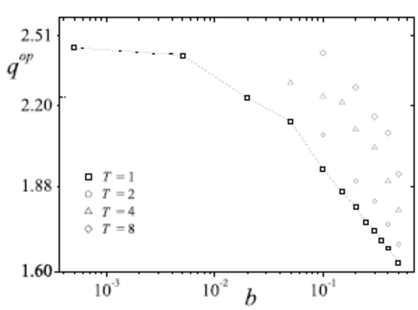


Figure 8 Representation of q^{op} vs. b (*ARCH* (1) process) for returns delayed of T=1,2,4,8. Dashed lines are presented as mere indication.



It has also been verified (see fig. 8) for *ARCH* (1) that the degree of dependence varies visibly with b (or q given eq. (29))and with the lag T between returns. In fig. 8 we observe that returns z_t and z_{t+8} from an *ARCH* (1) with b=0.1 present basically the same degree of dependence as immediate returns,

z_t and z_{t+1}, generated from an *ARCH* (1) with b=0.005, for which q is very close to 1.

B. Stochastic differential model for high-frequency returns

The introduction of stochastic differential equations for the description of price fluctuation dynamics can be assigned to Louis Bachelier. This approach is nowadays used in both academic and commercial applications. The first differential picture based on phenomenological aspects to be published was made by J.P. Bouchaud and R. Cont [48]. It uses the simple and plausible points:

- Instantaneous price changes, $\frac{dS}{dt}$, defined as instantaneous revenue, $r(t)$, are directly proportional to the imbalance between supplied volume and demanded volume, $\Delta\varpi$,

$$r = \frac{\Delta\varpi}{\delta}, \tag{30}$$

- where δ is the *market depth*, i.e., the number of shares needed to make price moves in one unit. This hypothesis describes the traditional relation between price behaviour and imbalance between demand and supply.
- The volume on the book depends on the superposition of satisfied orders and the introduction of new orders. The imbalance rate, by introduction of new orders $\frac{d\Delta\varpi^{(i)}}{dt}$ ((i) stands for incoming), depends itself on several factors:
  - A value which reflects de mean evolution, r_0, for a financial market, that is estimated around 0.01% per day;
  - The value of instantaneous revenue (or return), $r(t)$. Price rise induces a general augment of demand and the opposite for supply. Given that humans are mainly risk-averse, this rate is naturally larger for drop cases. Hence, the contribution for $\frac{d\Delta\varpi^{(i)}}{dt}$ can be written as:

$$\tilde{\alpha}'(r)\, r \approx \left(\tilde{\alpha} - \tilde{\alpha}_1 r - \alpha_2 r^2\right) r$$

However, we would like to stress that, the expansion of $\tilde{\alpha}'(r)$ must go beyond first order, so that price rises always persuades reduction in offer ($\tilde{\alpha} > \alpha_1 > \alpha_2 > 0$). This term also assures that, for large values of return, propensity to buy decreases strongly due to risk- aversion. For practical matters, α_2 aims to reproduce the effects of a stop order.

- The volatility, σ, which can be written proportional to r^2. It is an decreasing(increasing) function of demand(supply);
- Deviation between current, $S(t)$, and estimated worth, which is generally specified by risk agencies $\bar{S}'$.

Combining all these factors, the stochastic equation for imbalance is written as,

$$\frac{d\Delta\varpi}{dt} = \Delta\varpi_0 - \gamma'\Delta\varpi + \left(\tilde{\alpha}\, r - \tilde{\alpha}_1 r - \tilde{\alpha}_2 r^2\right) r - \tilde{\beta}\,\sigma - \tilde{\omega}\left(S - \bar{S}'\right) + \tilde{\eta}, \tag{31}$$

where $-\gamma'\Delta\varpi$ corresponds to effects caused by the absorption of orders, and η to a noise which reproduces random causes that affect market. Using instantaneous return definition, eq. (30), and the fact that "fair" price might include delta $\Delta\varpi_0$, we obtain,

$$\frac{d^2S}{dt^2} = -k\frac{dS}{dt} - \beta\left(\frac{dS}{dt}\right)^2 - \alpha_2\left(\frac{dS}{dt}\right)^3 - \omega\left(S - \bar{S}\right) + \eta, \tag{32}$$

with,

$$k = \gamma' - \frac{\tilde{\alpha}}{\delta}, \quad \beta = \frac{\tilde{\beta} + \tilde{\alpha}_1}{\delta^2}, \quad \alpha_2 = \frac{\tilde{\alpha}_2}{\delta^3},$$

$$\omega = \frac{\tilde{\omega}}{\delta}, \quad \bar{S} = \bar{S}' + \frac{\Delta\varpi_0}{\omega}, \quad \eta = \frac{\tilde{\eta}}{\delta}.$$

When risk-aversion is not taken into account, $\beta = \alpha_2 = 0$, eq. (31) becomes,

$$\frac{d^2S}{dt^2} = -k\frac{dS}{dt} - \omega\left(S - \bar{S}\right) + \eta. \tag{33}$$

Considering case in which η only depends on time, we can appraise price evolution by

$$\begin{aligned} S(t) = {} & \exp\left[-(k+K)t\right]C_1 + \exp\left[-(k-K)t\right]C_2 - \\ & \bar{S}\left\{\exp\left[-\frac{1}{2}(t-t')(k+K)\right] - 1 + \frac{k}{K}\exp\left[-\frac{1}{2}(t-t')(k-K)\right]\right\} - \\ & \frac{1}{K}\left\{\int_{t_0}^{t}\exp\left[-\frac{1}{2}(t-t')(k+K)\right]\eta(t')\,dt' - \right. \\ & \left. \int_{t_0}^{t}\exp\left[-\frac{1}{2}(t-t')(k-K)\right]\eta(t')\,dt'\right\}, \end{aligned}$$

where

$$K = \sqrt{k^2 - 4\omega}.$$

And $C_{1,2}$ are constants obtained from initial conditions $S(t)_0$ and $\left.\frac{dS}{dt}\right|_{t=t_0}$. For $k > 0$ case, market presents stationarity. This implies $\gamma' > \frac{\tilde{\alpha}}{\delta}$ and it is equivalent to say that either the absorption rate is greater than the rate for which new orders are introduced or it is necessary a huge traded volume in order that a relevant price movement appears. Both of the features are present in liquid markets like NYSE and NASDAQ. It is easy to show that the second term only has a significant influence when big differences $S-\overline{S}$ appear. This usually corresponds to long time variation. Therefore, for small time scale, on liquid markets, $-\varpi(S-\overline{S})$ can be neglected.

Making use of ϖ=1 approximation we have

$$K \approx k - 2\frac{\omega}{k}.$$

Hence,

$$k + K \cong 2k - 2\frac{\omega}{k} \approx 2k$$

and

$$k - K \cong 2\frac{\omega}{k}.$$

In other words, $k+K$, corresponds to a fast relaxation with a characteristic time around $\tau_1=K$ and $k-K$ corresponds to a second time scale, $\tau_2 = \frac{\tau_1}{\omega}$, within which ϖ contribution can be disregarded. During the period of our high—frequency time series, DJIA went from 10334.14 to 10783.01, i.e., a difference of 4.8%. In view of the fact that, usually, price fluctuations above 10% around fair price are used by risk agencies to launch their alerts [49], we can say a price/index fluctuation of that order corresponds to a time horizon of about one year, much greater than the scale of minute that we are going to treat.

1. Stable market

a. *Without aversion.* In this case, $\omega=\beta=\alpha_2=0$, and because of that, the equation for price is easily transformed into an equation for return. Bearing in mind that $r(t)=\frac{dS}{dt}$, eq.(33) can be written as,

$$\frac{dr}{dt} = -kr + \eta.$$

Let us discuss stochastic term, η. The simplest solution is to consider it as Gaussian noise with a width D [48]. However, such an assumption leads to a Gaussian distribution for price fluctuation, which has proved inadequate. Usually, in the physical analysis of systems with a large number of degrees of freedom, the stochastic term intents to simulate microscopic response to a set of factors [53]. This response, specifically its magnitude, is influenced in a natural way by return. Large (small) values of r , induce, in general, large (small) values of η . As large values of r are the lesser frequent values, it can be said that η is inversely proportional to return probability [32][50]. Thus, η can be written as

$$\eta = \sqrt{\theta}\left[p(r,t)\right]^{\frac{1-q}{2}}\xi(t) \qquad (q>1), \quad (34)$$

where θ is a constant value, named *volatility* constant, and ξ a white noise associated to a Gaussian of unitary variance. For this case, 1 minute price fluctuation dynamics[2] is given by,

$$dr = -kr\,dt + \sqrt{\theta}\left[P(r,t)\right]^{\frac{1-q}{2}}dW_t. \quad (35)$$

Probability density function, $p(r,t)$, is obtained from the following non-linear Fokker-Planck equation [28]

$$\frac{\partial p(r,t)}{\partial t} = \frac{\partial}{\partial r}\left[k\,r\,p(r,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left\{\theta\left[p(r,t)\right]^{(2-q)}\right\}, \quad (36)$$

whose solution is [51]

$$p(r,t) = \frac{1}{Z_q(t)}\exp_q\left[-\beta(t)x^2\right], \quad (37)$$

with

$$\frac{\beta(t)}{\beta(t_0)} = \left[\frac{Z_q(t_0)}{Z_q(t)}\right]^2,$$

$$Z_q(t) = Z_q(t_0)\left[\left(1-\frac{1}{K}\right)e^{-t/\tau} + \frac{1}{K}\right]^{1/(3-q)},$$

and

$$K = \frac{k}{\beta(t_0)\theta(2-q)\left[Z_q(t_0)\right]^{q-1}}.$$

The relaxation of normalisation constant, Z_q, occurs with the characteristic time τ

$$\tau = \frac{1}{k(3-q)},$$

Which is of the order of $1/k$, since $0<q<3$, so that $p(r,t)$ is normalisable. All correlations for this process are due to the drift term, and because of that, correlations are fast decaying. The form of eq. (35) corresponds to an equation in which variance is not constant. The time dependence of variance leads to the emergence of an asymptotic power-law behaviour for probability density function [52].

When k is positive, like it is for liquid and stable markets with $k(t-t_0) << 1$, $p(r,t)$ is infinitesimally distant from the stationary solution of eq. (36),

$$p_s(r) = \frac{1}{Z}\exp_q\left[-\frac{k}{(2-q)Z^{q-1}\theta}r^2\right], \quad (38)$$

where

$$Z = \left\{\frac{\sqrt{2\pi}\,\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2q-2}\right]}\sqrt{\frac{(2-q)\theta}{2k(q-1)}}\right\}^{\frac{2}{3-q}}.$$

2 Return for larger horizons are obtained by the addition of $r(t)$. This sum leads to the well-known approach towards Gaussian distribution.

Besides it provides an adequate description for price fluctuation PDF, it can be verified that dynamics (35) describes very well Kramers-Moyal moments, $M_n\left(x',t,\tau\right)$ analytically defined as [53]

$$M_n\left(x',t,\tau\right)=\left[\left\langle y\left(t+\tau\right)-y\left(t\right)\right\rangle\right]^n\Big|_{y(t)=x'}$$
$$=\int\left(x-x'\right)^n P\left(x,t+\tau\mid x',t\right)dx, \qquad (39)$$

particularly for case n=2 which is important to evaluate the accuracy of a possible dynamics. Explicitly, moment M_2 is proportional to the square of the coefficient of the stochastic term in standard stochastic differential equations [53]. Considering a stationary regime, $t_0=-\infty << -k^{-1}<<0$, and regarding eq. (39) we have

$$M_2\approx\tau\,\theta\left[p(r)\right]^{(1-q)}=\tau\,\frac{k}{2-q}\left[(5-3q)\sigma^2+(q-1)r^2\right], \qquad (40)$$

which is a second order polynomial.

Still other differential dynamical equations can reproduce the same distribution only processes like eq. (35) are also capable of reproducing Kramers-Moyal moments. Furthermore, this form emerges as a consequence of the dynamics, imposed a priori, and not as the outcome of a fitting procedure without some grounding as it occurs for other approaches [54-56].

It is worth remember that, although the matching of KM moments reduces the infinite set of dynamics compatible to a certain probability distribution, there is a large set of valid proposes yet. As an example, and for statistical purposes, we can unfold multiplicative noise in eq. (34), as the sum of uncorrelated additive and multiplicative noises in such a way that equation

$$dr = -k\,r\,dt+\sqrt{\theta}\left[f(t)\right]^{\frac{q-1}{2}}dW_t + \sqrt{(q-1)\theta\beta(t_0)}\,Z(t_0)\left[f(t)\right]^{-\frac{1}{2}}r(t)\,dW'_t \qquad (41)$$

where

$$f(t) = \left\{\exp\left[-k(3-q)t\right]+(2-q)\frac{\left(1-\exp\left[-k(3-q)t\right]\right)\left[Z(t_0)\right]^{q-1}\beta(t_0)\theta}{k}\right\}^{\frac{1}{3-q}},$$

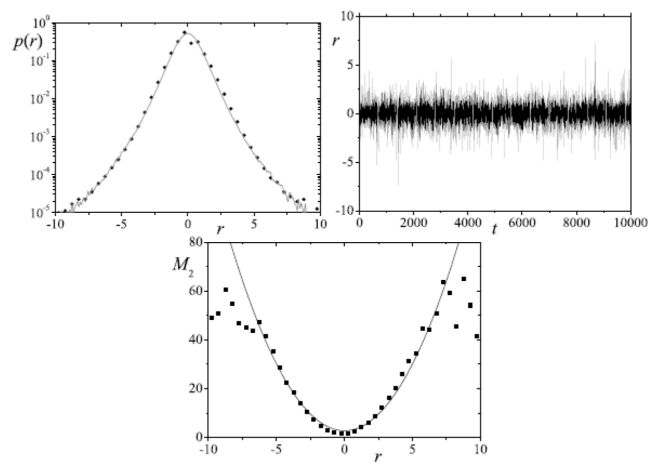
presents the same Fokker-Planck equation, and hence the same distribution [53] [57]. If eq. (35) permits an immediate relation between q and the response of the system to its own dynamics, eq. (41) allows a direct relation between q and the magnitude of multiplicative noise so that, for q =1, Ornstein-Uhlembeck process [52] is recovered as well as Gaussian distribution.

In order to determine values for parameters q, θ, and k, it has been used a numerical adjust for both the average distribution of DJIA components, with eq. (37), and KM second order with eq. (40), see fig. 9. The values for 1 minute returns are q=1.31±0.02, σ=0.930±0.08, and k=2.40±0.04. From equality between distributions (38) and (37) we obtain

$$\theta=\frac{k\left(5-3q\right)\sigma^2\left\{\frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2q-2}\right]}\sqrt{\frac{(q-1)}{(5-3q)\pi\sigma^2}}\right\}^{q-1}}{2-q}, \qquad (42)$$

for volatility constant. Parameter k=2.40±0.04 corresponds to a relaxation time of about 20 seconds, which is compatible with efficient market hypothesis. In fig. 9 we depict an excerpt of a numerical simulation of eq. (35). From eq. (42) the value for volatility constant is θ =2.67.

Figure 9 Upper panels: Symbols represent distribution vs. r for the set of companies which constitute DJIA, and line represents the returns distribution for the time series generated from eq. (35) that is depicted in right panel Lower panel: Second order moment $M_2\approx\tau\,\theta\left[p(r)\right]^{(1-q)}=\tau\,\frac{k}{2-q}\left[(5-3q)\sigma^2+(q-1)r^2\right]$ **which allows the determination of k. Parameters values: τ=1min, k=2.40±0.04, σ=0.930±0.08 e q=1.31±0.02.**



Analysing persistence for both simulation and real time series, by DFA-1 method, we verify a reasonable concordance between them (see fig. 10). Specifically, we have a Hurst exponent of 0.54±0.02 and 0.52±0.02 up to 20 minutes. Beyond this time scale the exponent is 0.513±0.004. These results are in agreement with the well-known absence of meaningful persistence for return time series. In fig. 11 we can verify that, although correlations are fast decaying, the model captures the famous volatility clustering.

Figure 10 Function *F* obtained using DFA-1 vs. window size *T* for the generated time series (circles) and data (squar es). Slopes correspond to Hurst exponent. For *T*20 minutes we have *H*=0.54±0.02 for replica e *H*=0.52±0.02 for data. After this scale the exponent decays to 0.513±0.004 for both of the curves.

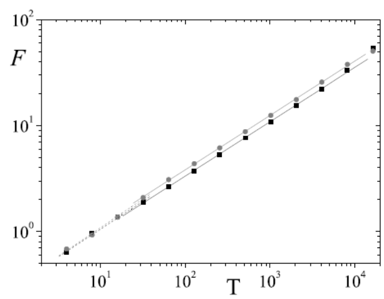
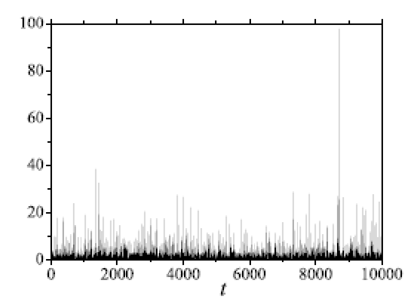


Figure 11 r^2 (a possible measure for volatility) vs. *t* for the excerpt presented in fig. 9.



b. With aversion. In spite of the fact that the last propose provides a satisfactory description for return, the point is that, even at normal regime, stable markets present left skewed distributions. Hence, for a reliable description we must consider $\beta \neq 0$ and $\alpha_2 \neq 0$. In a previous work [48], in which Gaussian noise and were considered, the confining potential is

$$V(r) = \frac{a}{2}r^2 + \frac{b}{3}r^3,$$

which presents a maximum for $r^* = -\frac{a}{b}$. This potential implies that, after passing this barrier, return is able to reach $r = -\infty$ within a finite interval of time. The introduction of $\alpha_2 \neq 0$, to reproduce stop orders, guarantees that market is going to take an infinite time to experience an infinity drop. For this case, price fluctuation is given by

$$dr = -\left(k\,r + \beta\,r^2 + \alpha_2\,r^3\right)dt + \sqrt{\theta}\left[p(r,t)\right]^{\frac{1-q}{2}} dW_t. \quad (43)$$

Its associated Fokker-Plack equation

$$\frac{\partial p(r,t)}{\partial t} = \frac{\partial}{\partial r}\left[\left(k\,r + \beta\,r^2 + \alpha_2\,r^3\right)p(r,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left\{\theta\left[p(r,t)\right]^{(2-q)}\right\}, \quad (44)$$

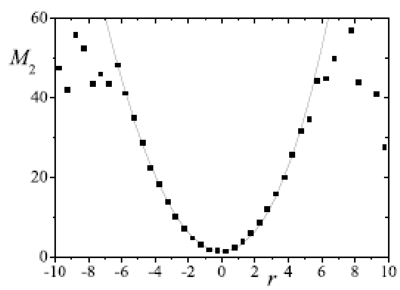
has as stationary distribution

$$p_s(r) = \frac{1}{Z}\exp_q\left[-\frac{Z^{1-q}}{(2-q)\theta}\left(\frac{k}{2}r^2 + \frac{\beta}{3}r^3 + \frac{\alpha_2}{4}r^4\right)\right]. \quad (45)$$

Using eq. (45), we can estimate values for parameters of "confining potential"

$$V(r) = \frac{k}{2}r^2 + \frac{\beta}{3}r^3 + \frac{\alpha_2}{4}r^4. \quad (46)$$

Figure 12 Symbols represent second order Kramers-Moyal moments for data, and line the fit considering stationary case eq. (45). Parameter used: τ=1min, $\theta \approx 2.67$, $q \approx 1.3$, *k*=5.5, β=0.12 e α_2=0.02



Given the quality of the adjust of both PDF and second order KM, β and α_2 should be small and compatible to small corrections. Considering values of $\theta \approx 2.67$ and $q \approx 1.3$ we obtain $k \approx 5.5$, $\beta \approx 0.12$, $\alpha_2 \approx 0.02$. With these values, we verify that $V(r)$ does not present a local maximum. As a matter of fact, the existence of such a maximum would introduce a signature in the market that it could make possible an estimate of crashes from first passage times. Consequently, the scenario that we have presented is consistent with the hypothesis that large drops are related to circumstances which cannot be explained within the same dynamics that rules normal regimes.

c. Illiquidity state. So far we have analysed cases in which k is positive, typical of a liquid market. However, it might occur pression on the market in such a form that the rate of introduction of new orders becomes greater than the rate of order absorption, i.e., $\gamma < \frac{\tilde{\alpha}}{\delta}$. This inequality implies loose of market stability. When k<0, r=0 becomes a local maximum instead of being a local minimum as it is for k<0. The other two extremes are local minimum,

$$r^{+} = \beta \, \frac{\sqrt{1+4|k|\alpha_2} - 1}{2\alpha_2},$$

and absolute minimum

$$r^{-} = -\beta \, \frac{\sqrt{1+4|k|\alpha_2} + 1}{2\alpha_2}.$$

This case is equivalent to a speculative time or economic bubble, in which a market presents a mean return r^{+}. However, this situation is unstable, because there is a strong tendency of fall. In other words, the time between successive rises (dynamics around r^{-}) and falls (dynamics around r^{-}) can be estimated by computing the mean passage time from r^{+} to 0.

When market is near stationarity, $\frac{\partial p(r,t)}{\partial t} \cong 0$, this time can be easily obtained from expressions (5.2.160), (5.2.157) and (5.2.144) of reference [52],

$$T\left(r^{+} \to 0\right) \simeq (2-q)B\left\{\int_{r^{+}}^{0} \exp_q\left[-BV(x)\right]^{2-q} dx\right\} \int_{-\infty}^{x} \exp_q\left[BV(y)\right] dy$$

with

$$B = \frac{Z^{1-q}}{(2-q)\theta}.$$

This conjecture takes place effectively in markets and corresponds to cases in which it happen a set of consecutive rises after that agents try to profit.

Let us conclude this section by saying that, although these models do not capture the empirically observed long-lasting r^2 self-correlations, this property can be introduced when we combine our dynamical proposes with the superstatistics approach [2][55]. Hence, instead constant in time, θ can be a quantity which slowly varies in time and it presents long-lasting correlations. Furthermore, theta is associated to some distribution $P(\theta)$. Possible candidates for $P(\theta)$ are: the log-normal distribution, x^2 -distribution and the F-distribution. In this approach, volatility fluctuations are, alongside with q, responsible for the non-Gaussian form of price fluctuation distribution. We must emphasise that the contribution from volatilities *cannot imply a transformation of* $q \neq 1$ *into* $q=1$. This transformation would lead to an independence on price fluctuation of Kramers-Moyal second order moment which is not compatible with measures made until now.

C. Dynamical approach to high-frequency traded volume

A price fluctuation of certain equity, and subsequently of an index, is intimately related to transaction of that equity. For this reason, traded volume is a crucial quantity in financial markets dynamics. Its importance can be checked in the old Wall Street saying "It takes volume to make prices move".

Several studies related to traded volume have been recently presented. The first thorough study on the statistical properties of traded volume (or just volume for short) was performed by members of Center for Polymer Studies (Boston University) who obtained the following results [58]:

- Probability distribution for volume presents an asymptotic power-law regime;
- The dynamics for volume is characterised by long-lasting correlations with Hurst exponents around 0.8.

Concerning the analysis of probability distribution the study made in was later extended for whole range of v [59]. In this case it was shown that the traded volume PDF is very well described by the following ansatz distribution

$$P(v) = \frac{1}{Z}\left(\frac{v}{\phi}\right)^{\rho} \exp_q\left(-\frac{v}{\phi}\right), \quad (47)$$

where v represents the traded volume expressed in its mean value unit $\langle V \rangle$, i.e., $v=V/\langle V \rangle$, ρ and ϕ are parameters, and $Z = \int_0^{\infty} \left(\frac{v}{\phi}\right)^{\rho} \exp_q\left(-\frac{v}{\phi}\right) dv$.

The probability density function (1) has recently been obtained from a mesoscopic dynamical scenario [60,61] based in the following multiplicative noise stochastic differential equation

$$dv = -\gamma\,(v - \frac{\omega}{\alpha})\,dt + \sqrt{2\frac{\gamma}{\alpha}}\, v\, dW_t, \quad (48)$$

where W_t is a regular Wiener process following a normal distribution, and $v \geq 0$. The right-hand side terms of eq. (48) represent inherent mechanisms of the system in order to keep v close to some "normal" value, ω/α, and to mimic microscopic effects on the evolution of v, like a multiplicative noise commonly used in intermittent processes. This dynamics, and the corresponding Fokker-Planck equation [53], leads to the following inverted Gamma stationary distribution:

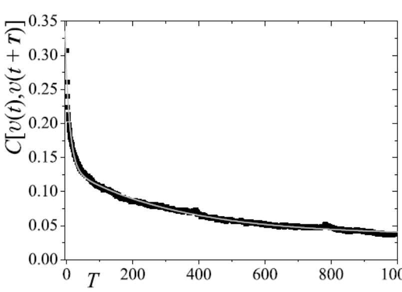
$$f(v) = \frac{1}{\omega\,\Gamma[\alpha+1]}\left(\frac{v}{\omega}\right)^{-\alpha-2}\exp\left[-\frac{\omega}{v}\right]. \quad (49)$$

Consider now, that instead of being a constant, ω is a time dependent quantity which evolves on a time scale T much larger than the time scale of order γ^{-1} required for eq. (48) to reach stationarity [35,60]. This time dependence is, in the present model, associated to changes in the volume of activity (number of traders that performed transactions) and empirically justified by the analysis of the self-correlation function for volume. In fig. 13 we have verified that the correlation function is very well described by

$$C\left[v(t), v(t+\tau)\right] = C_1\, e^{-\tau/T_1} + C_2\, e^{-\tau/T_2} \quad (50)$$

with $T_2 = 847 >> T_1 = 27$. In other words, there is, in a first phase, a fast decay of $C\left[v(t), v(t+\tau)\right]$, related to local equilibrium, and then a much slower decay for larger τ. This constitutes a necessary condition for the application of a superstatistical model [35].

Figure 13 Symbols represent the average correlation function for the 30 time series analysed and the line represents a double exponential fit with characteristic times of T_1=27 and T_1=844 yielding a ratio about 32 between the two time scales Eq. (4) (R^2=0.981, χ^2=2×10^{-2}, and time is expressed in minutes).



If we assume that ω follows a Gamma PDF, i.e.,

$$P(\omega) = \frac{1}{\lambda\Gamma[\delta]}\left(\frac{\omega}{\lambda}\right)^{\delta-1}\exp\left[-\frac{\omega}{\lambda}\right], \quad (51)$$

then, the long-term distribution of v is given by $p(v) = \int f(v) P(\omega)\, d\omega$. This yields

$$p(v) = \frac{1}{Z}\left(\frac{v}{\theta}\right)^{-\alpha-2}\exp_q\left[-\frac{\theta}{v}\right], \quad (52)$$

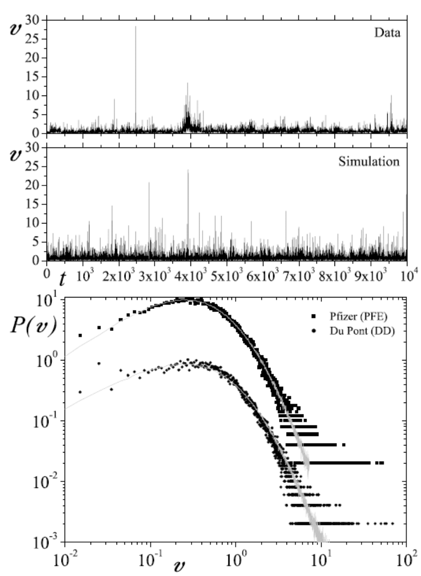
where $\lambda = \theta(q-1)$, $\delta = \frac{1}{q-1} - \alpha - 1$. Bearing in mind that, for $q>1$,

$$x^a e_q^{-\frac{x}{b}} = \left[\frac{b}{q-1}\right]^{\frac{1}{q-1}} x^{a-\frac{1}{q-1}}\, e_q^{-\frac{b/(q-1)^2}{x}}, \quad (53)$$

we can redefine our parameters and obtain the q -Gamma PDF (47).

In fig. 14 we compare traded volume time series of Pfizer stocks with a replica of that time series obtained using this dynamical proposal. As it can be easily verified, the agreement is remarkable.

Figure 14 Upper panel: Excerpt of the time series generated by our dynamical mechanism (simulation) to replicate 1 min traded volume of Citigroup stocks at NYSE (data). Lower panel: 1 min traded volume of Citigroup stocks probability density function vs. traded volume. Symbols are for data, and solid line for the replica. Parameter values: θ=0.212±0.003, ρ=1.35±0.02, and q=1.15±0.02 (χ^2=3.6×10^{-4}, R^2=0.994).



Two key parameters for this superstatistical approach are the local relaxation, γ^{-1}, and the time scale at which ω is updated, T. If the former can be easily related to the first relaxation scale for self-correlation function, the fact is that, up to now, and to the best of our knowledge, there has not been presented an effective procedure to evaluate the latter scale *T*. Along these lines, we have opted for the simplest approximation, i.e., we match large relaxation scale, T_2, with *T*. In fig. 15 we present the self-correlation function for Pfizer. In fig. 14 we compare probability distributions from real and simulated time series. For a better judgement of our proposal we also present our worst result which has been obtained for Du Pont.

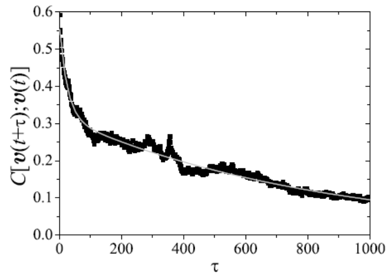
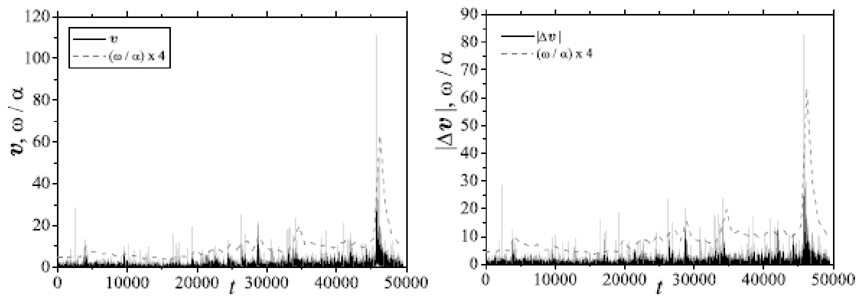


FIG. 15: Symbols represent self-correlation function for Pfizer (PFE) traded volume, and line the numerical adjust with eq. (50): C_1=0.24, T_1=25, C_2=0.32, T_2=825. χ^2=1.4×10^{-4} e R^2=0.9789.

With an estimated value for *T*, we can outline the time evolution for omega by performing averages of v within non-overlapping windows of length *T*. The time series for is depicted in fig. 16.

Figure 16 Left panel: The full line corresponds to PFE traded volume time series and the dashed line corresponds to the time evolution of local mean traded volume (multiplied by four for better visibility) which is calculated using windows of length $T^*=T_2$=825. Right panel: The full line corresponds to 1 minute traded volume fluctuations (in modulus), and the dashed line is the same as for left panel.



In simulations presented herein, stochastic differential processes have been used to generate ω. Albeit this procedure has shown to be, so far, useful, other mechanisms to obtain ω are certainly plausible. For instance, ω might be obtained by the sum

$$\sum_{i=1}^{n} X_i^2,$$

where X represents a set of n independent variables which are associated to a Gaussian. In this case, X_i^2 could be interpreted as the mean volume traded by an agent *i* over time interval *T*. Moreover, for each agent *i* the value of *Xi* would be strongly correlated. The distribution for omega is, as wanted, the Gamma distribution.

6. Empirical analysis within Non-Extensive formalism

Since long time it has been used in finance distributions related to Sq, namely, q-Gaussians under the designation of t-Student distributions. Apparently, the first application of eq. (9) is due to F.M. Ramos et al. [62] in the context of the analogy between price fluctuations and velocity differences made by S. Ghashghaie et al. [63]. After this, many works have been published [64]. The choice for a non-extensive approach, instead of a Lévy truncated, is related to the fact that, for the former, there is no need of any a priori cut-off on distribution domain to obtain accordance between theoretical distribution and tail exponent of the distribution.

A. Distribution for price fluctuations of liquid markets

Let us first analyse price fluctuation distribution for different time horizons. To that, we use a daily close value time series of NYSE index between the 2nd May 1966 and the 14th August 2003. From this time series we have computed daily return,

$$\ln S_t - \ln S_{t-1} = \ln(\frac{S_t - S_{t-1} + S_{t-1}}{S_{t-1}}) =$$

$$\ln(1 + \frac{S_t - S_{t-1}}{S_{t-1}}) = \ln(1 + r_1(t)) \cong r_1(t), \quad (54)$$

From these daily return time series we have determined aggregated return for several time horizons using,

$$r_T(t) \equiv \sum_{t'=1}^{T} r_1\left(t+t'-1\right), \tag{55}$$

For each time horizon, T, we have removed the normal bias, and we have also normalised returns in standard deviation units. From these time series have constructed a set of histograms, $p_d(r_T)$ (d stands for data) and we have adjusted each one for q-Gaussians

$$p(r) = \frac{\Gamma\left[\frac{5-3q}{2-2q}\right]}{\Gamma\left[\frac{2-q}{1-q}\right]} \sqrt{\frac{1-q}{\pi}\frac{1}{(3-q)\bar{\sigma}_q^2}} \left[1+(q-1)\frac{1}{(3-q)\bar{\sigma}_q^2}r^2\right]^{\frac{1}{1-q}}, \tag{56}$$

To that, we have used the following procedure:

Compute partial second-order moment,

$$\sigma^2_{\left(r_T^-,r_T^+\right)} = \sum_{r_T=r_T^-}^{r_T^+} r_T^2 \; p_d\left(r_T\right), \tag{57}$$

And partial fourth-order moment,

$$l_{\left(r_T^-,r_T^+\right)} = \sum_{r_T=r_T^-}^{r_T^+} r_T^4 \; p_d\left(r_T\right), \tag{58}$$

Where $r_T^{+(-)}$ represents the maximum (minimum) value of the interval considered This truncation occurs to avoid the of error because of bad statistics.

Compute partial second-order moment,

$$\tilde{\sigma}^2_{\left(r_T^-,r_T^+\right)} = \sum_{r_T=r_T^-}^{r_T^+} r_T^2 \int_{r_T-\delta}^{r_T+\delta} p\left(r_T'\right) dr_T', \tag{59}$$

And the partial fourth-order moment,

$$\tilde{l}_{\left(r_T^-,r_T^+\right)} = \sum_{r_T^-}^{r_T^+} r_T^4 \int_{r_T-\delta}^{r_T+\delta} p\left(r_T'\right) dr_T', \tag{60}$$

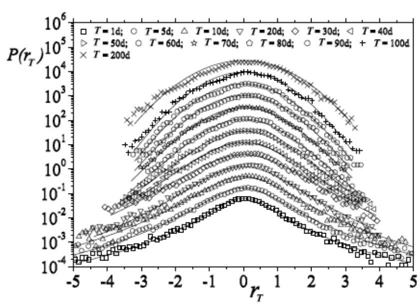
where $\delta = 5\times10^{-3}$ and $p(r'_T)$ for is given by eq. (56);

Set $\sigma^2_{\left(r_T^-,r_T^+\right)} = \tilde{\sigma}^2_{\left(r_T^-,r_T^+\right)}$ and $l_{\left(r_T^-,r_T^+\right)} = \tilde{l}_{\left(r_T^-,r_T^+\right)}$. Solving numerically these equations, we have obtained values for q and σ_q^2 which characterise distribution (56). The results are shown in Table I and depicted in fig. 17. As it can be verified, there is a slow convergence to Gaussian distribution.

Table I: Results obtained from the fitting procedure for NYSE returns.

T (dias)	$r_T^-;r_T^+$	q	σ_q^2	χ^2
1	–4.12;4.22	1.488	0.455	1.33×10^{-6}
5	–3.89;3.77	1.419	0.535	2.02×10^{-6}
10	–3.85;3.41	1.369	0.595	3.60×10^{-6}
20	–3.49;3.21	1.287	0.657	4.52×10^{-6}
30	–3.43;3.21	1.233	0.705	7.14×10^{-6}
40	–3.37;3.27	1.192	0.710	1.77×10^{-5}
50	–3.08;3.13	1.175	0.664	3.66×10^{-5}
60	–3.03;3.08	1.151	0.646	4.13×10^{-5}
70	–3.03;3.05	1.135	0.694	5.56×10^{-5}
80	–3.02;3.04	1.114	0.735	6.95×10^{-5}
90	–3.01;3.03	1.085	0.752	7.87×10^{-5}
100	–3.02; 3.04	1.073	0.772	9.26×10^{-5}

Figure 17 Symbols represent histograms made from NYSE returns time series and lines represent the corresponding adjusts tab. I. The curves are shifted along the ordinate by a factor of 3 better them for clarity.



Aiming to quantify this slow convergence, we remind that returns at horizon T can be obtained by

$$\ln S_t - \ln S_{t-T} = \ln\left(\frac{S_t - S_{t-T} + S_{t-T}}{S_{t-T}}\right) = \ln\left(1+\frac{S_t - S_{t-T}}{S_{t-T}}\right) = \ln\left(1+r_T(t)\right)\;;\; r_T(t), \tag{61}$$

distribution $p(r_T)$ can be obtained from,

$$p(r_T) = \sum_t P\left(S(t), S(t-T)\right)\delta\left(\log\frac{S(t)}{S(t-T)} - r_T\right), \tag{62}$$

where $P(S(t), S(t–T))$ represents the joint probability of having a value $S(t)$, at time t, and a value $S(t–T)$, at time $t–T$. As $S(t)$ and

$S(t-T)$ are elements of the same time series, $P(S(t),S(t-T))$ is obviously related to the same dependence degree between the values of NYSE index at different times. Since $p(r_T)$ depends functionally on $P(S(t),S(t-T))$, it is plausible to say that entropic index q, which defines the $p(r_T)$ fitting, therefore the kurtosis, is related to the degree of dependence between $S(t)$ and $S(t-T)$ which explicitly defines the form of $P(S(t),S(t-T))$.

To assess dependence between prices we have used the normalised q-generalised form of Kullback-Leibler, eq. (17), between $P(S(t),S(t-T))$ and

$$P'\left(S(t),S(t-T)\right)=P_1\left(S(t)\right)P_1\left(S(t-T)\right)\approx\left[P_1(S)\right]^2$$

as indicated previously. In fig. 18 we exhibit several curves of R_q vs. q, and in Table II the values for inflection points, q^{op}. When we plot $1-q^{op}$ vs. T, we verify that points adjust for the following power law,

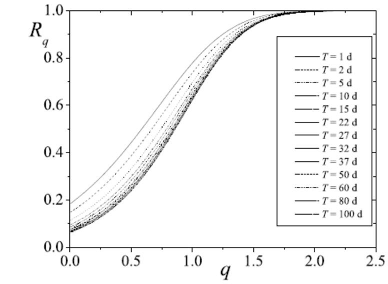
$$1-q^{op}\sim T^{-\nu},$$

with $\nu=0.417\pm0.006$ (see fig. 19).

Table II: Representation of q^{op} vs. T where q^{op} is the inflexion point os R_q curves for NYSE index.

T(dias)	q^{op}
1	0.730 ± 10^{-3}
2	0.795 ± 10^{-3}
5	0.866 ± 10^{-3}
10	0.898 ± 10^{-3}
15	0.916 ± 10^{-3}
20	0.930 ± 10^{-3}
22	0.932 ± 10^{-3}
27	0.939 ± 10^{-3}
32	0.945 ± 10^{-3}
37	0.945 ± 10^{-3}
50	0.950 ± 10^{-3}
80	0.957 ± 10^{-3}
100	0.961 ± 10^{-3}

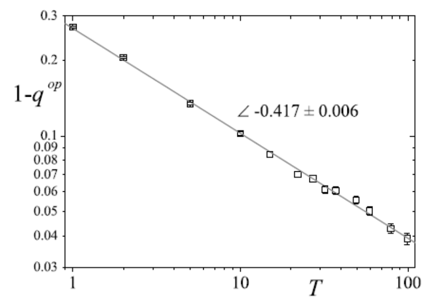
Figure 18 Representation of R_q vs. q for indice NYSE and for several time horizons.



The same analysis was also performed for DJIA time series and it was obtained a similar value for the exponent, $\nu=0.393\pm0.003$. It is noteworthy that exponents obtained by this method are very close to kurtosis relaxation exponents obtained by different procedures for $SP500$ index [55], German interest rate fluctuations and German Mark and US Dollar exchange rate fluctuations [2]. Although all these studies are conceptually different, both of them are in some way related to the decrease of non-Gaussianity. In this way, we might consider these results as a sign of the existence of universal behaviour down to the number of methods and consistency of exponents.

The kurtosis decrease is intimately related to the convergence towards Gaussian distribution, and, consequently, it is related to the famous Central Limit Theorem. For short, and omitting some details connected to the rigour of mathematical formalism, we can state that Lévy-Gnedenko Central Limit Theorem establishes that:

Figure 19 Representation, in log–log of $1-q^{op}$ vs. T for index NYSE according with the values in tab. II. The full line represents the best fit with slope -0.417 ± 0.006.



When a set of N random independent and identically distributed variables, related to a distribution $P_1(x)$, are summed, the distribution for the new variable $X_N=\sum_{i=1}^{N}x_i$, $P_N(X)$. Distribution $P_N(X)$, is given by the convolution of N $P_1(x)$ distributions. The stable distributions for the addition of random variables depend on the finiteness of $P_1(x)$ variance, σ^2. If σ^2 is finite, then the stable distribution $P_N(X)$ is a Gaussian, whereas for infinite variance the stable distribution is of Lévy form. As verified, and since price fluctuations are uncorrelated beyond a scale of few minutes, the sum of daily returns leads into the emergence of a Gaussian. This happens because q_1<5/3. To compare evolution of q in NYSE returns to the evolution for Central Limit Theorem we apply Béry-Esséen Theorem [65]. According to this theorem, the distance between distributions $P_T(X)$, which results from the sum of T independent variables (with $\left\langle |x|^3 \right\rangle < \infty$), and a Gaussian, $G(X)$ goes as

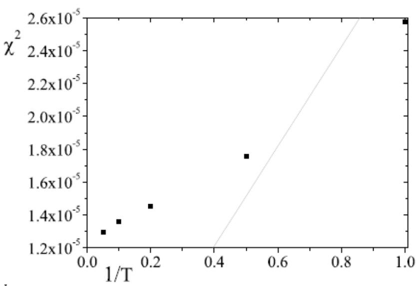
$$\Delta P_T \equiv | P_T(X) - G(X) | \approx \frac{1}{\sqrt{T}}.$$

An equivalent approach is to consider the behaviour of error χ^2 function, when we fit $P_T(r)$ for a Gaussian. It is simple to verify that χ^2 is nothing more than an average of $[\Delta P_T]^2$, hence,

$$\chi_T^2 \approx \frac{1}{T}.$$

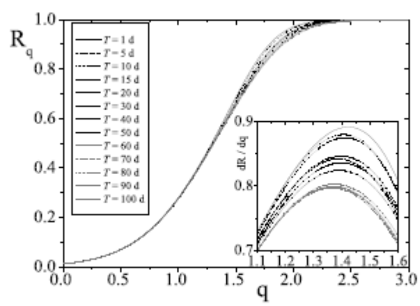
From figure 20 it is straightforward to see that χ^2 does not follow a linear relation with T^{-1}.

Figure 20 Representation of χ_T^2 vs. T^{-1} (symbols). Line represents $\chi_T^2 = 3\times10^{-5}\, T^{-1}$, where χ^2 is the error when one fits $P(r_T)$ for a Gaussian. This error should tend to zero on a linear way with T^{-1} when $T \to \infty$. The fact that the approximation to the limit is slower is an indication of dependence among variables



Besides the extent of linear correlations between returns is insufficient to provide a stable non-Gaussian distribution, such extension is not able to explain the slow convergence to Gaussian either. Based on that, we have applied R_q again, but this time to quantify the degree of dependence between daily returns delayed by a time interval T. In fig. 21 it is possible to verify that q^{op}, which is a measure of the dependence, remains almost constant. This result can be understood if we take, for comparison, *ARCH*(1)-like processes results. In that kind of processes return depends on volatility which, by its turn, depends on the past price fluctuations. Since the dependence of volatility on return is of short kind, we have an exponential decay for volatility self-correlation function, and as a consequence, the dependence degree is fast decaying too. As we increase the interval of past prices used to compute volatility, we both augment return dependence on past values, and increase the characteristic time for the volatility self-correlation function which becomes asymptotically a power-law when an infinite interval is considered. Hence, we can say that the existence of a long-lasting correlation function for volatilities (introduced by its long dependence on past prices) is the main responsible for the slow approach of price fluctuations distribution to the Gaussian.

Figure 21 Representation of R_q vs. q for NYSE index fluctuations. The in-set shows the derivative of R_q in order to q.



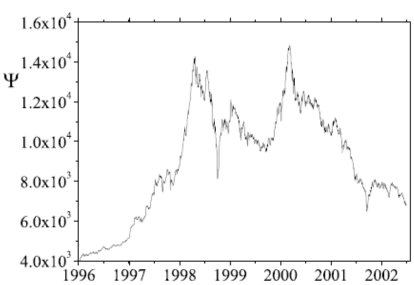
B. Illiquid markets: Lisbon Stock Market case

Up to now, we have been dealing with index/price fluctuations, at several horizons, of markets with high liquidity and which are considered as stable, in the sense of low volatile. However, it is easy to understand that by far all markets can be classified as liquid and/or stable. The performance of a financial market is mandatory related to the vigour of the economy to which it is linked, and it is also associated to factors like socio-political maturity that affects in strong manner volatility. This combination of factors turn out markets as the Americans, the German, the English, the French, and the Japanese present very similar features.

On the other way, countries like Brazil, that despite their industrial power, have debilities in their socio-political system, which make financial quantities present properties that are usually different from the observed in liquid markets.

Outside liquid pattern, we can also refer Indian case for which it has been verified that price fluctuations (at Mumbai stock market) follow an exponential distribution [66]. Another different class can be found for price fluctuations in Portuguese stock market [67]. Although Portugal presents both development indices within Western Europe standards and a stable political situation, it has a small economy, in large part because of population (around 10 million), but also because of the Latin influence of a familiar economy and a small/medium corporation culture. Thereinafter we are going to analyse intra-day PSI-20 index fluctuations of Lisbon Stock Market, presently Euronext-Lisbon, between February 1996 and June 2002 in a total of 4 million points (approximately) (see fig. 22). This number of points corresponds to an index update every 10 seconds (roughly).

Figure 22 Evolution of index PSI-20 between February 1996 and June 2002.



Analysing numerically the distribution for PSI-20 index fluctuations we have verified that none of the distributions currently used to describe index/price fluctuations fits for this particular case. To find an appropriate distribution we have used an analogy between differential equations and non-extensive formalism.

Exponential and Gaussian distribution, related to optimisation of SBG entropy, are also obtainable from,

$$\frac{dp(x)}{dz} = -\beta\; p(x), \qquad (63)$$

Where p is the probability and z some functional of x. When $z=|x|$, the distribution is exponential, while for $z=x^2$ the distribution is Gaussian.

The previous equation can be generalised yielding,

$$\frac{dp(x)}{dz} = -\beta_q \left[p(x)\right]^q, \qquad (64)$$

whose solutions are the q-exponential

$$p(x) \approx \left[1-(1-q)\beta_q\,|x|\right]^{\frac{1}{1-q}}$$

and the q-Gaussian

$$p(x) \approx \left[1-(1-q)\beta_q x^2\right]^{\frac{1}{1-q}}$$

whether we use $z=|x|$ and $z=x^2$. Equation (64) contains (63) as a particular case. In the spirit of works on protein folding [68], energy distribution in cosmic rays [69], we consider a more general form for Eq. (64), namely,

$$\frac{dp(x)}{dz} = -\beta_{q'}\left[p(x)\right]^{q'} - \left(\beta_q - \beta_{q'}\right)\left[p(x)\right]^q. \qquad (65)$$

Considering only positive values for $z=|x|$, i.e., $z=x$, the solution for eq. (65) is

$$x = \frac{1}{\beta_{q'}}\left\{\frac{\left[p(x)\right]^{-(q'-1)}}{q'-1} - \frac{\left(\beta_q/\beta_{q'}\right)-1}{1+q-2q'}\times \left[J\left(1;q-2q',q-q',\left(\beta_q/\beta_{q'}\right)-1\right) - J\left(p(x);q-2q',q-q',\left(\beta_q/\beta_{q'}\right)-1\right)\right]\right\} \qquad (66)$$

where $J(p(x);a,b,c) = \left[p(x)\right]^{1+a} F\left(\frac{1+a}{b};\frac{1+a+b}{c};\left[p(x)\right]^b c\right)$ and F is the hipergeometric function. For $1<q'<q$ and $\beta_{q'} << \beta_q$, three different regions can be observed. The first, in which q dominates, for $0 \leq x << x^*$ where

$$x^* = \frac{1}{(q-1)\beta_q}. \qquad (67)$$

An intermediate part influenced by both q and q' for

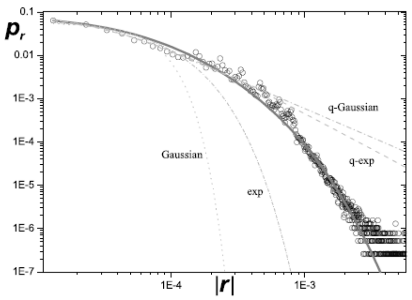
$$x^* << x << x^{**} \equiv \frac{\left[(q-1)\beta_q\right]^{\frac{q-1}{q-q'}}}{\left[(q'-1)\beta_{q'}\right]^{\frac{q-1}{q-q'}}} \qquad (68)$$

and finally a region in which q' prevails, $x >> x^{**}$.

Solving numerically eq. (65) we have found a solution which adjusts nicely for data distribution. The best values describing PSI-20 index fluctuations are q'=1.076, q=1.534 for entropic indices, and $\beta_{q'} = 659 \times 10^3$

$\beta_q = 747 \times 10^4$ (see fig. 23).

Figure 23 Probability density function (PDF) for PSI-20 index tick by tick return. The full line represents the solution of Eq. (3) and it is clearly best approach to PDF from data (circles) when compared with the best Gaussian, *q*-Gausssian (*q* =2.51), exponential and q-exponential, (*q* =1.59) fits that are also plotted.



The result we have just presented is very interesting indeed, because it represents the emergence of a new kind of distribution for price/index fluctuations in financial markets. Moreover, this distribution implies that Heston model is inadequate for the mimicry of this class of markets. For the Heston model, the functional form of $r(t)$ distributions goes from an exponential, for short-to-medium times (e.g. 1 day), to a Gaussian for long times. Besides, it does not capture multiplicative noise character verified in return time series.

7. Final remarks

In this article we have presented a set of results from the analysis of financial quantities, namely, price fluctuations and traded volume within non-extensive statistical mechanics formalism based on entropy Sq. The basis for the application of this formalism is due to the analysis of statistical features in which it is possible to verify the existence of asymptotic power-law behaviour, long-lasting correlations, and multi-fractal structure. These properties are considered as paradigmatic in order to classify a system as non-extensive. In this context we have shown that the time evolution of high-frequency price fluctuations can be made considering a generalised version of Ornstein-Uhlembeck stochastic differential equation in which the stochastic term depends on a quantity inversely proportional to the probability density function. With that, we have been able to provide a dynamical meaning to entropic index q: it reflects the effects of price fluctuations on the microscopic behaviour of the system which is mimicked by the stochastic term. In view if that, when q equal 1, the system is microscopically insensitive to price fluctuations, which corresponds to the regular Ornstein-Uhlembeck is obtained as well as the Gaussian distribution. For the case of traded volume we have been able to characterise it by stochastic differential equations enclosed in Feller class of processes. Together, we have assumed that the mean local value varies on a large time scale. These fluctuations have been associated with variations in the volume of activity which is a variable with long-lasting correlations. In this form, it has been possible to obtain the F-distribution (or q-Gamma) for which q represents a measure of fluctuations in activity volume, in such a way that q=1 indicates steady traded volume leading to a Gamma distribution. Furthermore, and along superstatistical concept also applied for traded volume, we have been able to obtain, for paradigmatic heteroskedastic processes, an expression which relates dynamical parameters, noise nature, and the entropic index that characterises the nature of stationary distribution. Besides, the analysis of liquid markets, we have also shown that non-extensive formalism can be useful in the study of illiquid markets. Let us conclude by emphasising that the discussions presented in this review have been done at a mesoscopic scale. The determination, from more microscopic models, of the parameters used at the mesoscopic scale is certainly welcome.

References

[1] P. Bak, How Nature Works: *The Science of Self Organized Criticality* (Copernicus, Oxford, 1997)

[2] J.-P. Bouchaud and M. Potters, *Theory of Financial Risks: From Statistical Physics to Risk Management* (Cambridge University Press, Cambridge, 2000)

[3] R.N. Mantegna and H.E. Stanley, *An introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambrigde, 1999)

[4] J. Voit, *The Statistical Mechanics of Financial Markets* (Springer-Verlag, Berlin, 2003)

[5] L. Boltzmann, *Lectures on Gas Theory*, (Dover, New York, 1995)

[6] K. Huang, *Statistical Mechanics*, (John Wiley & Sons, New York, 1963).

[7] A.I. Khinchin, *Mathematical Foundations of Information Theory* (Dover, New York, 1957) and *Mathematical Foundations of Statistical Mechanics* (Dover, New York, 1960).

[8] K.L. Volkovyski and Ya. G. Sinai, Funct. Anal. Appl. **5**, 185 (1971)

[9] C. Tsallis, J. Stat. Phys. **52**, 479 (1988). Bibliography URL: http://tsallis.cat.cbpf.br/biblio.htm.

[10] E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992).

[11] C. Tsallis , R.S. Mendes and A.R. Plastino, Physica A **261**, 534 (1998).

[12] B. Lesche, J. Stat. Phys. **27**, 419 (1982)

[13] V. Latora and M. Baranger, Phys. Rev. Lett. **273**, 97 (1999).

[14] C. Tsallis, in *Complexity, Metastability and Nonextensivity*, edited by C. Beck, G. Benedek, A. Rapisarda and C. Tsallis (World Scientific, Singapore, 2005), page 13; Y. Sato and C. Tsallis, Proceedings of the Summer School and Conference on Complexity (Patras and Olympia, July 2004), edited by T. Bountis, G. Casati and I. Procaccia, International Journal of Bifurcation and Chaos (2006), in press [cond-mat/0411073]; C. Tsallis, Milan Journal of Mathematics **73**, 145 (2005); C. Tsallis, M. Gell-Mann and Y. Sato, Proc. Natl. Acad. Sci. USA **102**, 15377 (2005); C. Tsallis, M. Gell-Mann and Y. Sato, Europhysics News **36**, 186 (2005).

[15] *Nonextensive Statistical Mechanics and its Applications*, edited by S. Abe and Y. Okamoto, Lecture Notes in Physics **560** (Springer-Verlag, Heidelberg, 2001); *Non-Extensive Thermodynamics and Physical Applications*, edited by G. Kaniadakis, M. Lissia, and A. Rapisarda [Physica A **305** (2002)]; *Anomalous Distributions, Nonlinear Dynamics and Nonextensivity*, edited by H. L. Swinney and C. Tsallis [Physica D **193** (2004)]; *Nonextensive Entropy - Interdisciplinary Applications*, edited by M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004); *Complexity, Metastability and Nonextensivity*, edited by C. Beck, G. Benedek, A. Rapisarda and C. Tsallis (World Scientific, Singapore, 2005); *Nonextensive statistical mechanics: new trends, new perpectives*, edited by J.P. Boon and C. Tsallis, Europhys. News **36** (6) (2005).

[16] A. Admati and P. Pfleiderer, Rev. Fin. Stud. **1**, 3 (1988); Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng and H.E. Stanley, Phys. Rev. E **60**, 1390 (1999).

[17] Kahneman D. and Tversky A., Econometrica **47**, 263 (1979); Tversky A. and Kahneman D., Journal of Risk and Uncertainty **5**, 297 (1992); Anteneodo C., Tsallis C., and Martinez A.S., Europhys. Lett. **59**, 635 (2002).

[18] C. Tsallis, S.V.F Levy, A.M.C. de Souza and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); Erratum: *ibid.* **77**, 5442 (1996).

[19] L.G. Moyano, C. Tsallis, and M. Gell-Mann, Europhys. Lett. , 813 (2006)

[20] S. Uramov, C. Tsallis, and S. Steinberg, arXiv:cond-mat/0603593 (manuscrito, 2006); S. Uramov, C. Tsallis, M. Gell-Mann, and S. Steinberg, arXiv:cond-mat/0606038 (manuscrito, 2006)

[21] R. Rosenfeld (unpublished, 2006)

[22] S. Kullback and R.A. Leibler, Ann. Math. Stat. **22**, 79 (1961)

[23] C. Granger and J. Lin, J. Time Ser. Anal. , 371 (1994); G. Darbellay, *An Adaptative Histogram Estimator for the Mutual Estimator*, UTIA Research Report nr. 1889 (Academy of Science, Prague, 2002); P. M. Robinson, Rev. Econ. Studies , 437 (1991)

[24] C. Tsallis, Phys. Rev. E **58**, 1442 (1998)

[25] L. Borland, A.R. Plastino and C. Tsallis, J.Math. Phys. **39**, 6490 (1998); [Errata: J. Math. Phys. **40**, 2196 (1999)]

[26] L. Bachelier, *Théorie de la spéculation*, Ann. Sci. École Norm. Sup. **III-17**, 21 (1900).

[27] A. Einstein, Ann. der Phys. **17**, 549 (1905).

[28] L. Borland, Phys. Rev. E **57**, 6634 (1998).

[29] D. O. Cajueiro and B. M. Tabak, Physica A, DOI:10.1016/j.physa.2006.05.054 (2006)

[30] L. Borland, Phys. Rev. Lett. **89**, 098701 (2002), L. Borland, Quantitat. Finance **2**, 415 (2002).

[31] L. Borland and J.P. Bouchaud, Quantitatit. Finance , 499 (2004).

[32] S.M. Duarte Queirós, Quantit. Finance **5**, 475 (2005); S.M. Duarte Queirós, working paper CBPF-NF-027/05.

[33] S.L. Heston, Rev. Fin. Stud. **6**, 327 (1993).

[34] G. Wilk, Z. Włodarczyk, Phys. Rev. Lett. **84**, 2770 (2000)

[35] C. Beck and E.G.D. Cohen, Physica A **322**, 267 (2003)

[36] E.F. Fama, J. Bus. **38**, 34 (1965).

[37] A. Lo, Econometrica **59**, 1279 (191); Z. Ding, C.W.J. Granger amd R.F. Engle, J. Empirical Finance **1**, 83 (1993).

[38] R.F. Engle, Econometrica **50**, 987 (1982).

[39] J. Hull and A. White, J. Fin. **42**, 281 (1987); E.M. Stein and J.C. Stein, Rev. Fin. Stud. **4**, 727 (1991); M. Potters, R. Cont and J.-P. Bouchaud, Europhys. Lett. **41**, 239 (1998).

[40] J.P. Fouque, G. Papanicolaou and K.R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, (Cambridge University Press, Cambridge) 2000.

[41] T. Bollerslev, R.Y. Chou and K.F. Kroner, J. of Econometrics **52**, 5 (1992).

[42] P. Embrechts, C. Kluppelberg and T. Mikosch, *Modelling Extremal Events for Insurance and Finance (Applications of Mathematics)*, (Springer-Verlag, Berlin) 1997.

[43] T. Bollerslev, J. of Econometrics **31**, 307 (1986).

[44] R.F. Engle and J. Russel, Econometrica **66**, 1127 (1998).

[45] S.M. Duarte Queirós and C. Tsallis, Europhys. Lett. **69**, 893 (2005).

[46] S.M. Duarte Queirós and C. Tsallis, Eur. Phys. J. B **48**, 139 (2005).

[47] S.M. Duarte Queirós, Physica A **344**,619 (2004).

[48] J.P. Bouchaud and R. Cont, Eur. Phys. J. B **6**, 543 (1998).

[49] P. Willmott, *On Quantitative Finance*, 2^{nd} edition (John Wiley & Sons, New York, 2000).

[50] S.M. Duarte Queirós, L.G. Moyano, J. de Souza, and C. Tsallis, Eur. Phys. J. B, DOI: 10.1140/epjb/e2006-00205-y (2006).

[51] C. Tsallis, D.J. Bukman, Phys. Rev. E **54**, R2197 (1996).

[52] C.W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* Third Edition (Springer-Verlag, Berlin, 2004).

[53] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications*, 2^{nd} edition (Springer-Verlag, Berlin, 1989).

[54] R. Friedrich, J. Peinke, and Ch. Renner, Phys. Rev. Lett. **54**, 5224 (2000).

[55] M. Ausloos and K. Ivanova, Phys. Rev. E **68**, 046122 (2003).

[56] K. Ivanova, M. Ausloos, and H. Takayasu em: *The Applications of Econophysics*, editado por H. Takayasu (Springer Verlag, Berlin, 2004), 161.

[57] C. Anteneodo and C. Tsallis, J. Math. Phys. **44**, 5194 (2003).

[58] P. Gopikrishnan, V. Plerou, X. Gabaix, and H.E. Stanley, Phys. Rev. E **62**, R4493 (2000).

[59] R. Osorio, L. Borland, and C. Tsallis em: *Nonextensive Entropy - Interdisciplinary Applications,* editado por: M. Gell-Mann and C. Tsallis (Oxford University Press, New York, 2004).

[60] S.M. Duarte Queirós, Europhys. Lett. **71**, 339 (2005).

[61] J. de Souza, L.G. Moyano, and S.M. Duarte Queirós, Eur. Phys. J. B **50**, 165 (2006)

[62] ver em: C. Tsallis, Braz. J. Phys. **29**, 1 (1999)

[63] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner and Y. Dodge, Nature **381**, (1996) 767.

[64] Referências de [1108] a [1129] de http://tsallis.cat.cbpf.br/biblio.htm.

[65] W. Feller, *An Introduction to Probability Theory and its Applications* (Whiley, New York, 1971)

[66] K. Matia, M. Pal, H.E. Stanley, and H. Salunkay, Europhys. Lett. , 909 (2004).

[67] S.M. Duarte Queirós, Physica A **344**, 279 (2004).

[68] C. Tsallis, G. Bemski, and R.S. Mendes, Phys. Lett. A **257**, 93 (2003).

[69] C. Tsallis, J. Anjos, and E.P. Borges, Phys. Lett. A **310**, 372 (2003).