Resistive internal kink modes in a differentially rotating cylindrical plasma

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The resistive internal kink modes in a differentially rotating cylindrical plasma column are studied. It is shown that the Velikhov effect, which causes the magnetorotational instability in astrophysics, contributes to the magnetic hill/well and thereby enhances or suppresses these modes, depending on the character of radial profile of the rotation frequency. It is pointed out that, in the case of unfavorable rotation frequency profile, such a rotation-induced magnetic hill can exceed the hill effect due to the plasma pressure gradient. Under this condition, there appears a new variety of resistive-interchange modes, which are referred to as rotational modes. On the other hand, for a favorable rotation frequency profile, the Velikhov effect suppresses the resistive-interchange modes. These results concern the $m>1$ modes, where m is the poloidal mode number. In the case of perturbations with $m=1$, the favorable rotation frequency profile leads to decreasing the growth rate of the reconnecting mode. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2804700]

I. INTRODUCTION

The ideal internal kink modes in differentially rotating cylindrical plasmas were investigated in Ref. 1, which is hereinafter referred to as paper I. The main contribution of I was to incorporate the Velikhov effect[2] into the standard theory of these modes, as summarized in Ref. 3, stimulated by its astrophysical applications introduced in the classical work of Balbus and Hawley.[4] These authors have shown that the Velikhov effect can drive the magnetorotational instability, which can be the underlying physical mechanism responsible for anomalous viscosity in accretion disks.[5]

The excitation of this instability depends on the sign of the plasma rotation frequency profile. In fact, for instability, it is required that $d\Omega^2/d\ln r<0$, where $\Omega=\Omega(r)$ is the rotation frequency and r is the radial coordinate, whereas in the opposite case, when $d\Omega^2/d\ln r>0$, the Velikhov effect leads to stabilization. It was explained in I that the Velikhov effect can be treated as a magnetic hill/well effect, depending on the sign of the value $d\Omega^2/d\ln r$. The essence of I was to study the role of this effect for ideal modes; the Suydam modes, i.e., the modes with $m\gg 1$, and the internal kink mode with $m=1$ were analyzed, where m is the poloidal mode number. Nonetheless, in addition to the ideal modes, the stability theory of magnetic confinement systems involves also a family of resistive internal kink modes,[6–8,3] making it of interest to extend the study of the Velikhov effect to these modes. This is the goal of the present paper. We restrict ourselves to the simplest case of cylindrically symmetric magnetic confinement configurations, having in sight the generalization of the results to toroidal geometry. A recipe for such a generalization can be found in Ref. 3.

Although the investigation of resistive modes in cylindrical plasma column with rotation has been continued in Ref. 9, the well/hill effect was neglected in this work. Recently resistive modes in cylindrical geometry and in the presence of plasma flows were considered in Refs. 10 and 11. However, it was assumed in both these references that the flow is directed along the equilibrium magnetic field and, under such an assumption and in the cylindrical approximation, the Velikhov effect cannot be properly investigated.

We note that the well/hill effect can be caused also by the plasma pressure gradient. In general, the resistive modes are studied using a singular perturbation mathematical procedure; the relevant equations are solved inside the singular layer, where higher order derivatives associated with small resistive diffusion are taken into account, and the solutions matched to those in the outside ideal region, where they are neglected. Such a procedure was initially performed in Ref. 6, where the resistive-interchange modes generated by the hill effect, and the tearing modes, driven by the unfavorable current profile, have been discovered. We remark that the so-called "constant-psi" approximation in the singular layer used in this reference is valid only for a sufficiently small plasma pressure gradient, as explained in Ref. 3.

As for the physical aspects of the problem of resistive-interchange modes considered in Ref. 6, it has been pointed out that, in addition to the ideal interchange modes revealed as the Suydam modes, there is their resistive analog in the case of a not too large magnetic hill. More detailed discussion of inter-relation between the interchange and resistive-interchange modes will be given below.

The hill/well in the singular layer was first explicitly accounted for in Ref. 12, in which it is elaborated a mathematical procedure to find the solution of the resistive magnetohydrodynamic equations in this layer and to match it to that in the ideal region, obtained in the approximation $m \gg 1$. Thereby, a general dispersion relation for resistive modes with $m \gg 1$ has been derived in Ref. 12. The procedure of that reference was generalized in Refs. 13 and 14 to the case of tokamak geometry with due allowance for the effects of neoclassical viscosity.

References 7 and 8, which initially studied the $m=1$ internal resistive and reconnecting modes, also neglected the well/hill effect in the singular layer. The role of this effect for $m=1$ was initially analyzed in Refs. 15 and 16. A further improvement of the procedure of Refs. 15 and 16 can be found in Ref. 3, in which the hill/well in the ideal region has been properly taken into account. The fact is that they dealt with the parameter λ_H entering also the problem of the ideal $m=1$ mode. This parameter depends not only the current profile but also on the plasma pressure gradient. The latter can be associated with the hill/well effect. Therefore, the modes studied in Refs. 7 and 8 are also interesting for our topic.

Another family of internal modes subjected to the hill/well effect is that of the internal modes with finite $m>1$. Since these modes are nonlocal, their analytical theory for arbitrary profiles of current, plasma pressure, and rotation frequency seems to be problematic. Nevertheless, there is a particular case of this theory for a nonrotating plasma column assuming the current and plasma pressure to have parabolic radial profiles.[15,16] This subtrend of resistive internal modes goes back to Ref. 17 addressed to the study of these modes for vanishing plasma pressure gradient.

In Sec. II the basic equations are given and the essence of our approach is explained. The procedure for obtaining the solution of plasmadynamical equations in the singular layer is presented in Sec. III. In Sec. IV the resistive-interchange modes with $m \gg 1$ are studied. These modes are local in the ideal region and are characterized by the ideal asymptotic solutions obtained in I. In contrast to this, the modes with finite m are nonlocal in the ideal region. The solution of plasmadynamical equations for such modes, in the ideal region in the approximation of parabolic profiles of the current and plasma pressure gradient and constant $d\Omega^2/d\ln r$, are found in the Appendix. The results of the Appendix are used in Sec. V for derivation of dispersion relation for the modes with finite m. The investigation of the resistive-interchange modes for finite $m>1$ is performed in Sec. VI. The modes with $m=1$ are analyzed in Sec. VII. Discussions of the results are given in Sec. VIII.

II. BASIC EQUATIONS AND THE ESSENCE OF THE APPROACH

A. Basic equations

The main difference between the problem formulated here and that investigated in I is the inclusion of finite resistivity in the generalized Ohm's law:

$$\mathbf{E} + \frac{1}{c}[\mathbf{V} \times \mathbf{B}] = \frac{\mathbf{j}}{\sigma}. \tag{1}$$

Here, σ is the plasma conductivity, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively, $\mathbf{V}$ is the plasma velocity, and c is the velocity of light. Using Maxwell's equation

$$\partial \mathbf{B}/\partial t = -c\,\nabla \times \mathbf{E} \tag{2}$$

and Ampere's law

$$\mathbf{j} = c\,\nabla \times \mathbf{B}/(4\pi) \tag{3}$$

leads to the magnetic diffusion equation in the form

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{B}] + \frac{c^2}{4\pi\sigma}\nabla \times \nabla \times \mathbf{B} = 0. \tag{4}$$

Our remaining basic equations are the same as in I; i.e., the plasma equation of motion

$$\rho d\mathbf{V}/dt = -\nabla p + [\mathbf{j} \times \mathbf{B}]/c, \tag{5}$$

the continuity equation

$$d\rho/dt + \rho \nabla \cdot \mathbf{V} = 0, \tag{6}$$

the adiabatic equation

$$d(p\rho^{-\Gamma})/dt = 0, \tag{7}$$

and the Maxwell equation

$$\nabla \cdot \mathbf{B} = 0. \tag{8}$$

Here, ρ is the plasma mass density, Γ is the adiabatic exponent, and p is the plasma pressure, and

$$d/dt = dt + \mathbf{V} \cdot \nabla. \tag{9}$$

The equilibrium state considered is the same as in I. The confinement equilibrium is assumed to have cylindrical symmetry characterized by the coordinates r, θ, and z (the radial, poloidal, and longitudinal coordinates, respectively). The plasma column rotates in the poloidal direction with the angular frequency Ω and there is an equilibrium magnetic field $\mathbf{B}_0=(0,B_{0\theta},B_{0z})$. As a result of plasma rotation, there appears an equilibrium electric field $\mathbf{E}_0=(E_0,0,0)$, where

$E_0=-\Omega r B_{0z}/c$. The plasma has the equilibrium mass density ρ_0 and pressure p_0. The functions Ω, ρ_0, p_0, $B_{0\theta}$, and B_{0z} are assumed to be dependent on r. They are related by the pressure balance equation

$$p_0' = \rho_0 r\Omega^2 - \frac{1}{4\pi}\left[B_{0z}\frac{\partial B_{0z}}{\partial r} + \frac{B_{0\theta}}{r}\frac{\partial}{\partial r}(rB_{0\theta})\right], \tag{10}$$

where the prime denotes the radial derivative. In addition, there are poloidal and longitudinal equilibrium currents ($j_{0\theta}$ and j_{0z}, respectively), given by

$$(j_{0\theta}, j_{0z}) = \frac{c}{4\pi}\left(-\frac{\partial B_{0z}}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rB_{0\theta})\right). \tag{11}$$

B. Perturbations

The perturbed quantities are assumed to be dependent on t, θ, and z as $\exp(-i\omega t + im\theta + ik_z z)$, where ω is the mode frequency, m is the poloidal mode number, and k_z is the longitudinal projection of the wave vector. In addition to m, we use the poloidal projection of the wave vector $k_y = m/r$. The perturbed magnetic field $\tilde{\mathbf{B}}$ is characterized by the components $\tilde{B}_r$, $\tilde{B}_\theta$, and $\tilde{B}_z$. The perturbed velocity $\tilde{\mathbf{V}}$ is expressed is terms of the perturbed plasma displacement $\boldsymbol{\xi}$ defined by $\boldsymbol{\xi} = i\tilde{\mathbf{V}}/\tilde{\omega}$, where $\tilde{\omega} \equiv \omega - m\Omega$ is the Doppler-shifted mode frequency. Using Eqs. (6) and (7), the perturbed plasma pressure $\tilde{p}$ is expressed in terms of $\boldsymbol{\xi}$ by

$$\tilde{p} = -Xp_0' - \Gamma p_0 \nabla \cdot \boldsymbol{\xi}, \tag{12}$$

where $X \equiv \xi_r$.

1. Perturbations far from the singular layer

We assume that, far from the singular layer, the resistivity is unimportant. Similar to I, we then have

$$\tilde{B}_r = i\left(k_z B_{0z} + \frac{m}{r}B_{0\theta}\right)X, \tag{13}$$

$$\tilde{B}_\theta = ik\hat{Y} - \frac{\partial}{\partial r}(XB_{0\theta}), \tag{14}$$

$$\tilde{B}_z = -\frac{im}{r}ik\hat{Y} - \frac{1}{r}\frac{\partial}{\partial r}(rB_{0z}X). \tag{15}$$

Here,

$$\hat{Y} = Y + iX\frac{B_{0z}}{\tilde{\omega}}\frac{d\Omega}{d\ln r}, \tag{16}$$

$$Y = \xi_\theta B_{0z} - \xi_z B_{0\theta}. \tag{17}$$

In addition, as in I, the term with $\nabla\cdot\boldsymbol{\xi}$ in Eq. (12) is assumed to be unimportant, so that

$$\tilde{p} = -Xp_0'. \tag{18}$$

Starting with the linearized version of Eq. (5), we then obtain two equations for X and $\hat{Y}$. We solve the equation for $\hat{Y}$ in the large aspect ratio approximation, implying $m \gg k_z r$, and $B_{0\theta}/B_{0z} \simeq k_z r/m$. As a result, we arrive at the following equation for X derived in I:

$$\frac{1}{r}\frac{d}{dr}\left[r^3\rho_0(v_A^2 k_\parallel^2 - \tilde{\omega}^2)\frac{dX}{dr}\right] - U^V X = 0. \tag{19}$$

Here, $k_\parallel = (k_z B_{0z} + mB_{0\theta}/r)/B_0$ means physically the projection of the wave vector on the direction of equilibrium magnetic field,

$$U^V = U^{(0)} + U^{(r)}, \tag{20}$$

where

$$U^{(0)} = \frac{1}{r^2}(m^2 - 1 + k_z^2 r^2)(mB_{0\theta} + rk_z B_{0z})^2 + 8\pi k_z^2 r p_0' + \frac{2k_z^2}{m^2}(r^2 k_z^2 B_{0z}^2 - m^2 B_{0\theta}^2), \tag{21}$$

$$U^{(r)} = 4\pi\rho_0(k_z^2 r^2 + h_\theta^2)d\Omega^2/d\ln r, \tag{22}$$

$h_\theta = B_{0\theta}/B_0$. The superscript "V" of U denotes the first letter of the name of the author of Ref. 2. The superscripts "(0)" and "(r)" designate the nonrotational and rotational parts of the function U^V, respectively.

2. Perturbations in the singular layer

Allowing for resistivity in the vicinity of the point $r = r_0$, where $k_\parallel(r_0) = 0$, we assume the radial derivatives of the perturbed quantities to be large compared with their derivatives over θ and z, $\partial/\partial r \gg (r^{-1}\partial/\partial\theta, \partial/\partial z)$. Equation (4) then reduces to

$$\hat{D}\tilde{\mathbf{B}} = \nabla \times \boldsymbol{\xi} \times \mathbf{B}_0. \tag{23}$$

Here the operator $\hat{D}$ means

$$\hat{D} = 1 - \frac{c^2}{4\pi\sigma\gamma}\frac{\partial^2}{\partial x^2}, \tag{24}$$

$\gamma = \mathrm{Im}\,\omega$, $x = r - r_0$. Instead of Eqs. (13)–(15), one has from Eq. (23)

$$\tilde{B}_r = \hat{D}^{-1}\left(\frac{B_{0\theta}}{r}\frac{\partial}{\partial\theta} + B_{0z}\frac{\partial}{\partial z}\right)X, \tag{25}$$

$$\tilde{B}_\theta = \hat{D}^{-1}\left[\frac{\partial\hat{Y}}{\partial z} - \frac{\partial}{\partial r}(B_{0\theta}X)\right], \tag{26}$$

$$\tilde{B}_z = -\hat{D}^{-1}\left\{\frac{1}{r}\left[\frac{\partial\hat{Y}}{\partial\theta} + \frac{\partial}{\partial r}(rB_{0z}X)\right]\right\}, \tag{27}$$

where $\hat{D}^{-1}$ is the operator inverse to $\hat{D}$.

In accordance with Ref. 3, for calculating the function $\hat{Y}$, we use the following consequence of Eqs. (5):

$$i\left(\frac{m}{r}\tilde{B}_z - k_z\tilde{B}_\theta\right) = \frac{j_{0z}}{B_{0z}}\tilde{B}_r + \frac{imp_0'}{rB_{0z}}X. \tag{28}$$

We then find that

$$\hat{Y} = \hat{Y}^{(0)} + \hat{Y}^{(\sigma)}, \tag{29}$$

where $\hat{Y}^{(0)}$ is the ideal part of $\hat{Y}$ represented in I, while the expression for $\hat{Y}^{(\sigma)}$ has the form

$$\hat{Y}^{(\sigma)} = \frac{imr}{B_{0z}(m^2 + k_z^2 r^2)}(\hat{D} - 1)X. \tag{30}$$

By means of Eqs. (29) and (30), we arrive at the following equation for the function X in the singular layer (for further details, see Ref. 3):

$$x\hat{D}^{-1}(xX)'' - U_0^V X + \frac{r_0^4 \gamma^2}{S^2 k_z^2 v_A^2} X'' = 0. \tag{31}$$

Here, $v_A^2 = B_0/(4\pi\rho_0)^{1/2}$ is the Alfvén velocity, S is the magnetic shear at $r=r_0$ defined by

$$S = (rq'/q)_{r=r_0}, \tag{32}$$

where q is the safety factor defined by $q = rB_{0z}/(RB_{0\theta})$, $R = L/(2\pi)$, L is the cylinder length, and

$$U_0^V = \frac{8\pi}{S^2}\left(8\pi r p_0' + \rho_0 r^2 \frac{d\Omega^2}{d\ln r}\right)_{r=r_0}. \tag{33}$$

III. THE SOLUTION IN THE SINGULAR LAYER

In solving Eq. (31), we use the Fourier representation taking $X(x)$ in the form

$$X(x) = \int \exp(ik_x x) X(k_x) dk_x. \tag{34}$$

Thereby, we will deal with the function $X(k_x)$ and the variable k_x. Instead of k_x, we introduce the dimensionless variable t defined by

$$t = k_x/k_y. \tag{35}$$

Equation (31) then reduces to

$$\frac{d}{dt}\left(\frac{t^2}{1+z^2}\frac{dX}{dt}\right) - [s(s+1) + \lambda^2 t^2]X = 0. \tag{36}$$

Here the variable z means

$$z = k_x c/(4\pi\gamma\sigma)^{1/2} \equiv t(\gamma_R/\gamma)^{1/2}, \tag{37}$$

where γ_R is the characteristic resistive decay rate defined by

$$\gamma_R = c^2 k_y^2/(4\pi\sigma). \tag{38}$$

The parameter s is introduced by

$$s = -\frac{1}{2} + \left(\frac{1}{4} + U_0^V\right)^{1/2}. \tag{39}$$

The parameter λ is the dimensionless growth rate defined by

$$\lambda = \gamma/\omega_A, \tag{40}$$

where $\omega_A = S v_{A\theta}/r_0$ is the characteristic Alfvén frequency, $v_{A\theta} = (B_\theta/B_0)v_A$.

Turning to Eqs. (39) and (20)–(22), one can see that the Velikhov effect is manifested in terms of the parameter s.

Equation (36) was originally solved in Ref. 12 and then in Refs. 13 and 14. The function X was represented in these references in the form

$$X \sim z^s \exp(-Mz^2/2)\hat{X}(\zeta), \tag{41}$$

where

$$\zeta = Mz^2, \tag{42}$$

$$M = Q^{3/2}, \tag{43}$$

$$Q = (\lambda^2 \gamma/\gamma_R)^{1/3} \equiv \gamma/(\omega_A^2 \gamma_R)^{1/3}. \tag{44}$$

Equation (36) then reduces to

$$\zeta \hat{X}'' + \left(\frac{1}{1+\zeta/M} + \tau - \zeta\right)\hat{X}' - \left[p + \frac{M - M_1}{2M(1+\zeta/M)}\right]\hat{X} = 0. \tag{45}$$

Here the prime indicates the derivative with respect to ζ,

$$p = (M - M_1)(M - M_2)/4M, \tag{46}$$

$$\tau = s + 1/2, \tag{47}$$

$$M_1 = -s, \tag{48}$$

$$M_2 = -(s+1). \tag{49}$$

The solution of Eq. (45) is the function

$$\hat{X} = U'(p,\tau,\zeta) - \frac{M - M_2}{2M} U(p,\tau,\zeta), \tag{50}$$

where U the confluent hypergeometrical function[18] satisfying the equation

$$\zeta U'' + (\tau - \zeta)U' - pU = 0. \tag{51}$$

Using Eq. (50) and the asymptotic formulae for the confluent hypergeometrical functions, we find that for $\zeta \ll 1$ and $s < 1/2$, the function (41) is of the form

$$X \sim t^s(1 + t^{-(2s+1)}\Delta_R). \tag{52}$$

Here,

$$\Delta_R = f(s)(\gamma/\gamma_R)^{1/2+s} h(M), \tag{53}$$

where

$$h(M) = \frac{1}{M^{1/2+s}} \frac{M - M_1}{M + M_2} \frac{\Gamma(p + 1/2 - s)}{\Gamma(p+1)}, \tag{54}$$

$$f(s) = \frac{\Gamma(s + 1/2)}{\Gamma(-s - 1/2)}, \tag{55}$$

where Γ is the gamma function. Note that Eq. (6.16) of Ref. 3 contains a misprint: in the denominator of the right-hand side of this equation, one should substitute $M - M_2$ by $M + M_2$.

In order to derive a dispersion relation, the asymptotic given by Eq. (52) should be matched with that of the solution in the ideal region. Such a solution proves to be dependent on the value of the poloidal mode number m.

IV. RESISTIVE-INTERCHANGE MODES WITH $m \gg 1$

A. Dispersion relation

For $m \gg 1$, Eq. (19) describing the perturbations in the ideal region reduces to (cf. I)

$$(x^2 X')' - U_0^V X = 0, \tag{56}$$

where the prime is the derivative with respect to x.

By means of Eq. (34), we introduce the Fourier component $X(k_x)$ and turn to the variable t defined by Eq. (35). Equation (56) then yields

$$\frac{d}{dt}\left[(t^2+1)\frac{dX}{dt}\right] - s(s+1)X = 0. \tag{57}$$

The general solution of Eq. (57) finite at $t \to 0$ is of the form

$$X(t) = AX_+(t) + BX_-(t). \tag{58}$$

Here, A and B are arbitrary constants, and the functions X_+ and X_- are given by

$$X_+ = F\left(-\frac{s}{2}, \frac{1+s}{2}; \frac{1}{2}, -\xi^2\right), \tag{59}$$

$$X_- = \xi F\left(-\frac{1-s}{2}, 1+\frac{s}{2}; \frac{3}{2}, -\xi^2\right), \tag{60}$$

where F is the hypergeometrical function.[19] The function X_+ corresponds to the even solutions, and X_- to the odd solutions.

The asymptotic of $X_\pm$ at $t \gg 1$ is of the form

$$X_\pm \sim t^s(1 + t^{-(2s+1)}\Delta_\pm). \tag{61}$$

Here,

$$\Delta_+ = \frac{1}{f(s)}\frac{\Gamma^2[(1+s)/2]}{\Gamma^2(-s/2)}, \tag{62}$$

$$\Delta_- = \frac{1}{f(s)}\frac{\Gamma^2(1+s/2)}{\Gamma^2[(1-s)/2]}. \tag{63}$$

Matching Eq. (61) with Eq. (52), we arrive at the dispersion relation

$$\Delta_R = \Delta_\pm. \tag{64}$$

It can be seen that really we have two dispersion relations corresponding to even and odd modes, respectively. Using Eq. (53), Eq. (64) reduces to

$$(\gamma/\gamma_R)^{1/2+s} h(M) = \Delta_\pm / f(s), \tag{65}$$

where $h(M)$ is given by Eq. (54).

We are interested in the perturbations with $\gamma \gg \gamma_R$. In this case, in accordance with Eq. (54), the dispersion relation (65) contains the small parameter $(\gamma_R/\gamma)^{1/2+s}$. Therefore, it is approximately satisfied if

$$h(M) = 0. \tag{66}$$

Equation (66) is the dispersion relation for resistive-interchange modes with $m \gg 1$.

B. General analysis of dispersion relation

According to Eq. (54), Eq. (65) has the solutions

$$M = M_1, \tag{67}$$

and

$$p = -l, \tag{68}$$

where $l = 1, 2, 3, \ldots$ is an integer. It was explained in Ref. 3 that the solution (67) corresponds to the perturbations of the ground (the lowest) energy level, while the solution (68) to the lth level. According to Eqs. (43), (44), (48), and (67), the perturbations of the ground level are unstable if

$$s < 0. \tag{69}$$

It follows from Eq. (68) that the growth rate of the instability considered is given by

$$\gamma = (-s)^{2/3}(\omega_A^2 \gamma_R)^{1/3}. \tag{70}$$

According to Eq. (68), the perturbations with $l \neq 0$ are characterized by the dispersion relation

$$M^2 + M(2s + 1 + 4l) + s(s+1) = 0. \tag{71}$$

It can be seen that, in the condition (69), one of the roots of Eq. (71) is positive ($M > 0$). This means that, as in the case $l=0$, the perturbations with $l \neq 0$ are also unstable. For $l \gg 1$, the growth rate of the perturbations is given by

$$\gamma = \left[-\frac{s(s+1)}{4l}\right]^{2/3}(\omega_A^2 \gamma_R)^{1/3}. \tag{72}$$

It can be seen that this growth rate is small compared with Eq. (70) as $l^{-2/3}$.

In finding Eq. (72), we have not assumed the parameter s to be small. Let us note that for small s ($s \ll 1$), Eq. (71) for unstable modes reduces to

$$M = -s/(4l). \tag{73}$$

Instead of Eq. (72), we then arrive at

$$\gamma = [-s/(4l)]^{2/3}(\omega_A^2 \gamma_R)^{1/3}. \tag{74}$$

C. The standard resistive-interchange instability and its inter-relation with the ideal interchange instability

Turning to Eq. (39), one can see that in neglecting the Velikhov effect, the condition (69) means

$$p_0' < 0, \tag{75}$$

which is nothing but the condition of the standard decreasing profile of plasma pressure. Equations (70)–(72) describe the standard resistive-interchange instability. This instability is essentially weaker than the Suydam instability driven for $s < -1/2$, i.e., for (see I for further details)

$$\frac{S^2}{4} + \frac{8\pi r_0 p_0'}{B_0^2} < 0. \tag{76}$$

At the same time, comparing Eqs. (75) and (76), one can see that the condition of driving the resistive-interchange modes is essentially weaker than that of ideal interchange modes: in the former case the stabilizing effect by the magnetic shear proves to be "switched off."

D. Rotational resistive-interchange instability

If the Velikhov effect exceeds the plasma pressure gradient effect, i.e.,

$$\left|\frac{d\Omega^2}{d\ln r}\right| > \frac{|p_0'|}{r_0\rho_0}, \tag{77}$$

Eqs. (70)–(72) can describe a new variety of resistive instabilities that can be called the rotational resistive-interchange instability. The condition of this instability is

$$d\Omega^2/d\ln r < 0. \tag{78}$$

The ideal analog of this instability is the ideal rotational interchange instability, pointed out in I, driven for the condition [cf. Eq. (76)]

$$\frac{S^2}{4} + \frac{2r_0^2}{v_A^2}\frac{d\Omega^2}{d\ln r} < 0. \tag{79}$$

E. Suppression of the resistive-interchange instability by the Velikhov effect

For positive gradient of the plasma rotation frequency,

$$d\Omega^2/d\ln r > 0, \tag{80}$$

the Velikhov effect suppresses the standard resistive-interchange instability. It follows from Eq. (39) that the condition of such a suppression is

$$\frac{d\Omega^2}{d\ln r} > -\frac{p_0'}{r_0\rho_0}. \tag{81}$$

V. DISPERSION RELATION FOR NONLOCAL RESISTIVE MODES

To derive the dispersion relation for nonlocal resistive modes, we should match the asymptotic ideal solution given by Eq. (A16) with the corresponding one at the resistive layer. The last is given by the function $X(t)$ of the form Eq. (52) in the Fourier representation. Therefore, we should preliminarily construct the function $X(\hat{x})$ in the usual space corresponding to the Fourier transform $X(t)$. We then use the formulae[20]

$$\int_0^\infty t^s \cos(t\hat{x})dt = \{\pi^{1/2}\Gamma[(1+s)/2]/2\Gamma(-s/2)\}\left(\frac{\hat{x}}{2}\right)^{-(s+1)}, \tag{82}$$

$$\int_0^\infty t^s \sin(t\hat{x})dt = [\pi^{1/2}\Gamma(1+s/2)/\Gamma(1/2-s/2)]\left(\frac{\hat{x}}{2}\right)^{-(s+1)}. \tag{83}$$

As a result, we find the resistive asymptotic of the form

$$X_\pm \sim \hat{x}^{-(s+1)}(1+\Delta_R^{(\pm)}\hat{x}^{2s+1}). \tag{84}$$

Here,

$$\Delta_R^{(\pm)} = \nu_\pm(s)\Delta_R, \tag{85}$$

$$\nu_\pm(s) = \Delta/\Delta_\pm. \tag{86}$$

Note that, allowing for the formula for convolution of Γ functions, Eq. (86) for $\nu_\pm(s)$ can be represented in the form

$$\nu_\pm(s) = \frac{\pi}{2\Gamma^2(1+s)}\begin{cases}1/\sin^2(\pi s/2),\\ 1/\cos^2(\pi s/2).\end{cases} \tag{87}$$

The matching explained leads to the dispersion relation

$$\Delta_p\Delta_c - \tfrac{1}{2}(\Delta_R^{(+)}+\Delta_R^{(-)})(\Delta_p+\Delta_c) + \Delta_R^{(+)}\Delta_R^{(-)} = 0. \tag{88}$$

We note that, if $m \gg 1$, Eq. (88) reduces to the double dispersion relations for the even and odd modes. In contrast to this, at finite m the perturbed displacement of the resistive modes is a combination of even and odd parts.

VI. RESISTIVE-INTERCHANGE MODES WITH FINITE $m>1$

The dispersion relation (66) for the resistive-interchange modes is valid for $m \gg 1$. In order to elucidate behavior of these modes for finite $m>1$ we turn to general dispersion relation (88). For simplicity, we assume $s \ll 1$; then

$$\Delta_R^{(+)} = \Delta_R/(\pi s^2), \tag{89}$$

$$\Delta_R^{(-)} = \pi\Delta_R/2, \tag{90}$$

$$\Delta_c = -\Delta_p = -2/(sm). \tag{91}$$

As a result, Eq. (88) reduces to

$$\Delta_R^2 = 4/m^2. \tag{92}$$

It follows that

$$\Delta_R = \pm 2/m, \tag{93}$$

i.e., allowing for Eq. (53),

$$h(M) = \mp 4(\gamma_R/\gamma)^{1/2}/m. \tag{94}$$

The signs $\pm$ in the right-hand side of Eq. (94) can be interpreted as consequences of different parity of the modes considered. It is evident that for $m\to\infty$, Eq. (94) transits to Eq. (66). The effect of finite $1/m$ modifies both the roots given by Eqs. (67) and (68). We restrict ourselves to the case of the root (68). The idea of our analysis of the finite $1/m$ effect is based on the method of successive approximations. We represent the growth rate γ in the form

$$\gamma = \gamma^{(0)} + \gamma^{(1)}, \tag{95}$$

where $\gamma^{(0)}$ is given by Eq. (74), while $\gamma^{(1)}$ is the part of the growth rate of the order of $1/m$.

In the scope of this idea, the function $h(M)$, introduced by Eq. (54), reduces to

$$h(M) = (4l-1)\Gamma(1/2-l)p^{(1)}. \tag{96}$$

Here,

$$p^{(1)} = -\frac{4l+1}{4s}M^{(1)}, \tag{97}$$

$$M^{(1)}=(\gamma^{(1)})^{3/2}/(\omega_A^2\gamma_R)^{1/2}. \quad (98)$$

Substituting Eq. (96) into Eq. (94) and allowing for Eq. (74), we arrive at

$$\frac{\gamma^{(1)}}{\gamma^{(0)}}=\left[\mp\frac{4}{ml(4l+1)\Gamma(1/2-l)(-1)^{l-1}(l-1)!}\right]^{2/3}\times\left(-\frac{s}{4l+1}\right)^{-2/9}\left(\frac{\gamma_R}{\omega_A}\right)^{2/9}. \quad (99)$$

The ratio γ_R/ω_A is the main small parameter of the problem of the resistive modes. Thereby, Eq. (99) evidences that, as a result of finite $1/m$, the growth rate of resistive-interchange modes differs from that of the modes with $m \gg 1$ on a small value. Thus, we have shown that the theory of the $m \gg 1$ resistive-interchange modes is valid also for finite $m>1$.

VII. THE MODES WITH $m=1$

A. Dispersion relation

Considering the modes with $m=1$, we restrict ourselves to the case $s \ll 1$. Moreover, the parameter s is assumed to be so small that the inequality

$$|s \ln \Delta\hat{x}| \ll 1 \quad (100)$$

is satisfied, where $\Delta\hat{x}$ is the characteristic length of the singular layer.

1. The approximation of parabolic profiles

Let us appeal to the particular case of parabolic profiles studied in the Appendix and turn to the ideal asymptotic given by Eq. (A16). We note that for not large m, this asymptotic is true only for negative s, $s<0$. In the contrary case, instead of Eq. (A16), from Eqs. (A9) and (A14) we find the asymptotic of the form for such m

$$X \sim |\hat{x}|^{-(1+s)}\left[1+\left(\frac{\Delta_c}{\Delta_p}\right)|\hat{x}|^{2s+1}\right]-\frac{|\hat{x}|^{-s}}{m}\left[\frac{AB/s+m+1}{B(B-C+1)/s+m+1}\right]. \quad (101)$$

It can be seen that for finite m and $s>0$, terms of order $|\hat{x}|^{-s}$ appear on the right-hand side of Eq. (101). Similar terms in the resistive asymptotic Eq. (52) are absent. As a result of this, matching the asymptotic Eq. (101) with Eq. (52) proves to be problematic.

The additional terms in Eq. (101) contain the part proportional to $1/s$. Therefore, one can suggest that it is impossible to perform in Eq. (101) the limiting transition to the case $s\to 0$. However, according to Eqs. (A17) and (A18), for $s \to 0$ the values Δ_c and Δ_p are given by approximate expressions

$$\Delta_c \to -2/(sm), \quad (102)$$

$$\Delta_p \to 2/(sm). \quad (103)$$

In addition, turning to Eqs. (A10)–(A12), we obtain that for $s \to 0$,

$$AB/s \to -2/s, \quad (104)$$

$$B(B-C+1)/s \to 2/s. \quad (105)$$

Equation (101) then takes the form (for further details, see Ref. 3)

$$X \sim |\hat{x}^{-1}|\left[1+\left(\frac{\tilde{\Delta}_c}{\tilde{\Delta}_p}\right)|\hat{x}| \mp \frac{2}{sm}(|\hat{x}|^{s}-|\hat{x}|^{-s})\right], \quad (106)$$

where $\tilde{\Delta}_c$ and $\tilde{\Delta}_p$ are quantities defined by the relations

$$\tilde{\Delta}_c=\Delta_c+2/(sm)+1/m, \quad (107)$$

$$\tilde{\Delta}_p=\Delta_p-2/(sm)-1/m. \quad (108)$$

According to Ref. 3, the explicit forms of $\tilde{\Delta}_c$ and $\tilde{\Delta}_p$ are the following:

$$\tilde{\Delta}_c=4/s, \quad (109)$$

$$\tilde{\Delta}_p=2[1-2\psi(1)], \quad (110)$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the psi function.

On the other hand, it was explained in Ref. 3 that in the case of parabolic profiles considered,

$$s=-4\lambda_H/\pi, \quad (111)$$

where λ_H is the dimensionless growth rate of the $m=1$ ideal kink modes given by (for further details, see Ref. 1)

$$\lambda_H=-\pi s/4. \quad (112)$$

Allowing for Eq. (111), the dispersion relation (88) for the $m=1$ resistive kink modes reduces to

$$\Delta_R=-1/\lambda_H, \quad (113)$$

where Δ_R is given by Eq. (53) for $s \ll 1$.

2. General profiles of plasma pressure and rotation frequency

Following Ref. 3, one can show that Eq. (113) is valid for arbitrary profiles of plasma pressure and rotation frequency if, instead of Eq. (112), one takes

$$\lambda_H=-\frac{\pi}{S^2B_\theta^2 n}\int_0^{r_0} rU^{\mathrm{V}}dr, \quad (114)$$

where S and B_θ are calculated for $r=r_0$. In the explicit form, Eq. (114) means

$$\lambda_H=\lambda_H^{(0)}+\lambda_H^{(r)}, \quad (115)$$

where

$$\lambda_H^{(0)}=-\frac{\pi k_z^2}{S^2B_\theta^2 n}\int_0^{r_0} r[8\pi r p_0'-B_{0\theta}^2(1-nq)(1+3nq)]dr, \quad (116)$$

$$\lambda_H^{(r)}=-\frac{4\pi^2}{S^2B_\theta^2 n}\int_0^{r_0} r(k_z^2r^2+h_\theta^2)\frac{d\Omega^2}{d\ln r}dr. \quad (117)$$

B. Rotational reconnecting modes

Let us consider Eq. (113) in the case $M \simeq s$. It then follows from Eq. (53) that

$$\Delta_R = \frac{1}{2}\left(\frac{\gamma}{\gamma_R M}\right)^{1/2}(M+s)\frac{\Gamma[3/4+s/(4M)]}{\Gamma[5/4+s(4M)]}. \qquad (118)$$

For $s \ll M$, Eq. (118) reduces to

$$\Delta_R = \frac{1}{2}\left(\frac{\gamma M}{\gamma_R}\right)^{1/2}\frac{\Gamma(3/4)}{\Gamma(5/4)}. \qquad (119)$$

In the case $\lambda_H < 0$ and $|\lambda_H| \gg \varepsilon_R^{1/3}$, where $\varepsilon_R = \gamma_R/\omega_A$, it follows from Eqs. (113) and (116) that

$$\gamma \simeq \omega_A \varepsilon_R^{1/3}|\lambda_H|^{-4/3}. \qquad (120)$$

Neglecting the Velikhov effect, this solution characterizes the so-called reconnecting mode,[7,8] which is an analog of the tearing mode.[6] In the contrary case, when the Velikhov effect is overwhelming, so that $\lambda_H = \lambda_H^{(r)}$, Eq. (120) yields

$$\gamma \simeq \omega_A \varepsilon_A^{1/3}|\lambda_H^{(r)}|^{-4/3}. \qquad (121)$$

The unstable mode described by this dispersion relation can be called the rotational reconnecting mode.

VIII. DISCUSSION

The recent stability theory of magnetic confinement systems, dealing with the magnetic well/hill effect, has an evident drawback, on that, in studying the problems of differentially rotating plasma, it does not allow for the Velikhov effect revealed as a rotation-induced magnetic well/hill. The first step to eliminate this drawback has been made in Ref. 21 initially incorporating the Velikhov effect into the theory of the Suydam modes (the ideal interchange modes) in the cylindrical plasma). The incorporation has been continued in the paper I considering both the Suydam modes and the $m = 1$ ideal internal kink mode. Nonetheless, there is a family of internal resistive kink modes also sensitive to the hill/well effect, which are the subject of the present paper. In agreement with the paper I, we have explained that the Velikhov effect is revealed as the magnetic hill effect in the case of decreasing rotation frequency profile, $d\Omega^2/d\ln r < 0$. Such a profile is treated as unfavorable. It enhances the resistive modes. In the opposite case of increasing rotation frequency profile, i.e., $d\Omega^2/d\ln r > 0$, the Velikhov effect leads to the magnetic well effect suppressing these modes.

Based on the idea that the Velikhov effect is revealed as a contribution into the magnetic hill/well, we have analyzed the internal resistive modes sensitive to the hill/well. We have pointed out a family of such modes in a rotating plasma driven by the Velikhov effect. They are the rotational resistive interchange modes with $m \gg 1$ (see Sec. IV D) and with the finite $m > 1$ (see Sec. VI) as well as the rotational reconnecting modes [see Eq. (121)].

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APPENDIX: NONLOCAL IDEAL SOLUTION FOR PARABOLIC PROFILE OF CURRENT AND CONSTANT MAGNETIC HILL/WELL

Let us take the equilibrium longitudinal current j_{0z} to be parabolically distributed along the cylinder radius,

$$j_{0z} = (1 - \alpha_J r^2/r_*^2)j^{(0)}. \qquad (A1)$$

Here, $j^{(0)}$ is a constant, α_J is one more constant assuming to be a small parameter, i.e., $\alpha_J \ll 1$, and r_* is the casing (the wall) coordinate. In this case,

$$\frac{1}{q} - \frac{n}{m} = \frac{n}{m}\frac{S(r_0)}{2}\left(1 - \frac{r^2}{r_0^2}\right), \qquad (A2)$$

where $n = -k_z R$ is an integer called the toroidal mode number, the parameter $S(r_0)$ is given by

$$S(r_0) = \alpha_J r_0^2/r_*^2. \qquad (A3)$$

The equilibrium plasma pressure $p_0(r)$ is taken to be parabolic and given by

$$p_0(r) = (1 - r^2/r_*^2)p^{(0)}, \qquad (A4)$$

where $p^{(0)}$ is a constant. The rotation frequency profile $\Omega(r)$ is approximated by

$$\rho_0\left(k_z^2 + \frac{h_\theta^2}{r^2}\right)\frac{d\Omega^2}{d\ln r} = \text{const}. \qquad (A5)$$

The value U_0^V introduced by Eq. (33) can then be approximated by a constant,

$$U_0^V = \text{const}. \qquad (A6)$$

As a result, Eq. (19) with $\tilde{\omega}^2 \to 0$ reduces to

$$\frac{d}{dz}\left[z^2(1-z)^2\frac{dX}{dz}\right] - \left[\frac{m^2-1}{4}(1-z)^2 + U_0^V z\right]X = 0, \qquad (A7)$$

where

$$z = r^2/r_0^2. \qquad (A8)$$

Equation (A7) has a precise solution expressed in terms of the hypergeometrical functions. In the central region, i.e., $r < r_0$, the solution finite at $r = 0$ is of the form

$$X = z^{(m-1)/2}(1-z)^{-(s+1)}F(A,B;C;z). \qquad (A9)$$

Here,

$$A = (m - 2s - \bar{m})/2, \qquad (A10)$$

$$B = (m - 2s + \bar{m})/2, \qquad (A11)$$

$$C = m + 1, \qquad (A12)$$

$$\bar{m} = (m^2+8)^{1/2}, \tag{A13}$$

and the parameter s is given by Eq. (39). In the peripheral region, $r_0 < r \leqslant r_*$, the solution of Eq. (A7) vanishing in the casing, i.e., at $r = r_*$, is of the form

$$X = z^{-(\bar{m}+3)/2}(1-1/z)^{-(s+1)}[F(B,B-C+1;B-A+1;1/z) + Dz^{\bar{m}}F(A,A-C+1;A-B+1;1/z)], \tag{A14}$$

where

$$D = -\left(\frac{r_0}{r_*}\right)^{2\bar{m}} \frac{F(B,B-C+1;B-A+1;r_0^2/r_*^2)}{F(A,A-C+1;A-B+1;r_0^2/r_*^2)}. \tag{A15}$$

In accordance with Ref. 3, using Eqs. (A9) and (A14), the asymptotic of X near the singular layer in the approximation $r_0/r_* \ll 1$ is given by

$$X \sim |\hat{x}|^{-(1+s)}\left[1 + \left(\frac{\Delta_c}{\Delta_p}\right)|\hat{x}|^{2s+1}\right]. \tag{A16}$$

Here, $\hat{x} = m(r-r_0)/r_0$,

$$\Delta_c = \left(\frac{2}{m}\right)^{2s+1} \frac{\Gamma(-1-2s)}{\Gamma(1+2s)} \frac{\Gamma(1+2s+B)\Gamma(1+2s+A)}{\Gamma(A)\Gamma(B)}, \tag{A17}$$

$$\Delta_p = \left(\frac{2}{m}\right)^{2s+1} \frac{\Gamma(-1-2s)}{\Gamma(1+2s)} \frac{\Gamma(1-A)\Gamma(1+2s+B)}{\Gamma(B)\Gamma(-A-2s)}. \tag{A18}$$

[1] A. B. Mikhailovskii, J. G. Lominadze, A. P. Churikov, N. N. Erokhin, S. V. Konovalov, V. S. Tsypin, and R. M. O. Galvao, "Ideal internal kink modes in a differentially rotating cylindrical plasma," Plasma Phys. Rep. (submitted).
[2] E. P. Velikhov, Sov. Phys. JETP **36**, 995 (1959).
[3] A. B. Mikhailovskii, *Instabilities in a Confined Plasmas* (Institute of Physics, Bristol, 1998).
[4] S. A. Balbus and J. F. Hawley, Astrophys. J. **376**, 214 (1991).
[5] N. I. Shakura and R. A. Sunyaev, Astron. Astrophys. **24**, 337 (1973).
[6] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[7] B. Coppi, R. Galvao, R. Pellat, M. N. Rosenbluth, and P. H. Rutherford, Sov. J. Plasma Phys. **2**, 533 (1976).
[8] G. Ara, B. Basu, B. Coppi, G. Laval, M. N. Rosenbluth, and B. V. Waddell, Ann. Phys. (N.Y.) **112**, 443 (1978).
[9] A. Bondeson and H. X. Xie, Phys. Plasmas **4**, 2081 (1997).
[10] A. D. Wood, E. O'Riordan, N. Sweeney, and R. B. Paris, J. Plasma Phys. **70**, 155 (2004).
[11] R. Coelho and E. Lazzaro, Phys. Plasmas **14**, 012101 (2007).
[12] D. Correa-Restrepo, Z. Naturforsch. A **37**, 848 (1982).
[13] A. B. Mikhailovskii, B. N. Kuvshinov, V. P. Lakhin, S. V. Novakovskii, A. I. Smolyakov, S. E. Sharapov, and A. P. Churikov, Plasma Phys. Controlled Fusion **31**, 1741 (1989).
[14] A. B. Mikhailovskii, B. N. Kuvshinov, V. P. Lakhin, S. V. Novakovskii, S. E. Sharapov, A. I. Smolyakov, and A. P. Churikov, Plasma Phys. Controlled Fusion **31**, 1759 (1989).
[15] B. N. Kuvshinov, A. B. Mikhailovskii, and E. G. Tatarinov, Sov. J. Plasma Phys. **14**, 239 (1988).
[16] B. N. Kuvshinov and A. B. Mikhailovskii, Sov. J. Plasma Phys. **16**, 639 (1990).
[17] L. E. Zakharov, Preprint IAE-3540/16, Moscow, Kurchatov Institute, 1982 (in Russian).
[18] L. J. Slater, *Handbook of Mathematical Functions*, edited by M. Abramovitz and I. A. Stegun (Dover, New York, 1968), p. 503.
[19] F. Oberhettinger, *Handbook of Mathematical Functions*, edited by M. Abramovitz and I. A. Stegun (Dover, New York, 1968), p. 585.
[20] I. S. Gradshtein and I. M. Ryzhik, *Table of Iintegrals, Series, and Products* (Academic, New York, 1965).
[21] A. Bondeson, R. Iakono, and A. Bhattacharjee, Phys. Fluids **30**, 2167 (1987).