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Nonlinear quantum equations: Classical field theory

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An exact classical field theory for nonlinear quantum equations is presented herein. It has been applied recently to a nonlinear Schrödinger equation, and it is shown herein to hold also for a nonlinear generalization of the Klein-Gordon equation. These generalizations were carried by introducing nonlinear terms, characterized by exponents depending on an index q, in such a way that the standard, linear equations, are recovered in the limit $q \to 1$. The main characteristic of this field theory consists on the fact that besides the usual $\Psi(\vec{x}, t)$, a new field $\Phi(\vec{x}, t)$ needs to be introduced in the Lagrangian, as well. The field $\Phi(\vec{x}, t)$, which is defined by means of an additional equation, becomes $\Psi^*(\vec{x}, t)$ only when $q \to 1$. The solutions for the fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$ are found herein, being expressed in terms of a q-plane wave; moreover, both field equations lead to the relation $E^2 = p^2c^2 + m^2c^4$, for all values of q. The fact that such a classical field theory works well for two very distinct nonlinear quantum equations, namely, the Schrödinger and Klein-Gordon ones, suggests that this procedure should be appropriate for a wider class nonlinear equations. It is shown that the standard global gauge invariance is broken as a consequence of the nonlinearity. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4824129]

I. INTRODUCTION

An intense effort has been done lately in the study of deformations of some mathematical structures[1–3] and several physical systems.[4] Different approaches were used in considering these deformations, going from a noncommutative space-time, motivated by fundamental issues in gravity,[5] to phenomenological procedures in nuclear physics.[6] Very often, these deformations lead to nonlinear (NL) equations, like the recent generalizations of the main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations.[7]

In the recent years several complex phenomena in nature demanded NL equations for their description.[8–10] As a consequence, one frequently finds difficulties in solving these equations exactly, so that approximative analytical methods, or numerical procedures have been used and in some situations new algorithms were developed. In these later cases, many researches benefited from the recent advances in computer technology, leading to considerable progresses in many areas of physics that use these types of equations, like nonlinear optics, superconductivity, plasma physics, and nonequilibrium statistical mechanics.

Many NL equations appear as generalizations of linear ones, so that the later may be recovered as particular cases. Among many possible ways, these generalizations have been carried in the literature mostly by considering two different procedures: (i) the addition of new NL terms to a linear equation and (ii) the modification of the exponent of existing linear terms. Many NL Schrödinger equations were constructed according to the first procedure:[10] the introduction of an extra cubic term in the wave function is responsible for the modulation of the wave function for some particular type of solution. The second one has been much used in nonextensive statistical mechanics;[11,12] this theory emerged from the generalization of the Boltzmann-Gibbs entropy,

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by introducing a real index q, such as to recover the former in the limit $q \to 1$.[13] A power $(2 - q)$ in the probability of the diffusion term has led to a NL Fokker-Planck equation[14,15] capable of explaining many interesting physical phenomena related to anomalous diffusion[16] (for which $q \neq 1$). Moreover, the generalizations of the main equations of quantum physics done in Ref. 7 followed this second procedure. Apart from those, linear inhomogeneous equations have been proposed as well, and some were shown to be related to nonextensive statistical mechanics, like inhomogeneous Fokker-Planck[17] and Schrödinger[18] equations.

The linear Fokker-Planck equation is usually associated to the Boltzmann-Gibbs entropy either through the H-theorem, or by comparing its solution with the distribution that comes from an extremization of this entropy.[19,20] In a similar manner, the above-mentioned generalized nonadditive entropy is connected to the NL Fokker-Planck equation of Refs. 14 and 15 through the proof of an H-theorem.[21–23] Moreover, the extremization of this entropy leads to the q-Gaussian distribution, which represents a generalization of the standard Gaussian; this distribution solves the corresponding NL Fokker-Planck equation and appears also in generalized forms of the Central Limit Theorem.[24,25] The q-Gaussian distribution has been very useful for experiments in real systems;[11,12] among many, one could highlight: (i) the velocities of cold atoms in dissipative optical lattices;[26] (ii) the velocities of particles in quasi two-dimensional dusty plasma;[27] (iii) single ions in radio frequency traps interacting with a classical buffer gas;[28] (iv) the relaxation curves of Ruderman-Kittel-Kasuya-Yosida (RKKY) spin glasses, like CuMn and AuFe;[29] (v) transverse momenta distributions at Large Hadron Collider (LHC) experiments;[30] and (vi) the overdamped motion of interacting vortices in type II superconductors.[31–33]

In a recent work,[34] a classical field theory for the NL Schrödinger equation (NLSE) introduced in Ref. 7 was developed. It was shown that besides the usual $\Psi(\vec{x}, t)$, a new field $\Phi(\vec{x}, t)$ has to be considered in order to fulfill the equations of motion; curiously, this new field becomes $\Psi^*(\vec{x}, t)$ only when $q \to 1$. In the present work, we follow a similar procedure towards the formulation of a classical field theory for the NL Klein-Gordon equation (NLKGE) of Ref. 7; analogously, we show that an additional field needs also to be introduced. In Sec. II, we present the NLSE and NLKGE, together with their common solution, and its main properties; the classical-field-theory approach for the NLSE is briefly reviewed. In Sec. III, we derive the classical field theory for NLKGE; in Sec. IV, we present our main conclusions.

II. NONLINEAR SCHRÖDINGER AND KLEIN-GORDON EQUATIONS: COMMON SOLUTION AND PROPERTIES

Proposals for nonlinear generalizations of the main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations, were presented in Ref. 7 in the case of a free particle of mass m. Herein, we will be concerned with two of these generalizations, i.e., the NLSE,

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = -\frac{1}{2-q} \frac{\hbar^2}{2m} \nabla^2 \left[\Psi(\vec{x}, t)\right]^{2-q} \tag{1}$$

and the NLKGE,

$$\nabla^2 \Psi(\vec{x}, t) = \frac{1}{c^2} \frac{\partial^2 \Psi(\vec{x}, t)}{\partial t^2} + q \frac{m^2 c^2}{\hbar^2} \left[\Psi(\vec{x}, t)\right]^{2q-1}. \tag{2}$$

The equations above are valid for a general d-dimensional position vector $\vec{x}$ and are written herein in terms of a dimensionless wave function $\Psi(\vec{x}, t)$ (corresponding to the one defined in Ref. 7 scaled by its amplitude). They recover the corresponding linear equations in the limit $q \to 1$ and present the following q-plane wave as a common solution,

$$\Psi(\vec{x}, t) = \exp_q \left[\frac{i}{\hbar}(\vec{p} \cdot \vec{x} - Et)\right] \qquad (q \in \Re), \tag{3}$$

expressed in terms of the q-exponential function $\exp_q(u)$ that emerges in nonextensive statistical mechanics.[11] This function generalizes the standard exponential, and for a pure imaginary iu, it is defined as the principal value of

$$\exp_q(iu) = [1 + (1-q)iu\,]^{\frac{1}{1-q}}\,; \qquad \exp_1(iu) \equiv \exp(iu), \tag{4}$$

where we used $\lim_{\epsilon \to 0}(1 + \epsilon)^{1/\epsilon} = e$. One may verify that the q-plane wave of Eq. (3) satisfies both quantum equations above, preserving the corresponding energy-momentum relations for all q,

$$E = \frac{p^2}{2m} \quad \text{(NLSE)}; \qquad E^2 = p^2c^2 + m^2c^4 \quad \text{(NLKGE)}. \tag{5}$$

For $q \neq 1$ the q-exponential $\exp_q(iu)$ exhibits an oscillatory behavior characterized by an amplitude $\rho_q \neq 1$,[35–37]

$$\exp_q(\pm iu) = \cos_q(u) \pm i\,\sin_q(u), \tag{6}$$

$$\cos_q(u) = \rho_q(u)\cos\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \tag{7}$$

$$\sin_q(u) = \rho_q(u)\sin\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \tag{8}$$

$$\rho_q(u) = \left[1 + (1-q)^2u^2\right]^{1/[2(1-q)]}, \tag{9}$$

from which one sees that $\rho_q(u)$ decreases for increasing arguments, if $q > 1$. From Eqs. (6)–(9) one notices that $\cos_q(u)$ and $\sin_q(u)$ cannot be zero simultaneously, so that $\exp_q(\pm iu) \neq 0$. Moreover, $\exp_q(iu)$ presents further peculiar properties,

$$[\exp_q(iu)]^* = \exp_q(-iu) = [1 - (1-q)iu\,]^{\frac{1}{1-q}}, \tag{10}$$

$$\exp_q(iu)[\exp_q(iu)]^* = [\rho_q(u)]^2 = \left[1 + (1-q)^2u^2\right]^{\frac{1}{1-q}}, \tag{11}$$

$$\exp_q(iu_1)\exp_q(iu_2) = \exp_q[iu_1 + iu_2 - (1-q)u_1u_2], \tag{12}$$

$$\left\{[\exp_q(iu)]^\alpha\right\}^* = \left\{[\exp_q(iu)]^*\right\}^\alpha = [\exp_q(-iu)]^\alpha, \tag{13}$$

for any α real. By integrating Eq. (11) from $-\infty$ to $+\infty$, one obtains,[36]

$$\mathcal{I}_q = \int_{-\infty}^{\infty} du\,[\rho_q(u)]^2 = \left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{3-q}{2(q-1)}\right)}{(q-1)\,\Gamma\left(\frac{1}{q-1}\right)}\right]^{1/2} \tag{14}$$

leading to the physically important property of square integrability for $1 < q < 3$; as some simple typical examples, one has $\mathcal{I}_{3/2} = \mathcal{I}_2 = \sqrt{\pi}$. One should notice that this integral diverges in both limits $q \to 1$ and $q \to 3$, as well as for any $q < 1$. Since $\mathcal{I}_q$ is related to the norm of the wave function,[38] its divergence in the limit $q \to 1$ reflects the nonintegrability of the standard free particle solution (plane wave) in open space, e.g., its norm is not defined. As a consequence, in this case, the free particle is only treatable inside a box of finite dimensions.

An exact classical field theory associated with Eq. (1) was developed in Ref. 34. One curious aspect concerns the fact that a Lagrangian density constructed only in terms of $\Psi(\vec{x}, t)$ and $\Psi^*(\vec{x}, t)$ (as usually done in the case $q = 1$[39–41]) does not lead to the NLSE of Eq. (1) and its complex conjugate; for this reason, a second field, $\Phi(\vec{x}, t)$, had to be introduced by means of a Lagrangian density,

$$\mathcal{L} \equiv \mathcal{L}\left(\Psi, \vec{\nabla}\Psi, \dot{\Psi}, \Phi, \vec{\nabla}\Phi, \dot{\Phi}, \Psi^*, \vec{\nabla}\Psi^*, \dot{\Psi}^*, \Phi^*, \vec{\nabla}\Phi^*, \dot{\Phi}^*\right), \tag{15}$$

which depends also on the spatial and time derivatives of these fields. The equations of motion for the classical fields were derived in the usual way, through the principle of stationary action; the Euler-Lagrange equation for the field Φ yielded the NLSE of Eq. (1). Similarly, the Euler-Lagrange equation for the field $\Psi(\vec{x}, t)$ led to

$$i\hbar \frac{\partial \Phi(\vec{x}, t)}{\partial t} = \frac{\hbar^2}{2m} [\Psi(\vec{x}, t)]^{1-q} \, \nabla^2 \Phi(\vec{x}, t), \tag{16}$$

as an additional equation, associated with Eq. (1); similar frameworks hold for the complex-conjugate fields, $\Psi^*(\vec{x}, t)$ and $\Phi^*(\vec{x}, t)$. One should notice that Eq. (16) becomes the complex conjugate of Eq. (1) only for $q = 1$, in which case $\Phi(\vec{x}, t) = \Psi^*(\vec{x}, t)$. For all $q \neq 1$, one has a field $\Phi(\vec{x}, t)$ distinct from $\Psi^*(\vec{x}, t)$, with $\Phi(\vec{x}, t)$ and $\Psi(\vec{x}, t)$ being related through Eq. (16). Substituting the solution of Eq. (1) [i.e., the q-exponential of Eq. (3)] in Eq. (16), one finds,

$$\Phi(\vec{x}, t) = \left\{ \exp_q \left[\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right] \right\}^{-q} = [\Psi(\vec{x}, t)]^{-q}. \tag{17}$$

From the fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$ above the following quantities and properties related to the NLSE were obtained:[34] (i) the canonical conjugate fields leading to the corresponding Hamiltonian density; (ii) the energy spectrum of a free particle; (iii) a probability density satisfying a continuity equation; and (iv) the physically important property that the global gauge invariance does not hold in the present formalism. Considering the form of the solution in Eq. (17), such transformations are given by

$$\begin{aligned} &\Psi(\vec{x}, t) \to \mathrm{e}^{i\alpha} \Psi(\vec{x}, t); \quad \Psi^*(\vec{x}, t) \to \mathrm{e}^{-i\alpha} \Psi^*(\vec{x}, t), \\ &\Phi(\vec{x}, t) \to \mathrm{e}^{-iq\alpha} \Phi(\vec{x}, t); \quad \Phi^*(\vec{x}, t) \to \mathrm{e}^{iq\alpha} \Phi^*(\vec{x}, t); \quad (\alpha \in \Re), \end{aligned} \tag{18}$$

where $\exp(\pm i\alpha)$ and $\exp(\pm iq\alpha)$ represent constant phase factors. Therefore, the Lagrangian density is not invariant under a constant phase change in the fields; as a consequence, all equations and corresponding physical properties derived from this Lagrangian are not invariant under global gauge transformations.

In Sec. III, we apply the procedure described above for the NLKGE of Eq. (2), showing that it leads to many common properties as compared to those found in Ref. 34.

III. CLASSICAL FIELD THEORY FOR THE NONLINEAR KLEIN-GORDON EQUATION

Although Eq. (2) and its solution in Eq. (3) are valid for a general d-dimensional position vector $\vec{x}$, herein, for simplicity, we will restrict ourselves to a three-dimensional vector $\vec{x}$; let us then introduce the four-dimensional space-time operators,[39–41]

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \equiv \left\{ \frac{\partial}{\partial (ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right\}; \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \equiv \left\{ \frac{\partial}{\partial (ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}, \tag{19}$$

whereas the same notation applies for the spatial derivatives, ∂^j and ∂_j. Therefore, Eq. (2) may be rewritten as

$$\partial^\mu \partial_\mu \Psi(\vec{x}, t) + q \, \frac{m^2 c^2}{\hbar^2} \, \left[\Psi(\vec{x}, t)\right]^{2q-1} = 0. \tag{20}$$

Similar to the case of the NLSE,[34] below we develop a field theory in terms two classical fields, $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$; then, we write the following Lagrangian density,

$$\begin{aligned} \mathcal{L} = A \Bigg\{ &\Phi(\vec{x}, t)[\partial^\mu \partial_\mu \Psi(\vec{x}, t)] + q \, \frac{m^2 c^2}{\hbar^2} \Phi(\vec{x}, t)[\Psi(\vec{x}, t)]^{(2q-1)} \\ + \; &\Phi^*(\vec{x}, t)[\partial^\mu \partial_\mu \Psi^*(\vec{x}, t)] + q \, \frac{m^2 c^2}{\hbar^2} \Phi^*(\vec{x}, t)[\Psi^*(\vec{x}, t)]^{(2q-1)} \Bigg\}, \end{aligned} \tag{21}$$

with the multiplicative factor, $A \equiv \hbar^2 c^2 / [2(1 - 3q)EV]$, depending on the total energy E, and as usual, the fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$ are confined to a finite volume V.[41] The Lagrangian above is equivalent to

$$\mathcal{L} = A \Bigg\{ -[\partial_\mu \Phi(\vec{x},t)][\partial^\mu \Psi(\vec{x},t)] + q\,\frac{m^2c^2}{\hbar^2}\Phi(\vec{x},t)[\Psi(\vec{x},t)]^{(2q-1)} + \partial_\mu[\Phi(\vec{x},t)\partial^\mu \Psi(\vec{x},t)]$$

$$- \; [\partial_\mu \Phi^*(\vec{x},t)][\partial^\mu \Psi^*(\vec{x},t)] + q\,\frac{m^2c^2}{\hbar^2}\Phi^*(\vec{x},t)[\Psi^*(\vec{x},t)]^{(2q-1)} + \partial_\mu[\Phi^*(\vec{x},t)\partial^\mu \Psi^*(\vec{x},t)] \Bigg\}, \quad (22)$$

where the total derivative terms are relevant for $q \neq 1$ (contrary to what happens in the linear case), as will be seen later on. One should notice that the above Lagrangian density presents higher-order derivatives, as compared to the one of Eq. (15), i.e., it depends on the fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$, as well as on their first and second derivatives,

$$\mathcal{L} \equiv \mathcal{L}\left(\Psi, \partial_\mu\Psi, \partial_\mu\partial_\nu\Psi, \Phi, \partial_\mu\Phi, \partial_\mu\partial_\nu\Phi, \Psi^*, \partial_\mu\Psi^*, \partial_\mu\partial_\nu\Psi^*, \Phi^*, \partial_\mu\Phi^*, \partial_\mu\partial_\nu\Phi^*\right). \quad (23)$$

In this case, the Euler-Lagrange equations should take into account higher-order terms, in such a way that for the field Φ one has,[42,43]

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\right] + \partial_\mu\partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu \Phi)}\right] = 0. \quad (24)$$

Substituting the Lagrangian density of Eq. (22) in the above Euler-Lagrange equation, one obtains the NLKGE of Eq. (20); carrying the same procedure in the Euler-Lagrange equation for the field Ψ,

$$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}\right] + \partial_\mu\partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu \Psi)}\right] = 0, \quad (25)$$

one obtains an additional equation for the field Φ,

$$\nabla^2\Phi(\vec{x},t) = \frac{1}{c^2}\frac{\partial^2\Phi(\vec{x},t)}{\partial t^2} + q(2q-1)\,\frac{m^2c^2}{\hbar^2}\,\Phi(\vec{x},t)\left[\Psi(\vec{x},t)\right]^{2(q-1)}. \quad (26)$$

The equation above becomes the complex conjugate of Eq. (2) for $q \to 1$ through the identification $\Phi(\vec{x},t) = \Psi^*(\vec{x},t)$; substituting the q-exponential of Eq. (3) in Eq. (26), one finds,

$$\Phi(\vec{x},t) = \left\{\exp_q\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right]\right\}^{-(2q-1)} = [\Psi(\vec{x},t)]^{-(2q-1)}. \quad (27)$$

One important aspect of Eq. (26) may be seen by substituting the above solutions for $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ in order to reproduce the same energy-momentum relation of Eq. (5), for all q.

Let us now define the Hamiltonian density by[43]

$$\mathcal{H} = \Pi_\Psi\dot{\Psi} + \Pi_{\Psi^*}\dot{\Psi}^* + \Pi_\Phi\dot{\Phi} + \Pi_{\Phi^*}\dot{\Phi}^* + \Pi_{\dot{\Psi}}\ddot{\Psi} + \Pi_{\dot{\Phi}}\ddot{\Phi} + \Pi^*_{\dot{\Psi}^*}\ddot{\Psi}^* + \Pi^*_{\dot{\Phi}^*}\ddot{\Phi}^* - \mathcal{L}, \quad (28)$$

where the canonical momenta conjugated to the fields Ψ, Φ, $\dot{\Psi}$, and $\dot{\Phi}$ are

$$\Pi_\Psi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} - \frac{\partial}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\ddot{\Psi}} - 2\partial_j\frac{\partial\mathcal{L}}{\partial(\partial_j\dot{\Psi})} = -A\dot{\Psi}, \quad (29)$$

$$\Pi_\Phi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} - \frac{\partial}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\ddot{\Phi}} - 2\partial_j\frac{\partial\mathcal{L}}{\partial(\partial_j\dot{\Phi})} = 0, \quad (30)$$

$$\Pi_{\dot{\Psi}} = \frac{\partial\mathcal{L}}{\partial\ddot{\Psi}} = \Psi, \quad (31)$$

$$\Pi_{\dot{\Phi}} = \frac{\partial\mathcal{L}}{\partial\ddot{\Phi}} = 0, \quad (32)$$

with similar results holding for the corresponding complex conjugates. The Hamiltonian density becomes

$$\mathcal{H} = A\left\{-\frac{1}{c^2}\frac{\partial\Phi(\vec{x},t)}{\partial t}\frac{\partial\Psi(\vec{x},t)}{\partial t} + \Phi\nabla^2\Psi(\vec{x},t) - q\,\frac{m^2c^2}{\hbar^2}\Phi(\vec{x},t)[\Psi(\vec{x},t)]^{(2q-1)}\right.$$
$$\left. -\ \frac{1}{c^2}\frac{\partial\Phi^*(\vec{x},t)}{\partial t}\frac{\partial\Psi^*(\vec{x},t)}{\partial t} + \Phi^*\nabla^2\Psi^*(\vec{x},t) - q\,\frac{m^2c^2}{\hbar^2}\Phi^*(\vec{x},t)[\Psi^*(\vec{x},t)]^{(2q-1)}\right\}. \quad (33)$$

Considering the solutions for the fields Ψ and Φ according to Eqs. (3) and (27), the integral of the above Hamiltonian density in the volume V yields the energy,

$$\int d^3x\ \mathcal{H} = -\frac{2AV}{\hbar^2c^2}[(2q-1)E^2 + qp^2c^2 + qm^2c^4] = -\frac{2AV}{\hbar^2c^2}[(3q-1)E^2] = E, \quad (34)$$

which is valid for all values of q. In the calculations above one notices the relevance of the contour terms introduced in Eq. (22) for obtaining the correct energy.

For a Lagrangian density with a dependence like the one in Eq. (23), the components of the energy-momentum tensor may be calculated as[42]

$$T^{\alpha\mu} = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Psi)} - \partial_\beta\,\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Psi)}\right]\partial^\mu\Psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Psi)}\,\partial_\beta\partial^\mu\Psi$$
$$+ \left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi)} - \partial_\beta\,\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Phi)}\right]\partial^\mu\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Phi)}\,\partial_\beta\partial^\mu\Phi$$
$$+ \left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Psi^*)} - \partial_\beta\,\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Psi^*)}\right]\partial^\mu\Psi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Psi^*)}\,\partial_\beta\partial^\mu\Psi^*$$
$$+ \left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi^*)} - \partial_\beta\,\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Phi^*)}\right]\partial^\mu\Phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\Phi^*)}\,\partial_\beta\partial^\mu\Phi^* - g^{\alpha\mu}\mathcal{L}, \quad (35)$$

where $g^{\alpha\mu}$ denotes components of the metric tensor ($g^{\alpha\mu} = 0$, for $\alpha \neq \mu$, and $g^{00} = 1$, $g^{jj} = -1$). Using the Lagrangian density of Eq. (22), one gets

$$T^{\alpha\mu} = A\left\{-[\partial^\alpha\Phi(\vec{x},t)][\partial^\mu\Psi(\vec{x},t)] + \Phi(\vec{x},t)\partial^\alpha\partial^\mu\Psi(\vec{x},t) - g^{\alpha\mu}\Phi(\vec{x},t)\partial^\rho\partial_\rho\Psi(\vec{x},t)\right.$$
$$-\ [\partial^\alpha\Phi^*(\vec{x},t)][\partial^\mu\Psi^*(\vec{x},t)] + \Phi^*(\vec{x},t)\partial^\alpha\partial^\mu\Psi^*(\vec{x},t) - g^{\alpha\mu}\Phi^*(\vec{x},t)\partial^\rho\partial_\rho\Psi^*(\vec{x},t)$$
$$\left. -\ q\,\frac{m^2c^2}{\hbar^2}\Phi(\vec{x},t)[\Psi(\vec{x},t)]^{(2q-1)} - q\,\frac{m^2c^2}{\hbar^2}\Phi^*(\vec{x},t)[\Psi^*(\vec{x},t)]^{(2q-1)}\right\}. \quad (36)$$

As expected, the component T^{00} yields the Hamiltonian density, i.e., $T^{00} = \mathcal{H}$, whereas the momentum density components are given by

$$T^{0j} = \frac{A}{c}\left\{[\partial_t\Phi(\vec{x},t)][\partial_j\Psi(\vec{x},t)] - \Phi(\vec{x},t)\partial_j[\partial_t\Psi(\vec{x},t)]\right.$$
$$\left. +\ [\partial_t\Phi^*(\vec{x},t)][\partial_j\Psi^*(\vec{x},t)] - \Phi^*(\vec{x},t)\partial_j[\partial_t\Psi^*(\vec{x},t)]\right\}. \quad (37)$$

Integrating over the volume V and taking the solutions for the fields Ψ and Φ of Eqs. (3) and (27), one obtains the momentum components,

$$\int d^3x\ T^{0j} = -\frac{2AV}{c}\left[(3q-1)\frac{Ep^j}{\hbar^2}\right] = cp^j, \quad (38)$$

for all values of q.

Apart from satisfying the energy-momentum relation of Eq. (5), the present field equations [Eqs. (2) and (26)] are invariant under Lorentz transformations: since this property is directly related to the terms containing derivatives[41]—which were not changed in the present proposal—these equations preserve the Lorentz invariance. It should be mentioned that the fields Ψ and Φ above do not satisfy a continuity equation; this comes as a consequence of the probability definition, which is troublesome already in the linear case ($q = 1$), where it may take negative values.[41]

IV. CONCLUSIONS

We have presented a classical field theory associated with a recently proposed nonlinear Klein-Gordon equation. This nonlinearity was introduced by means of an exponent, depending on an index q, in the field $\Psi(\vec{x}, t)$ that appears in the corresponding mass term of the well-known linear equation; in this way, the linear equation is recovered in the limit $q \to 1$. Analogously to a recent classical field theory for a similar generalization of the Schrödinger equation,[34] herein we have shown that besides the usual $\Psi(\vec{x}, t)$, a new field $\Phi(\vec{x}, t)$, defined by means of an additional equation, must be introduced; this later field becomes $\Psi^*(\vec{x}, t)$ only when $q \to 1$.

The equations for the fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$ present simple solutions, written in terms of the q-plane wave, $\exp_q[i(\vec{p}\cdot\vec{x} - Et)/\hbar]$. By using these solutions, both equations lead to the Einstein relation, $E^2 = p^2c^2 + m^2c^4$, for all values of q. For $1 < q < 3$ the q-plane wave presents the property that its amplitude decreases when its argument $(\vec{p}\cdot\vec{x} - Et)$ increases, and so, this new type of solution describes a typical nonlinear oscillatory phenomenon. Due to this modulation, one may characterize a given physical system by a single value of q; therefore, by specifying the rate of decay of the q-plane wave amplitude (which should be a characteristic of a given physical system), we may determine the appropriate value of q. Hence, this type of solution can be considered as a good candidate for describing nonlinear physical phenomena characterized by oscillatory motion with modulation in both space and time, like those appearing in superconductivity, plasma physics, nonlinear optics, and lattice dynamics of solids. Exploratory efforts in possible applications to such nonlinear physical phenomena would be welcome.

A further interesting aspect concerns the fact the Lagrangian density proposed herein is not invariant under global gauge transformations; considering the form of solutions presented, such transformations are given by

$$\begin{aligned} &\Psi(\vec{x}, t) \to \mathrm{e}^{i\alpha}\Psi(\vec{x}, t); \quad \Psi^*(\vec{x}, t) \to \mathrm{e}^{-i\alpha}\Psi^*(\vec{x}, t), \\ &\Phi(\vec{x}, t) \to \mathrm{e}^{-i(2q-1)\alpha}\Phi(\vec{x}, t); \quad \Phi^*(\vec{x}, t) \to \mathrm{e}^{i(2q-1)\alpha}\Phi^*(\vec{x}, t); \quad (\alpha \in \Re), \end{aligned} \tag{39}$$

where $\exp(\pm i\alpha)$ and $\exp[\pm i(2q-1)\alpha]$ represent constant phase factors. This result implies that all equations derived from this Lagrangian also violate this symmetry. One should remind that, in standard field theory, the invariance of the Lagrangian under global gauge transformations is directly related to the property that the corresponding fields may be associated to charged particles that interact with light.

An important aspect concerns the identification of how the breakdown of global gauge invariance in Eq. (39) appears in the above formalism. Below we present a simple argument indicating that this violation comes as a direct consequence of the nonlinearity introduced. Let us consider a weak nonlinear contribution, $q = 1 + \epsilon/2$ (ϵ small, positive or negative), in the term $\left[\Psi(\vec{x}, t)\right]^{2q-1}$ that appears in Eq. (2). Moreover, we allow small fluctuations η around a value $\tilde{\Psi}$, such that $\Psi = \tilde{\Psi} + \eta$; hence, this nonlinear contribution may be expanded,

$$\begin{aligned} \Psi^{2q-1} &= (\tilde{\Psi} + \eta)^{1+\epsilon} = \exp\left[\ln(\tilde{\Psi} + \eta)^{1+\epsilon}\right] = (\tilde{\Psi} + \eta)\exp\left[\epsilon\ln(\tilde{\Psi} + \eta)\right] \\ &= (\tilde{\Psi} + \eta)\left[1 + \epsilon\ln(\tilde{\Psi} + \eta) + \cdots\right] = (\tilde{\Psi} + \eta)\left[1 + \epsilon\left(\ln\tilde{\Psi} + \frac{\eta}{\tilde{\Psi}} - \frac{1}{2}\frac{\eta^2}{\tilde{\Psi}^2}\right) + \cdots\right] \\ &= \tilde{\Psi} + \eta(1 + \epsilon + \epsilon\ln\tilde{\Psi}) + \epsilon\tilde{\Psi}\ln\tilde{\Psi} + \frac{\epsilon\,\eta^2}{2\tilde{\Psi}} + \cdots . \end{aligned} \tag{40}$$

The linear contribution in η can be interpreted as a modification in the mass of the system; moreover, for $\epsilon \neq 0$ the gauge symmetry is broken by the contribution $O(\eta^2)$ and only for $\epsilon = 0$ is that this symmetry is restored. The same analysis may be carried in the equation for the field $\Phi(\vec{x}, t)$ [Eq. (26)], leading to an analogous conclusion. This breaking of global gauge invariance may be useful for generating a mass for the electromagnetic field, describing a procedure similar to the Higgs mechanism; this can be analyzed by modifying the Lagrangian density of Eq. (21) such as to introduce an interaction with the electromagnetic field.[44]

Since a similar procedure has been proposed recently for the nonlinear Schrödinger equation, this work suggests that the approach presented herein should be rather general, in the sense that a wide class of classical field theories, associated with nonlinear equations, may be formulated by means of Lagrangian densities depending on two (or even more) classical fields. As natural extension of this work, the quantization and renormalization of the present classical field theory appear to be nontrivial and constitute goals for future investigations.

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