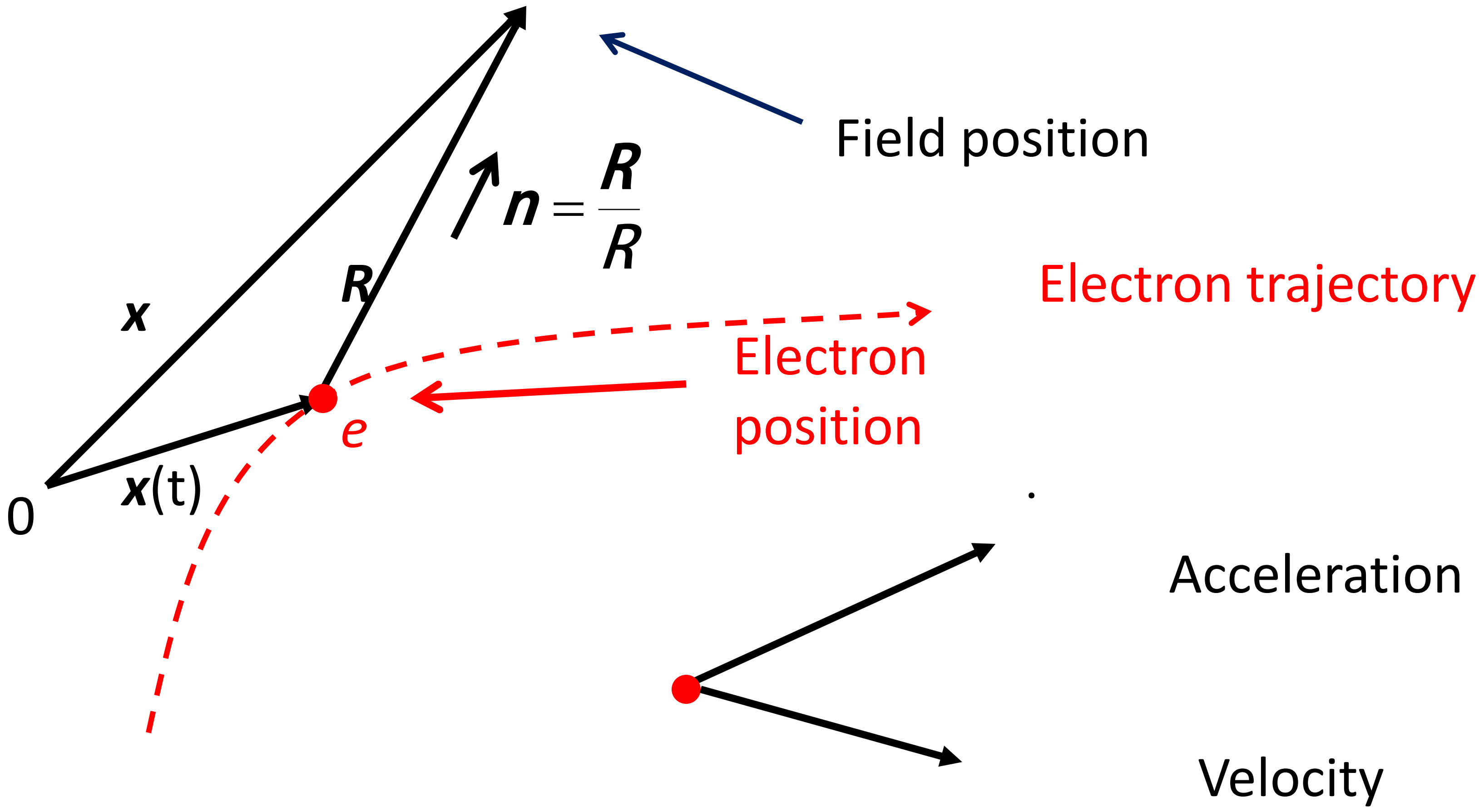
Spontaneous Undulator Radiation

Vector definitions



Lienard-Wiechert Retarded Fields

Velocity field

Radiation field

$$\boldsymbol{E}(\boldsymbol{x},t) = -\frac{|e|}{4\pi\varepsilon_0}\left[\frac{1}{s^3}\left\{\frac{\boldsymbol{n}-\quad}{\gamma^2 R^2} + \frac{\boldsymbol{n}\times\left[(\boldsymbol{n}-\quad)\times\dot{\ }\right]}{cR}\right\}\right]_{t-R/c}$$

$$\boldsymbol{B}(\boldsymbol{x},t) = \frac{\boldsymbol{n}\big|_{t-R/c}\times\boldsymbol{E}(\boldsymbol{x},t)}{c} \qquad s = (1-\boldsymbol{n}\cdot\quad)_{t-R/c}$$

$$\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}(t) = \sqrt{(x-x(t))^2 + (y-y(t))^2 + (y-y(t))^2}$$

Unbounded Solution of Maxwell’s Equations in vacuum for a single electron

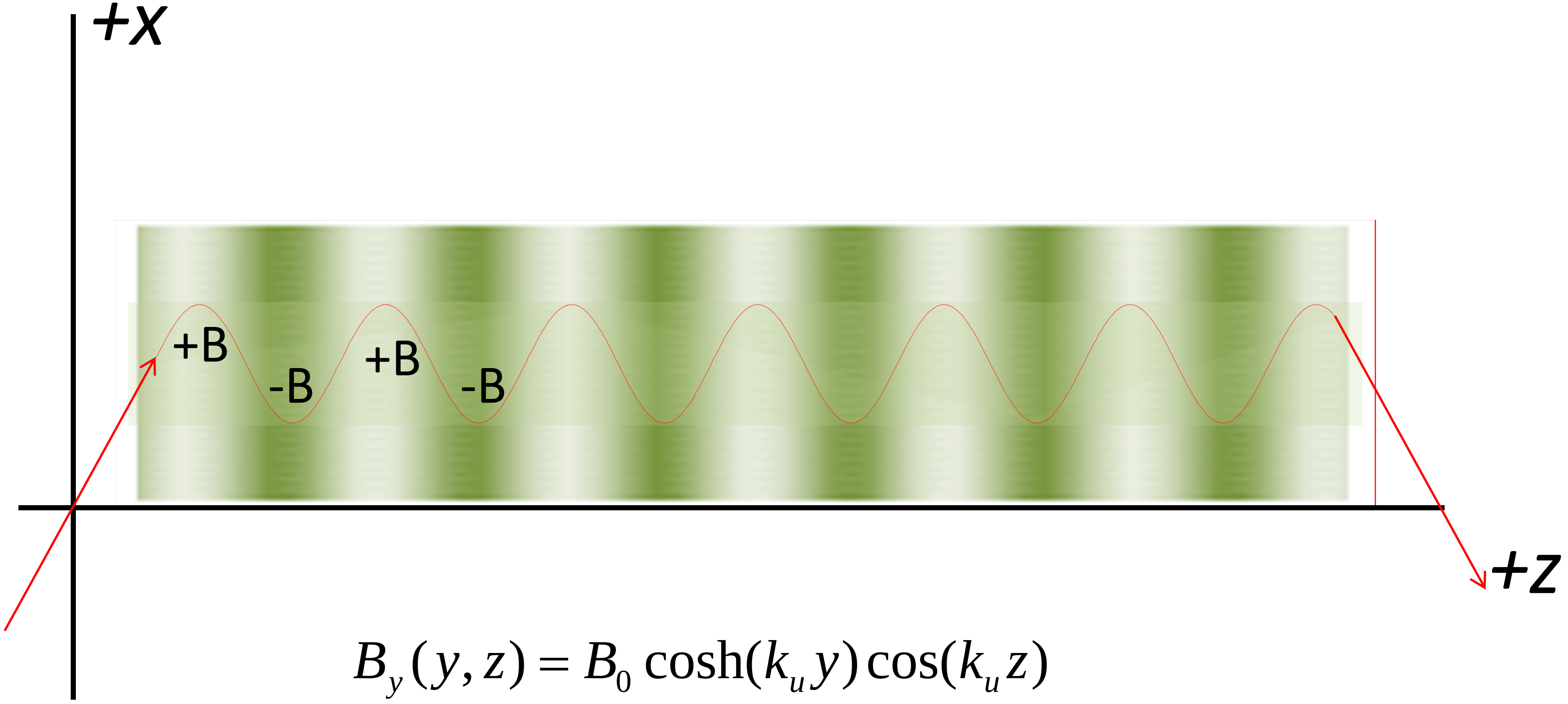
Radiation LW Fields

$$\boldsymbol{E}(\boldsymbol{x},t)=-\frac{|e|}{4\pi\varepsilon_0}\left[\frac{1}{s^3}\,\frac{\boldsymbol{n}\times\left[(\boldsymbol{n}-\quad)\times\dot{\ }\;\right]}{cR}\right]_{t-R/c}$$

$$\boldsymbol{B}(\boldsymbol{x},t)=\frac{\boldsymbol{n}\big|_{t-R/c}\times\boldsymbol{E}(\boldsymbol{x},t)}{c}\qquad\qquad s=(1-\boldsymbol{n}\cdot\quad)_{t-R/c}$$

$$\boldsymbol{R}=\boldsymbol{x}-\boldsymbol{x}(t)=\sqrt{(x-x(t))^2+(y-y(t))^2+(y-y(t))^2}$$

Electron trajectory in a planar undulator

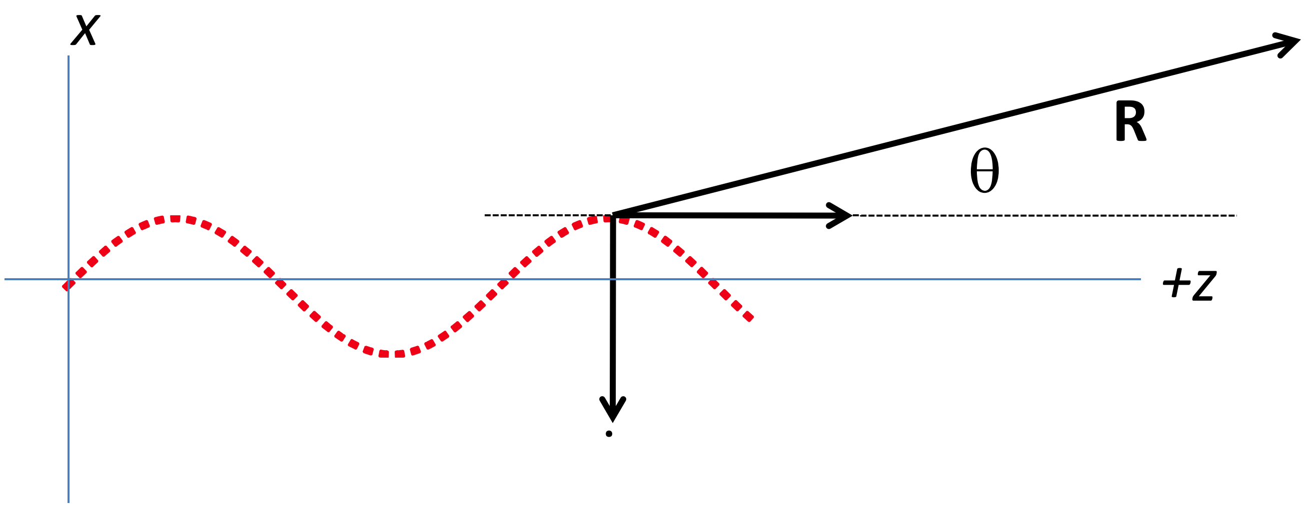


$$B_y(y,z) = B_0 \cosh(k_u y)\cos(k_u z)$$

$$B_z(y,z) = B_0 \sinh(k_u y)\sin(k_u z)$$

$$B_x = 0$$

Maximum Undulator LW Fields on undulator plane

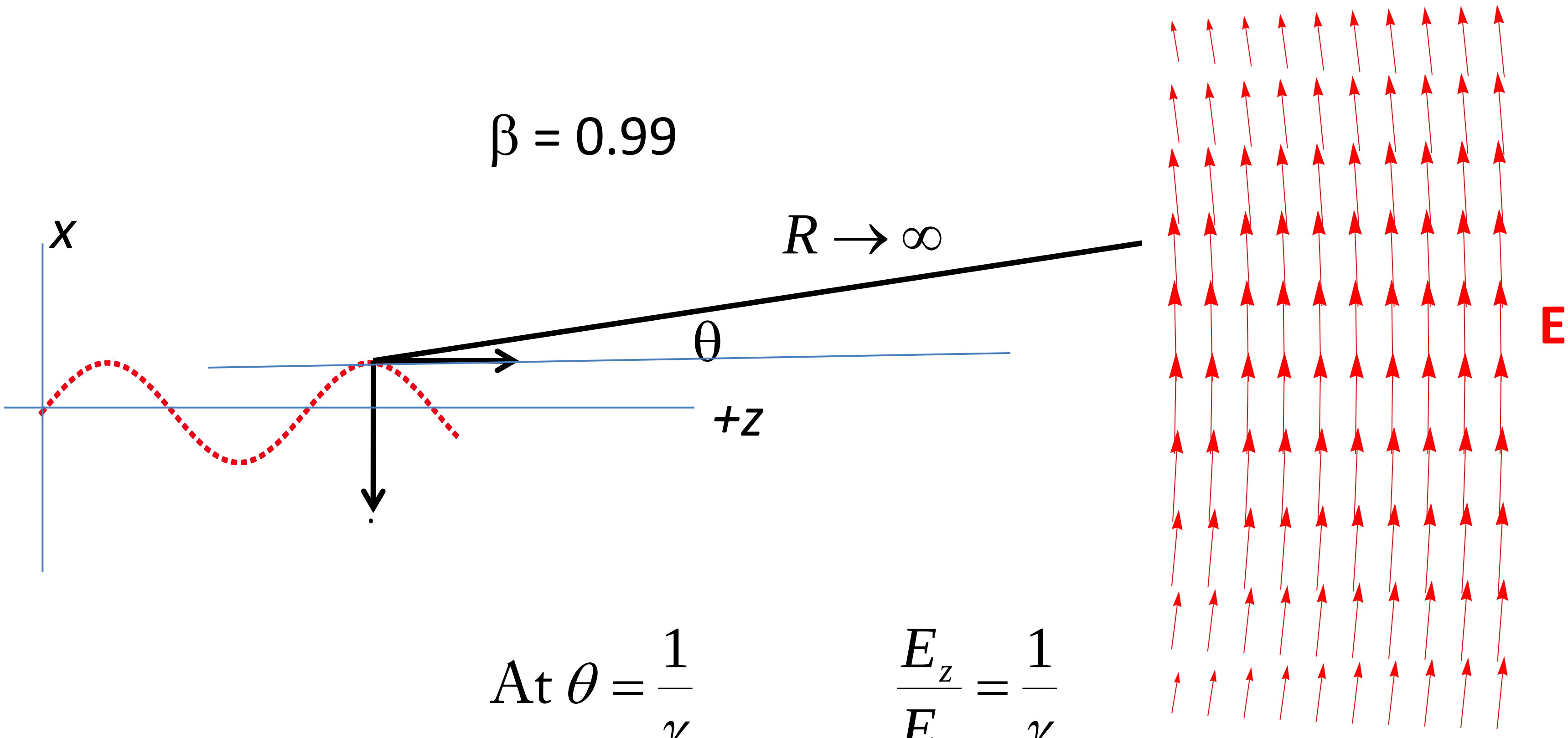


$$\boldsymbol{n} = \boldsymbol{e}_x \sin\theta\cos\phi + \boldsymbol{e}_y \sin\theta\sin\phi + \boldsymbol{e}_z \cos\theta$$

$$= \beta\boldsymbol{e}_z \qquad \dot{} \; = -\dot{\beta}\boldsymbol{e}_x$$

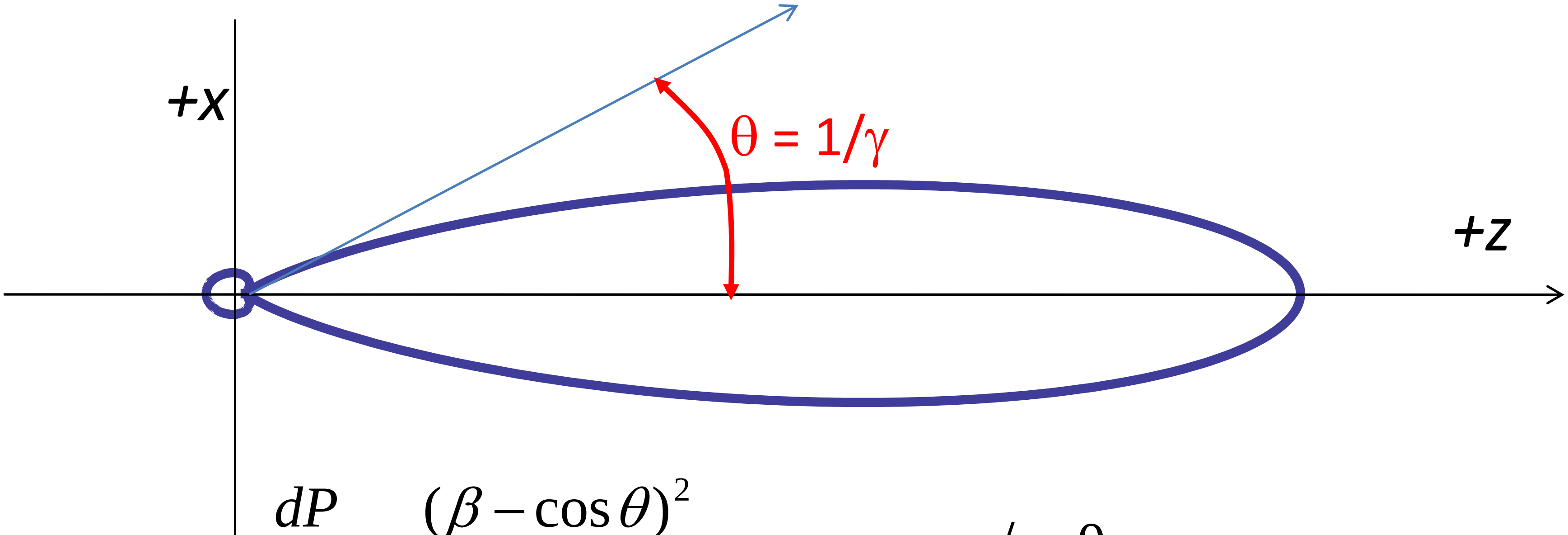
$$\boldsymbol{E} \propto \frac{\boldsymbol{n}\times\left[\left(\boldsymbol{n} - \quad\right)\times \;\dot{}\;\right]}{1-\boldsymbol{n}\cdot} = \dot{\beta}\,\frac{\left(\cos\theta-\beta\right)\left\{\cos\theta\boldsymbol{e}_x - \sin\theta\boldsymbol{e}_z\right\}}{\left(1-\beta\cos\theta\right)^3}$$

Far Electric Field on x-z plane



$$\text{At } \theta = \frac{1}{\gamma} \qquad \frac{E_z}{E_x} = \frac{1}{\gamma}$$

Angular distribution on undulating plane ($\beta = 0.5$)



$$\frac{dP}{d\Omega} \propto \frac{(\beta - \cos\theta)^2}{(1 - \beta\cos\theta)^5}$$

$$\phi = 0$$

Instantaneous radiated power by a single electron

From Lienard's result

$$P_e = \frac{3}{2c}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \gamma^6 \left[\dot{\ }^2 - \left| \ \times \dot{\ } \right|^2 \right]$$

In a planar undulator

$$P_e = \sigma_{TH}\left(\frac{B_u^2}{2\mu_0}\right)c\gamma^2$$

$$\sigma_{TH} = \text{Thomson cross section} = 0.66\times 10^{-28}\, m^2$$

Spontaneous radiation power by a beam of electrons

$$W_e = \sigma_{TH}\left(\frac{B_u^2}{2\mu_0}\right)c\gamma^2\frac{L}{c}$$ ← Radiated energy per electron

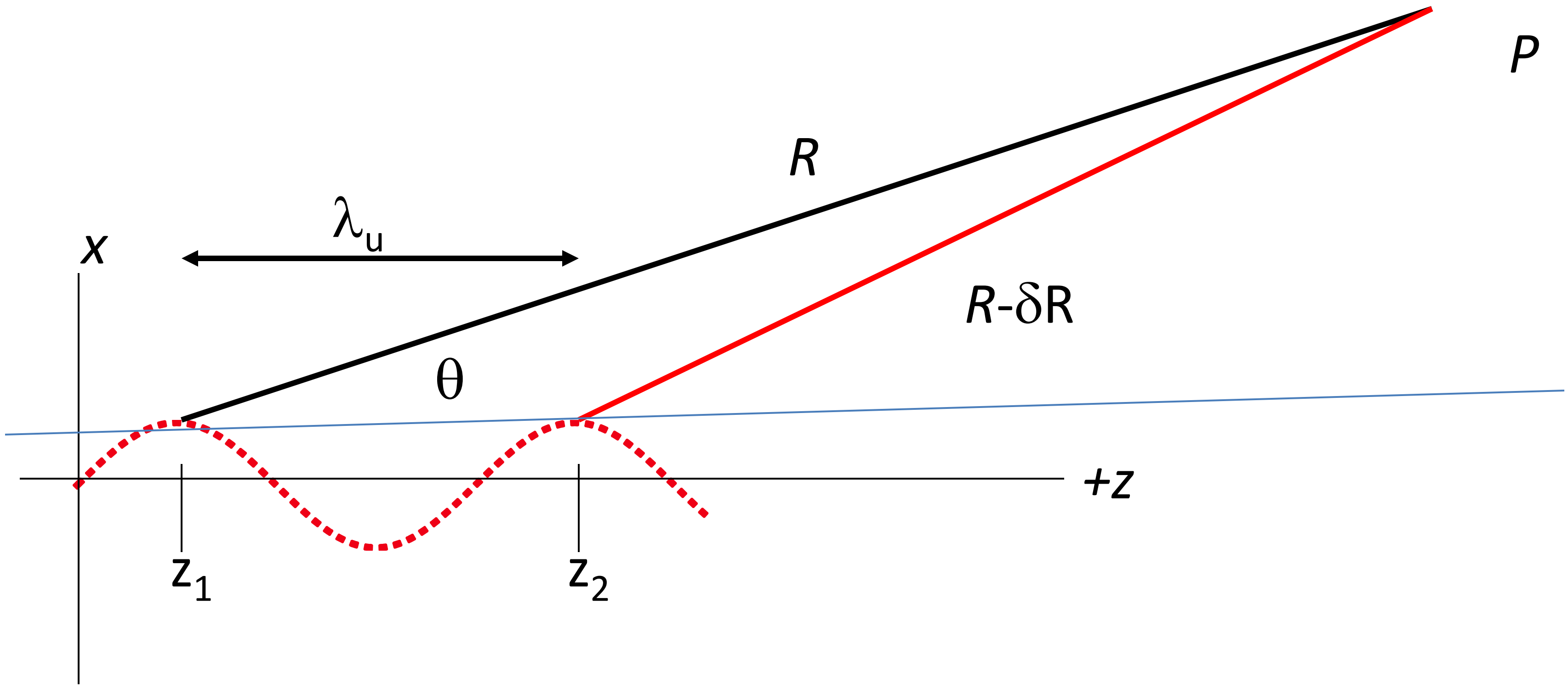
$$P_s = \frac{I}{|e|}\frac{L}{c}\sigma_{TH}\left(\frac{B_u^2}{2\mu_0}\right)c\gamma^2$$ ← Radiation electron beam power, assuming random position of electrons in the beam

Synchrotron radiation !!

L = Undulator length

I = Beam current

Undulator radiation wavelength



t_1 = Time of arrival at P of signal from electron position 1

t_2 = Time of arrival at P of signal from electron position 2

Radiation Wavelength

$$t_1 = \frac{R}{c} \qquad t_2 = \frac{\lambda_u}{c\overline{\beta}_z} + \frac{R - \delta R}{c}$$

$$\delta t = t_2 - t_1 = \frac{\lambda_u}{c\overline{\beta}_z} - \frac{\delta R}{c} = \frac{\lambda_u}{c\overline{\beta}_z} - \frac{\lambda_u \cos\theta}{c}$$

$$\lambda = \lambda_u \left[\frac{1}{\overline{\beta}_z} - \cos\theta \right]$$

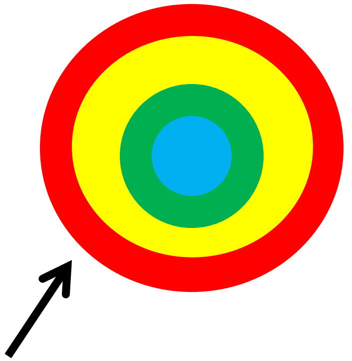
For small angles θ and β near 1

$$\cos\theta \cong 1 - \frac{\theta^2}{2}$$

$$\beta_z = \sqrt{\beta^2 - \beta_x^2} = \sqrt{1 - \frac{1}{\gamma^2} - \frac{a_u}{\gamma^2}(\sin k_u z)^2}$$

$$\overline{\beta}_z \cong 1 - \frac{1}{2\gamma^2}\left(1 + \frac{a_u^2}{2}\right)$$

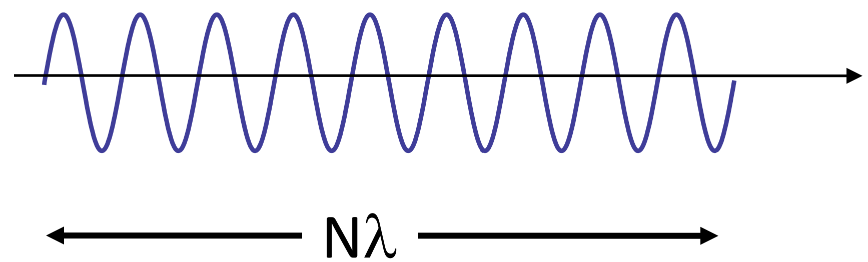
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left[1 + \frac{a_u^2}{2} + (\gamma\theta)^2\right]$$



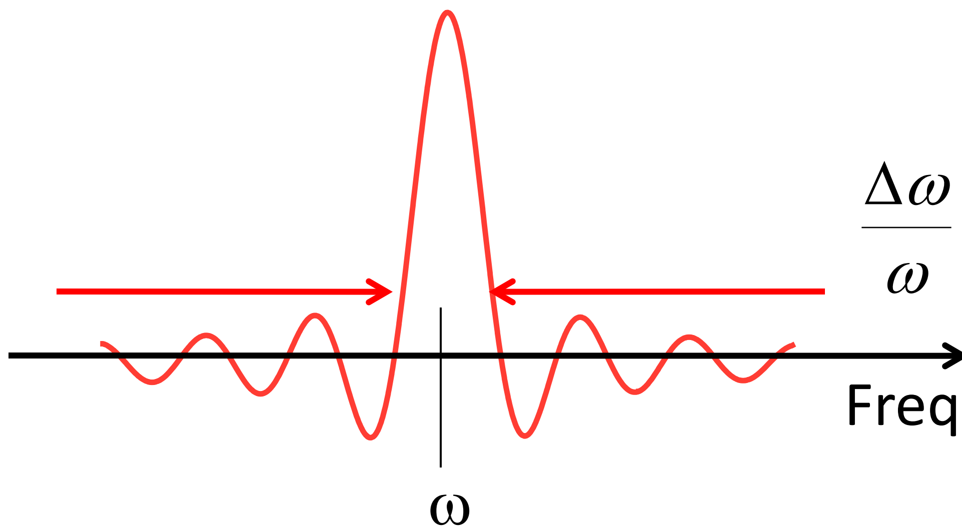
As seen along direction opposite to the instantaneous electron velocity

Undulator radiation spectral distribution

Radiation pulse on undulator plane at fixed angle θ

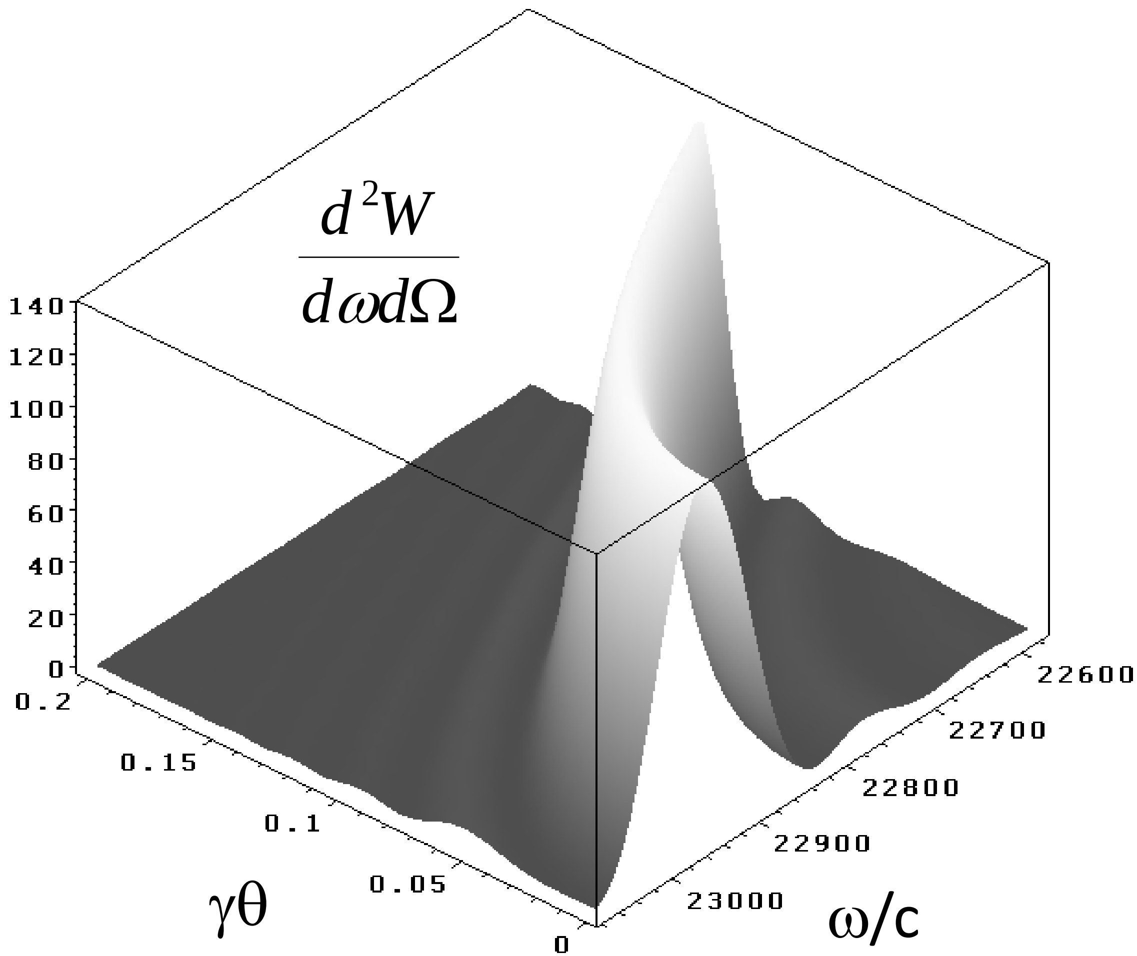


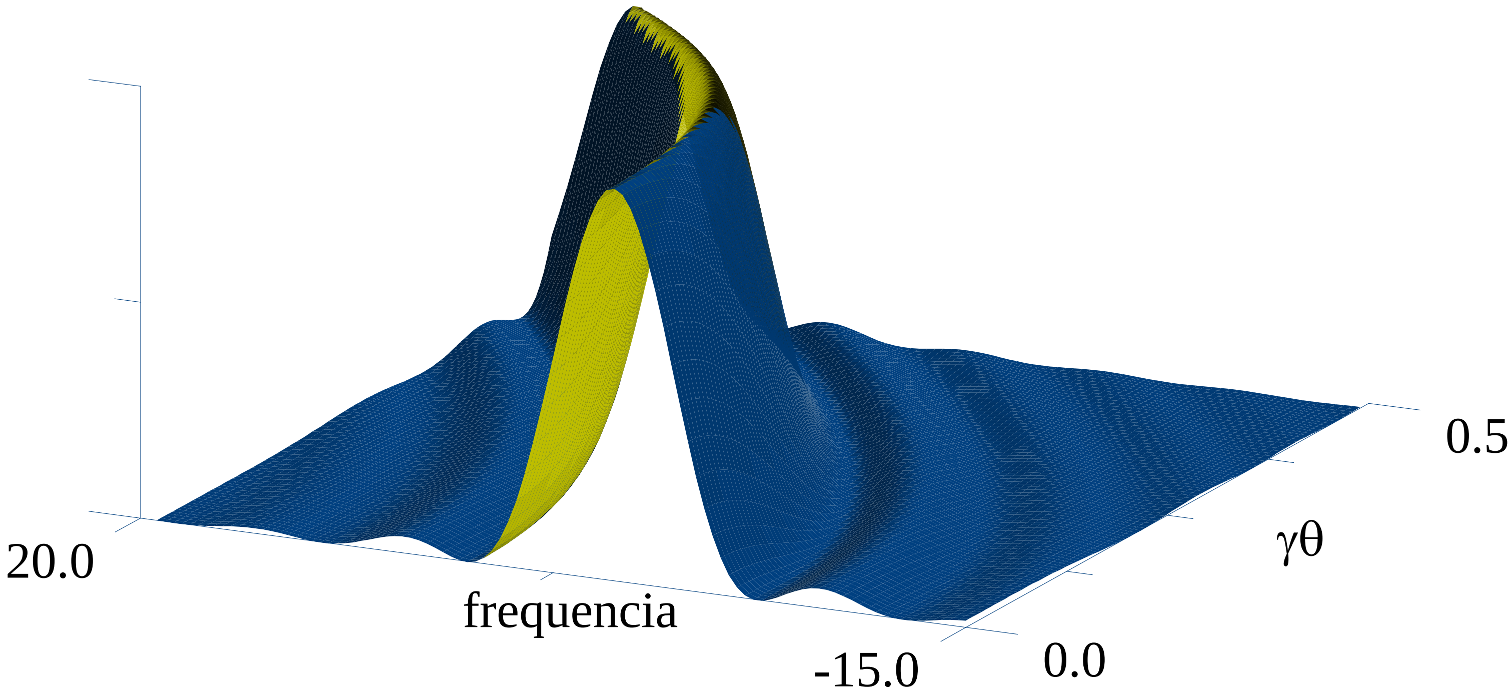
Angular Frequency spectrum



$$\frac{\Delta\omega}{\omega} \cong \frac{1}{2N}$$

Angular and spectral distribution of undulator radiation





20.0
frequencia
-15.0
0.0
γθ
0.5

UH/CBPF undulator radiation

λ_u	*8 mm*
a_u	*0.14*
N	150
L	1.6 m
γ	< 5
Radiation Cone angle $1/\gamma$	< 200 mrad
Fractional spectral BW	0.66 %
Beam current	0.3 A
Spontaneous radiation power	5 μW

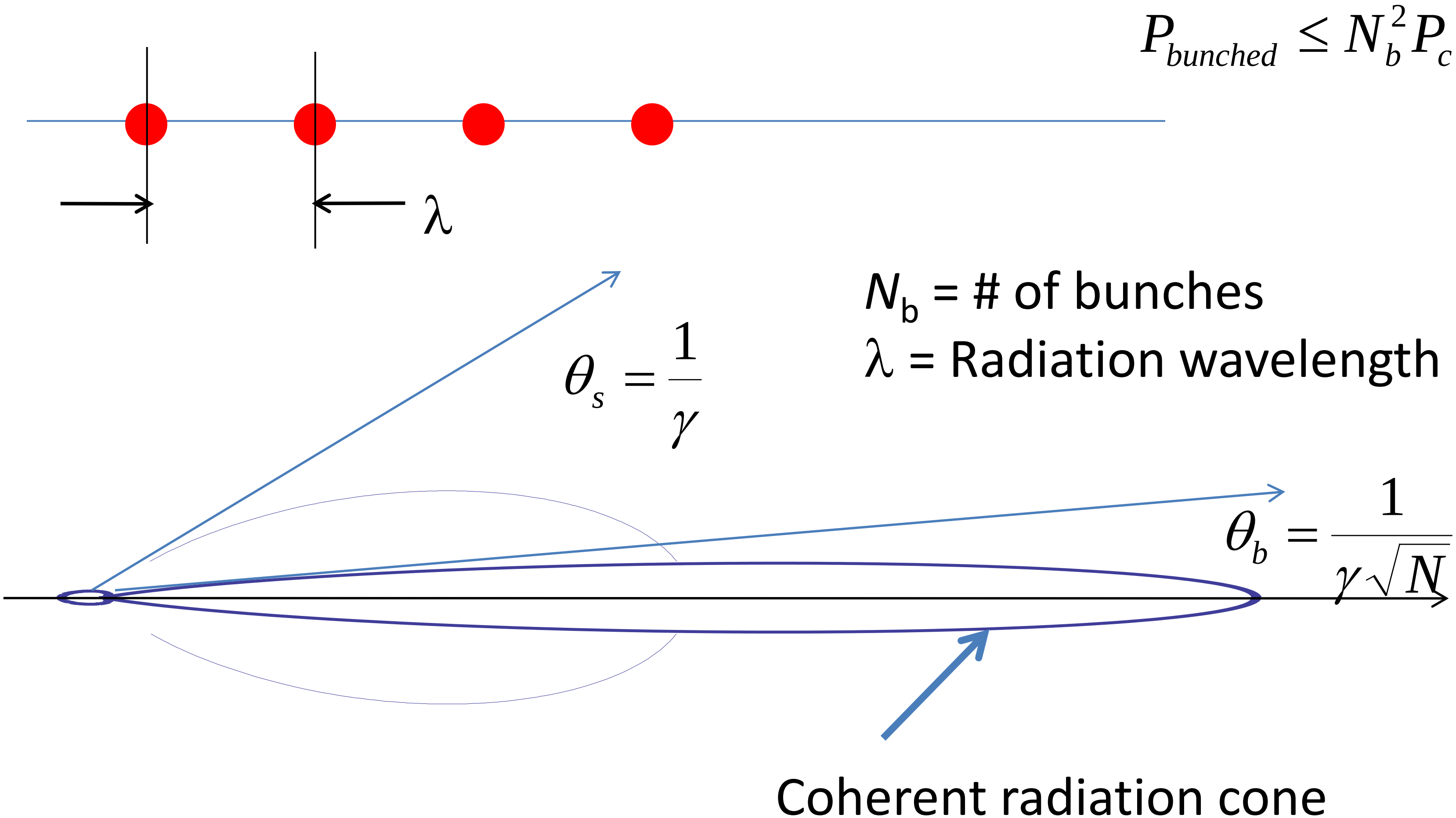
Coherent spontaneous radiation

If N_e electrons are located within one bunch whose dimensions are much smaller then the radiation wavelength then the coherent radiation power is given by

$$P_c = N_e^2 P_s$$

The angular and spectral characteristics of the radiation are similar to those of a single electron

Coherent spontaneous radiation from a periodic bunched beam



- The coherent radiation cone angle is smaller than the single electron radiation cone

$$\frac{1}{\gamma\sqrt{N}} < \frac{1}{\gamma}$$

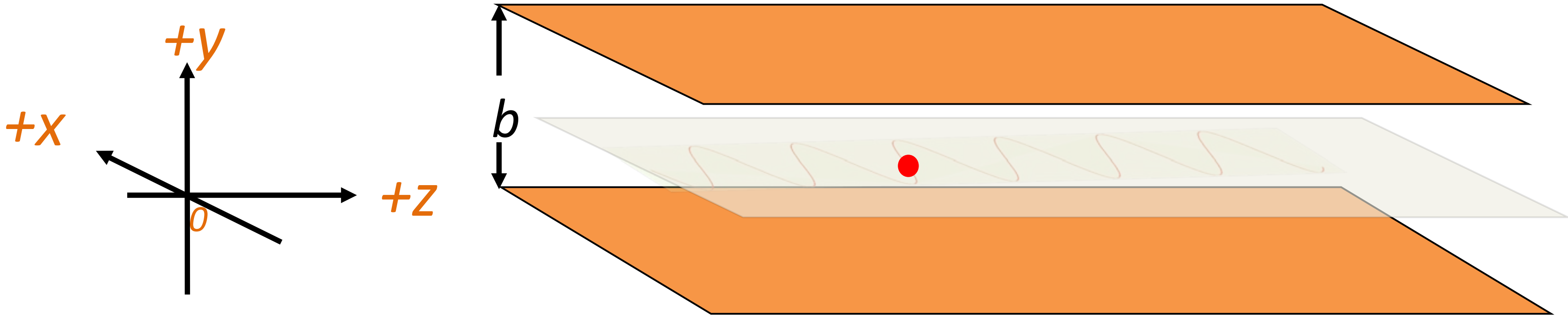
- The coherent radiation fractional bandwidth is smaller than that of single electron

$$\left[\frac{\Delta\omega}{\omega}\right]_{bunched} \leq \frac{1}{N+N_b}$$

Undulator Radiation in a Parallel Plate Waveguide

Undulator radiation in a parallel plate waveguide

Undulator electron trajectory on the mid-plane between parallel conducting plates



TE$_n$ waveguide modes

Vector potential

$$\boldsymbol{A}_s(y,z,t) = \boldsymbol{e}_z A_s \sin\left(\frac{n\pi y}{b}\right)\cos\left(k_{zn} z - \omega_s t\right)$$

Propagation constant

$$k_{zn} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)}$$

Phase and group velocities

$$\frac{v_\phi^{(n)}}{c} = \frac{1}{\sqrt{1-\left(\frac{n\pi}{k_s b}\right)^2}}$$

$$\frac{v_g^{(n)}}{c} = \sqrt{1-\left(\frac{n\pi}{k_s b}\right)^2}$$

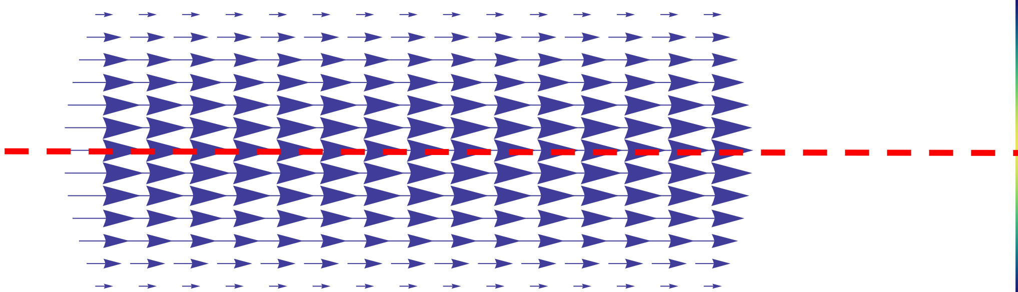
Electric and Magnetic fields

$$\boldsymbol{E}_s(y,z,t) = -\boldsymbol{e}_x A_s \omega_s \sin\left(\frac{n\pi y}{b}\right)\sin\left(k_{zn} z - \omega_s t\right)$$

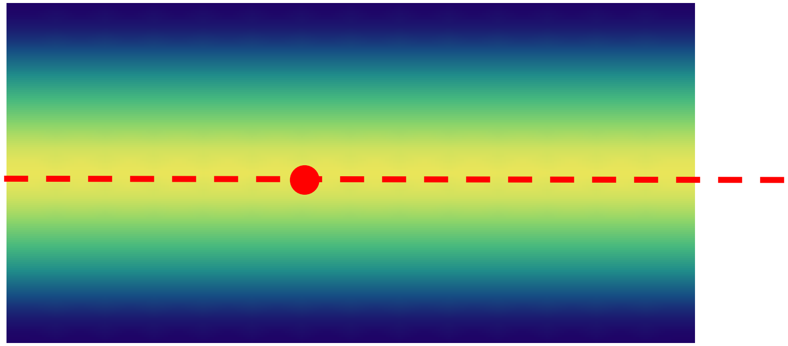
$$\boldsymbol{B}_s(y,z,t) = -A_s \left[\begin{array}{l} \boldsymbol{e}_y k_n \sin\left(\frac{n\pi y}{b}\right)\sin\left(k_{zn} z - \omega_s t\right) \\ + \boldsymbol{e}_z \frac{n\pi}{b}\cos\left(\frac{n\pi y}{b}\right)\cos\left(k_{zn} z - \omega_s t\right) \end{array} \right]$$

TE_1 Mode

Strong coupling to electron beam



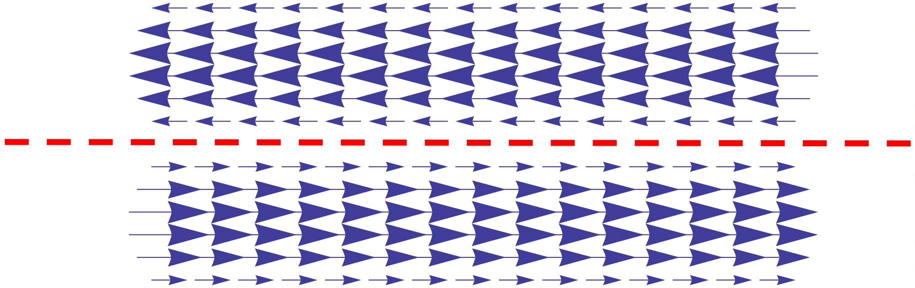
Lines of Electric field

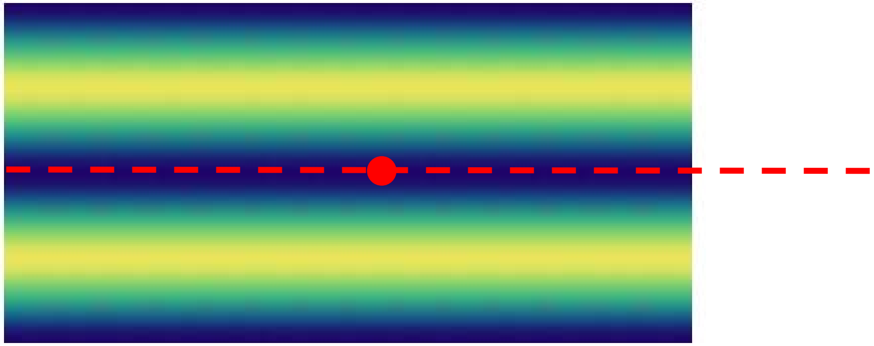


Electric field density plot

TE_2 Mode

Weak coupling to electron beam



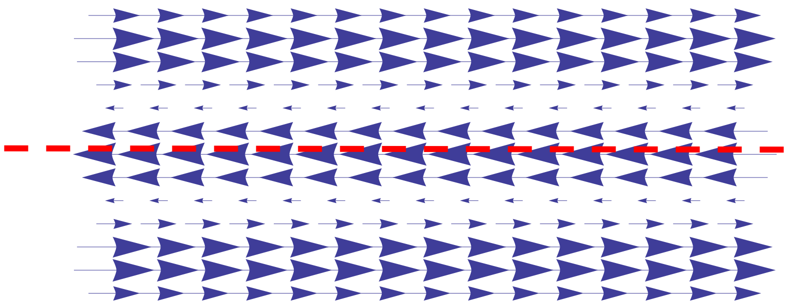


Lines of Electric field

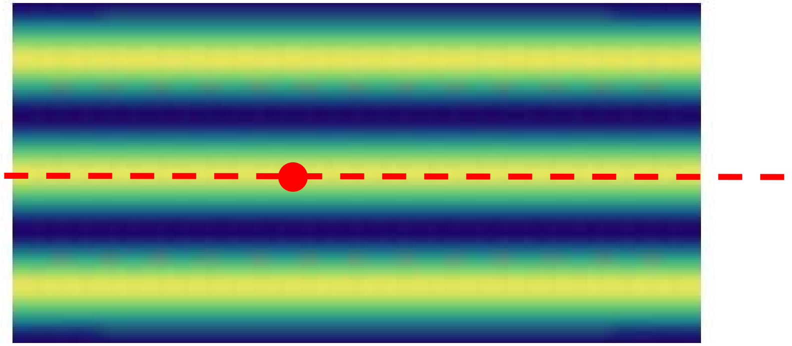
Electric field density plot

TE_3 Mode

Strong coupling to electron beam



Lines of Electric field



Electric field density plot

Spectral and Angular Distribution of Radiation

$$\frac{d^2W}{d\omega d\phi} \propto \sum_{n=1}^{n_{\max}} \left[\sin k_n y_0 \frac{\gamma_n}{k_u} J_0\left(\frac{\omega}{k_u v_z}\right) J_0\left(\frac{\gamma_n}{k_u}\alpha \sin\phi\right) \frac{\sin\nu}{\nu}\cos\phi \right]^2$$

$$\alpha = \frac{a_u}{\gamma\beta} \qquad \gamma_n \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad \nu = 1 + \frac{\gamma_n}{k_u}\cos\phi - \frac{\omega}{k_u c \overline{\beta}_z}$$

Peak frequency at a fixed angle ϕ

At fixed angle solve

$$\nu = 1 + \frac{\gamma_n}{k_u}\cos\phi - \frac{\omega}{k_u c \overline{\beta}_z} = 0$$

$$k_n = \frac{n\pi}{b} \qquad \gamma_n = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_n^2}$$

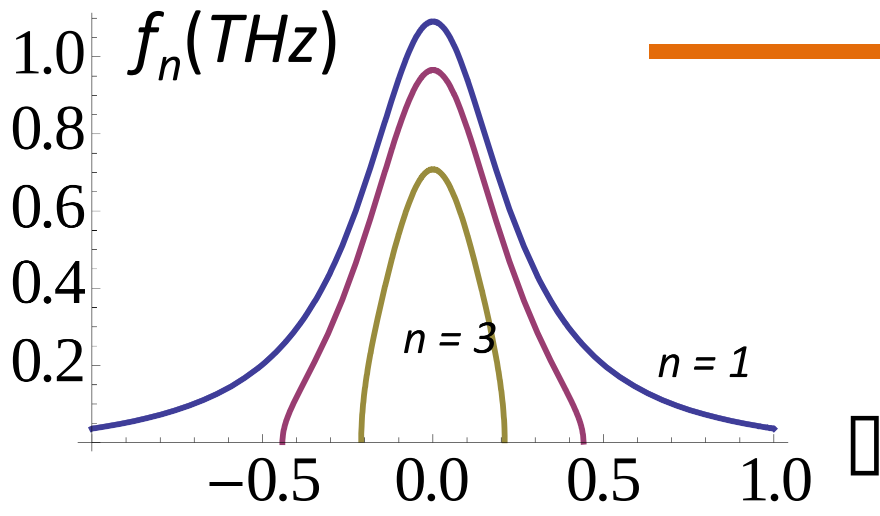
$$\omega^{\pm} = \frac{rck_u}{\overline{\beta}_z \eta^2}\left[1 \mp \overline{\beta}_z \cos\phi \sqrt{1 - \left(\frac{k_n \eta}{rk_u}\right)^2}\,\right] \qquad \eta = \sqrt{\frac{1}{\overline{\beta}_z^{\,2}} - (\cos\phi)^2}$$

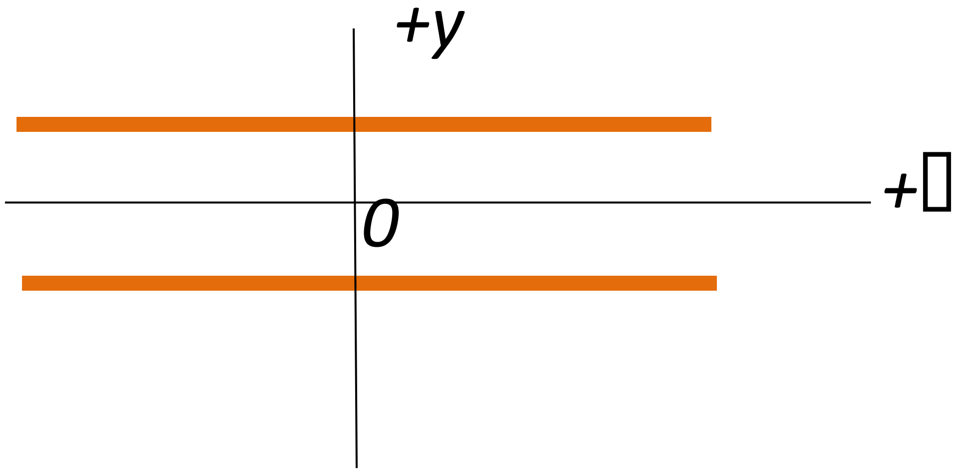
r = harmonic number   n = mode index

UH/CBPF FEL Data

[illegible]	2 - 4.5
$[illegible]_{u}$	0.008 m
a_{u}	0.15
N_{u}	180
[illegible]	< 0.03
b	0.004 m

Angular Dependence of radiation fundamental frequency for n =1,2,3





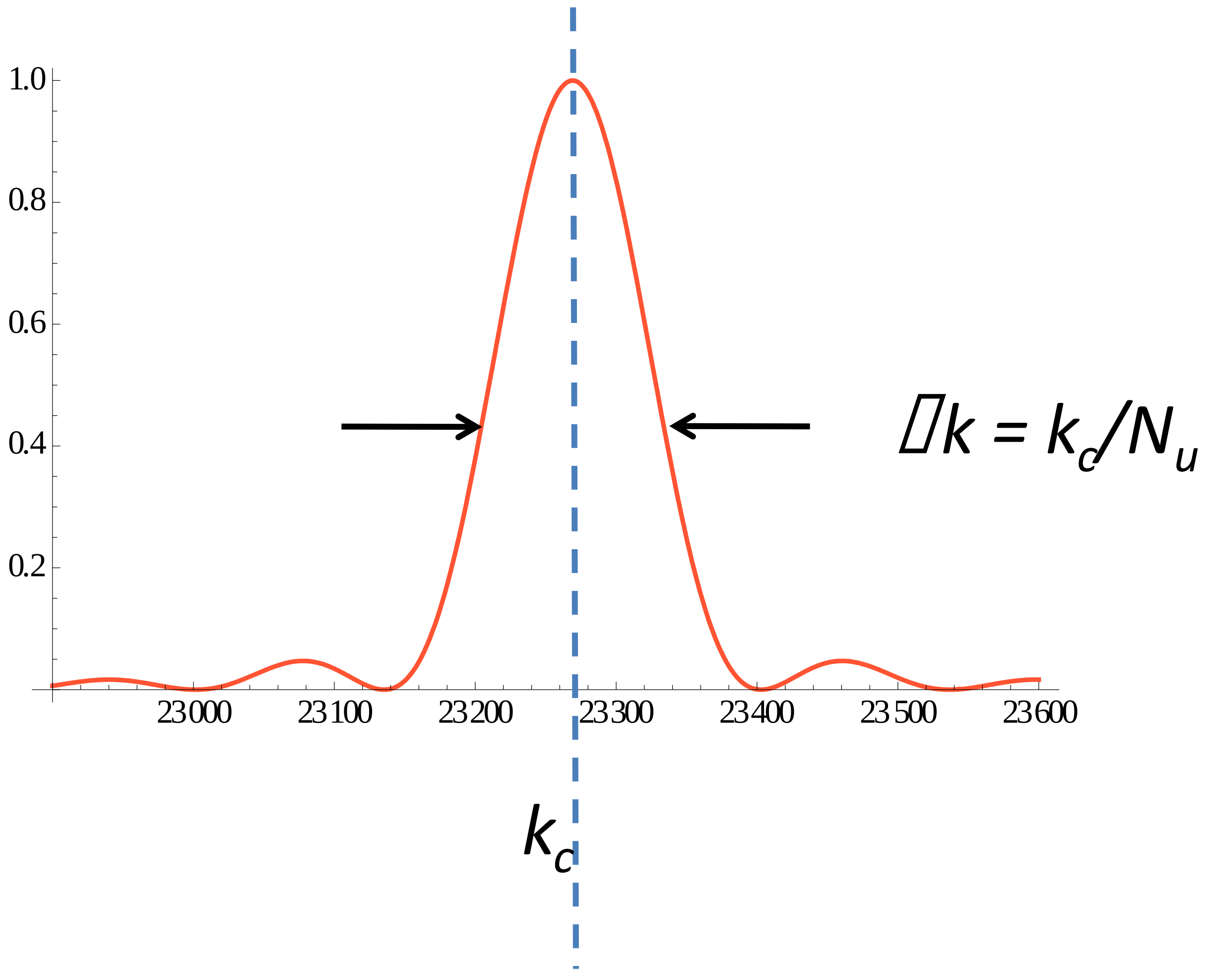
= 4.5

Frequency Spectrum

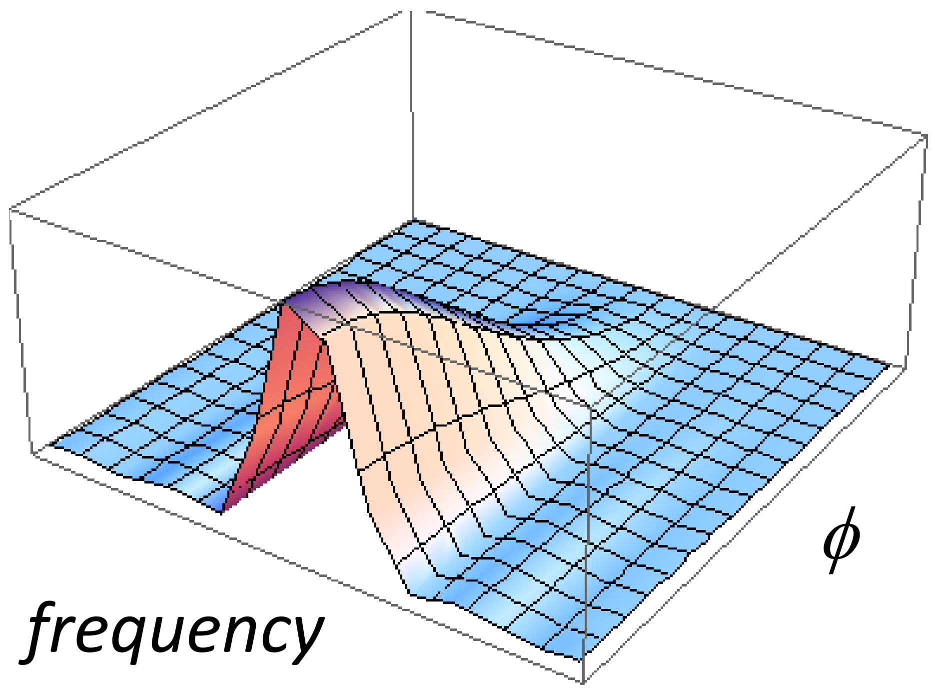
$$s(k) = \left[1 + \frac{\sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2}}{k_u} - \frac{k}{k_u \overline{\beta}_z} \right] N_u \pi$$

$$\frac{dW}{dk} \propto \left[\frac{\sin s}{s} \right]^2$$

Frequency spectrum on axis (n = 1)



Angular and Spectral Distribution of radiated energy



Maximum number of modes filled by spontaneous undulator radiation

$$\omega^{\pm} = \frac{rck_u}{\overline{\beta}_z \eta^2}\left[1 \mp \overline{\beta}_z \cos\phi \sqrt{1-\left(\frac{k_n \eta}{rk_u}\right)^2}\right] \qquad \eta = \sqrt{\frac{1}{\overline{\beta}_z^{\,2}} - (\cos\phi)^2}$$

$$\frac{k_n \eta}{rk_u} \leq 1 \quad \rightarrow \quad n \leq \frac{2b}{\lambda_u}\overline{\beta}_z \overline{\gamma}_z \quad \rightarrow \quad n \leq 4 \text{ For the UH/CBPF FEL}$$