Quantum wave kinetics of high-gain free-electron lasers

A. Serbeto,[1,a)] J. T. Mendonça,[2] K. H Tsui,[1] and R. Bonifacio[3,4]
[1]*Instituto de Física, Universidade Federal Fluminense, 24210-340 Rio de Janeiro, Brazil*
[2]*GoLP and CFIF, Instituto Superior Técnico, 1049-001 Lisbon, Portugal*
[3]*Instituto Nazionale di Fisica Nucleare, Via Celoria 16, I-20133 Milano, Italy*
[4]*Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 160, 22290-180 Rio de Janeiro, Brazil*

(Received 31 July 2007; accepted 18 December 2007; published online 29 January 2008)

A wave kinetic equation equivalent to the Schrodinger-like equation for the electrons is derived from relativistic theory in the eikonal approximation. This equation is used to study the interaction of the relativistic electron beam in a wiggler field in the quantum regime, i.e., when the normalized quantum free-electron laser parameter $\bar{\rho} \leqslant 1$. A general quantum dispersion relation and an expression for free-electron laser instabilities are derived. Conditions for kinetic instability, which takes into account the Landau damping effect, are established. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2833591]

I. INTRODUCTION

The free-electron laser (FEL) is today a very important area of research. This device, which can generate high-power coherent radiation using a relativistic electron beam moving in a periodic magnetostatic field (wiggler), is capable of operating in a broad band of the electromagnetic spectrum even at wavelengths not accessible to conventional lasers. This tunable feature of the FEL is due to the fact that the wavelength of the coherent radiation, λ_r, is mainly determined by the relativistic electron beam energy and the wavelength or period of the wiggler magnetic field, λ_w, which satisfy the approximate resonant condition $\lambda_r \cong \lambda_w / 2\gamma_r^2$, where γ_r is the normalized resonant electron beam energy. This characteristic of the FEL has mainly motivated the construction of very large experimental systems that aim to produce intense radiation sources in the ultraviolet and x-ray domains.[1,2] Recently, Bonifacio *et al.*[3] proposed and examined the FEL operation in a self-amplified spontaneous emission (SASE) regime, in which the electron motion is quantized and the propagation or slippage effects are considered with startup from noise. In an x-ray FEL, the quantum regime occurs when a quantum free-electron laser parameter, which measures the average number of photons scattered per particle, $\bar{\rho} \leqslant 1$. This condition means that one photon momentum recoil $\hbar k$ is larger than the momentum spread.

In the present work, we return to the quantum theory of the free-electron laser by using a different perspective. We propose to establish a quantum theory for the electron beam evolution in a wiggler field by using wave kinetics, which is based on the evolution equation for the Wigner function of the system,[4] coupled with the interacting field by an adequate ponderomotive potential. In recent years, considerable progress has been made in wave kinetics, when applied to different problems, in both the classical and the quantum domain, such as plasma turbulence,[5,6] nonlinear optics,[7] Bose–Einstein condensates,[8,9] or neutrino physics.[10,11] Here, we also show that it is possible to derive a Schrodinger type of equation for the relativistic electron beam in the wiggler field, by starting from the total energy E of an electron given by the relativistic theory and using the eikonal approximation. In this paper, the wave kinetic equation for the Wigner function is derived as in Ref. 12, where, in contrast with the present work, the Schrodinger type of equation is not explicitly derived. This wave kinetic equation is coupled to the radiation field through the ponderomotive potential due to the beating between the radiation and the wiggler fields. Using a perturbative approach, we derive a kinetic dispersion relation for the bunching oscillations of the electron beam, taking into account the beam energy spread, which can lead to a kinetic instability associated with a negative Landau damping. This kinetic approach generalizes previous work.[13]

II. BASIC EQUATIONS

We first show that, starting from the exact relativistic energy of an electron beam, we can derive an approximate form of the wave equation that is formally identical to a Schrodinger equation. In a free-electron laser, the electrons move in a ponderomotive potential $V(\vec{r},t)$, which is due to the beating of the static periodic magnetic field (wiggler field, in short) and the radiation field produced by the beam electrons themselves. The single-electron total relativistic energy in this ponderomotive potential is

$$E = \sqrt{p^2c^2 + m_e^2c^4} + V(\vec{r},t). \tag{1}$$

In the absence of the potential, $V=0$, the electrons would have a constant energy $E=\hbar\omega_e$, and a constant momentum $\vec{p}=\hbar\vec{k}_e$. But, for $V\neq 0$, these quantities will slowly vary in space and time, and in order to account for such changes we have to make the replacements

$$E \rightarrow \hbar\left(\omega_e + i\frac{\partial}{\partial t}\right), \quad \vec{p}_e \rightarrow \hbar(\vec{k}_e - i\,\nabla), \tag{2}$$

which is valid in the eikonal approximation.[15–17] This is equivalent to the slowly varying envelope approximation (SVEA),[18]

[a)]Electronic mail: serbeto@if.uff.br.

$$\left|\frac{\partial\psi}{\partial t}\right| \ll \omega_e|\psi|, \quad |\nabla\psi| \ll k_e|\psi|, \tag{3}$$

and following Refs. 15–17 we get from Eqs. (1)–(3) the following wave equation:

$$i\left(\frac{\partial}{\partial t} + \vec{v}_e\cdot\nabla\right)\psi + \frac{c^2}{2\omega_e}\left[\nabla^2 - \left(1 - \frac{m_e^2c^4}{\hbar^2\omega_e^2}\right)(\hat{u}\cdot\nabla)^2\right]\psi = \frac{1}{\hbar}V(\vec{r},t)\psi, \tag{4}$$

where $\hat{u}=\vec{k}_e/k_e$. Here, ω_e and $\vec{k}_e$ are the frequency (energy) and wave vector (momentum) of the electron in the absence of interaction, respectively. The electron unperturbed velocity $\vec{v}_e$ associated with the energy(ω_e) and momentum($\vec{k}_e$) spectra is defined by

$$\vec{v}_e = c\vec{k}_e\left(k_e^2 + \frac{m_e^2c^2}{\hbar^2}\right)^{-1/2} \approx \frac{\hbar\vec{k}_e}{m_e} \tag{5}$$

for the Compton wavelength of the electron $\lambda_c=h/m_ec \gg 2\pi/k_e$. Here, we are assuming that the N-electron system can be described as a quantum ensemble represented by the macroscopic matter-wave function $\psi(\vec{r},t)$,[12] where $|\psi|^2$ is the probability amplitude of the electron ensemble normalized in such a way that the electron density is given by $n_e=|\psi|^2$. We should point out that this normalization is valid since we are considering that the electrons in the ensemble interact only collectively through an electromagnetic potential, i.e., we are assuming a pure state ensemble of electron (perfectly coherent particle sample) to represent a relativistic electron beam in a FEL. Defining new variables as

$$\vec{\xi} = \vec{r} - \vec{v}_e t, \quad t' = t, \tag{6}$$

and making $\hbar\omega_e=\gamma m_ec^2$, with γ being the relativistic factor associated with the electron velocity $\vec{v}_e$, the wave equation (4) reduces to

$$i\frac{\partial\psi}{\partial t'} + \frac{\hbar}{2\gamma m_e}\left(\frac{1}{\gamma^2}\frac{\partial^2}{\partial\xi_\parallel^2} + \frac{\partial^2}{\partial\xi_\perp^2}\right)\psi = \frac{V}{\hbar}\psi, \tag{7}$$

where $\vec{\xi}=\xi_\parallel\hat{u}+\vec{\xi}_\perp$ has been used. Neglecting transverse effects ($\partial^2/\partial\xi_\perp^2=0$), we obtain the following Schrodinger-like equation of the form

$$i\frac{\partial\psi}{\partial t'} + \frac{\hbar}{2m_{\rm eff}}\frac{\partial^2\psi}{\partial\xi^2} = \frac{V}{\hbar}\psi, \tag{8}$$

where $\xi=z-v_zt$ and $m_{\rm eff}=\gamma^3m_e$ have been used. If we now apply this equation to describe the quantum state of nearly resonant electrons interacting with the ponderomotive potential associated with a given wiggler field, we have to use

$$v_z = v_r \equiv c\frac{k}{k+k_w}, \quad \gamma = \gamma_r \equiv \sqrt{\frac{k}{2k_w}(1+a_w^2)}, \tag{9}$$

where k, k_w, and v_r are the radiation, the wiggler field wavenumbers, and the beam resonant velocity, respectively. Here, $a_w=eA_w/m_ec^2$ is the normalized amplitude of the wiggler field and γ_r is the associated normalized resonant energy. It is also useful to introduce the following dimensionless variables:

$$\tau = 2\rho k_w ct', \quad \theta = (k+k_w)z - \omega t', \tag{10}$$

where $\theta \simeq k\xi$ is the electron phase for $k \gg k_w$, and ρ is the classical free-electron laser parameter[20] defined as

$$\rho = \frac{1}{\gamma_r}\left(\frac{a_w\omega_p}{4ck_w}\right)^{2/3}. \tag{11}$$

Replacing these new variables in the one-dimensional wave equation (8), we obtain

$$i\frac{\partial\psi}{\partial\tau} + \frac{1}{2\bar{\rho}}\frac{\partial^2\psi}{\partial\theta^2} = V_p\psi, \tag{12}$$

where we have also used the new quantities

$$\bar{\rho} = \left(\frac{\gamma_r m_e c}{\hbar k}\right)\rho, \quad V_p = \frac{V}{2\hbar\rho k_w c}. \tag{13}$$

In order to derive Eq. (12), the FEL resonant condition $k = 2k_w\gamma_r^2$ has been used. Note that this equation, which was derived here in a simple way, is exactly the same equation as given by a much heavier quantum field calculation.[14] Although these new variables and parameters are not strictly necessary, they are used here for comparison with previous work on the quantum theory of the free-electron laser, where the electron beam is described by the matter-wave function $\psi(\tau,\theta)$. Equation (12) has to be solved with the following expression for the ponderomotive potential:

$$V_p = -i\bar{\rho}[A(\tau)e^{i\theta} - \text{c.c.}] \tag{14}$$

as given in Ref. 12 with $A=a/\sqrt{\bar{\rho}N}$ being the dimensionless radiation amplitude such that $\bar{\rho}|A|^2$ estimates the ratio between the number of photons and electrons in the beam. Finally, the evolution equation for the radiation field A can be written in dimensionless form as[12]

$$\left(\frac{\partial}{\partial\tau} - i\bar{\delta}\right)A = \int_0^{2\pi}|\psi(\tau,\theta)|^2 e^{-i\theta}\frac{d\theta}{2\pi}, \tag{15}$$

where $\bar{\delta}=(\gamma_0-\gamma_r)/\rho\gamma_r$ is the detuning parameter. This wave equation is valid when the slippage effect is negligible. This completes our discussion of the basic equations for the free-electron laser.

III. WAVE KINETIC DESCRIPTION

Next we derive the equivalent wave kinetic equation for a Wigner distribution function, W, to describe the ensemble of the particles in the relativistic beam. This approach replaces the Schrodinger description of a pure state ensemble of particles (coherent particle sample) given by the matter-wave function $\psi(\tau,\theta)$ to a description that will also allow a mixed-state sample of particles as in a real FEL relativistic electron beam. Let us first rewrite Eq. (12) in the standard form

$$i\hbar\frac{\partial}{\partial\tau}\psi=\left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial\theta^2}+\tilde{V}(\tau,\theta)\right)\psi\equiv H\psi \tag{16}$$

with

$$\mu=\bar{\rho}\hbar,\quad \tilde{V}\equiv\hbar V_p=-i\hbar\bar{\rho}[A(\tau)e^{i\theta}-\text{c.c.}]. \tag{17}$$

Let us now define the autocorrelation function for the wave function as

$$C(\tau,\theta,s)=\psi^*(\tau,\theta-s/2)\psi(\tau,\theta+s/2). \tag{18}$$

From Eq. (16), it is then possible to derive an evolution equation for the autocorrelation function in the form

$$\left[i\hbar\frac{\partial}{\partial\tau}+\frac{\hbar^2}{\mu}\frac{\partial^2}{\partial\theta\partial s}-2\sinh\left(\frac{s}{2}\frac{\partial}{\partial\theta}\tilde{V}\right)\right]C=0. \tag{19}$$

We now define the Wigner function $W(\tau,\theta,p)$, which is the Fourier transform of the autocorrelation function

$$W(\tau,\theta,p)=\int_{-\infty}^{\infty}C(\tau,\theta,s)e^{-ips}ds. \tag{20}$$

It is obvious from here that the integral of this function over the variable p is equal to the quantum probability, assuming that we are using a normalized wave function

$$\int_{-\infty}^{\infty}W(\tau,\theta,p)\frac{dp}{2\pi}=|\psi(\tau,\theta)|^2. \tag{21}$$

Inserting this new function in Eq. (19), we obtain the desired wave kinetic equation in the form

$$i\hbar\left(\frac{\partial}{\partial\tau}+\frac{\hbar p}{\mu}\frac{\partial}{\partial\theta}\right)W=\int\tilde{V}[W_- - W_+]e^{iq\theta}\frac{dq}{2\pi}, \tag{22}$$

where we assumed that $W_\pm=W(p\pm q/2)$, and we used the Fourier transform of the ponderomotive potential as defined by the integral

$$\tilde{V}(\tau,\theta)=\int\tilde{V}(\tau,q)e^{iq\theta}\frac{dq}{2\pi}. \tag{23}$$

Equation (22) is exactly equivalent to the Schrodinger equation (12) and can be used to study the interaction of the relativistic electron beam with the nearly resonant ponderomotive potential in a free-electron laser. Now, for the free-electron laser, this potential takes the simple form given by Eq. (17). In this case, the wave kinetic equation, which is the evolution equation for the Wigner function, reduces to

$$\left(\frac{\partial}{\partial\tau}+\frac{\hbar p}{\mu}\frac{\partial}{\partial\theta}\right)W=(\tilde{A}e^{i\theta}+\text{c.c.})(W_- - W_+), \tag{24}$$

where $\tilde{A}=\bar{\rho}A(\tau)$ is the potential amplitude. This kinetic equation reduces to the Vlasov equation, which gives the classical FEL limit under $\bar{\rho}\gg 1$, which means that the quantum recoil effect due to single-photon process is negligible.

As a matter of fact, due to the periodicity in θ between $(0,2\pi)$, one should use the discrete Wigner function as described in Ref. 19. This is because the momentum recoil is not a continuous spectrum but a discrete one whose eigenvalue is a multiple of the photon momentum $\hbar k$. This discreteness can, however, be neglected in the quasiclassical limit considered here, when the initial momentum spread is larger than $\hbar k$.

As we know, the free-electron laser instability corresponds to a radiation-induced electron bunching, and can be studied by using the above wave kinetic equation, coupled with the radiation field equation. In order to study this instability, let us consider a perturbation of the form

$$A(\tau)=\tilde{a}\exp(-i\Omega\tau),\quad W=W_0(\theta,p)+\tilde{W}\exp(iq\theta-i\Omega\tau). \tag{25}$$

It is obvious from the above equations that coupling between the perturbations $\tilde{W}$ and $\tilde{a}$ can only occur for $q=\pm 1$. After linearization with respect to the perturbed quantities, we obtain from Eq. (24)

$$\tilde{W}=i\frac{\bar{\rho}\tilde{a}}{(\Omega-q\bar{v})}(W_+ - W_-) \tag{26}$$

and, from the radiation field equation,

$$\tilde{a}=\frac{i}{(\Omega+\bar{\delta})}\int\frac{dp}{2\pi}\int d\theta\tilde{W}. \tag{27}$$

From these two equations, we can then obtain the dispersion relation for perturbations oscillating with frequency Ω and wavenumber q, both in the electron beam and in the radiation field. This relation takes the form

$$1+\frac{\bar{\rho}}{(\Omega+\bar{\delta})}\int\frac{(G_{0+}-G_{0-})}{(\Omega-q\bar{v})}dp=0, \tag{28}$$

where we have used the auxiliary function

$$G_0(p)=\frac{1}{2\pi}\int_0^{2\pi}W_0(\theta,p)d\theta. \tag{29}$$

We have also used $G_{0\pm}=G_0(p\pm q/2)$. It is obvious that $G_0(p)$ is the unperturbed Wigner function of the electron beam averaged over the angular variable θ. In order to understand the meaning of the dispersion relation (28) and its physical consequences, we first consider the case of a monoenergetic electron beam, such that we can write

$$G_0(p)=G_0\delta(p-p_0). \tag{30}$$

Integration of Eq. (28) then leads to

$$1+\frac{\bar{\rho}}{(\Omega+\bar{\delta})}\left(\frac{1}{(\Omega_+-q\bar{v}_0)}-\frac{1}{(\Omega_--q\bar{v}_0)}\right)=0, \tag{31}$$

where we have used

$$\bar{v}_0=\frac{\hbar p_0}{\mu},\quad \Omega_\pm=\Omega\pm\frac{\hbar q^2}{2\mu}. \tag{32}$$

This can also be written in the alternative form

$$(\Omega+\bar{\delta})\left[(\Omega-q\bar{v}_0)^2-\left(\frac{\hbar q^2}{2\mu}\right)^2\right]=\frac{\hbar q^2}{\mu}\bar{\rho}. \tag{33}$$

This expression shows the coupling between two distinct modes, with dispersion relations given by

$$(\Omega+\bar{\delta})=0, \quad (\Omega-q\bar{v}_0)=\frac{\hbar q^2}{2\mu}. \tag{34}$$

The first mode, in the absence of detuning, is just given by $\Omega=0$. The second mode looks like a Doppler-shifted mode with a photon recoil term. They can be seen as a field and a beam mode with nearby frequencies, coupled through the free-electron laser parameter $\bar{\rho}$. It should be noticed that, if we take $\bar{v}_0=0$, for a Dirac delta function centered at $p_0=0$, and $\hbar q^2/\mu=1/\bar{\rho}$, which means $q=\pm 1$ as pointed out above, we are left with an equation for Ω of the form

$$(\Omega+\bar{\delta})\left(\Omega^2-\frac{1}{4\bar{\rho}^2}\right)=1. \tag{35}$$

This equation coincides with that obtained in Ref. 3 for the quantum free-electron laser if we replace $\Omega=-\lambda$. It is also formally identical to the well known classical dispersion relation, if we assume a classical distribution with a rectangular shape and a width equal to $2\bar{\rho}$.[20] It has already been shown that this equation has solutions with $\Im(\Omega)>0$, corresponding to the free-electron laser instability. We should point out that when the number of photons per particle is much larger than unity, i.e., when $\bar{\rho}\gg 1$, this quantum FEL dispersion relation reduces to the classical one with a maximum growth rate given by $\Im(\Omega)=\sqrt{3}/2$ at $\bar{\delta}=0$, as pointed out in Ref. 3. The wave kinetic formulation is, therefore, able to reproduce such results. But its main interest is related to the study of a more general situation in which the simple distribution (30) is not valid. Let us go back to the general quantum dispersion relation, given by Eq. (28). This can be rewritten in the form

$$1+\chi_r(\Omega,q)+i\chi_i(\Omega,q)=0, \tag{36}$$

where the real part of the susceptibility function $\chi(\Omega,q)$ is given by

$$\chi_r(\Omega,q)=-\frac{\bar{\rho}}{(\Omega+\bar{\delta})}\frac{\mu}{\hbar q}\mathbf{P}\int G_0(z)\left(\frac{1}{(z-\Omega_+)}-\frac{1}{(z-\Omega_-)}\right)dz, \tag{37}$$

where $\mathbf{P}\int$ represents the principal part of the integral, and the imaginary part of the susceptibility function contains the contribution from the singularities,

$$\chi_i(\Omega,q)=-\frac{\pi\bar{\rho}}{(\Omega+\bar{\delta})}\frac{\mu}{\hbar q}[G_0(z=\Omega_+)-G_0(z=\Omega_-)]. \tag{38}$$

In these equations, we have used a new variable defined as $z=q\bar{v}$, where $\bar{v}=\hbar p/\mu$. If the energy spread of the relativistic electron beam is not very significant, Eq. (37) will lead to a dispersion relation, $1+\chi_r(\Omega,q)=0$, which is not very much different from the previous one given by Eq. (31) or Eq. (33). In addition, however, we have an extra source of instability provided by the imaginary part of the susceptibility. This leads to a kinetic damping coefficient determined by

$$\Gamma=-\frac{\chi_i(\Omega_r,q)}{(\partial\chi_r/\partial\Omega)_{\Omega=\Omega_r}}, \tag{39}$$

where we have taken $\Omega=\Omega_r+i\Gamma$. This shows that the free-electron laser can also become unstable outside the range of classical instability described by Eq. (31), in cases in which $\Gamma>0$. This kinetic instability condition can more explicitly be written as

$$\Gamma=\pi\bar{\rho}\frac{\mu}{\hbar q}\frac{[G_0(\Omega_+)-G_0(\Omega-)]}{(\partial\chi_r/\partial\Omega)_{\Omega=\Omega_r}}. \tag{40}$$

Taking $\bar{v}_0=0$, we can approximately write

$$\frac{\partial\chi_r}{\partial\Omega}\simeq\frac{1}{(\Omega+\bar{\delta})}+\frac{2\Omega}{(\Omega^2-\hbar^2q^4/4m^2)}. \tag{41}$$

This means that, even if the free-electron laser is classically stable, the wave kinetic description of its quantum behavior predicts a kinetic instability for $\Gamma>0$. This can occur for an appropriate inversion of population in the p states, and shown by the numerator of Eq. (40). Equation (41) generalizes the results pointed out in Ref. 13 where χ_r is assumed to be constant. In this approximation, the Landau damping effect becomes negligible. We should point out that this nondissipative kinetic effect can also be neglected if the denominator of Eq. (40) is large enough so that the kinetic instability is suppressed.

IV. CONCLUSION

In conclusion, a wave kinetic theory was applied to formulate the quantum free-electron laser theory. We have first shown how a Schrodinger type of equation can be derived from the relativistic theory, in the eikonal approximation, for an electron beam, moving in the ponderomotive potential that results from the beating between the wiggler field and the radiation field. We have then established the wave kinetic equation for the relativistic electron beam, which is exactly equivalent to the one-dimensional Schrodinger type of equation describing the electron bunching in a wiggler. We have then used this wave kinetic equation, coupled with the envelope equation for the radiation field, to derive a general kinetic dispersion relation for perturbations in the beam. We also have shown that, in the particular case of a monoenergetic electron beam, this dispersion relation reduces to the well known quantum results. However, for a beam with a finite-energy distribution, new domains of instability can be shown to exist. The present approach, therefore, leads to a quite general description of the free-electron laser instability taking into account the kinetic nondissipative effect known as Landau damping. The present theoretical model can also be adapted to the case of free-electron laser effects in plasmas.[21–24]

ACKNOWLEDGMENTS

One of the authors (A.S.) would like to thank the hospitality of GoLP/IST (Lisbon) during the final stages of this work.

This work was supported by the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq; The Brazilian National Council of Scientific and Technological Developments).

[1] J. Andruszkov, B. Aune, V. Ayvazyan *et al.*, Phys. Rev. Lett. **85**, 3825 (2005).
[2] Z. Huang and K.-J. Kim, Phys. Rev. ST Accel. Beams **10**, 034801 (2007).
[3] R. Bonifacio, N. Piovella, and G. R. M. Robb, Nucl. Instrum. Methods Phys. Res. A **543**, 645 (2005).
[4] M. Hillary, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[5] J. T. Mendonça and A. Serbeto, Phys. Plasmas **13**, 102109 (2006).
[6] J. T. Mendonça, P. Norreys, R. Bingham, and J. R. Davies, Phys. Rev. Lett. **94**, 245002 (2005).
[7] L. O. Silva and J. T. Mendonça, Opt. Commun. **196**, 285 (2001).
[8] J. T. Mendonça, P. K. Shukla, and R. Bingham, Phys. Lett. A **340**, 355 (2005).
[9] J. T. Mendonça, R. Bingham, and P. K. Shukla, JETP **201**, 942 (2005).
[10] J. T. Mendonça, P. K. Shukla, R. Bingham, J. M. Dawson, and L. O. Silva, J. Plasma Phys. **64**, 97 (2000).
[11] L. O. Silva, R. Bingham, J. M. Dawson, J. T. Mendonça, and P. K. Shukla, Phys. Rev. Lett. **83**, 2703 (1999).
[12] R. Bonifacio, M. M. Cola, N. Piovella, and G. R. M. Ross, Europhys. Lett. **69**, 55 (2005).
[13] N. Piovella and R. Bonifacio, Nucl. Instrum. Methods Phys. Res. A **560**, 240 (2006).
[14] G. Preparata, Phys. Rev. A **38**, 233 (1988).
[15] V. I. Karpman, *Nonlinear Waves in Dispersive Media* (Pergamon, Oxford, 1975).
[16] A. Hasegawa, *Plasma Instabilities and Nonlinear Effects* (Springer, Berlin, 1975).
[17] N. L. Tsintsadze, J. T. Mendonça, and L. N. Tsintsadze, Phys. Plasmas **5**, 3512 (1998).
[18] F. T. Arecchi and R. Bonifacio, IEEE J. Quantum Electron. **1**, 169 (1965).
[19] N. Piovella, M. M. Cola, L. Volpe, R. Gaiba, A. Schiavi, and R. Bonifacio, Opt. Commun. **274**, 347 (2007).
[20] R. Bonifacio, R. Corsini, L. De Salvo, P. Pierini, and N. Piovells, Riv. Nuovo Cimento **15**, 1 (1992).
[21] A. Serbeto and M. Virginia Alves, IEEE Trans. Plasma Sci. **21**, 243 (1993).
[22] V. Petrillo, A. Serbeto, C. Maroli, R. Parrella, and R. Bonifacio, Phys. Rev. E **51**, 6293 (1995).
[23] K. H. Tsui, A. Serbeto, and J. B. D'Olival, Phys. Rev. E **57**, 1029 (1998).
[24] K. H. Tsui and A. Serbeto, Phys. Rev. E **58**, 5013 (1998).