Nonlinear Schroedinger Equation in the Presence of Uniform Acceleration

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We consider a recently proposed nonlinear Schroedinger equation exhibiting soliton-like solutions of the power-law form $e_q^{i(kx-wt)}$, involving the q-exponential function which naturally emerges within nonextensive thermostatistics $[e_q^z \equiv [1+(1-q)z]^{1/(1-q)}$, with $e_1^z = e^z]$. Since these basic solutions behave like free particles, obeying $p = \hbar k$, $E = \hbar\omega$ and $E = p^2/2m$ $(1 \leq q < 2)$, it is relevant to investigate how they change under the effect of uniform acceleration, thus providing the first steps towards the application of the aforementioned nonlinear equation to the study of physical scenarios beyond free particle dynamics. We investigate first the behaviour of the power-law solutions under Galilean transformation and discuss the ensuing Doppler-like effects. We consider then constant acceleration, obtaining new solutions that can be equivalently regarded as describing a free particle viewed from an uniformly accelerated reference frame (with acceleration a) or a particle moving under a constant force $-ma$. The latter interpretation naturally leads to the evolution equation $i\hbar\frac{\partial}{\partial t}\left(\frac{\Phi}{\Phi_0}\right) = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\Phi}{\Phi_0}\right)^{2-q}\right] + V(x)\left(\frac{\Phi}{\Phi_0}\right)^q$ with $V(x) = max$. Remarkably enough, the potential V couples to Φ^q, instead of coupling to Φ, as happens in the familiar linear case $(q = 1)$.

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The importance of symmetries in physics can hardly be overestimated. A proper understanding of the relevant symmetries exhibited by a physical phenomenon encapsulates some of its most essential features, and usually leads to the most elegant and compact mathematical formulation of the associated physical laws. Many important symmetries are directly related to changes in the reference frame employed to describe physical systems. Such is the case of the dynamical symmetries associated with the Galilean transformation between inertial frames, playing a central role in Newtonian mechanics and nonrelativistic quantum mechanics; the Lorentz transformation plays an analogous role in special relativity, both at the classical and at the quantum levels. We can also mention the invariance of statistical mechanics and thermodynamics under uniform translation of the energy spectrum, which leads to the full freedom of choosing the zero level of energies in any physical system (the above list of examples is, of course, far from complete). A remarkable feature of symmetry or invariance principles is that they constitute powerful heuristic tools to extend knowledge about particular or restricted instances of the behaviour of physical systems to more general situations or scenarios. The most spectacular example of this approach was probably Einstein's use of his Principle of Equivalence in the development of General Relativity.

Symmetries with important physical implications are also observed in pure mathematics. We may focus, for example, on the Central Limit Theorem: the stability of Gaussians in the space of distributions with regard to adding (into the system) a finite or even infinite number of nearly independent random variables with finite variance is at the heart of their robustness, and therefore of their ubiquity in nature. The expansibility property of an entropic functional (that is, its invariance when adding new microscopic configurations with zero probability) constitutes another relevant example. It plays a special role within the (Shannon-Khinchine) set of axioms that uniquely determine the mathematical form of the Boltzmann-Gibbs (BG) additive entropy, basis of BG statistical mechanics.

Symmetry considerations arise naturally in the study of both linear and nonlinear dynamics. In the latter case, however, the analysis of the relevant symmetries is usually much more difficult. The physically relevant nonlinear evolution equations are highly diverse and describe wide classes of physical systems: see for instance [1–4]. Among the most studied nonlinear differential equations we have nonlinear versions of the Schroedinger [1, 5, 6] and Fokker-Planck [2, 7, 8] ones. Here we focus primarily on the soliton-like solutions of the recently proposed nonlinear Schroedinger equation [5] inspired by nonextensive statistical mechanics and the associated nonadditive entropies [9–11]. The nonlinear evolution equation advanced in [5] (and the above mentioned soliton-like solution) are implicitly assumed to hold with repect to an inertial reference frame. Our main aim in the present work is to investigate the form of these solutions when described with respect to a uniformly accelerated reference frame. We consider first the behavior of the soliton-like solutions under Galilean transformations connecting inertial frames. We then tackle the case of accelerated frames.

The aforementioned soliton-like solutions are referred to as q-plane waves and may be relevant in diverse areas of physics, including nonlinear optics, superconductivity, plasma physics, and dark matter [6, 12]. The theory within which q-plane waves emerged generalizes the BG entropy and statistical mechanics, through the introduction of an index q ($q \to 1$ recovers the BG case). Along this line, considerable progress was achieved, leading to generalized functions, distributions, various equations of physics, and new forms of the Central Limit Theorem [13]. In particular, the q-Gaussian distribution, which generalizes the standard Gaussian, appears naturally by extremizing the q-entropy [9], or from the solution of the corresponding nonlinear Fokker-Planck equation [14], and has been successfully applied to the analysis of recent experimental results in various fields [10]. Among others, we may mention: (i) The velocities of cold atoms in dissipative optical lattices [15]; (ii) The velocities of particles in quasi-two dimensional dusty plasma [16]; (iii) Single ions in radio frequency traps interacting with a classical buffer gas [17]; (iv) The relaxation curves of RKKY spin glasses, like CuMn and AuFe [18]; (v) Transverse momenta distributions at LHC experiments [19].

Herein we discuss the following q-generalized Schroedinger equation for a d-dimensional free particle of mass m [5]:

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = -\frac{1}{2-q}\;\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2-q} \quad (q \geq 1)\,, \tag{1}$$

where Φ_0 guarantees the correct physical dimensionalities for all terms (this scaling becomes irrelevant only for the linear equation, i.e., $q = 1$). Its solutions are expressed in terms of the q-exponential function $\exp_q(u)$ which, for a pure imaginary iu, is defined as the principal value of

$$\exp_q(iu) = [1 + (1-q)iu\,]^{\frac{1}{1-q}}\,;\ \exp_1(iu) \equiv \exp(iu). \tag{2}$$

The above function satisfies [20],

$$\begin{aligned}
\exp_q(\pm iu) &= \cos_q(u) \pm i\sin_q(u)\ , \\
\cos_q(u) &= \rho_q(u)\cos\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}\ , \\
\sin_q(u) &= \rho_q(u)\sin\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}\ , \\
\rho_q(u) &= \left[1+(1-q)^2u^2\right]^{1/[2(1-q)]}\ ,
\end{aligned} \tag{3}$$

$$\begin{aligned}
\exp_q(iu)\exp_q(-iu) &= [\rho_q(u)]^2 = \exp_q(-(q-1)u^2), \\
\exp_q(iu_1)\exp_q(iu_2) &\neq \exp_q[i(u_1+u_2)]\,,\ (q \neq 1)
\end{aligned} \tag{4}$$

As a consequence of Eqs. (3-4), a q-exponential with a pure imaginary argument, $\exp_q(iu)$, presents an oscillatory behavior with a u-dependent amplitude $\rho_q(u)$. The function $\exp_q(iu)$ complies with the physically important property of square integrability for $1 < q < 3$, whereas the concomitant integral diverges in both limits $q \to 1$ and $q \to 3$ and also for $q < 1$ [21].

The d-dimensional q-plane wave is given by

$$\Phi(\vec{x},t) = \Phi_0\ \exp_q\left[i(\vec{k}\cdot\vec{x} - \omega t)\right]\ , \tag{5}$$

If we take into account that $d\exp_q(z)/dz = [\exp_q(z)]^q$ and $d^2\exp_q(z)/dz^2 = q[\exp_q(z)]^{2q-1}$ we obtain, for the d-dimensional Laplacian,

$$\nabla^2\left(\frac{\Phi}{\Phi_0}\right) = -q\left(\sum_{n=1}^{d}k_n^2\right)\left(\frac{\Phi}{\Phi_0}\right)^{2q-1}\ . \tag{6}$$

Now, inserting the q-plane wave ansatz (5) into the NL Schroedinger Eq. (1), we verify that the q-plane wave is indeed a solution provided that the frequency ω and the momentum k satisfy the relation $\omega = \frac{\hbar k^2}{2m}$. Equivalently, if one makes [5], according to the celebrated de Broglie and Planck relations, the identifications $\vec{k} \to \vec{p}/\hbar$ and $\omega \to E/\hbar$, one verifies that the q-plane wave is a solution of equation (1) with $E = p^2/2m$, thus *preserving the energy spectrum of the free particle for all values of* q.

Eq. (1) differs from previous formulations [1, 22] where new nonlinear terms (usually a cubic nonlinearity in the wave function) are added to the two existing linear terms. The main differences between Eq. (1) and other proposals for NL Schroedinger equations are: (i) Instead of adding an extra term in which the nonlinearity is introduced, we generalize the spatial second-derivative term; (ii) The equation, together with the proposed solution, are easily extended from one to d dimensions; (iii) The corresponding solution of Eq. (1) manifests nonlinearity in both space and time, through a modulation in these two variables, which keeps the norm finite for all $(\vec{x},t)$; (iv) The well-known energy spectrum of a free particle is preserved for all q. Therefore Eq. (1), together with its solution Eq. (5), can be considered as candidates for describing interesting types of physical phenomena.

Let us now investigate how the q-plane wave solution is transformed under two basic types of changes in the reference frame. As already mentioned, we consider first a Galilean transformation connecting inertial reference frames. Then, we analyze the aspect of the q-plane wave solutions when "viewed" from an uniformly accelerated reference frame. We shall consider the one dimensional nonlinear Schroedinger equation. The extension to the d-dimensional case is straightforward.

We shall assume that the nonlinear Schroedinger equation,

$$i\hbar\frac{\partial}{\partial t'}\left[\frac{\Psi(x',t')}{\Psi_0}\right] = -\frac{1}{2-q}\;\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\left[\frac{\Psi(x',t')}{\Psi_0}\right]^{2-q} \quad (q \geq 1), \tag{7}$$

holds in an inertial reference frame characterized by the spatio-temporal coordinates (x', t') and that the system under consideration is described by a q-plane wave solution

$$\Psi(x',t') = \Psi_0 \, \exp_q\left[i(kx' - \omega t')\right]. \qquad (8)$$

Let us consider now a Galilean transformation

$$t = t'; \quad x = x' - vt', \qquad (9)$$

relating the original inertial frame (x', t') with a second inertial frame (x, t) that moves with respect to the former one with a uniform velocity v. To obtain a "new" solution $\Phi(x, t)$ to the nonlinear Schroedinger equation expressed in terms of the spatio-temporal coordinates (x, t), that corresponds to the "old" solution (8), one may naively just re-express (8) in terms of the new variables (that is, substitute in (8) the old variables (x', t') by their expressions in terms of the new variables (x, t)),

$$\begin{aligned}\Psi(x',t') \rightarrow \Phi(x,t) &= \Psi(x+vt,t) \\ &= \Phi_0 \exp_q\left[i(k(x+vt) - \omega t)\right], \;\; \Phi_0 = \Psi_0 \end{aligned} \qquad (10)$$

However, this procedure leads to a function of x and t that does not satisfy the nonlinear Schroedinger equation (as expressed in terms of the new variables (x, t)). In order to obtain a valid solution it is necessary to add an extra term to the argument of the q-exponential. Indeed, it can be verified after some algebra that,

$$\frac{\Phi}{\Phi_0} = \left[1 - i(1-q)\left\{\omega t - k(x+vt) + \frac{1}{\hbar}\left(mvx + \frac{1}{2}mv^2 t\right)\right\}\right]^{\frac{1}{1-q}}. \qquad (11)$$

does satisfy the nonlinear Schroedinger equation. The extra term $\frac{1}{\hbar}\left(mvx + \frac{1}{2}mv^2 t\right)$ admits a clear physical interpretation. If we recast (11) under the guise,

$$\frac{\Phi}{\Phi_0} = \left[1 - i(1-q)\left\{\left(\omega - kv + \frac{mv^2}{2\hbar}\right)t - \left(k - \frac{mv}{\hbar}\right)x\right\}\right]^{\frac{1}{1-q}}, \qquad (12)$$

it is plain that (12) has the form of a q-plane wave with frequency $\tilde{\omega}$ and wave number $\tilde{k}$ respectively given by

$$\tilde{\omega} = \omega - kv + \frac{mv^2}{2\hbar}; \quad \tilde{k} = k - \frac{mv}{\hbar}. \qquad (13)$$

Now, as shown in [5], the q-plane wave solutions to the nonlinear Schroedinger equation are compatible with the Planck and de Broglie relations connecting respectively frequency and wave number with energy and momentum: $E = \hbar\omega$ and $p = \hbar k$. Combining equations (13) with the Planck and de Broglie relations we obtain $\tilde{E} = E - pv + \frac{mv^2}{2}$ and $\tilde{p} = p - mv$, which are the correct Galilean transformations for the kinetic energy and momentum of a particle of mass m obeying the (nonrelativistic) energy-momentum relation $E = p^2/2m$. These considerations reinforce the validity of the Planck and de Broglie relations for the q-plane wave solutions of Eq. (1).

Taking the limit $q \rightarrow 1$ of the transformed solution (11), on sees that the relation between the original solution $\Psi(x', t')$ and the transformed one $\Phi(x, t)$ becomes,

$$\begin{aligned}\Psi(x',t') &\rightarrow \Phi(x,t) \\ &= \exp\left[-\frac{i}{\hbar}\left(\frac{mv^2}{2}t + mvx\right)\right]\Psi(x+vt,t),\end{aligned} \qquad (14)$$

thus recovering the transformation rule corresponding to the linear Schroedinger equation [23].

Let us now consider a uniformly accelerated reference frame. The corresponding spatio-temporal coordinates (x, t) are

$$t = t'; \quad x = x' - \frac{1}{2}at'^2 = x' - \frac{1}{2}\frac{F}{m}t'^2, \qquad (15)$$

where (x', t') are the variables associated with an inertial frame, a is the constant acceleration of reference frame (x, t), and $a = \frac{F}{m}$. As in the previous discussion, we assume that the nonlinear Schroedinger equation (7) holds in the inertial frame (x', t'), and also that in this frame our system is described by the q-plane wave solution (8). Again, simply re-writting the q-plane wave solution (8) in terms of the new variables (x, t) does not yield a solution of the nonlinear Schroedinger equation. As in the above Galilean transformation case, new terms are needed in the argument of the q-exponential to obtain a valid solution. Let us consider the ansatz,

$$\frac{\Phi}{\Phi_0} = \left[1 - i(1-q)\left\{\omega t - k\left(x + \frac{Ft^2}{2m}\right) + \frac{F}{\hbar}\left(xt + \frac{Ft^3}{6m}\right)\right\}\right]^{\frac{1}{1-q}}. \qquad (16)$$

Inserting (16) in the right and the left hand sides of the nonlinear Schroedinger equation yields

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\Phi}{\Phi_0}\right) = \left[\hbar\omega - \frac{\hbar kFt}{m} + Fx + \frac{F^2t^2}{2m}\right]\left(\frac{\Phi}{\Phi_0}\right)^q, \qquad (17)$$

and

$$-\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\Phi}{\Phi_0}\right)^{2-q}\right] = \left[\frac{\hbar^2k^2}{2m} - \frac{\hbar kFt}{m} + \frac{F^2t^2}{2m}\right]\left(\frac{\Phi}{\Phi_0}\right)^q. \qquad (18)$$

Comparing (17) with (18) one verifies that the ansatz (16) satisfies the nonlinear equation,

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\Phi}{\Phi_0}\right) = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\Phi}{\Phi_0}\right)^{2-q}\right] + V(x)\left(\frac{\Phi}{\Phi_0}\right)^q, \qquad (19)$$

where $V(x) = Fx$. The nonlinear Eq. (19) can be interpreted as describing the motion of a particle of mass m under a constant force $-F$ (with the associated potential function $V = Fx$). This is consistent with the well-known fact that the behavior of a free particle with respect to a uniformly accelerated reference frame is equivalent to the the behavour of a particle in an inertial reference frame moving under the effect of a constant force.

In the limit $F \to 0$, Eq. (19) reduces to the nonlinear Schroedinger equation for a free particle introduced in [5], and solution (16) reduces to the corresponding q-plane wave solution. Also, $q \to 1$ in Eq. (19) corresponds to the standard linear Schroedinger equation for a particle of mass m moving under a constant force $-F$. An interesting feature of equation (19) is that the potential V couples to Φ^q, instead of coupling to Φ, as happens in the standard linear case ($q = 1$). Consistently, the q-plane wave $\Phi(x,t) = \Phi_0 \exp_q[i(kx - \omega t)]$ is not only a solution of the free-particle nonlinear Schroedinger equation (when $\hbar\omega = \frac{\hbar^2 k^2}{2m}$), but also of the nonlinear equation

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\Phi}{\Phi_0}\right) = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\Phi}{\Phi_0}\right)^{2-q}\right] + V_0\left(\frac{\Phi}{\Phi_0}\right)^q, \tag{20}$$

with a constant potential V_0, provided that $\hbar\omega = \frac{\hbar^2 k^2}{2m} + V_0$, which, using the Planck and de Broglie relations, becomes $E = \frac{p^2}{2m} + V_0$, as expected.

Considering now the limit $q \to 1$ of the transformed solution (16), we verify that the original solution $\Psi(x', t')$ and the transformed one $\Phi(x,t)$ are linked through

$$\Psi(x', t') \to \Phi(x,t) = \exp\left[-\frac{i}{\hbar}\left(Fxt + \frac{F^2 t^3}{6m}\right)\right]\Psi\left(x + \frac{Ft^2}{2m}, t\right), \tag{21}$$

thus recovering the transformation rule associated with the linear Schroedinger equation [23].

We have investigated the effect of uniform acceleration on q-plane waves, soliton-like solutions of a recently proposed non-linear Schroedinger equation associated with non-extensive thermostatistics. We first studied the behaviour of these solutions under Galilean transformations relating different inertial frames and obtained the transformation rule satisfied by the q-plane waves. This rule turns out to be fully consistent with the de Broglie and Planck relations, thus providing further support to the validity of these relations for the q-plane wave solutions. Then we derived the transformation law yielding new solutions corresponding to the aforementioned q-plane waves when "viewed" from uniformly accelerated frames. In the limit $q \to 1$ the transformation laws advanced here reduce to those associated with time dependent solutions of the standard, linear Schroedinger equation. The accelerated q-plane wave solutions investigated here admit two possible interpretations: they can be viewed as describing a free particle as "seen" from a uniformly accelerated frame or, alternatively, as describing a particle moving under the effect of a constant force. In fact, the non-linear Schroedinger equation governing these solutions (when expressed in terms of the accelerated frame's coordinates) incorporates a new term involving a potential function $V(x)$. This equation indicates that, within the present generalization of Schroedinger equation, the potential $V(x)$ "couples" to a power of the wave function, Φ^q, instead of coupling just to the wave function Φ, as happens with the linear Schroedinger equation. This result opens the door for the study of a variety of simple potentials, which should result in physical applications.

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