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A procedure for obtaining general nonlinear Fokker–Planck equations

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Abstract

A procedure for deriving general nonlinear Fokker–Planck equations (FPEs) directly from the master equation is presented. The nonlinear effects are introduced in the transition probabilities, which present a dependence on the probabilities for finding the system in a given state. It is shown that the FPEs, obtained from master equations describing transitions among discrete and continuous sets of states, are identical. Within such a procedure, we construct nonlinear FPEs that appear to be very general. Our general FPEs recover, as particular cases, nonlinear FPEs investigated previously by many authors, introduced on a purely phenomenological basis, and they lead to the possibility of more complete and complex diffusive equations.

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1. Introduction

One of the fundamental equations of nonequilibrium statistical mechanics is certainly the master equation, which describes the evolution, in phase space, of a given physical system following a Markovian stochastic process [1–3]. More specifically, it describes how the probability for finding the system in a given state changes in time due

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to transitions between states. Considering a system specified by a set of discrete stochastic variables, we shall define $P(n,t)$ as the probability to find the system in a state characterized by the variable n at time t. Basically, the master equation expresses the simple fact that $P(n,t)$ tends to increase with time due to transitions from other states to n, whereas it decreases due to transitions from state n to other states, i.e.,

$$\frac{\partial P(n,t)}{\partial t} = \sum_{m=-\infty}^{\infty} [P(m,t)w_{m,n}(t) - P(n,t)w_{n,m}(t)] . \tag{1.1}$$

In the equation above, $w_{k,l}(t)$ represents the transition probability rate from state k to l [i.e., $w_{k,l}(t)\,\mathrm{d}t$ is the probability for a transition from state k to l to occur during the time interval $t \to t + \mathrm{d}t$]. In most of the cases—basically Hamiltonian systems—the transitions between states are essentially instantaneous, in such a way that the transition rates $w_{k,l}(t)$ do not depend on the probabilities for finding the system in either one of the states k or l, leading to a master equation that is usually considered as a linear equation. The master equation may be written also for the case of continuous stochastic variables $\{x\}$,

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}x'[P(x',t)w(x'|x) - P(x,t)w(x|x')] , \tag{1.2}$$

where $w(y|z)$ represents the transition rate from state y to z.

The standard, linear Fokker–Planck equations (FPE), may be obtained by choosing conveniently the transition rates (with no dependence on the probabilities for finding the system on the states before and after the transition), from both forms of the master equation (Eqs. (1.1) and (1.2)) [2,3]. Although the linear FPE is widely accepted as an appropriate equation for the description of many natural phenomena—mainly those within the class of normal diffusion—it is well-established nowadays that such an equation is not suitable to describe more complicated diffusion processes, like those inserted in the class of anomalous-diffusion problems [4,5]. Among such processes, one may single out the transport of a fluid in porous media [6], dynamics of surface growth [6], diffusion of polymer-like breakable micelles [7], evolution of the density of flux lines in disordered superconductors [8], correlations in heartbeat interval increments [9], and financial transactions [10,11].

The introduction of nonlinearities in FPEs has appeared recently as a possible procedure to describe anomalous-transport processes properly. Essentially, two approaches have been used for such a purpose. The first approach consists in considering a linear theory, and introduce the anomalous nature of the process through correlations expressed in non-local operators; this has been much considered recently, through the study of the fractional FPE (see Ref. [12] for a review), which may also be derived from a generalized master equation [13]. In the second approach, one uses nonlinear FPEs [4–7,14–22], some of which appear as simple phenomenological generalizations of the usual linear FPE [14–18], presenting the power-like probability distribution as a solution, which maximizes the entropy proposed by Tsallis [23–26].

Recently, nonlinear FPEs were derived directly from the master equation, by introducing transition rates with a dependence on the probabilities for finding the system on the states before and after the transition [27]. Such a scheme should apply to systems that evolve in time through gradual transitions (rather than instantaneous transitions, as in standard Hamiltonian systems), in such a way that the transition rate should depend on the above-mentioned probabilities. One could also mention the fuzzy-logic systems, for which there is always a pertinence for the system to be in more than one state at the same time. In addition to that, as argued in Ref. [27], we could mention, as good candidates for the application of such a transition rate, those systems following anomalous-diffusion processes, like the ones indicated above, e.g., particle diffusion in a porous media, surface growth in fractals, flux lines in disordered superconductors, and financial transactions.

For a system described in terms of discrete stochastic variables, the following nonlinear FPE has been derived [27],

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + a\,\frac{\partial^2 P^{\mu}(x,t)}{\partial x^2} + bP^{\nu-1}(x,t)\,\frac{\partial^2 P(x,t)}{\partial x^2} - bP(x,t)\,\frac{\partial^2 P^{\nu-1}(x,t)}{\partial x^2}\;. \tag{1.3}$$

The constants a and b, may depend, in principle, on the system under consideration, $F(x)$ is an external force, and the exponents μ and ν are real numbers. It is important to remind that Eq. (1.3) reduces to the usual (linear) FPE [2,3] either for $(a=D, b=0, \mu=1)$, $(b=D, a=0, \nu=1)$, or $(a+b=D, \mu=\nu=1)$, with D representing the diffusion constant.

For the case of continuous stochastic variables, the corresponding nonlinear FPE obtained was [27],

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + a\,\frac{\partial^2 P^{\mu}(x,t)}{\partial x^2} + b\,\frac{\partial^2 P^{\nu}(x,t)}{\partial x^2} - 2b\,\frac{\partial P(x,t)}{\partial x}\,\frac{\partial P^{\nu-1}(x,t)}{\partial x} - 2bP(x,t)\,\frac{\partial^2 P^{\nu-1}(x,t)}{\partial x^2}\;. \tag{1.4}$$

Once again, the above equation recovers the standard FPE in the particular limits $(a=D,\ b=0,\ \mu=1)$, $(b=D,\ a=0,\ \nu=1)$, or $(a+b=D,\ \mu=\nu=1)$.

It is possible to show that Eqs. (1.3) and (1.4) are identical, reducing to the simple form (see the appendix),

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\,\frac{\partial P(x,t)}{\partial x}\right\}\;, \tag{1.5a}$$

$$\Omega[P(x,t)] = a\mu P^{\mu-1}(x,t) + b(2-\nu)P^{\nu-1}(x,t)\;. \tag{1.5b}$$

It should be noticed that Eqs. (1.5) recover the nonlinear FPE proposed, on a phenomenological basis, by many authors [14–17,19,20], either for $b=0$, $a=0$ $(\nu\neq 2)$, or $\mu=\nu$. In such cases, relating the exponents μ and ν to Tsallis's entropic index q [23,26], $\mu=2-q$, $\nu=2-q$, and $\mu=\nu=2-q$, respectively, it is possible to show

[14,15,27] that the time-dependent Tsallis's probability distribution,

$$P(x,t) = B(t)[\xi(x,t)]^{1/(1-q)} , \tag{1.6a}$$

$$\xi(x,t) = 1 + \beta(t)(q-1)[x - x_0(t)]^2 \quad (1 < q < 3) , \tag{1.6b}$$

is a solution of Eqs. (1.5), provided that one considers a harmonic external force, $F(x) = k_1 - k_2 x$ (k_1 and k_2 constants, $k_2 \geqslant 0$).

It is important to remind also that Eqs. (1.5) reduce, for the particular case $\mu=2$ and $v=1$, to the nonlinear diffusion equation proposed for the description of the evolution of the density of flux lines in disordered superconductors [8].

In the present work we introduce, into the master equation, transition probabilities that generalize those of Ref. [27], deriving FPEs for both discrete and continuous sets of states, and show that such equations are identical. The FPEs obtained herein are much more general than the one of Eqs. (1.5). In the next section, we consider transition probabilities with a polynomial dependence on the probabilities for finding the system in a given state, whereas in Section 3 we deal with transition probabilities of a general kind. Finally, in Section 4 we present our conclusions.

2. Polynomial transition rates

In Ref. [27] we have considered transition rates which depended on either one of the probabilities for finding the system on the state before or after the transition at time t. However, for the kind of systems mentioned in the previous section, it is quite reasonable to expect, as well, a dependence of the transition rate on both probabilities simultaneously. For that, one should consider contributions which take into account products of such probabilities; in general, one may have several contributions, with different powers each.

Let us first consider the case of discrete stochastic variables, described by the master equation of Eq. (1.1); for a random walk in which the step size is given by Δ, one has that

$$\frac{\partial P(n\Delta,t)}{\partial t} = \sum_{m=-\infty}^{\infty} [P(m\Delta,t)w_{m,n}(\Delta) - P(n\Delta,t)w_{n,m}(\Delta)] . \tag{2.1}$$

Now, we shall introduce a transition rate with a polynomial dependence on the probabilities for finding the system on a given state, i.e.,

$$w_{k,l}(\Delta) = -\frac{1}{\Delta}\,\delta_{k,l+1}F(k\Delta) + \frac{1}{\Delta^2}\,(\delta_{k,l+1} + \delta_{k,l-1}) \times \sum_{i=1}^{M} c_i P^{\eta_i - 1}(k\Delta,t)P^{\rho_i - 1}(l\Delta,t) , \tag{2.2}$$

where c_i $(i = 1,2,\ldots,M)$ are constants, the exponents η_i and ρ_i are real numbers, and M denotes an arbitrary number of terms (which could be infinite, in principle). One may see easily that the corresponding transition rate defined in Ref. [27] appears to be a particular case of the one defined above, for $M=2$ and $(c_1=a, c_2=b)$, $(\eta_1=\mu, \eta_2=1)$, $(\rho_1=1, \rho_2=\nu)$.

Substituting the transition rate defined above in Eq. (2.1), carrying out the sums, and defining $x=n\Delta$, one gets

$$\frac{\partial P(x,t)}{\partial t} = -\frac{1}{\Delta}\,[P(x+\Delta,t)F(x+\Delta) - P(x,t)F(x)] + \frac{1}{\Delta^2}\sum_{i=1}^{M} c_i P^{\rho_i-1}(x,t)[P^{\eta_i}(x+\Delta,t) + P^{\eta_i}(x-\Delta,t)] - \frac{1}{\Delta^2}\sum_{i=1}^{M} c_i P^{\eta_i}(x,t)[P^{\rho_i-1}(x+\Delta,t) + P^{\rho_i-1}(x-\Delta,t)]\ . \tag{2.3}$$

Considering the limit $\Delta \to 0$, one gets the nonlinear FPE,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + \sum_{i=1}^{M} c_i P^{\rho_i-1}(x,t)\frac{\partial^2 P^{\eta_i}(x,t)}{\partial x^2} - \sum_{i=1}^{M} c_i P^{\eta_i}(x,t)\frac{\partial^2 P^{\rho_i-1}(x,t)}{\partial x^2}\ . \tag{2.4}$$

Let us now turn to the case of continuous stochastic variables; introducing the variable $y=x-x'$, the master equation of Eq. (1.2) becomes,

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}y[P(x-y,t)w(x-y|x) - P(x,t)w(x|x+y)]\ , \tag{2.5}$$

where we have changed $y \to -y$ in the first integral. Defining $\tau(x,y)=w(x|x+y)$, as the transition rate between states x and $x+y$, Eq. (2.5) may be written as

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}y[P(x-y,t)\tau(x-y,y) - P(x,t)\tau(x,y)]\ . \tag{2.6}$$

Considering $\tau(x,y)$ sharply peaked around $y=0$, one may expand Eq. (2.6) [3,28,29],

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}y \left\{ \sum_{n=1}^{\infty} \frac{(-y)^n}{n!}\,\frac{\partial^n[P(x,t)\tau(x,y)]}{\partial^n x} \right\}\ . \tag{2.7}$$

Similarly to what was done in the discrete case [see Eq. (2.2)], let us now define the transition rate for a transition between states x and $x+y$,

$$\tau(x,y) = \gamma_1(x,y) + \gamma_2(x,y)\sum_{i=1}^{M} c_i P^{\eta_i-1}(x,t)P^{\rho_i-1}(x+y,t)\ , \tag{2.8}$$

where

$$\gamma_1(x,y) = \begin{cases} \dfrac{F(x)}{\Delta^2} & \text{if } 0 \leqslant y \leqslant \sqrt{2}\Delta\ , \\ 0 & \text{otherwise} \end{cases} \tag{2.9a}$$

and

$$\gamma_2(x,y)=\begin{cases}\dfrac{1}{2\sqrt{6}\Delta^3} & \text{if } -\sqrt{6}\Delta \leqslant y \leqslant \sqrt{6}\Delta\ ,\\ 0 & \text{otherwise}\ .\end{cases} \tag{2.9b}$$

Again, by considering in the definition above, $M=2$ and $(c_1=a, c_2=b)$, $(\eta_1=\mu, \eta_2=1)$, $(\rho_1=1, \rho_2=\nu)$, the corresponding transition rate of Ref. [27] is recovered. Substituting the transition rate of Eq. (2.8) in Eq. (2.7), expanding $P^{\rho_i-1}(x+y,t)$ for y small, and taking the limit $\Delta\to 0$, one gets the following nonlinear FPE,

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} = &-\frac{\partial[F(x)P(x,t)]}{\partial x}+\sum_{i=1}^{M} c_i\,\frac{\partial^2 P^{\eta_i+\rho_i-1}(x,t)}{\partial x^2}\\ &-2\sum_{i=1}^{M} c_i\,\frac{\partial P^{\eta_i}(x,t)}{\partial x}\,\frac{\partial P^{\rho_i-1}(x,t)}{\partial x}\\ &-2\sum_{i=1}^{M} c_i P^{\eta_i}(x,t)\,\frac{\partial^2 P^{\rho_i-1}(x,t)}{\partial x^2}\ .\end{aligned} \tag{2.10}$$

It is important to stress that the procedure used for deriving Eq. (2.10) is very general, and depends only on how the moments of $\gamma_1(x,y)$ and $\gamma_2(x,y)$ scale with Δ and not on their specific forms [2,3]. One may show that Eqs. (2.4) and (2.10) are identical (see the appendix); they both reduce to the simple form of Eq. (1.5a), with

$$\Omega[P(x,t)]=\sum_{i=1}^{M} c_i(\eta_i-\rho_i+1)P^{\eta_i+\rho_i-2}(x,t)\ . \tag{2.11}$$

As expected, Eq. (2.11) recovers Eq. (1.5b) for $M=2$, $(c_1=a, c_2=b)$, $(\eta_1=\mu, \eta_2=1)$, and $(\rho_1=1, \rho_2=\nu)$. There are several possibilities where Eqs. (1.5a) and (2.11) lead to the nonlinear FPE of Refs. [14–17,19,20], for which Tsallis's time-dependent probability distribution is a solution [14,15,27], e.g., if one considers $\eta_i+\rho_i-2=\mu-1$ $(\mu=2-q)$ $(\forall i)$. In addition to that, the nonlinear FPE proposed for describing the evolution of the density of flux lines in disordered superconductors [8] may be obtained as a particular case of Eqs. (1.5a) and (2.11) for $M=2$, with $\eta_1+\rho_1=3$ $(\eta_1\neq\rho_1)$, and $\eta_2+\rho_2=2$ $(\eta_2\neq\rho_2)$.

In the next section we will derive a very general nonlinear FPE, by introducing transition probabilities with an arbitrary dependence on the probabilities for finding the system on a given state.

3. Transition rates of a general kind

Let us now introduce transition rates with a general dependence on the probabilities for finding the system on a given state. For the case of discrete stochastic variables, one may write Eq. (2.2) in the following general form,

$$w_{k,l}(\Delta) = -\frac{1}{\Delta}\,\delta_{k,l+1}F(k\Delta) + \frac{1}{\Delta^2}\,(\delta_{k,l+1} + \delta_{k,l-1})\;\Gamma[P(k\Delta,t),Q(l\Delta,t)]\;, \tag{3.1}$$

whereas for continuous variables, a similar procedure should change Eq. (2.8) into

$$\tau(x,y) = \gamma_1(x,y) + \gamma_2(x,y)\;\Gamma[P(x,t),Q(x+y,t)]\;. \tag{3.2}$$

For the sake of the calculations, we will distinguish the probabilities P and Q; obviously, one should have $Q(k\Delta,t) \equiv P(k\Delta,t)$ and $Q(x,t) \equiv P(x,t)$, for the cases of discrete and continuous stochastic variables, respectively.

In the latter case, one may use the expansion,

$$\begin{aligned}\Gamma[P(x,t),Q(x+y,t)] = \Bigg[&\Gamma[P(x,t),Q(x,t)] + \left(y\,\frac{\partial Q(x,t)}{\partial x}\right.\\ &\left.+\frac{y^2}{2}\,\frac{\partial^2 Q(x,t)}{\partial x^2}\right)\frac{\partial\Gamma[P,Q]}{\partial Q}\\ &+\frac{y^2}{2}\left(\frac{\partial Q(x,t)}{\partial x}\right)^2\frac{\partial^2\Gamma[P,Q]}{\partial Q^2} + \cdots\Bigg]_{Q=P}\end{aligned} \tag{3.3}$$

in such a way that substituting the transition rate of Eq. (3.2) into Eq. (2.7), and taking the limit $\Delta \to 0$, one gets the FPE,

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + \Bigg[&\frac{\partial^2\{P(x,t)\Gamma[P,Q]\}}{\partial x^2}\\ &-2\,\frac{\partial}{\partial x}\left(P(x,t)\,\frac{\partial\Gamma[P,Q]}{\partial Q}\,\frac{\partial Q(x,t)}{\partial x}\right)\Bigg]_{Q=P}\;.\end{aligned} \tag{3.4}$$

The above equation may be easily rewritten in the form of Eqs. (1.5), i.e.,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\,\frac{\partial P(x,t)}{\partial x}\right\}\;, \tag{3.5a}$$

$$\Omega[P(x,t)] = \left[\Gamma[P,Q] + P(x,t)\left(\frac{\partial\Gamma[P,Q]}{\partial P} - \frac{\partial\Gamma[P,Q]}{\partial Q}\right)\right]_{Q=P}\;, \tag{3.5b}$$

where we have used the fact that $(\partial P(x,t)/\partial x) \equiv (\partial Q(x,t)/\partial x)$.

A similar result applies for the case of discrete stochastic variables, i.e., if one applies the same procedure of the previous section for the transition rate of Eq. (3.1), one ends up with the same nonlinear FPE defined by Eqs. (3.5). This represents the most general type of nonlinear FPE that one can derive from the master equation, by using transition rates that depend on the probabilities for finding the system on a given state. In particular, if one considers

$$\Gamma[P(x,t),Q(x+y,t)] = \sum_{i=1}^{M} c_i P^{\eta_i - 1}(x,t) Q^{\rho_i - 1}(x+y,t) , \tag{3.6}$$

one recovers the form of $\Omega[P(x,t)]$ of the previous section [cf. Eq. (2.11)].

Due to its generality, the nonlinear FPE of Eqs. (3.5) is expected to present a wide range of applicability, covering many different types of nonlinear diffusive equations, in such a way that each physical system should be characterized by its own form of $\Gamma[P,Q]$. Since the simpler form in Eqs. (1.5) covers already a large variety of nonlinear Fokker–Planck-like equations known in the literature [6–8,14–17,19,20], the general form in Eqs. (3.5) should lead to even more complex diffusive equations.

4. Conclusion

We have obtained general nonlinear FPEs through approximations to the master equation. The nonlinear effects were introduced by considering, inside the transition probabilities of the master equation, arbitrary dependences on the probabilities for finding the system in a given state. Such a procedure was carried for both cases of discrete and continuous sets of states, and we have shown that the FPEs obtained are identical. We have argued that this kind of transition probability should be relevant for the description of many real phenomena included in the class of anomalous-diffusion problems. Due to their generality, the nonlinear FPEs derived herein cover many different types of diffusive equations, introduced previously in the literature, in a purely phenomenological basis. These equations are expected to present a wide range of applicability in the area of anomalous diffusion, opening the way to even more complex diffusive equations.

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Appendix A

In this appendix, we will show that the FPEs (2.4) and (2.10) [and consequently, Eqs. (1.3) and (1.4)], derived from the master equation for the cases of discrete and continuous sets of states, respectively, are identical. We shall consider the properties for the derivatives,

$$\frac{\partial P^{\kappa}(x,t)}{\partial x} = \kappa P^{\kappa-1}(x,t)\,\frac{\partial P(x,t)}{\partial x}\ , \tag{A.1a}$$

$$\frac{\partial^2 P^{\kappa}(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\,\frac{\partial P^{\kappa}(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[\kappa P^{\kappa-1}(x,t)\,\frac{\partial P(x,t)}{\partial x}\right]\ . \tag{A.1b}$$

The term of Eq. (2.4),

$$\mathscr{I}_1 = \sum_{i=1}^{M} c_i \left[P^{\rho_i-1}(x,t)\,\frac{\partial^2 P^{\eta_i}(x,t)}{\partial x^2} - P^{\eta_i}(x,t)\,\frac{\partial^2 P^{\rho_i-1}(x,t)}{\partial x^2}\right]\ , \tag{A.2a}$$

$$\mathscr{I}_1 = \sum_{i=1}^{M} c_i \,\frac{\partial}{\partial x}\left[P^{\rho_i-1}(x,t)\,\frac{\partial P^{\eta_i}(x,t)}{\partial x} - P^{\eta_i}(x,t)\,\frac{\partial P^{\rho_i-1}(x,t)}{\partial x}\right]\ , \tag{A.2b}$$

becomes, after using the properties of Eqs. (A.1),

$$\mathscr{I}_1 = \sum_{i=1}^{M} c_i(\eta_i - \rho_i + 1)\,\frac{\partial}{\partial x}\left[P^{\eta_i+\rho_i-2}(x,t)\,\frac{\partial P(x,t)}{\partial x}\right]\ . \tag{A.3}$$

Let us now consider the term of Eq. (2.10),

$$\mathscr{I}_2 = \sum_{i=1}^{M} c_i \left[\frac{\partial^2 P^{\eta_i+\rho_i-1}(x,t)}{\partial x^2} - 2\,\frac{\partial P^{\eta_i}(x,t)}{\partial x}\,\frac{\partial P^{\rho_i-1}(x,t)}{\partial x} - 2P^{\eta_i}(x,t)\,\frac{\partial^2 P^{\rho_i-1}(x,t)}{\partial x^2}\right]\ , \tag{A.4a}$$

$$\mathscr{I}_2 = \sum_{i=1}^{M} c_i \left[\frac{\partial^2 P^{\eta_i+\rho_i-1}(x,t)}{\partial x^2} - 2\,\frac{\partial}{\partial x}\left(P^{\eta_i}(x,t)\,\frac{\partial P^{\rho_i-1}(x,t)}{\partial x}\right)\right]\ , \tag{A.4b}$$

that, after using Eqs. (A.1), may be written as

$$\mathscr{I}_2 = \sum_{i=1}^{M} c_i(\eta_i - \rho_i + 1)\,\frac{\partial}{\partial x}\left[P^{\eta_i+\rho_i-2}(x,t)\,\frac{\partial P(x,t)}{\partial x}\right]\ . \tag{A.5}$$

Since $\mathscr{I}_1 = \mathscr{I}_2$, one concludes that Eqs. (2.4) and (2.10) are identical, and may be written in the form

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial [F(x)P(x,t)]}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\,\frac{\partial P(x,t)}{\partial x}\right\}\ , \tag{A.6a}$$

$$\Omega[P(x,t)] = \sum_{i=1}^{M} c_i(\eta_i - \rho_i + 1)P^{\eta_i+\rho_i-2}(x,t)\ . \tag{A.6b}$$

Since Eqs. (2.4) and (2.10) reduce, respectively, to Eqs. (1.3) and (1.4), for the particular case $M=2$, $(c_1=a, c_2=b)$, $(\eta_1=\mu, \eta_2=1)$, and $(\rho_1=1, \rho_2=\nu)$, one obviously concludes that Eqs. (1.3) and (1.4) are identical as well. For this particular case, the corresponding FPE is given by Eqs. (1.5), which follow trivially from the above equations.

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