From worldline to quantum superconformal mechanics with and without oscillatorial terms: $D(2,1;\alpha)$ and $sl(2|1)$ models

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In this paper, we quantize superconformal σ models defined by worldline supermultiplets. Two types of superconformal mechanics, with and without a De Alfaro Fubini Furlan (DFF) term, are considered. Without a DFF term (Calogero potential only), the supersymmetry is unbroken. The models with a DFF term correspond to deformed (if the Calogero potential is present) or undeformed oscillators. For these (un) deformed oscillators, the classical invariant superconformal algebra acts as a spectrum-generating algebra of the quantum theory. Besides the $osp(1|2)$ examples, we explicitly quantize the superconformally invariant worldline σ models defined by the $\mathcal{N}=4$ (1, 4, 3) supermultiplet [with $D(2,1;\alpha)$ invariance, for $\alpha\neq 0,-1$] and by the $\mathcal{N}=2$ (2, 2, 0) supermultiplet [with two-dimensional target and $sl(2|1)$ invariance]. The parameter α is the scaling dimension of the (1, 4, 3) supermultiplet and, in the DFF case, has a direct interpretation as a vacuum energy. In the DFF case, for the $sl(2|1)$ models, the scaling dimension λ is quantized (either $\lambda=\frac{1}{2}+\mathbb{Z}$ or $\lambda=\mathbb{Z}$). The ordinary two-dimensional oscillator is recovered, after imposing a superselection restriction, from the $\lambda=-\frac{1}{2}$ model. In particular, a single bosonic vacuum is selected. The spectrum of the unrestricted two-dimensional theory is decomposed into an infinite set of lowest-weight representations of $sl(2|1)$. Extra fermionic raising operators, not belonging to the original $sl(2|1)$ superalgebra, allow (for $\lambda=\frac{1}{2}+\mathbb{Z}$) to construct the whole spectrum from the two degenerate (one bosonic and one fermionic) vacua.

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I. INTRODUCTION

In this paper, we quantize superconformal σ models defined by worldline supermultiplets. We consider two types of superconformal mechanics, parabolic or trigonometric [1], namely, in the absence or in the presence, respectively, of an oscillatorial DFF term [2].

In the absence of a DFF term, the systems under consideration possess only a Calogero potential [3]; they are supersymmetric and have a continuous spectrum. In the presence of a DFF term, they correspond to deformed (if the Calogero potential is present) or undeformed oscillators with a discrete spectrum, bounded from below. For these (un)deformed oscillators, the classical invariant superconformal algebra acts as a spectrum-generating algebra of the quantum theory.

We illustrate at first our method with two $osp(1|2)$-invariant examples, the ordinary one-dimensional harmonic oscillator being recovered in the trigonometric case. Later, we explicitly quantize the superconformally invariant worldline σ models defined by:

(i) the $\mathcal{N}=4$ (1, 4, 3) supermultiplet with scaling dimension $\alpha\neq 0,-1$ [these models are classically invariant under the exceptional $D(2,1;\alpha)$ Lie superalgebra]

(ii) the $\mathcal{N}=2$ (2, 2, 0) supermultiplet of scaling dimension λ [these models present a two-dimensional target and classical $sl(2|1)$-invariance].

For the (1, 4, 3) supermultiplet, at the special $\alpha=-\frac{1}{2}$ value, the Calogero potential terms are vanishing. For this value, the invariant superalgebra is $D(2,1;-\frac{1}{2})=D(2,1)\approx osp(4|2)$.

The results about the quantum parabolic $D(2,1;\alpha)$ models coincide with those obtained, with different methods, in Ref. [4]. The new feature, in the present paper, is the construction of the quantum trigonometric models that, so far, have not been investigated. An interesting result, in the (1, 4, 3) trigonometric case, consists in the direct and simple interpretation of α as a vacuum energy (if α is regarded as an external control parameter, it determines the Casimir energy of the system).

For the $sl(2|1)$ models, the scaling dimension λ is quantized (either $\lambda=\frac{1}{2}+\mathbb{Z}$ or $\lambda=\mathbb{Z}$). In the trigonometric case, the ordinary two-dimensional oscillator (without Calogero potential terms) is recovered from the special $\lambda=-\frac{1}{2}$ value after a superselection of the spectrum, defined by a projection operator, is imposed. The restriction implies, in particular, that a single bosonic vacuum is obtained. The spectrum of the unrestricted theory turns out to be decomposed into an infinite set of lowest-weight representations of $sl(2|1)$. By construction, the role of $sl(2|1)$ as a spectrum-generating algebra is expected. It is unexpected the further result that extra fermionic raising

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operators, not belonging to the $sl(2|1)$ superalgebra, allow to construct, for $\lambda = \frac{1}{2} + \mathbb{Z}$, the whole spectrum from the two degenerate (one bosonic and one fermionic) vacua (in Appendix A, this action is visualized in diagrams).

Models of superconformal mechanics have been investigated in Refs. [5–13] (see, e.g., the review [14] and references therein). For superconformal actions with oscillator potentials, see Refs. [1,15,16]. (Super)conformal mechanics is currently a very active area of research; among the motivations for this interest, one can mention the AdS_2/CFT_1 correspondence [17,18] or the possibility to apply it to test particles moving in the proximity of the horizon of certain black holes (see Ref. [12]).

$\mathcal{N} = 4$ superconformal models based on the exceptional (see Ref. [19]) Lie superalgebra $D(2,1;\alpha)$ were investigated in Refs. [20–26]. The models considered in those works, mostly classical, are supersymmetric; for that reason, they do not allow the presence of the oscillatorial DFF terms (in Appendix C, we comment about the "soft" supersymmetry property of the oscillatorial models). The recognition in Ref. [27] that conformal mechanics could allow new potentials permitted the introduction in Ref. [1] of the trigonometric (read: oscillatorial) classical $D(2,1;\alpha)$ models.

The scheme of the paper is the following. Sections II, III, and IV are propaedeutic. In Sec. II, we discuss the change of coordinates from linear to nonlinear realizations of the superconformal algebras (the "constant kinetic basis"), which allows us to present the worldline superconformal σ models in the Hamiltonian framework. A detailed description of the passage from classical Lagrangians to Hamiltonians is given in Sec. III. In Sec. IV, the quantization procedure and the construction of the Noether charges is explained for two examples, the parabolic and trigonometric $osp(1|2)$-invariant σ models. Section V contains the main results for the quantization of the parabolic (i.e., both superconformal and supersymmetric) quantum models with $D(2,1;\alpha)$ invariance, based on the $\mathcal{N} = 4$ worldline supermultiplet (1, 4, 3), and $sl(2,1)$ invariance, based on the $\mathcal{N} = 2$ (2, 2, 0) worldline supermultiplet. In Sec. VI, the main results of their quantum trigonometric versions are derived. These systems contain DFF terms and are "softly supersymmetric." They correspond to (un)deformed oscillators. The main results are the derivation of the vacuum energy in terms of the α scaling dimension for the (1, 4, 3) supermultiplet and the derivation of the spectrum-generating superalgebra for the (un)deformed two-dimensional oscillator with quantized scaling dimension λ. In Appendix A, diagrams are presented, illustrating the decomposition of the two-dimensional oscillators in terms of the $sl(2|1)$ lowest-weight representations, interconnected by the extra fermionic raising and lowering operators introduced in Sec. VI. For completeness, in Appendix B, the classical version of the trigonometric $\mathcal{N} = 2$ (2, 2, 0) superconformal σ model is presented. Finally, in Appendix C, we discuss the soft supersymmetry of the (un)deformed oscillators and the role, for these theories, of the spectrum-generating superalgebras. In the Conclusions, we present the open questions raised by our analysis.

II. WORLDLINE (SUPER)CONFORMAL σ MODELS IN CONSTANT KINETIC BASIS

A convenient approach, in constructing one-dimensional superconformal σ models, consists in starting from a linear D-module representation of the superconformal algebra. Once such a representation is known, the Lagrangian defining the superconformally invariant action can be systematically constructed by applying fermionic generators to a prepotential function that depends only on the propagating bosons. The requirement of superconformal invariance, imposed as a constraint, determines the specific form of the prepotential. This method (and its applications) has been discussed in Ref. [1].

The kinetic term $\Phi(\vec{x})\frac{1}{2}\delta_{ij}(\dot{x}_i\dot{x}_j + \ldots)$ of the derived Lagrangian is an ordinary constant kinetic term multiplied by a conformal factor $\Phi(\vec{x})$, which is a function of the propagating bosons. To apply the standard methods of quantization, we need to reabsorb the conformal factor. One way to do this consists in introducing a new set of fields. In the new basis of fields, the kinetic term is expressed as a constant coefficient (hence the name "constant kinetic basis" given in Ref. [1]); the superalgebra, on the other hand, is realized nonlinearly.

In Ref. [1], the procedure of changing the basis (from the "linear" to the constant kinetic basis) was sketched for certain D-module representations acting on supermultiplets consisting of a single propagating boson. We discuss it here in a more general framework.

Let us consider a D-module irreducible representation of an $\mathcal{N}$-extended superconformal algebra (for our purposes, $\mathcal{N} = 1, 2, 4, 8$) acting on a $(k, \mathcal{N}, \mathcal{N} - k)$ supermultiplet [28–31] (namely, k propagating bosons, $\mathcal{N}$ fermions, and $\mathcal{N} - k$ bosonic auxiliary fields). In the linear basis, the propagating bosons are labeled as $x_1, \ldots, x_k$; the fermions are labeled as $\psi_1, \ldots, \psi_{\mathcal{N}}$; and the auxiliary bosons are labeled as $b_1, \ldots, b_{\mathcal{N}-k}$. The kinetic term in the Lagrangian is given by

$$\frac{1}{2} r^{-\frac{1+2\lambda}{\lambda}}(\dot{x}_m\dot{x}_m + i\omega\psi_\beta\dot{\psi}_\beta - \omega^2 b_n b_n). \quad (1)$$

In the above equation, the summation over the repeated indices is implied. The constant ω is dimensionless (and can be set equal to unity) in the parabolic case, while it is dimensional, see Ref. [1], in the hyperbolic/trigonometric case. The function r is $r = (x_m x_m)^{\frac{1}{2}}$, and the parameter λ is the scaling dimension of the supermultiplet. At $\lambda = -\frac{1}{2}$, the kinetic term is constant. For the remaining $\lambda \neq -\frac{1}{2}$ values, a change to a constant kinetic basis is required in order to present a kinetic term with constant coefficients. Let us denote the propagating bosons in the constant kinetic basis as $y_1, \ldots, y_k$; the fermions as $\chi_1, \ldots, \chi_{\mathcal{N}}$; and the auxiliary bosons as $a_1, \ldots, a_{\mathcal{N}-k}$. The transformations passing from the linear to the constant kinetic basis are given by the following:

(i) For the $(1, \mathcal{N}, \mathcal{N}-1)$ supermultiplets, we have

$$y = -2\lambda x^{-\frac{1}{2\lambda}}, \qquad \chi_\beta = x^{-\frac{1+2\lambda}{2\lambda}}\psi_\beta, \qquad a_n = x^{-\frac{1+2\lambda}{2\lambda}} b_n; \tag{2}$$

in terms of the new fields, Eq. (1) is expressed as

$$\frac{1}{2}(\dot{y}\dot{y} + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n) \tag{3}$$

(ii) When $\mathcal{N} \geq 2$, for the $(2, \mathcal{N}, \mathcal{N}-2)$ supermultiplets, it is convenient to use a complex notation for the propagating bosons and set

$$y = -2\lambda(x_1 + ix_2)^{-\frac{1}{2\lambda}}, \qquad y^* = -2\lambda(x_1 - ix_2)^{-\frac{1}{2\lambda}},$$
$$\chi_\beta = r^{-\frac{1+2\lambda}{2\lambda}}\psi_\beta, \qquad a_n = r^{-\frac{1+2\lambda}{2\lambda}} b_n \tag{4}$$

so that the kinetic term can be expressed as

$$\frac{1}{2}(\dot{y}\dot{y}^* + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n). \tag{5}$$

(iii) When $\mathcal{N} = 4, 8$ it is possible to construct a constant kinetic basis for any $(k, \mathcal{N}, \mathcal{N}-k)$ supermultiplet at the specific $\lambda = 1/2$ value of the scaling dimension via the transformations

$$y_m = \frac{x_m}{r^2}, \qquad \chi_\beta = \frac{\psi_\beta}{r^2}, \qquad a_n = \frac{b_n}{r^2}, \tag{6}$$

leading to the kinetic term

$$\frac{1}{2}(\dot{y}_m\dot{y}_m + i\omega\chi_\beta\dot{\chi}_\beta - \omega^2 a_n a_n). \tag{7}$$

For $\mathcal{N} = 4$ and $k \neq 2$, irreps of the exceptional superalgebras $D(2,1;\alpha)$ are recovered, see Refs. [1,25,26], from the $(k, 4, 4-k)$ supermultiplets according to the relation

$$\alpha = (2-k)\lambda. \tag{8}$$

At the special $\lambda = \frac{1}{2}$ value, the associated superalgebra is $A(1,1)$ for the (4,4,0) supermultiplet and $D(2,1)$ for the (3,4,1) supermultiplet.

For $\mathcal{N} = 8$ and $k \neq 4$, irreps of superconformal algebras are recovered for each supermultiplet $(k, 8, 8-k)$ at the critical values of the scaling dimension given by

$$\lambda_k = \frac{1}{k-4}. \tag{9}$$

The special value $\lambda = \frac{1}{2}$ yields an irrep of $A(3,1)$ acting on the supermultiplet (6,8,2). The reader is referred to Refs. [25,26] for a detailed discussion on the criticality of the scaling dimension of the $\mathcal{N} = 4$, 8 superconformal algebras.

III. FROM LAGRANGIANS TO CLASSICAL HAMILTONIANS: AN APPLICATION TO THE *osp*(1|2)-INVARIANT σ MODELS

The quantization of the one-dimensional superconformal σ models follows the canonical procedure formalized by Dirac and based on the classical Hamiltonian formalism. Since these σ models have fermionic degrees of freedom, the passage from the Lagrangian to the classical Hamiltonian formalism requires the use of Dirac brackets (see, e.g., Ref. [32]). The need for Dirac brackets becomes clear after inspecting Eqs. (3), (5), and (7); it is due to the fact that the linear dependence on the fermionic velocities $\dot{\chi}_\beta$ forces us to extend the phase space of the system and treat the fermionic canonical momenta as constraints in this extended phase space. In Dirac's language, these constraints are both *primary* (they hold even without using the equations of motion) and *second class* (namely, a constraint that has nonvanishing Poisson brackets with at least one of the constraints).

This procedure, used throughout the paper, will be illustrated in detail for the simplest possibility given by the $osp(1|2)$-invariant σ models (their two variants, *parabolic* and *hyperbolic/trigonometric*; see Ref. [1]). In the parabolic case, the Hamiltonian is identified with a bosonic root of the superconformal algebra, while in the hyperbolic/trigonometric case, it is associated with a Cartan element. The parabolic D-module reducible representations describe systems that are supersymmetric, while the hyperbolic/trigonometric reps furnish only a *soft* version of supersymmetry; see the discussion in the Introduction. The hyperbolic and trigonometric models are interrelated via a Wick rotation of the dimensional parameter ω. The trigonometric case is emphasized here with respect to the hyperbolic one because it yields a Hamiltonian bounded from below.

In the rest of this section, we discuss in detail the Hamiltonian formulation of both parabolic and trigonometric $osp(1|2)$-invariant σ models. The method, notations, and conventions presented here are later applied to models with larger superconformal symmetry.

A. $osp(1|2)$-invariant parabolic σ model

In the constant kinetic basis, the generators of the $osp(1|2)$ parabolic D-module rep read as

$$H = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}, \qquad D = \begin{pmatrix} t\partial_t - \frac{1}{2} & 0 \\ 0 & t\partial_t \end{pmatrix},$$
$$K = \begin{pmatrix} t^2\partial_t - t & 0 \\ 0 & t^2\partial_t \end{pmatrix},$$
$$Q = \begin{pmatrix} 0 & 1 \\ i\partial_t & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & t \\ it\partial_t - i & 0 \end{pmatrix}. \tag{10}$$

The above generators act on the column vector supermultiplet $(y,\chi)^T$ possessing the scaling dimension $\lambda = -\frac{1}{2}$.

The bosonic generators H, D, and K span the $sl(2)$ Lie subalgebra, while the fermionic generators Q and $\bar{Q}$ span the odd sector of $osp(1|2)$.

The associated $osp(1|2)$-invariant action is simply

$$\mathcal{S} = \int dt\mathcal{L} = \int dt\frac{1}{2}(\dot{y}^2 + i\chi\dot{\chi}). \tag{11}$$

Unlike the $\mathcal{N} \geq 2$ superconformal algebras discussed in the following, for $osp(1|2)$, the same action is recovered by starting from a generic D-module rep with scaling dimension $\lambda \neq -\frac{1}{2}$ and applying the (2) change of basis.

For a theory possessing bosons and fermions, a conserved Noether charge is expressed, for a symmetry generator O, as

$$C_O = (\delta_O\phi_I)\frac{\partial\mathcal{L}}{\partial\dot{\phi}_I} - J_O, \tag{12}$$

where J_O stems from the variation $\delta_O\mathcal{L} = \frac{dJ_O}{dt}$; the sum over the repeated index I labeling the fields is understood. The given ordering of the right-hand side of Eq. (12) is essential in dealing with Grassmann variables and derivatives.

For the case at hand, the classical Noether charges are

$$C_H = \frac{\dot{y}^2}{2}, \qquad C_D = \frac{t\dot{y}^2}{2} - \frac{y\dot{y}}{2}, \qquad C_K = \frac{t^2\dot{y}^2}{2} - ty\dot{y} + \frac{y^2}{2},$$
$$C_Q = \dot{y}\chi, \qquad C_{\bar{Q}} = t\dot{y}\chi + y\chi. \tag{13}$$

The Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) \tag{14}$$

lead to the equations of motion

$$\ddot{y} = 0, \qquad \dot{\chi} = 0. \tag{15}$$

The Grassmann variable in the classical $osp(1|2)$ model is a constant and plays essentially no physical role besides ensuring the $osp(1|2)$ invariance.

To introduce the Hamiltonian formalism, we have to compute the conjugate momenta given by

$$p = \frac{\partial\mathcal{L}}{\partial\dot{y}} = \dot{y}, \qquad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = -\frac{i\chi}{2}. \tag{16}$$

In the Hamiltonian framework, the classical charges (13) are rewritten as

$$C_H = \frac{p^2}{2}, \qquad C_D = \frac{tp^2}{2} - \frac{yp}{2},$$
$$C_K = \frac{t^2p^2}{2} - typ + \frac{y^2}{2}, \qquad C_Q = p\chi,$$
$$C_{\bar{Q}} = tp\chi + y\chi. \tag{17}$$

The last step requires defining the Dirac brackets. The second equation in (16) makes clear why Dirac brackets need to be introduced. The conjugate momentum π to the Grassmann variable χ is not an invertible function of the velocity $\dot{\chi}$. The second equation in (16) should therefore be viewed as a second-class constraint on the phase space,

$$u = \pi + \frac{i\chi}{2}. \tag{18}$$

The super-Poisson bracket involving even or odd f and g functions is given by

$$\{f,g\}_P = \sum_I (-1)^{deg(f)\cdot deg(g)}\frac{\partial f}{\partial\phi_I}\frac{\partial g}{\partial\pi_I} - \frac{\partial f}{\partial\pi_I}\frac{\partial g}{\partial\phi_I}, \tag{19}$$

where the degree function deg is 0 if evaluated on bosons and 1 on fermions.

Denoting with u_i the set of all second-class constraints, the Dirac bracket reads as

$$\{f,g\}_D = \{f,g\}_P - \sum_{k,l}\{f,u_k\}_P U^{-1}_{kl}\{u_l,g\}_P, \tag{20}$$

where $U_{kl} = \{u_k,u_l\}_P$ is a matrix constructed from the super-Poisson brackets of all second-class constraints.

u entering (18) is a second-class constraint since it satisfies

$$\{u,u\}_P = -i.$$

A straightforward computation gives the nonvanishing Dirac brackets

$$\{y,p\}_D = 1, \qquad \{\chi,\chi\}_D = -i. \tag{21}$$

We can derive, with the use of the Dirac brackets, the equations of motion in the Hamiltonian formalism and compute [recovering $osp(1|2)$] the superalgebra satisfied by the (17) conserved charges.

In terms of Dirac brackets, Hamilton's equations are

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \{\phi, C_H\}_D. \tag{22}$$

For the case at hand, we get

$$\dot{p} = 0, \qquad \dot{\chi} = 0, \tag{23}$$

which, together with the $p = \dot{y}$ position, allow us to recover (15).

B. $osp(1|2)$-invariant trigonometric σ model

In the trigonometric case, the passage from the Lagrangian to the Hamiltonian formalism follows the same steps as before. We therefore skip unnecessary comments.

In the constant kinetic basis, the generators of the $osp(1|2)$ trigonometric D-module rep are

$$H = e^{i\omega t}\begin{pmatrix} \frac{1}{\omega}\partial_t - \frac{i}{2} & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{1}{\omega}\partial_t & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix},$$
$$K = e^{-i\omega t}\begin{pmatrix} \frac{1}{\omega}\partial_t + \frac{i}{2} & 0 \\ 0 & \frac{1}{\omega}\partial_t \end{pmatrix},$$
$$Q = e^{\frac{i\omega t}{2}}\begin{pmatrix} 0 & 1 \\ \frac{i}{\omega}\partial_t + \frac{1}{2} & 0 \end{pmatrix}, \qquad \bar{Q} = e^{-\frac{i\omega t}{2}}\begin{pmatrix} 0 & 1 \\ \frac{i}{\omega}\partial_t - \frac{1}{2} & 0 \end{pmatrix}. \tag{24}$$

The $osp(1|2)$-invariant action is

$$\mathcal{S} = \int dt\mathcal{L} = \int dt \frac{1}{2}\left(\dot{y}^2 + i\omega\chi\dot{\chi} - \frac{\omega^2}{8}y^2\right). \tag{25}$$

The derived conserved Noether charges are

$$C_H = e^{i\omega t}\left(\frac{1}{2\omega}\dot{y}^2 - \frac{i}{2}y\dot{y} - \frac{\omega}{8}y^2\right), \qquad C_D = \frac{1}{2\omega}\dot{y}^2 + \frac{\omega}{8}y^2,$$
$$C_K = e^{-i\omega t}\left(\frac{1}{2\omega}\dot{y}^2 + \frac{i}{2}y\dot{y} - \frac{\omega}{8}y^2\right),$$
$$C_Q = e^{\frac{i\omega}{2}t}\left(\dot{y}\chi - \frac{i\omega}{2}y\chi\right), \qquad C_{\bar{Q}} = e^{-\frac{i\omega}{2}t}\left(\dot{y}\chi + \frac{i\omega}{2}y\chi\right). \tag{26}$$

The Euler-Lagrange equations of motion are

$$\ddot{y} = -\frac{\omega^2 y}{4}, \qquad \dot{\chi} = 0. \tag{27}$$

The conjugate momenta are given by

$$p = \frac{\partial\mathcal{L}}{\partial\dot{y}} = \dot{y}, \qquad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = -\frac{i\omega\chi}{2}. \tag{28}$$

In the Hamiltonian formulation, the (26) conserved charges are

$$C_H = e^{i\omega t}\left(\frac{1}{2\omega}p^2 - \frac{i}{2}yp - \frac{\omega}{8}y^2\right), \qquad C_D = \frac{1}{2\omega}p^2 + \frac{\omega}{8}y^2,$$
$$C_K = e^{-i\omega t}\left(\frac{1}{2\omega}p^2 + \frac{i}{2}yp - \frac{\omega}{8}y^2\right),$$
$$C_Q = e^{\frac{i\omega}{2}t}\left(p\chi - \frac{i\omega}{2}y\chi\right), \qquad C_{\bar{Q}} = e^{-\frac{i\omega}{2}t}\left(p\chi + \frac{i\omega}{2}y\chi\right). \tag{29}$$

The second equation in (28) gives the constraint in phase space,

$$u = \pi + \frac{i\omega\chi}{2}, \tag{30}$$

which allows us to compute the Dirac brackets as before. The nonvanishing Dirac brackets are

$$\{y, p\}_D = 1, \qquad \{\chi, \chi\}_D = -\frac{i}{\omega}. \tag{31}$$

Hamilton's equations of motion are now written as

$$\dot{\phi} = \omega\{\phi, C_D\}_D + \frac{\partial\phi}{\partial t}. \tag{32}$$

One should note that, while in the parabolic σ model the charge C_H is the physical Hamiltonian and the symmetry operator H is the generator of the time translations, in the trigonometric σ model, the physical Hamiltonian is given by ωC_D, the Cartan generator ωD being the generator of the time translations. One can readily check that Eq. (32) leads to

$$\dot{p} = -\frac{\omega^2 y}{4}, \qquad \dot{\chi} = 0, \tag{33}$$

which reproduces (27) by taking into account that $p = \dot{y}$.

IV. QUANTIZATION: QUANTUM VS CLASSICAL NOETHER CHARGES AND THE $osp(1|2)$ MODELS

The canonical quantization of the models presented in Sec. III is realized by substituting the Dirac brackets by the appropriate (based on the superalgebra structure) (anti) commutators, which we will denote with the "[.,.}" symbol:

$$\{A,B\}_D \to \frac{1}{i\hbar}[A,B\}. \tag{34}$$

By applying Eq. (34) to Eqs. (21) and (31) we get, respectively, the parabolic and trigonometric $osp(1|2)$-invariant quantum superconformal models.

We point out that, since the observables must be Hermitian operators, the parabolic and trigonometric quantum models correspond to different real forms (read: conjugations) of the invariant superalgebra. We illustrate in detail this feature, which is also valid for $\mathcal{N} \geq 2$-invariant theories.

A. Parabolic $osp(1|2)$-invariant quantum σ model

The nonvanishing (anti)commutators recovered from (21) are

$$[\hat{y},\hat{p}] = i\hbar, \qquad \{\hat{\chi},\hat{\chi}\} = \hbar. \tag{35}$$

In the position-space representation, the above operators are given by

$$\hat{y} = y, \qquad \hat{p} = -i\hbar\partial_y, \qquad \hat{\chi} = \sqrt{\frac{\hbar}{2}}. \tag{36}$$

The last equation is particularly important because it tells us that the fermionic field χ, classically represented by a Grassmann variable, becomes a Clifford variable $\hat{\chi}$ in the quantum version. The choice in (36) of representing $\hat{\chi}$ as a real number is not unique. An alternative choice, which respects the $\mathbb{Z}_2$-graded structure of the supervector space acted upon by the operators $\hat{y}$, $\hat{p}$, $\hat{\chi}$, consists in picking $\hat{\chi}$ as the 2×2 matrix $\sqrt{\frac{\hbar}{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. In this $\mathbb{Z}_2$-graded representation, the operators $\hat{y}$, $\hat{p}$, $\hat{\chi}$ are

$$\hat{y} = \begin{pmatrix} y & 0\\ 0 & y\end{pmatrix}, \qquad \hat{p} = \begin{pmatrix} -i\hbar\partial_y & 0\\ 0 & -i\hbar\partial_y\end{pmatrix},$$
$$\hat{\chi} = \sqrt{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad N_f = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}, \tag{37}$$

while N_f is the fermion parity operator.

The possibility, offered by the $\mathbb{Z}_2$-graded structure, of doubling the vector space will be used in the following in constructing $\mathcal{N} = 2$, 4 quantum models.

It is worth pointing out that superalgebras admit super-representations acting on $\mathbb{Z}_2$-graded vector spaces. In some cases, superalgebra (anti)commutation relations are also realized on ordinary (not $\mathbb{Z}_2$-graded) vector spaces. This feature can be seen when realizing the $\chi^2 = \mathbb{I}$ equation either through $\chi = 1$ or the $\chi = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\mathbb{Z}_2$-graded solution [they induce a $Cl(1,0)$ Clifford algebra, which is respectively identified either with $Cl(1,0) \approx \mathbb{R}$ or with the split-complex numbers $Cl(1,0) \approx \tilde{\mathbb{C}}$]. Upon a convenient normalization, Eq. (36) corresponds to the first choice, while Eq. (37) corresponds to the second choice.

It is worth pointing out that the different quantum models derived from Eqs. (36) and (37) (only the latter one being supersimmetric) are both consistent. The Eq. (36) model can be derived from the Eq. (37) model after imposing a superselection rule induced by a projector (a similar projector inducing a superselection rule is introduced in Appendix A). For simplicity, we discuss in this section the parabolic [and its trigonometric counterpart; see Eq. (42)] model corresponding to the first choice. The $\mathbb{Z}_2$-graded choice is used in Secs. V and VI to derive $\mathcal{N} = 4$ and $\mathcal{N} = 2$ quantum models.

The Eq. (42) model coincides with the ordinary quantum oscillator [its connection with the $osp(1|2)$ superalgebra is elucidated in Appendix C].

The parabolic quantum $osp(1|2)$ superalgebra obtained by the (34) quantization of the classical counterpart leads to

$$[\hat{H},\hat{D}] = i\hbar\hat{H}, \qquad [\hat{H},\hat{K}] = 2i\hbar\hat{D}, \qquad [\hat{K},\hat{D}] = -i\hbar\hat{K}$$
$$[\hat{H},\hat{\tilde{Q}}] = i\hbar\hat{Q}, \qquad [\hat{K},\hat{Q}] = -i\hbar\hat{\tilde{Q}},$$
$$[\hat{Q},\hat{D}] = \frac{i\hbar}{2}\hat{Q}, \qquad [\hat{\tilde{Q}},\hat{D}] = -\frac{i\hbar}{2}\hat{\tilde{Q}}, \qquad \{\hat{Q},\hat{Q}\} = 2\hbar\hat{H},$$
$$\{\hat{Q},\hat{\tilde{Q}}\} = 2\hbar D, \qquad \{\hat{\tilde{Q}},\hat{\tilde{Q}}\} = 2\hbar K. \tag{38}$$

The remaining (anti)commutators are vanishing.

The above superalgebra is realized by the quantum charges

$$\hat{H} = \frac{1}{2}\hat{p}^2, \qquad \hat{D} = \frac{t}{2}\hat{p}^2 - \frac{1}{4}(\hat{y}\,\hat{p} + \hat{p}\,\hat{y}),$$
$$\hat{K} = \frac{t^2}{2}\hat{p}^2 - \frac{t}{2}(\hat{y}\,\hat{p} + \hat{p}\,\hat{y}) + \frac{1}{2}\hat{y}^2,$$
$$\hat{Q} = \hat{\chi}\,\hat{p}, \qquad \hat{\tilde{Q}} = t\hat{\chi}\,\hat{p} - \hat{y}\,\hat{\chi}\,. \tag{39}$$

They are, up to symmetrization, identical to the classical charges. This is a unique feature of the $\mathcal{N} = 1$ $osp(1|2)$-invariant models. From $\mathcal{N} \geq 2$, the models explicitly depend on the scaling dimension λ. As a result, the quantum versions of these theories require corrections that are traced backed to the mapping of the classical Grassmann variables into quantum Clifford generators.

The Hamiltonian $\hat{H}$ in (39) corresponds to the one-dimensional free particle. The operators $\hat{H}$, $\hat{D}$, and $\hat{K}$ close the $sl(2)$ bosonic symmetry algebra of the system. $\hat{H}$ and $\hat{Q}$ give the $\mathcal{N} = 1$ algebra of the supersymmetric quantum mechanics. In terms of the Eq. (36) realization ($\hat{\chi}$ is a real number), the parabolic $osp(1|2)$-invariant model admits no fermionic degrees of freedom. This is no longer the case (fermions are present) if the model is expressed via the Eq. (37) realization.

In the parabolic model, all charges entering Eq. (39) are observables. The superalgebra (38) can be reexpressed in terms of the canonical $osp(1|2)$ Cartan-Weyl basis H, $F^{\pm}$, $E^{\pm}$ (such that all the structure constants are real), see Ref. [19], through the identifications

$$\hat{H} = -E^-, \qquad \hat{D} = iH, \qquad \hat{K} = -E^+,$$
$$\hat{Q} = 2F^-, \qquad \hat{\tilde{Q}} = 2iF^+. \tag{40}$$

The computation of the $osp(1|2)$ structure constants in the new basis is immediate.

The superalgebra conjugation corresponding to (39) reads, in the Cartan-Weyl basis, as

$$(E^{\pm})^{\dagger} = E^{\pm}, \qquad H^{\dagger} = -H, \qquad (F^{\pm})^{\dagger} = \mp(F^{\pm}). \tag{41}$$

Concerning the dimensional analysis of the model, we can set, without loss of generality, $[\partial_t] = 1$. If we set the Planck constant $\hbar$ and the action $\mathcal{S}$ to be dimensionless, we therefore get $[\hat{y}] = -\frac{1}{2}$, $[\hat{p}] = \frac{1}{2}$, and $[\hat{\chi}] = [\mathcal{S}] = 0$.

B. Trigonometric $osp(1|2)$-invariant quantum σ model

The quantization of the trigonometric model follows the same lines of the parabolic one. Without loss of generality, we can set $\omega = 1$, reproducing the nonvanishing (anti) commutators (35) and the Eqs. (36) and (37) position-space representations for the operators $\hat{y}$, $\hat{p}$, $\hat{\chi}$. The quantum trigonometric generators, identical to the classical ones up to symmetrization, are

$$\hat{H} = e^{it}\left(\frac{1}{2}\hat{p}^2 - \frac{i}{4}(\hat{y}\,\hat{p} + \hat{p}\,\hat{y}) - \frac{1}{8}\hat{y}^2\right),$$
$$\hat{K} = e^{-it}\left(\frac{1}{2}\hat{p}^2 + \frac{i}{4}(\hat{y}\,\hat{p} + \hat{p}\,\hat{y}) - \frac{}{8}\hat{y}^2\right),$$
$$\hat{D} = \frac{1}{2}\hat{p}^2 + \frac{1}{8}\hat{y}^2, \qquad \hat{Q} = e^{\frac{it}{2}}\left(\hat{\chi}\,\hat{p} - \frac{i}{2}\hat{\chi}\,\hat{y}\right),$$
$$\hat{\tilde{Q}} = e^{-\frac{it}{2}}\left(\hat{\chi}\,\hat{p} + \frac{i}{2}\hat{\chi}\,\hat{y}\right). \tag{42}$$

In the Eq. (42) realization, the $osp(1|2)$ nonvanishing brackets read as

$$[\hat{H},\hat{D}] = \hbar\hat{H}, \qquad [\hat{H},\hat{K}] = 2\hbar\hat{D}, \qquad [\hat{K},\hat{D}] = -\hbar\hat{K},$$
$$[\hat{H},\hat{\tilde{Q}}] = \hbar\hat{Q}, \qquad [\hat{K},\hat{Q}] = -\hbar\hat{\tilde{Q}}, \qquad [\hat{Q},\hat{D}] = \frac{\hbar}{2}\hat{Q},$$
$$[\hat{\tilde{Q}},\hat{D}] = -\frac{\hbar}{2}\hat{\tilde{Q}}, \qquad \{\hat{Q},\hat{Q}\} = 2\hbar\hat{H},$$
$$\{\hat{Q},\hat{\tilde{Q}}\} = 2\hbar D, \qquad \{\hat{\tilde{Q}},\hat{\tilde{Q}}\} = 2\hbar K. \tag{43}$$

The $osp(1|2)$ Cartan-Weyl basis is recovered, from the Eq. (42) trigonometric charges, via the identifications

$$\hat{H} = E^-, \qquad \hat{D} = H, \qquad \hat{K} = -E^+,$$
$$\hat{Q} = 2iF^-, \qquad \hat{\tilde{Q}} = -2iF^+. \tag{44}$$

We obtain a different conjugation with respect to the parabolic case, given by

$$(E^{\pm})^{\dagger} = -E^{\mp}, \qquad H^{\dagger} = H, \qquad (F^{\pm})^{\dagger} = F^{\mp}. \tag{45}$$

In the trigonometric case, the Hamiltonian is given by the $osp(1|2)$ Cartan generator $\omega\hat{D}$.

By taking into account the presence of the dimensional parameter ω that we set, for convenience, equal to 1 in the formulas above, the dimensional analysis of the trigonometric model gives us the dimensions $[t] = -1$, $[\hat{y}] = -\frac{1}{2}$, $[\hat{p}] = \frac{1}{2}$, $[\hat{\chi}] = -\frac{1}{2}$, $[\omega] = 1$, and $[\mathcal{S}] = 0$.

V. SUPERCONFORMAL QUANTUM MECHANICS WITH CALOGERO POTENTIALS: 1D $D(2,1;\alpha)$ AND 2D $sl(2|1)$ MODELS

In this section, we quantize the worldline superconformal σ models recovered from the $\mathcal{N} = 4$ (1, 4, 3) (i.e., one-dimensional target) and $\mathcal{N} = 2$ (2, 2, 0) (i.e., two-dimensional target) parabolic supermultiplets. Unlike the $\mathcal{N} = 1$ parabolic model analyzed in Sec. IV, nontrivial potential terms and nontrivial quantum corrections to the classical Hamiltonians appear.

The $\mathcal{N} = 4$ (1, 4, 3) parabolic model possesses a $D(2,1;\alpha)$ invariance, in which $\alpha \neq 0, -1$ is identified with the scaling dimension of the supermultiplet. The Hamiltonian describes a particle moving on a line under an inverse square potential and includes spinlike degrees of freedom.

The $\mathcal{N} = 2$ (2, 2, 0) parabolic model possesses an $sl(2|1)$ invariance. Its Hamiltonian describes a particle moving on a plane under an inverse square potential and with a spin-orbit coupling.

A. $\mathcal{N} = 4$ (1, 4, 3) parabolic model with $D(2,1;\alpha)$ invariance

A discussion of the classical $\mathcal{N} = 4$ (1, 4, 3) superconformal worldline model can be found, e.g., in Ref. [1]. We present here the quantization of this model repeating the same steps discussed in Sec. IV for the $osp(1|2)$-invariant model. In this subsection, we recover, within a different framework, the models discussed in Ref. [4].

The nonvanishing (anti)commutators obtained from quantizing the Dirac brackets are

$$[\hat{y},\hat{p}] = i, \qquad \{\hat{\chi}_\alpha,\hat{\chi}_\beta\} = \delta_{\alpha\beta}, \tag{46}$$

with $\alpha, \beta = 0, \ldots, 3$. The above equations define the superalgebra $\mathfrak{h}_1 \oplus C_4$, with the one-dimensional Heisenberg algebra $\mathfrak{h}_1$ in its even sector and the four $C\ell(4,0)$ Clifford algebra gamma matrices in its odd sector. These gamma matrices can be expressed as 4×4 complex matrices. We choose to respect the $\mathbb{Z}_2$-graded structure of the superalgebra, block-antidiagonal gamma matrices, while representing the Heisenberg generators as block-diagonal operators.

The position-space representation of (46) is

$$\hat{y} = y\mathbb{I}_4, \qquad \hat{p} = -i\partial_y\mathbb{I}_4, \qquad \hat{\chi}_0 = \frac{1}{\sqrt{2}}\sigma_2\otimes\mathbb{I}_2,$$
$$\hat{\chi}_1 = -\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_1, \qquad \hat{\chi}_2 = -\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_2,$$
$$\hat{\chi}_3 = -\frac{1}{\sqrt{2}}\sigma_1\otimes\sigma_3, \tag{47}$$

where $\mathbb{I}_n$ is the $n\times n$ identity matrix and the σ_i's ($i=1,2,3$) are the Pauli matrices.

The quantum charges are given by

$$\hat{H} = \left(\frac{\hat{p}^2}{2}+\frac{(1+2\alpha)^2}{8\hat{y}^2}\right)\mathbb{I}_4+\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4,$$
$$\hat{D} = \left(\frac{t\hat{p}^2}{2}-\frac{1}{4}(\hat{y}\,\hat{p}+\hat{p}\,\hat{y})+\frac{t(1+2\alpha)^2}{8\hat{y}^2}\right)\mathbb{I}_4 + \frac{t(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,$$
$$\hat{K} = \left(\frac{t^2\hat{p}^2}{2}-\frac{t}{2}(\hat{y}\,\hat{p}+\hat{p}\,\hat{y})+\frac{\hat{y}^2}{2}+\frac{t^2(1+2\alpha)^2}{8\hat{y}^2}\right)\mathbb{I}_4 + \frac{t^2(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4,$$
$$\hat{Q}_0 = \hat{\chi}_0\hat{p}+\frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}},$$
$$\hat{Q}_i = \hat{\chi}_i\hat{p}-\frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}},$$
$$\hat{\tilde{Q}}_0 = t\hat{\chi}_0\hat{p}-\chi_0\hat{y}+\frac{it(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}},$$
$$\hat{\tilde{Q}}_i = t\hat{\chi}_i\hat{p}-\chi_i\hat{y}-\frac{it(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}},$$
$$\hat{J}_i = -i\left(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k+\hat{\chi}_0\hat{\chi}_i\right),$$
$$\hat{L}_i = -i\left(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k-\hat{\chi}_0\hat{\chi}_i\right). \tag{48}$$

In the above formulas, we used the Fermi parity operator $\mathcal{F}_4$, defined by $\mathcal{F}_{2n} = \begin{pmatrix}\mathbb{I}_n & 0\\ 0 & -\mathbb{I}_n\end{pmatrix}$. One should note that the quantum operators $\hat{H}$, $\hat{D}$, and $\hat{K}$ contain an Ehrenfest quantum correction term, proportional to $\frac{\hbar^2(1+2\alpha)^2}{\hat{y}^2}\mathbb{I}_4$, which is not present in the classical charges. Its appearance can be traced to the change from classical Grassmann to quantum Clifford variables.

At a given value $\alpha\neq 0,-1$, the above operators close the exceptional superalgebra $D(2,1;\alpha)$. The R-symmetry generators $\hat{J}_i$ and $\hat{L}_i$, $i=1,2,3$, close two independent ($[\hat{J}_i,\hat{L}_j]=0$) $su(2)$ subalgebras.

In the Cartan-Weyl basis, the nonvanishing $D(2,1;\alpha)$ brackets are given by

$$[H,E^\pm]=\pm E^\pm, \qquad [E^+,E^-]=2H,$$
$$[H,F^\pm_\beta]=\pm\frac{1}{2}F^\pm_\beta, \qquad [E^\pm,F^\mp_\beta]=-F^\pm_\beta,$$
$$\{F^\pm_0,F^\mp_j\}=-\frac{i}{4}(\lambda J_j+(1+\lambda)L_j),$$
$$\{F^+_j,F^-_k\}=\epsilon_{jkl}\left(-\frac{i\lambda}{4}J_l+\frac{i(\lambda+1)}{4}L_l\right)+\delta_{jk}\frac{H}{2},$$
$$\{F^+_0,F^-_0\}=\frac{H}{2}, \qquad \{F^\pm_\beta,F^\pm_\gamma\}=\pm\frac{1}{2}\delta_{\beta\gamma}E^\pm,$$
$$[J_j,F^\pm_0]=iF^\pm_j, \qquad [J_j,F^\pm_k]=i(-\delta_{jk}F^\pm_0+\epsilon_{jkl}F^\pm_l),$$
$$[L_j,F^\pm_0]=-iF^\pm_j, \qquad [L_j,F^\pm_k]=i(\delta_{jk}F^\pm_0+\epsilon_{jkl}F^\pm_l),$$
$$[J_j,J_k]=2i\epsilon_{jkl}J_l, \qquad [L_j,L_k]=2i\epsilon_{jkl}L_l. \tag{49}$$

The above superalgebra is realized by the (48) quantum operators via the identifications

$$\hat{H}=-E^-, \qquad \hat{D}=iH, \qquad \hat{K}=-E^+, \qquad \hat{Q}_\beta=2F^-_\beta,$$
$$\hat{\tilde{Q}}_\beta=2iF^+_\beta, \qquad \hat{J}_j=J_j, \qquad \hat{L}_j=L_j. \tag{50}$$

The Hamiltonian operator $\hat{H}$, explicitly written in 4×4 supermatrix form, is given by

$$\hat{H}=\left(\begin{array}{c|c}(\frac{\hat{p}^2}{2}+\frac{4\alpha^2+8\alpha+3}{8\hat{y}^2})\mathbb{I}_2 & 0\\ \hline 0 & (\frac{\hat{p}^2}{2}+\frac{4\alpha^2-1}{8\hat{y}^2})\mathbb{I}_2\end{array}\right). \tag{51}$$

It is the Hamiltonian of the $\mathcal{N}=4$ super-Calogero model with $D(2,1;\alpha)$ invariance.

It contains a (purely bosonic) Calogero Hamiltonian in both its upper and lower diagonal blocks. We recall that the Calogero Hamiltonian $\mathcal{H}_C$ is given by

$$\mathcal{H}_C=\frac{1}{2}\hat{p}^2+\frac{g^2}{\hat{y}^2}. \tag{52}$$

The self-adjointness of the Calogero Hamiltonian $\mathcal{H}_C$ depends on the value of the coupling parameter g. We refer to Refs. [33,34] for a thorough discussion of this subtle point.

For our purposes, it is important to note here the relation between the coupling constant g and the scaling dimension parameter α. From Ref. [33], we know that $\mathcal{H}_C$ is self-adjoint, provided that the inequality $g^2>-\frac{1}{8}$ is satisfied. Under this condition, the boundary value problem

$$\mathcal{H}_C\phi_k=E_k\phi_k, \qquad \phi_k(0)=0$$

gives a continuous positive spectrum, $0 \le E_k < \infty$, the eigenfunctions and eigenvalues being

$$\phi_k(y) = 2^{\mu-\frac{1}{2}}\Gamma\left(\mu+\frac{1}{2}\right)(ky)^{-(\mu-\frac{1}{2})}J_{\mu-\frac{1}{2}}(ky)y^{\mu},$$
$$E_k = \frac{1}{2}k^2,$$

for

$$g^2 = \frac{1}{2}\mu(\mu-1). \tag{53}$$

Let us set

$$g_b^2 = \frac{4\alpha^2+8\alpha+3}{8}, \qquad g_f^2 = \frac{4\alpha^2-1}{8} \tag{54}$$

for the Calogero parameters entering, respectively, the upper and lower diagonal blocks of the Eq. (51) Hamiltonian. It is quite rewarding that imposing, simultaneously, the g_b^2, $g_f^2 > -\frac{1}{8}$ condition we end up with the $\alpha \neq 0, -1$ inequality for the scaling dimension. The class of exceptional $D(2,1;\alpha)$ superalgebras guarantees the existence of a well-defined Hamiltonian with a continuous positive spectrum bounded from below.

At the special $\alpha = -\frac{1}{2}$ value, the Calogero potential terms (in both upper and lower blocks) vanish. Therefore, this special point corresponds to a free theory. At this given value, see Ref. [19], we have $D(2,1;-\frac{1}{2}) = D(2,1)$, so the invariant superalgebra coincides with the classical $D(2,1) \approx osp(4|2)$ superalgebra.

We can express, from (53), g_b and g_f in terms of their respective μ_b and μ_f parameters. From (54), μ_b and μ_f can be given in terms of α. The result is the linear relations

$$\mu_b = \frac{1}{2} \pm (\alpha+1), \qquad \mu_f = \frac{1}{2} \pm \alpha. \tag{55}$$

In quantum mechanics, the continuity conditions are also imposed on the probability currents. Since the zero-energy wave function is (up to a normalizing factor) $\phi_0(y) = y^{\mu}$, these conditions imply that both μ_b and μ_f must satisfy μ_b, $\mu_f > \frac{1}{2}$ to ensure continuity at the origin. Equations (55) show that any $\alpha \neq 0, -1$ is suitable to fulfill these constraints.

For a final comment, we point out that the energy levels of both bosonic (upper) and fermionic (lower) blocks are doubly degenerated. This degeneracy is removed by taking into account the Hermitian operators $\hat{J}_3$ and $\hat{L}_3$, which commute with $\hat{H}$. Indeed,

$$\hat{J}_3 = \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \hat{L}_3 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma_3 \end{array}\right) \tag{56}$$

are both diagonal and specify spinlike quantum numbers in the bosonic and fermionic sectors, respectively. We can say that the bosonic states have $\frac{1}{2}\,\hat{J}$ spin and $0\,\hat{L}$ spin, while the fermionic states have $0\,\hat{J}$ spin and $\frac{1}{2}\,\hat{L}$ spin.

B. $\mathcal{N}=2$ (2, 2, 0) parabolic model with $sl(2|1)$ invariance

The classical $sl(2|1)$-invariant action based on the parabolic D-module rep of the (2, 2, 0) supermultiplet is presented in Appendix B. Its quantization is performed with the techniques previously outlined (introduction of the constant kinetic basis, Dirac brackets, etc.). For this model, it is convenient to express the two propagating bosons in terms of a complex field y.

We obtain the nonvanishing (anti)commutators

$$[y^*, p_{y^*}] = [y, p_y] = i\hbar, \qquad \{\chi, \chi^\dagger\} = \frac{\hbar}{C}, \tag{57}$$

where $p_y = -i\hbar\partial_y$, $p_{y^*} = -i\hbar\partial_{y^*}$ and the fermions can be expressed as $\chi = \sqrt{\frac{\hbar}{C}}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\chi^\dagger = \sqrt{\frac{\hbar}{C}}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Let us fix, for simplicity, $\hbar = 1$ and $C = \frac{1}{2}$. Then, the quantum charges can be written as

$$\begin{aligned}
\hat{H} &= \left(2p_y p_{y^*} + \frac{(2\lambda+1)^2}{8yy^*}\right)\mathbb{I}_2 \\
&\quad + i\frac{2\lambda+1}{4}(\chi\chi^\dagger - \chi^\dagger\chi)\left(\frac{p_{y^*}}{y} - \frac{p_y}{y^*}\right), \\
\hat{D} &= t\hat{H} - \frac{1}{2}(y^* p_{y^*} + y p_y - i)\mathbb{I}_2, \\
\hat{K} &= t^2\hat{H} - t(y^* p_{y^*} + y p_y - i)\mathbb{I}_2 + \frac{1}{2}yy^*\mathbb{I}_2, \\
\hat{Q}_-^{(1)} &= -\frac{i}{2}\left(\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}} p_y + p_y\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}}\right)\chi \\
&\quad - \frac{i}{2}\left(\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}} p_{y^*} + p_{y^*}\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}}\right)\chi^\dagger, \\
\hat{Q}^{(2)}{}_- &= -\frac{1}{2}\left(\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}} p_y + p_y\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}}\right)\chi \\
&\quad + \frac{1}{2}\left(\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}} p_{y^*} + p_{y^*}\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}}\right)\chi^\dagger, \\
\hat{Q}^{(1)}{}_+ &= t\hat{Q}^{(1)}{}_- - \frac{1}{\sqrt{2}}\sqrt{yy^*}\left(\left(\frac{y}{y^*}\right)^{\lambda}\chi + \left(\frac{y^*}{y}\right)^{\lambda}\chi^\dagger\right), \\
\hat{Q}^{(2)}{}_+ &= t\hat{Q}^{(2)}{}_- - \frac{i}{\sqrt{2}}\sqrt{yy^*}\left(\left(\frac{y}{y^*}\right)^{\lambda}\chi - \left(\frac{y^*}{y}\right)^{\lambda}\chi^\dagger\right), \\
\hat{J} &= \frac{i}{2}\left(\frac{p_{y^*}}{y} - \frac{p_y}{y^*}\right) - \frac{1-2\lambda}{8}(\chi\chi^\dagger - \chi^\dagger\chi).
\end{aligned} \tag{58}$$

Here, $\hat{H}$ is the quantum Hamiltonian.

Using $p_y = -i\hbar\partial_y$, $p_{y^*} = -i\hbar\partial_{y^*}$, the quantum operators $\hat{Q}_-^{(1)}$ and $\hat{Q}_-^{(2)}$ turn out to be

$$\hat{Q}_-^{(1)} = i\begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix}, \qquad \hat{Q}_-^{(2)} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \tag{59}$$

where

$$A^\dagger = -\frac{i}{\sqrt{2}} e^{-i2\lambda\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r}\right),$$
$$A = -\frac{i}{\sqrt{2}} e^{i2\lambda\theta}\left(\partial_r - \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r}\right) \tag{60}$$

are expressed in polar coordinates ($y = re^{i\theta}$, $y^* = re^{-i\theta}$).

In the same way, the quantum Hamiltonian $\hat{H}$ can be expressed as

$$\hat{H} = \left[-\frac{1}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) + i\frac{(2\lambda+1)}{2r^2}\sigma_3\partial_\theta + \frac{(2\lambda+1)^2}{8r^2}\right]\mathbb{I}_2, \tag{61}$$

with σ_3 being the diagonal Pauli matrix. $\frac{(2\lambda+1)^2}{8r^2}$ is the Ehrenfest term resulting from quantization.

The nonvanishing (anti)commutators, closing the $sl(2|1)$ superalgebra, are ($m, n = 0, \pm 1$)

$$[\hat{L}_n, \hat{L}_m] = i(m-n)\hat{L}_{m+n}, \qquad [\hat{L}_0, \hat{Q}_\pm^I] = \pm\frac{i}{2}\hat{Q}_\pm^I,$$
$$[\hat{L}_{\pm 1}, \hat{Q}_\mp^I] = \mp\, i\hat{Q}_\pm^I, \qquad [\hat{J}, \hat{Q}_\pm^I] = \frac{i}{2}\epsilon_{IJ}\hat{Q}_\pm^J,$$
$$\{\hat{Q}_\pm^I, \hat{Q}_\pm^J\} = 2\delta_{IJ}\hat{L}_{\pm 1}, \qquad \{\hat{Q}_\pm^I, \hat{Q}_\mp^J\} = 2\delta_{IJ}\hat{L}_0 \pm 2\epsilon_{IJ}\hat{J}, \tag{62}$$

where $\hat{L}_{-1} = \hat{H}$; $\hat{L}_0 = \hat{D}$; $\hat{L}_1 = \hat{K}$; $I, J = 1, 2$; and $\epsilon_{12} = -\epsilon_{21} = 1$.

The eigenvalue equation $\hat{H}\psi_{E_{m\pm}} = E_{m\pm}\psi_{E_{m\pm}}$, for $E_{m\pm} > 0$, produces a continuum spectrum with eigenfunctions

$$\psi_{Em+}(r,\theta) = J_{|\frac{2\lambda+1}{2}-m|}(\alpha r)e^{im\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\psi_{Em-}(r,\theta) = J_{|\frac{2\lambda+1}{2}+m|}(\alpha r)e^{im\theta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{63}$$

where $J_{|\frac{2\lambda+1}{2}-m|}(\alpha r)$ and $J_{|\frac{2\lambda+1}{2}+m|}(\alpha r)$ are Bessel functions and $\alpha = \sqrt{2E}$.

To conclude the analysis of this model, we present it as supersymmetric quantum mechanics. Let us introduce

$$\hat{Q} = \frac{\hat{Q}_-^2 + i\hat{Q}_-^1}{2} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$
$$\hat{Q}^\dagger = \frac{\hat{Q}_-^2 - i\hat{Q}_-^1}{2} = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}. \tag{64}$$

We get $\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H}$ and $\hat{Q}^2 = (\hat{Q}^\dagger)^2 = 0$.

From the expressions (60), it follows that $\hat{Q}\psi_{E_{m-}} = \psi_{E_{(m+2\lambda)+}}$ and $\hat{Q}^\dagger\psi_{E_{m+}} = \psi_{E_{(m-2\lambda)-}}$. Since $m + 2\lambda$ and $m - 2\lambda$ need to be integer numbers, $\hat{Q}\psi_{E_{m-}}$ and $\hat{Q}^\dagger\psi_{E_{m+}}$ belong to the Hilbert space only if 2λ is an integer number. A supersymmetric pair is therefore only encountered for the quantized values of the scaling dimension, either $\lambda \in \frac{1}{2} + \mathbb{Z}$ or $\lambda \in \mathbb{Z}$.

VI. SUPERCONFORMAL QUANTUM MECHANICS WITH DFF OSCILLATOR POTENTIAL TERMS: 1D $D(2,1;\alpha)$ AND 2D $sl(2|1)$ MODELS

In this section, we quantize the worldline trigonometric σ models obtained from the $\mathcal{N} = 4$ (1, 4, 3) and $\mathcal{N} = 2$ (2, 2, 0) supermultiplets (see Appendix B). They contain (besides a Calogero potential) an oscillatorial (DFF) term that furnishes a discrete spectrum, bounded from below. The associated $D(2,1;\alpha)$ and $sl(2|1)$ superconformal algebras, respectively, act as spectrum-generating algebras for these models.

The $D(2,1;\alpha)$ (1, 4, 3) trigonometric σ models shed some new light on the results of Calogero [3] and de Alfaro *et al.* [2]. Indeed, their Casimir energy linearly depends (in two regions) on the scaling dimension parameter α (in contrast with the complicated dependence expressed in terms of the Calogero coupling constant; see Ref. [33]).

For what concerns the $sl(2|1)$ (2, 2, 0) trigonometric σ models, interesting features are also obtained. The scaling dimension λ needs to be quantized (either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda \in \mathbb{Z}$). At the special $\lambda = -\frac{1}{2}$ value, the ordinary two-dimensional oscillator (since the Calogero potential vanishes at this special point) can be recovered after performing a restriction induced by a superselection rule. The restriction selects, in particular, a single bosonic vacuum. The Hilbert space of the unrestricted two-dimensional models is decomposed into an infinite direct sum of $sl(2|1)$ lowest-weight representations. An unexpected feature is the existence of fermionic raising operators [not entering the $sl(2|1)$ superalgebra] that allow, together with the $sl(2|1)$ raising operators, for $\lambda = \frac{1}{2} + \mathbb{Z}$ to recover the whole Hilbert space of the theory from the two degenerate (one bosonic and one fermionic) vacua of the theory. The existence of these extra fermionic operators is traced to the presence of a discrete symmetry.

A. Quantum $D(2,1;\alpha)$ trigonometric model from $\mathcal{N}=4$ (1, 4, 3)

The quantization of this model follows the same steps as the quantization of the $osp(1|2)$-invariant trigonometric model described in Sec. IV. We end up, just like its $\mathcal{N}=4$ (1, 4, 3) parabolic counterpart of Sec. V, with (anti) commutators defining the $\mathfrak{h}_1 \oplus C_4$ superalgebra (46). We set, for convenience and without loss of generality, the dimensional parameter $\omega = 1$ (its presence in the equations can be restored by means of dimensional analysis).

The quantum operators are [$\mathcal{F}_4$ is the fermion parity operator introduced in (48)]

$$\begin{aligned}
\hat{H} &= e^{it}\left(\frac{\hat{p}^2}{2} - \frac{i}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}) - \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2}\right)\mathbb{I}_4 \\
&\quad + e^{it}\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4, \\
\hat{D} &= \left(\frac{\hat{p}^2}{2} + \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2}\right)\mathbb{I}_4 + \frac{(1+2\alpha)}{4\hat{y}^2}\mathcal{F}_4, \\
\hat{K} &= e^{-it}\left(\frac{\hat{p}^2}{2} + \frac{i}{4}(\hat{y}\hat{p}+\hat{p}\hat{y}) - \frac{\hat{y}^2}{8} + \frac{(1+2\alpha)^2}{8\hat{y}^2}\right)\mathbb{I}_4 \\
&\quad + e^{-it}\frac{1+2\alpha}{4\hat{y}^2}\mathcal{F}_4, \\
\hat{Q}_0 &= e^{\frac{it}{2}}\left(\hat{\chi}_0\hat{p} - \frac{i}{2}\hat{\chi}_0\hat{y} + \frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}}\right), \\
\hat{Q}_i &= e^{\frac{it}{2}}\left(\hat{\chi}_i\hat{p} - \frac{i}{2}\hat{\chi}_i\hat{y} - \frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}}\right), \\
\hat{\bar{Q}}_0 &= e^{-\frac{it}{2}}\left(\hat{\chi}_0\hat{p} + \frac{i}{2}\hat{\chi}_0\hat{y} + \frac{i(1+2\alpha)}{6}\epsilon_{ijk}\frac{\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k}{\hat{y}}\right), \\
\hat{\bar{Q}}_i &= e^{-\frac{it}{2}}\left(\hat{\chi}_i\hat{p} + \frac{i}{2}\hat{\chi}_i\hat{y} - \frac{i(1+2\alpha)}{2}\epsilon_{ijk}\frac{\hat{\chi}_0\hat{\chi}_j\hat{\chi}_k}{\hat{y}}\right), \\
\hat{J}_i &= -i\left(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k + \hat{\chi}_0\hat{\chi}_i\right), \\
\hat{L}_i &= -i\left(\frac{1}{2}\epsilon_{ijk}\hat{\chi}_j\hat{\chi}_k - \hat{\chi}_0\hat{\chi}_i\right).
\end{aligned} \tag{65}$$

The above operators realize the $D(2,1;\alpha)$ superalgebra (49) with the identifications

$$\hat{H} = E^-, \qquad \hat{D} = H, \qquad \hat{K} = -E^+, \qquad \hat{Q}_\beta = 2iF^-_\beta, \qquad \hat{\bar{Q}}_\beta = -2iF^+_\beta, \qquad \hat{J}_j = J_j, \qquad \hat{L}_j = L_j. \tag{66}$$

The quantum Hamiltonian $\hat{\mathcal{H}} \equiv \hat{D}$ is, explicitly,

$$\hat{D} = \left(\begin{array}{c|c} (\frac{\hat{p}^2}{2} + \frac{4\alpha^2+8\alpha+3}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 & 0 \\ \hline 0 & (\frac{\hat{p}^2}{2} + \frac{4\alpha^2-1}{8\hat{y}^2} + \frac{\hat{y}^2}{8})\mathbb{I}_2 \end{array}\right). \tag{67}$$

Both upper (bosonic) and lower (fermionic) diagonal blocks of $\hat{D}$ contain a Calogero Hamiltonian with the DFF oscillatorial potential,

$$\hat{\mathcal{H}}_{DFF} = \frac{1}{2}\hat{p}^2 + \frac{g^2}{\hat{y}^2} + \frac{\hat{y}^2}{8}. \tag{68}$$

A detailed analysis of this Hamiltonian can be found in Refs. [3,33]. Just like the parabolic case, the inequality $g^2 > -\frac{1}{8}$ guarantees the existence of physically acceptable solutions. The boundary value problem

$$\hat{\mathcal{H}}_{DFF}\phi_n = E_n\phi_n, \quad \phi_n(0)=0, \quad n=0,1,2,\ldots \tag{69}$$

implies the discrete spectrum

$$E_n = \frac{1}{2}(n+\nu+1), \tag{70}$$

with eigenfunctions given (up to normalization) by

$$\phi_n(y) = y^{\nu+\frac{1}{2}}\exp\left(-\frac{y^2}{4}\right)L_n^\nu\left(\frac{1}{2}y^2\right). \tag{71}$$

In the right-hand side, L_n^ν stands for the modified Laguerre polynomials. The parameter ν entering the Casimir energy $\frac{1}{2}(\nu+1)$ is

$$\nu = \frac{1}{2}(1+8g^2)^{\frac{1}{2}}. \tag{72}$$

Comparing Eqs. (67) and (68), we see that g_b and g_f are again given by Eqs. (54) so that $\alpha \neq 0, -1$ to ensure that both g_b^2 and g_f^2 are greater than $-\frac{1}{8}$.

Since the Hamiltonian is a Cartan generator of the (65) superalgebra, the whole spectrum can be recovered from a lowest-weight representation of $D(2,1;\alpha)$, where the Q_β's are the lowering and the $\bar{Q}_\beta$'s are the raising operators. The vacuum $|\Lambda\rangle$ is introduced by requiring

$$Q_\beta|\Lambda\rangle = 0, \qquad \beta = 0,1,2,3. \tag{73}$$

From the definition of the Q_β's in (65), the four differential equations (73) can be recast into the single differential equation

$$\left(\hat{p} - \frac{i}{2}\hat{y} - \frac{i(1+2\alpha)}{2\hat{y}}\mathcal{F}_4\right)|\Lambda\rangle = 0. \tag{74}$$

In position-space representation, Eq. (74) splits into two separate equations for the bosonic (+) and fermionic (−) subspaces, respectively,

$$\frac{d\phi_{0,\sigma}}{dy} = -\frac{1}{2}\left(y \pm \frac{1+2\alpha}{y}\right)\phi_{0,\sigma}. \tag{75}$$

The label σ accounts, just as in the parabolic case, for the $\hat{J}$,$\hat{L}$-spin degrees of freedom.

Integrating the above equation, we get, up to normalization, the vacuum solutions

$$\phi_{0,\sigma} = y^{\mp(\frac{1+2\alpha}{2})}\exp\left(-\frac{y^2}{4}\right). \tag{76}$$

This result is in agreement with (71), provided that we set

$$\nu_b = -(1+\alpha), \qquad \nu_f = \alpha. \tag{77}$$

This analysis forces us to conclude that two degenerate lowest-energy vacua exist for $\alpha \neq -\frac{1}{2}$. They are bosonic for $\alpha < -\frac{1}{2}$ and fermionic for $\alpha > -\frac{1}{2}$. This is implied by Eq. (71), which tells us that any bosonic (fermionic) vacuum should be such that $\nu_b + \frac{1}{2} > 0$ ($\nu_f + \frac{1}{2} > 0$).

At the special $\alpha = -\frac{1}{2}$ value, we have that $D(2,1;-\frac{1}{2}) \equiv D(2,1) \approx osp(4|2)$. The Calogero potential terms vanish in both the upper and lower diagonal blocks. At $\alpha = -\frac{1}{2}$, we recover four undeformed harmonic oscillator equations. All the states of the theory (including the minimal energy states) are four times degenerated, with two bosonic and two fermionic states of the same energy.

The energy levels of the system are given by

$$E_{b,n} = \frac{1}{2}(n-\alpha), \qquad E_{f,n} = \frac{1}{2}(n+\alpha+1),$$
$$n = 0,1,2,\ldots. \tag{78}$$

$E_{b,n}$ is the whole spectrum of energies recovered from a bosonic vacum ($\alpha < -\frac{1}{2}$). Conversely, $E_{f,n}$ is the whole spectrum when the vacuum is fermionic ($\alpha > \frac{1}{2}$).

For a bosonic (fermionic) vacuum, the energy of the two degenerate vacua is, respectively, given by

$$E_{b,vac} = -\frac{1}{2}\alpha, \qquad \left(\alpha \leq -\frac{1}{2}\right);$$
$$E_{f,vac} = \frac{1}{2}(\alpha+1), \qquad \left(\alpha \geq -\frac{1}{2}\right). \tag{79}$$

The scaling dimension α can be regarded as an external control parameter of the theory so that the vacuum energy can be interpreted as a Casimir energy. The Casimir energy of the (1, 4, 3) $D(2,1;\alpha)$ (un)deformed oscillator admits a very nice expression in terms of α, being simply given by

$$E_{vac} = \frac{1}{4}(1+|2\alpha+1|). \tag{80}$$

This expression should be compared with the much more complicated expression of the vacuum energy in terms of the Calogero coupling constant g and derived from (72). This result suggests that the scaling dimension α has a more direct physical interpretation of the Calogero coupling constant g. One should also note that, contrary to g, α directly enters the spectrum-generating superalgebra $D(2,1;\alpha)$.

B. $\mathcal{N}=2$ (2, 2, 0) trigonometric model with $sl(2|1)$ invariance

As in the parabolic case, we obtain from quantization the nonvanishing (anti)commutators

$$[y^*, p_{y^*}] = [y, p_y] = i\hbar, \qquad \{\chi, \chi^\dagger\} = \frac{\hbar}{\omega C}, \tag{81}$$

with $\chi = \sqrt{\frac{\hbar}{\omega C}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\chi^\dagger = \sqrt{\frac{\hbar}{\omega C}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We work with $\hbar = 1$, $C = \frac{1}{2}$, and $\omega = 2$. Therefore, the quantum operators of the superalgebra can be written as

$$\hat{H} = i\frac{e^{-2it}}{2}(\hat{\mathcal{H}} - yy^*\mathbb{I}_2 + i(y^*p_{y^*} + yp_y - i)\mathbb{I}_2),$$
$$\hat{D} = \frac{i}{2}\left(2p_yp_{y^*} + \frac{yy^*}{2} + \frac{(2\lambda+1)^2}{8yy^*}\right)\mathbb{I}_2 + i\frac{2\lambda+1}{4}(\chi\chi^\dagger - \chi^\dagger\chi)\left(\frac{p_{y^*}}{y} - \frac{p_y}{y^*}\right) = \frac{i}{2}\hat{\mathcal{H}},$$
$$\hat{K} = i\frac{e^{2it}}{2}(\hat{\mathcal{H}} - yy^*\mathbb{I}_2 - i(y^*p_{y^*} + yp_y - i)\mathbb{I}_2),$$
$$\hat{Q}^{(1)}{}_\pm = -ie^{\mp it}\left[\frac{1}{2}\left(\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}}p_y + p_y\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}}\right)\chi - \frac{1}{2}\left(\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}}p_{y^*} + p_{y^*}\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}}\right)\chi^\dagger \mp \frac{i}{2}(yy^*)^{\frac{1}{2}}\left(\left(\frac{y}{y^*}\right)^{\lambda}\chi - \left(\frac{y^*}{y}\right)^{\lambda}\chi^\dagger\right)\right],$$
$$\hat{Q}^{(2)}{}_\pm = e^{\mp it}\left[\frac{1}{2}\left(\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}}p_y + p_y\left(\frac{y}{y^*}\right)^{\frac{1+2\lambda}{2}}\right)\chi + \frac{1}{2}\left(\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}}p_{y^*} + p_{y^*}\left(\frac{y^*}{y}\right)^{\frac{1+2\lambda}{2}}\right)\chi^\dagger \mp \frac{i}{2}(yy^*)^{\frac{1}{2}}\left(\left(\frac{y}{y^*}\right)^{\lambda}\chi + \left(\frac{y^*}{y}\right)^{\lambda}\chi^\dagger\right)\right],$$
$$\hat{J} = \frac{i}{2}\left(\frac{p_{y^*}}{y} - \frac{p_y}{y^*}\right) - \frac{1-2\lambda}{8}(\chi\chi^\dagger - \chi^\dagger\chi). \tag{82}$$

The fermionic operators $\hat{Q}^{(I)}_\pm$, $I = 1, 2$, entering $sl(2|1)$, can also be expressed as

$$\hat{Q}^{(1)}_{\pm} = ie^{\mp it}\begin{pmatrix} 0 & -A_{\pm} \\ B_{\pm} & 0 \end{pmatrix}, \qquad \hat{Q}^{(2)}_{\pm} = e^{\mp it}\begin{pmatrix} 0 & A_{\pm} \\ B_{\pm} & 0 \end{pmatrix}, \tag{83}$$

where, using the polar coordinates as in the parabolic case, we have

$$A_{\pm} = -\frac{i}{2}e^{i2\lambda\theta}\left(\partial_r - \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r} \pm r\right),$$
$$B_{\pm} = -\frac{i}{2}e^{-i2\lambda\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{2\lambda+1}{2r} \pm r\right). \tag{84}$$

In the trigonometric case, the Hamiltonian $\hat{\mathcal{H}}$ is related to the Cartan generator $\hat{D}$. We have $\hat{\mathcal{H}} = -2i\hat{D}$ so that

$$\hat{\mathcal{H}} = \left[-\frac{1}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) + i\frac{(2\lambda+1)}{2r^2}\sigma_3\partial_\theta + \frac{(2\lambda+1)^2}{8r^2} + \frac{r^2}{2}\right]\mathbb{I}_2. \tag{85}$$

In the rhs, σ_3 is the diagonal Pauli matrix.

For later use, we also write the operator $\hat{J}$ as a differential operator,

$$\hat{J} = -\frac{i}{2}\mathbb{I}_2\partial_\theta - \frac{2\lambda-1}{4}\sigma_3. \tag{86}$$

One can check that the $sl(2|1)$ superalgebra is recovered from the (anti)commutators of the operators (82) using (81).

The differential equation for the radial part of the eigenfunctions $\psi = e^{im\theta}R_{\pm}(r)e_{\pm}$ of $\hat{\mathcal{H}}$, where $e_+ = \binom{1}{0}$ and $e_- = \binom{0}{1}$, is

$$\left[-\frac{1}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \frac{1}{2r^2}\left(m \mp \frac{2\lambda+1}{2}\right)^2 + \frac{r^2}{2} - E\right]R_{\pm}(r) = 0. \tag{87}$$

E is the energy. In Ref. [3], the same equation is found and solved for the problem of three bodies in a line. Furthermore, the issue of self-adjointness of the differential operator acting on $R_{\pm}$ was investigated in Ref. [35]; since $\sqrt{(m \pm \frac{2\lambda+1}{2})^2} \geq 0$, the existence of a self-adjoint extension for the Halmiltonian (85) is ensured.

The requirement of single-valuedness for the operators $\hat{Q}^{(I)}_{\pm}$ on the $\mathbb{R}^2$ plane implies, from the exponents in (84), that the constraint $4\lambda\pi = 2k\pi$, with k integer, must be satisfied. Therefore, the scaling dimension λ has to be quantized, either $\lambda = \frac{1}{2} + \mathbb{Z}$ or $\lambda = \mathbb{Z}$. We discuss in detail the half-integer case, with side remarks about the models with integer values of λ.

One should note that at $\lambda = -\frac{1}{2}$ one obtains (two copies of) the Hamiltonian of the undeformed two-dimensional bosonic oscillator.

For half-integer λ, the $\hat{Q}^{(I)}_{\pm}$ operators act as raising/lowering operators. Let us take, e.g., $\hat{Q}^{(2)}_{\pm}$; it follows, from the commutators $[\hat{\mathcal{H}}, \hat{Q}^{(2)}_{\pm}] = \mp \hat{Q}^{(2)}_{\pm}$, that an energy eigenstate ψ with eigenvalue E_n is mapped into an eigenstate $\hat{Q}^{(2)}_{\pm}\psi$ with eigenvalue $E_n \mp 1$ (provided that $E_n \mp 1 \neq 0$): $\hat{\mathcal{H}}\psi = E_n\psi \to \hat{\mathcal{H}}\hat{Q}^{(2)}_{\pm}\psi = (E_n \mp 1)\hat{Q}^{(2)}_{\pm}\psi$.

Therefore, starting from a lowest-weight state satisfying $\hat{Q}^{(2)}_{+}\psi = 0$, an infinite tower of higher-energy eigenstates are constructed by repeatedly applying $\hat{Q}^{(2)}_{-}$. The solutions of the lowest-weight equation $\hat{Q}^{(2)}_{+}\psi = 0$ are given by the eigenfunctions

$$\psi_{m+}(r,\theta) = A_m r^{(m-\frac{2\lambda+1}{2})}e^{-r^2}e^{im\theta}\begin{pmatrix}1\\0\end{pmatrix},$$
$$\psi_{m-}(r,\theta) = B_m r^{-(m+\frac{2\lambda+1}{2})}e^{-r^2}e^{im\theta}\begin{pmatrix}0\\1\end{pmatrix}, \tag{88}$$

where A_m and B_m are normalization constants given by

$$A_m = 2^{\frac{\alpha+1}{2}}\frac{1}{\sqrt{\pi\Gamma(\alpha+1)}}, \qquad \alpha = m - \frac{2\lambda+1}{2},$$
$$B_m = 2^{\frac{\beta+1}{2}}\frac{1}{\sqrt{\pi\Gamma(\beta+1)}}, \qquad \beta = -\left(m + \frac{2\lambda+1}{2}\right) \tag{89}$$

and Γ is the gamma function.

To have finite lowest-weight eigenfunctions at the origin, the integer m is constrained. From the bosonic states, the necessary condition is

$$m \geq \frac{2\lambda+1}{2}, \tag{90}$$

while from the fermionic states, the necessary condition is

$$m \leq -\frac{2\lambda+1}{2}. \tag{91}$$

The energy eigenvalue equation of the bosonic and fermionic lowest-weight eigenstates is, respectively, given by

$$\hat{\mathcal{H}}\psi_{m+} = \left(1 + m - \frac{2\lambda+1}{2}\right)\psi_{m+},$$
$$\hat{\mathcal{H}}\psi_{m-} = \left(1 - \left(m + \frac{2\lambda+1}{2}\right)\right)\psi_{m-}. \tag{92}$$

Two minimal vacua, one bosonic and the other fermionic, are obtained with vacuum energy 1. They are recovered from the "saturated" bosonic and fermionic lowest-weight eigenstates with, respectively, $m = \frac{2\lambda+1}{2}$ and $m = -\frac{2\lambda+1}{2}$.

The same set of lowest-weight states given by formula (88) is obtained from the lowest-weight condition associated with the lowering operator $Q_+^{(1)}$ ($Q_+^{(1)}\psi = 0$). The repeated application of the raising operator $Q_-^{(1)}$ applied to a lowest state reconstructs, up to a phase, the higher-energy states obtained from the raising operator $Q_-^{(2)}$.

The theory therefore possesses a degenerate vacuum, one vacuum state being bosonic and the other one being fermionic. As discussed in Appendix A, it is possible to impose a superselection rule, imposed by a projector, which selects half of the states being physical. The superselected theory possesses a unique bosonic vacuum, and for $\lambda = -\frac{1}{2}$, its spectrum coincides with the spectrum of the ordinary two-dimensional (undeformed) oscillator, which can therefore be recovered as the superselected, $\lambda = -\frac{1}{2}$, $sl(2|1)$ acting on (2, 2, 0), quantum trigonometric model.

We conclude this section with two important remarks. Contrary to the two vacua of the (not superselected) $\lambda = \frac{1}{2} + \mathbb{Z}$ theory, the $\lambda \in \mathbb{Z}$ quantum deformed oscillators possess four vacuum states (two bosonic and two fermionic states). The construction of the Hilbert space follows the same lines as the half-integer λ case. The main difference lies in the fact that the necessary conditions (90) and (91) for the integer m cannot be satisfied as equalities when $\lambda \in \mathbb{Z}$. It is beyond the scope of this work to present the detailed analysis of the $\lambda \in \mathbb{Z}$ deformed oscillators, which will be presented elsewhere.

The second important remark concerns the fact that, for the superselected $\lambda = \frac{1}{2} + \mathbb{Z}$ theory, the Hilbert space cannot be recovered by repeatedly acting with the $sl(2|1)$ raising operators from the vacuum state. The Hilbert space is decomposed (this point is discussed in Appendix A) in an infinite direct sum of the $sl(2|1)$ lowest-weight representations. This is in sharp contrast with respect to the one-dimensional harmonic oscillator, of which the single irreducible lowest-weight representation of the $osp(1|2)$ spectrum-generating superalgebra allows us to recover the whole Hilbert space.

One can note, however, that it is possible to construct an extra set of fermionic symmetry operators, $\bar{Q}_\pm^{(I)}$, which also act as raising/lowering operators. The construction goes as follows. At first, a discrete symmetry operator $\hat{C}$, playing the role of a charge conjugation operator, is introduced. It is given by

$$\hat{C} = \begin{pmatrix} 0 & e^{i(2\lambda+1)\theta} \\ e^{-i(2\lambda+1)\theta} & 0 \end{pmatrix}. \quad (93)$$

One can verify that $[\hat{\mathcal{H}}, \hat{C}] = 0$, where $\hat{\mathcal{H}}$ is given in (85), and that $\hat{C}^2 = \mathbb{I}_2$. The operator $\hat{C}$ also commutes with the $\hat{K}$ and $\hat{H}$ operators in (82). It does not commute, however, with $\hat{J}$ and the $sl(2|1)$ fermionic operators.

With the help of $\hat{C}$, we can introduce the new symmetry operators

$$\begin{aligned} \hat{C}\hat{Q}_\pm^{(1)}\hat{C} &= \bar{Q}_\pm^{(1)} = ie^{\mp it}\begin{pmatrix} 0 & C_\pm \\ -D_\pm & 0 \end{pmatrix}, \\ \hat{C}\hat{Q}_\pm^{(2)}\hat{C} &= \bar{Q}_\pm^{(2)} = e^{\mp it}\begin{pmatrix} 0 & C_\pm \\ D_\pm & 0 \end{pmatrix}, \end{aligned} \quad (94)$$

where

$$\begin{aligned} C_\pm &= -\frac{i}{2}e^{i2(\lambda+1)\theta}\left(\partial_r + \frac{i}{r}\partial_\theta - \frac{2\lambda+1}{2r} \pm r\right), \\ D_\pm &= -\frac{i}{2}e^{-i2(\lambda+1)\theta}\left(\partial_r - \frac{i}{r}\partial_\theta - \frac{2\lambda+1}{2r} \pm r\right), \end{aligned} \quad (95)$$

and

$$\hat{C}\hat{J}\hat{C} = \bar{J} = -\frac{i}{2}\partial_\theta - \frac{2\lambda+3}{4}\sigma_3. \quad (96)$$

Let us collectively denote as $\hat{g}_i$ ($i = 1, 2, \ldots, 8$) the $sl(2|1)$ operators entering (82). By construction, the operators $\bar{g}_i = \hat{C}\hat{g}_i\hat{C}^{-1}$, obtained through a similarity transformation, close as well the $sl(2|1)$ superalgebra. It is worth pointing out that this second set of $sl(2|1)$ operators cannot be expressed as a linear combination of the $\hat{g}_i$ set of $sl(2|1)$ operators. In particular, the (anti)commutators $[\hat{g}_i, \bar{g}_j\}$ produce new operators on the right-hand side. It is not clear which algebraic structure is induced by the combined set of $\hat{g}_i$ and $\bar{g}_j$ operators (see the comments in the Conclusions). An important feature, discussed in Appendix A, is the fact that we need rasing operators from both sets, $\hat{g}_i$ and $\bar{g}_j$, to produce *every* excited state of the theory by applying raising operators on the ground state(s). An exemplification of this is illustrated, e.g., by the Fig. 1 diagram of Appendix A. Both $\hat{Q}_\pm^{(I)}$ and $\bar{Q}_\pm^{(I)}$ act as rasing/lowering operators. The action of the $\hat{Q}_\pm^{(I)}$ raising operators is illustrated by the solid edges, while the action of the $\bar{Q}_\pm^{(I)}$ raising operators is illustrated by the dashed edges.

In terms of $\hat{C}$, we can also introduce the new quantum operators

$$\mathcal{J} = \hat{J} + \bar{J} = -i\partial_\theta - \frac{2\lambda+1}{2}\sigma_3, \qquad N_f = \sigma_3 = \hat{J} - \bar{J}, \quad (97)$$

which allows us to define the new quantum numbers (used in Appendix A; see Fig. 4):

$$\begin{aligned} &\hat{\mathcal{H}}|n,j,\epsilon\rangle = (n+1)|n,j,\epsilon\rangle, \qquad \mathcal{J}|n,j,\epsilon\rangle = j|n,j,\epsilon\rangle, \\ &\sigma_z|n,j,\epsilon\rangle = \epsilon|n,j,\epsilon\rangle. \end{aligned} \quad (98)$$

VII. CONCLUSIONS

In this paper, we presented a framework for quantizing the large class of classical worldline superconformal σ models derived from supermultiplets. These systems are defined in Refs. [25] (for the parabolic case) and [1] (for the trigonometric case). We applied the quantization prescription to derive explicitly the $\mathcal{N}=4$ (1, 4, 3) and the $\mathcal{N}=2$ (2, 2, 0) quantum superconformal mechanics [with $D(2,1;\alpha)$ and $sl(2|1)$ dynamical symmetry, respectively]. The parameter $\alpha \neq 0,-1$ is the scaling dimension of the (1, 4, 3) supermultiplet, while the scaling dimension of the (2, 2, 0) supermultiplet is quantized and given by $\lambda=\frac{1}{2}+\mathbb{Z}$ or $\lambda \in \mathbb{Z}$.

The results concerning the trigonometric models are particularly relevant. These systems are only softly supersymmetric; see the discussion in Appendix C. As such, they have not received much attention like the parabolic models. The trigonometric models correspond to superconformal mechanics in the presence of the DFF damping oscillatorial term; stated otherwise, they are oscillators in which Calogero potential terms are possibly present. Their spectrum is discrete and bounded from below.

For the (1, 4, 3) trigonometric models [i.e., the $D(2,1;\alpha)$ oscillators], we derive the following nice formula for the vacuum energy:

$$E_{vac}=\frac{1}{4}(1+|2\alpha+1|). \tag{99}$$

If α is interpreted as a physical external parameter, then (99) can be interpreted as a Casimir energy.

A restriction (obtained by imposing a superselection rule derived by a projector; see Appendix A) of the (2, 2, 0) trigonometric model at the special value $\lambda=-\frac{1}{2}$ allows us to recover the spectrum of the ordinary two-dimensional oscillator.

The (unrestricted) $\mathcal{N}=2$ (2, 2, 0) trigonometric models for the $\lambda \in \frac{1}{2}+\mathbb{Z}$ and $\lambda \in \mathbb{Z}$ quantized values of the scaling dimension possess an $sl(2|1)$ dynamical symmetry. As a consequence, their spectrum is a direct sum of an infinite tower of $sl(2|1)$ lowest-weight representations.

The surprising presence of an extra fermionic symmetry (discussed at length in Sec. VI and in Appendixes A and C) produces extra fermionic generators that act as raising and lowering operators. They allow us to reach each state belonging to the Hilbert space of the two-dimensional models by repeatedly applying the raising operators to the vacuum state.

This result seems to suggest the existence of a broader dynamical symmetry algebra (not necessarily a superalgebra, it could be, see Ref. [36], a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded dynamical symmetry algebra), which has to be introduced in order to recover the spectrum of the $\mathcal{N}=2$ (2, 2, 0) (deformed) oscillators from a single, irreducible, lowest-weight representation. We are planning to address this remarkable feature in our forthcoming investigations.

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APPENDIX A: DIAGRAMS OF THE SPECTRUM-GENERATING SUPERALGEBRA FOR THE $\mathcal{N}=2$, (2, 2, 0), $\lambda=\frac{1}{2}+\mathbb{Z}$ TRIGONOMETRIC CASES

It is convenient, for the two-dimensional cases based on the $\mathcal{N}=2$ (2, 2, 0) trigonometric reps, to encode in diagrams the action of the raising and lowering operators of the spectrum-generating superalgebra. We explicitly present three such diagrams, Figs. 1, 2 and 3, respectively, associated with three values of the scaling dimension, $\lambda=\frac{1}{2}$, $\lambda=-\frac{1}{2}$, and $\lambda=-\frac{3}{2}$. In a further diagram, the general features of the $\lambda=\frac{1}{2}+\mathbb{Z}$ case are presented.

In the diagrams, the bosonic (fermionic) states are denoted by white (black) dots. Gray dots denote the presence of both bosonic and fermionic states. The vertical axis represents the energy level, labeled by n, while the horizontal axis represents the angular momentum, labeled by m. We denote with ϵ the eigenvalues of the fermion number operator ($\epsilon=+1$ for bosons and $\epsilon=-1$ for fermions). Solid (dashed) lines represent states connected by $\hat{Q}_{\pm}^{(I)}$ (respectively, $\bar{Q}_{\pm}^{(I)}$) raising and lowering operators with $I=1,2$; see (83) and (94) (for simplicity, we drop the indices here).

The $sl(2|1)$ lowest-weight states appear, in the diagrams, as the dots where the solid lines originate (in the upward direction). In Figs. 2 and 4, the existence of such lowest-weight states is not immediately evident; this is, however, just a side effect of the condensed notation used (a gray dot being associated with two states).

The operators $\hat{Q}_{\pm}^{(1)}$, $\hat{Q}_{\pm}^{(2)}$ (and, similarly, $\bar{Q}_{\pm}^{(1)}$, $\bar{Q}_{\pm}^{(2)}$), applied to a $|n,m,\epsilon\rangle$ state that does not coincide with a

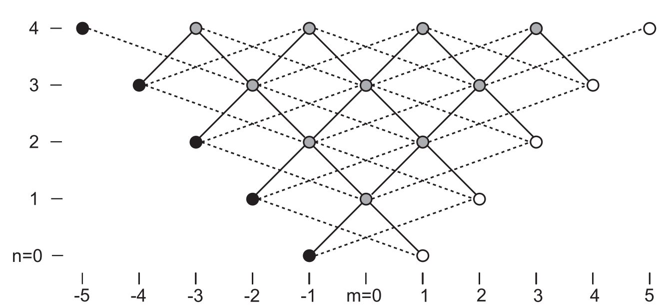

FIG. 1. $\lambda=\frac{1}{2}$ diagram of $\hat{Q}$'s and $\bar{Q}$'s raising and lowering operators.

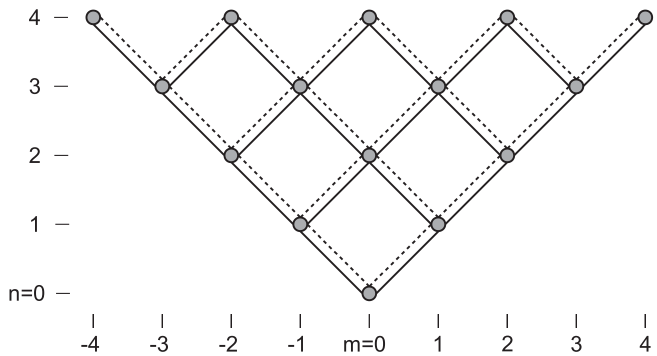

FIG. 2. $\lambda = -\frac{1}{2}$ diagram of $\hat{Q}$'s and $\bar{Q}$'s raising and lowering operators.

lowest-weight state produce, apart from a normalization factor, the same state. We can write, for $I = 1, 2$,

$$\hat{Q}_{\pm}^{(I)}|n,m,\epsilon\rangle \propto |n\mp 1, m-\epsilon 2\lambda, -\epsilon\rangle,$$
$$\bar{Q}_{\pm}^{(I)}|n,m,\epsilon\rangle \propto |n\mp 1, m-\epsilon 2(\lambda+1), -\epsilon\rangle. \quad \text{(A1)}$$

From the three diagrams, Figs. 1, 2, and 3, we can immediately read several important features. In particular, in all three cases, the $n > 0$ higher-energy states are produced via repeated applications of the $\hat{Q}$'s and $\bar{Q}$'s raising operators from the two (one bosonic and one fermionic) $n = 0$ fundamental level states. As a corollary, we need both types ($\hat{Q}$'s and $\bar{Q}$'s) of raising operators to recover the Hilbert space of the associated model. This means, stated otherwise, that the Hilbert space is *reducible* with respect to the $sl(2|1)$ superalgebra defined by the $\hat{Q}_{\pm}^{(I)}$ operators alone. In terms of a $sl(2|1)$ decomposition, an infinite tower (one state at each given integer value n) of lowest-weight states needs to be introduced to recover the Hilbert space of the theory. Therefore, to have an irreducible description, the $\bar{Q}_{+}^{(I)}$ operators need to enter the picture.

One should note that the $\lambda = -\frac{1}{2}$ case corresponds to the undeformed (namely, without the extra Calogero potential term) two-dimensional harmonic oscillator. The Hilbert space defined by Fig. 2 contains a double degeneracy. Two eigenstates (one bosonic and the other one fermionic) are

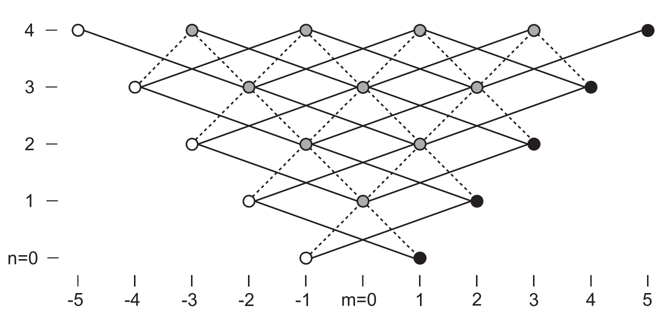

FIG. 3. $\lambda = -\frac{3}{2}$ diagram of $\hat{Q}$'s and $\bar{Q}$'s raising and lowering operators.

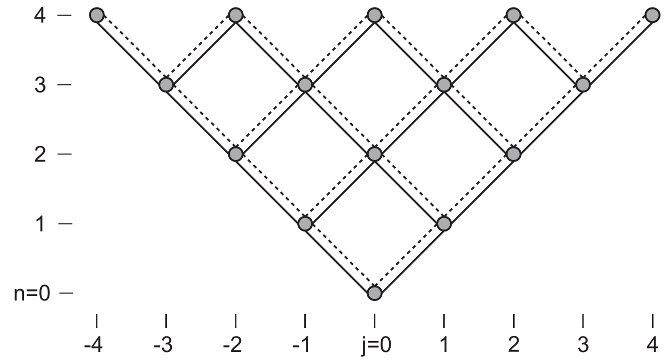

FIG. 4. The $\lambda = \frac{1}{2} + \mathbb{Z}$ general diagram.

associated with each n, m pair of eigenvalues. The introduction of a suitable projection allows us to remove the double degeneracy and recover the Hilbert space of the ordinary two-dimensional harmonic oscillator. The superselection rule is defined in terms of the projection operator $\hat{P}$ ($\hat{P}^2 = \mathbb{I}$), given by

$$\hat{P} = N_f e^{i\pi\mathcal{H}}, \quad \text{(A2)}$$

where N_f is the fermion parity operator and $\hat{\mathcal{H}} = -2i\hat{D}$ is the Hamiltonian (its eigenvalues are the non-negative integers n). The

$$\hat{P}|\Psi\rangle = |\Psi\rangle \quad \text{(A3)}$$

superselection rule implies that the Hilbert space of the superselected theory is given by bosonic states at even energy eigenvalues ($n = 2k$, with $k = 0, 1, 2, \ldots$) and fermionic states at odd energy eigenvalues ($n = 2k + 1$).

The superselection removes, in particular, the degeneracy of the vacuum, the single vacuum state being now bosonic. The spectrum of the ordinary two-dimensional harmonic oscillator is therefore recovered from the *superselected* $\mathcal{N} = 2$ (2, 2, 0) model at scaling dimension $\lambda = -\frac{1}{2}$.

For any half-integer value $\lambda = \frac{1}{2} + \mathbb{Z}$, the Hilbert space of the two-dimensional deformed (due to the presence, besides the quadratic potential, of a Calogero potential term) harmonic oscillator can be formally recovered from the $\lambda = -\frac{1}{2}$ Fig. 2 diagram, by replacing the angular momentum m with the j eigenvalues of the $\mathcal{J}$ operator introduced in (97) (this is also true for the $\lambda = \frac{1}{2}, -\frac{3}{2}$ cases explicitly introduced in Figs. 1 and 3).

Let us introduce the basis defined by the quantum numbers

$$\hat{\mathcal{H}}|n,j,\epsilon\rangle = (n+1)|n,j,\epsilon\rangle;$$
$$\hat{\mathcal{J}}|n,j,\epsilon\rangle = j|n,j,\epsilon\rangle, \quad (j\in\mathbb{Z});$$
$$N_f|n,j,\epsilon\rangle = \epsilon|n,j,\epsilon\rangle, \quad (\epsilon = \pm 1).$$

In this basis, the action of $\hat{Q}^{(I)}_\pm$, $\bar{Q}^{(I)}_\pm$ on a state that does not coincide with a lowest-weight state, reads as follows:

$$\hat{Q}^{(I)}_\pm |n,j,\epsilon\rangle \propto |n\mp 1, j+\epsilon, -\epsilon\rangle,$$
$$\bar{Q}^{(I)}_\pm |n,j,\epsilon\rangle \propto |n\mp 1, j-\epsilon, -\epsilon\rangle. \tag{A4}$$

The $\lambda=\frac{1}{2}+\mathbb{Z}$ associated diagrams are presented in Fig. 4.

This makes clear that the superselection rule induced by (A2) can be imposed on any $\lambda=\frac{1}{2}+\mathbb{Z}$ deformed oscillator, guaranteeing in all these cases the existence of a Hilbert space with a single bosonic vacuum.

APPENDIX B: THE CLASSICAL (2, 2, 0) $sl(2|1)$-INVARIANT MODELS

We present, for completeness, the construction of the $sl(2|1)$-invariant classical actions obtained from the parabolic and the trigonometric D-module reps acting on the (2, 2, 0) supermultiplet.

The parabolic D-module rep is given by the transformations

$$\begin{aligned}
&L_n x_i = t^n(t\dot{x}_i + (n+1)\lambda x_i),\\
&L_n\psi_i = t^n\left(t\dot{\psi}_i + (n+1)\left(\frac{2\lambda+1}{2}\right)\psi_i\right), \qquad n=0,\pm1;\\
&Jx_i = -\lambda\epsilon_{ij}x_j, \qquad J\psi_i = -\frac{2\lambda-1}{2}\epsilon_{ij}\psi_j,\\
&Q^1_\pm x_i = t^{\frac{1\pm1}{2}}\epsilon_{ij}\psi_j, \qquad Q^1_\pm\psi_i = -it^{\frac{1\pm1}{2}}\epsilon_{ij}(t\dot{x}_j + (1\pm1)\lambda x_j),\\
&Q^2_\pm x_i = t^{\frac{1\pm1}{2}}\psi_i, \qquad Q^2_\pm\psi_i = it^{\frac{1\pm1}{2}}(t\dot{x}_i + (1\pm1)\lambda x_i),
\end{aligned} \tag{B1}$$

where the x_i's ($i=1,2$) are the propagating bosons and the ψ_i's are the fermionic fields. The above transformations close the $sl(2|1)$ superalgebra.

The $sl(2|1)$-invariant action is obtained from the Lagrangian $\mathcal{L}=Q^2_+Q^1_+(\frac{1}{2}F\epsilon_{ij}\psi_i\psi_j)$, with the operators Q^2_+, Q^1_+ acting on the prepotential $F=C(x_ix_i)^{-\frac{2\lambda+1}{2\lambda}}$ (C is a normalization constant). Explicitly, the invariant action of the classical (2, 2, 0) parabolic model is

$$\mathcal{S}=\int dt\mathcal{L}=\int dt(F(\dot{x}_i\dot{x}_i - i\dot{\psi}_i\psi_i) - iF_i\dot{x}_j\psi_i\psi_j). \tag{B2}$$

The trigonometric D-module rep is given by the transformations

$$\begin{aligned}
&L_n x_i = \frac{e^{-in\omega t}}{-i\omega}(\dot{x}_i - in\lambda\omega x_i),\\
&L_n\psi_i = \frac{e^{-in\omega t}}{-i\omega}\left(\dot{\psi}_i - in\left(\frac{2\lambda+1}{2}\right)\omega\psi_i\right), \qquad n=0,\pm1;\\
&Jx_i = -\lambda\epsilon_{ij}x_j, \qquad J\psi_i = -\frac{2\lambda-1}{2}\epsilon_{ij}\psi_j,\\
&Q^1_\pm x_i = e^{\mp i\frac{\omega}{2}t}\epsilon_{ij}\psi_j, \qquad Q^1_\pm\psi_i = \frac{e^{\mp i\frac{\omega}{2}t}}{i\omega}\epsilon_{ij}(\dot{x}_j \mp i\lambda\omega x_j),\\
&Q^2_\pm x_i = e^{\mp i\frac{\omega}{2}t}\psi_i, \qquad Q^2_\pm\psi_i = \frac{e^{\mp i\frac{\omega}{2}t}}{-i\omega}(\dot{x}_i \mp i\lambda\omega x_i).
\end{aligned} \tag{B3}$$

Without loss of generality, we can set $\omega=1$. The classical action, $sl(2|1)$ invariant under the (B3) trigonometric transformations, is therefore given by

$$\begin{aligned}
\mathcal{S} &= \int dt\mathcal{L}\\
&= \int dt(F(\dot{x}_i\dot{x}_i - i\dot{\psi}_i\psi_i) - iF_i\dot{x}_j\psi_i\psi_j + C\lambda^2(x_ix_i)^{-\frac{1}{2\lambda}}).
\end{aligned} \tag{B4}$$

APPENDIX C: ON THE SOFT SUPERSYMMETRY OF THE OSCILLATORS

We make some comments here on the role of superalgebras applied to oscillators (either the ordinary quantum oscillators or the oscillators which are "deformed" by the presence of a Calogero potential term).

The starting point is the famous work of Wigner [37]. In modern terms, after the concept of superalgebra was introduced in mathematics, Wigner's results can be reinterpreted (see Ref. [38]) according to the following lines. For the ordinary quantum oscillator, with creation/annihilation operators a and $a^\dagger$ (satisfying $[a,a^\dagger]=1$) and symmetrized Hamiltonian $\mathcal{H}=\{a,a^\dagger\}$, we can assign odd grading to the operators a and $a^\dagger$ so that they belong to a set of five operators, a, $a^\dagger$, a^2, $(a^\dagger)^2$, and $\mathcal{H}=\{a,a^\dagger\}$, closing the $osp(1|2)$ superalgebra under (anti)commutations. The last three (bosonic) operators close the $sl(2)$ subalgebra. Under this construction, we have an alternative point of view for describing the computation of the spectrum of the ordinary (one-dimensional) harmonic oscillator: we can state that, instead of deriving it from the Fock vacuum $|0\rangle$, annihilated by a ($a|0\rangle=0$), the spectrum is obtained from a lowest-weight representation of $osp(1|2)$, the Hamiltonian being the Cartan element. By adopting this viewpoint, the superalgebra $osp(1|2)$ becomes a spectrum-generating superalgebra for the ordinary quantum oscillator, with its Hilbert space being recovered from a single, irreducible, $osp(1|2)$ lowest-weight representation.

One should note that the bosonic $sl(2)$ subalgebra also acts as a spectrum-generating algebra for the harmonic oscillator. The Hilbert space of the harmonic oscillator is, however, reducible under the $sl(2)$ decomposition. It is given by the direct sum of two irreducible $sl(2)$ lowest-weight representations. The first lowest state is the vacuum of the theory (proportional to the Gaussian e^{-x^2} under proper conventions and normalization). The other lowest state is the first excited state, with the eigenfunction proportional to xe^{-x^2} and having odd parity with respect to the $x \mapsto -x$ transformation. The two $sl(2)$ lowest-weight reps correspond to, respectively, the even-parity and the odd-parity energy eigenstates. The role of the fermionic operators in $osp(1|2)$ consists in connecting energy eigenstates of even and odd parity.

After the introduction and the subsequent classification of simple Lie superalgebras [39,40], the Wigner approach was advocated in Ref. [41], with special emphasis on parastatistics, prompting a series of investigations on lowest-weight representations of simple Lie superalgebras (for a recent review, see, e.g., Ref. [42]).

On a separate development, the DFF "trick" of introducing oscillator damping potentials in conformal mechanics relates oscillators (with/without the Calogero potential term) to conformal algebras.

It was recognized in Ref. [27] that, due to the DFF trick, the introduction of new potentials for conformal mechanics becomes possible. The two aspects, superalgebra vs conformal algebra, were reconciled in Ref. [1]. The notion of parabolic vs trigonometric/hyperbolic D-module reps of superconformal algebras was pointed out, with the latter class describing the (deformed or undeformed) oscillators and potentials bounded from below in the trigonometric case.

The main property shared by the two big classes of superconformal theories, parabolic vs trigonometric, is that at the classical level their respective actions are superconformally invariant. Concerning their differences, we have the following:

(i) The parabolic models are, both classically and quantum, superconformal and supersymmetric. The supersymmetry implies the existence of a symmetry operator $\mathcal{Q}$, which is the "square root" of the Hamiltonian $\mathcal{H}$, namely, $\mathcal{Q}^2 = \mathcal{H}$.

(ii) The trigonometric models, on the other hand, despite being superconformally invariant, are not supersymmetric. In this case, symmetry operators $\mathcal{Q}$ and $\mathcal{Z}$ exist such that $\mathcal{Q}^2 = \mathcal{Z}$. The key point is that the operator $\mathcal{Z}$ does not coincide with the Hamiltonian: $\mathcal{Z} \neq \mathcal{H}$.

One can easily say that the trigonometric models are "intermediate" between the supersymmetric and the non-supersymmetric theories. This "intermediate notion of supersymmetry," namely, $\mathcal{Q}^2 = \mathcal{Z} \neq \mathcal{H}$, has no special name in the literature. In Ref. [1], the notion of "weak supersymmetry" was employed, borrowing the term from a construction described in Ref. [43], which shares a similar feature. The use of the term weak supersymmetry, however, could be misleading since the models in Ref. [43] are not based on superconformal algebras. In that paper, a "weak supersymmetric oscillator" that has no relation with the oscillators derived from the trigonometric D-module reps of superconformal algebras is discussed.

For this reason, it seems more appropriate to denote this important class of trigonometric models (which include, as shown in this paper, the ordinary one-dimensional and two-dimensional harmonic oscillators) as softly supersymmetric. As far as we know, the term 'soft supersymmetry" has not been employed in a different context, making this term both suitable and available to describe the special properties of the trigonometric superconformal mechanics.

The softly supersymmetric trigonometric models are characterized by the following:

(i) There is classical superconformal invariance of the action.

(ii) There is spontaneous breaking of the superconformal invariance. Indeed, in the simplest application, the Fock vacuum $|0\rangle$ of the harmonic oscillator is annihilated by a and not by the Hermitian operator $a + a^\dagger$: $(a + a^\dagger)|0\rangle \neq 0$.

(iii) In the quantum case, the role of the superconformal algebra is that of a spectrum-generating superalgebra.

Concerning the last point, we indeed proved, see Appendix A, that the spectrum of the ordinary two-dimensional oscillator is decomposed into an infinite tower of $sl(2|1)$ irreducible lowest-weight representations. The puzzling presence of the extra fermionic generators (94) that connect eigenstates belonging to different lowest-weight reps reminds us of the role, just discussed above, played by the $osp(1|2)$ fermionic generators in connecting the two $sl(2)$ lowest-weight reps of the one-dimensional oscillator.

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