PHYSICS OF PLASMAS **12**, 122509 (2005)

Nonlinear viscosity and its role in drift-Alfvén modes

V. S. Tsypin
Physics Institute, University of São Paulo, Cidade Universitaria, 05508-900, São Paulo, Brazil

A. B. Mikhailovskii
Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia
and Nonlinear Physics Laboratory, Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi 141700, Moscow Region, Russia

M. S. Shirokov
Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia, and Moscow Engineering Physics Institute, Kashirskoe Shosse 31, Moscow 115409, Russia

E. A. Kovalishen
Nonlinear Physics Laboratory, Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi 141700, Moscow Region, Russia, and Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia

S. V. Konovalov
Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, Kurchatov Sq., 1, Moscow 123182, Russia

R. M. O. Galvão
Physics Institute, University of São Paulo, Cidade Universitaria, 05508-900, São Paulo, Brazil, and Brazilian Center for Research in Physics, Rua Xavier Sigaud, 150, 22290-180, Rio de Janeiro, Brazil

(Received 28 September 2005; accepted 22 November 2005; published online 23 December 2005)

The moment approach is used to analyze the part of the magnetized plasma viscosity related to the nonlinear character of the Landau collision integral in the Boltzmann kinetic equation (nonlinear viscosity), pointed out by Catto and Simakov [Phys. Plasmas **11**, 90 (2004)]. It is shown that the results of these authors, who have used an alternative procedure based on a more detailed analysis of the kinetic equation, correspond to a 15-moment approach. In comparison with the 13-moment approach (density, temperature, velocity, heat flux, and the viscosity tensor) of Grad, the 15-moment approach takes into account two higher-order moments, one of which is the vector-type moment similar to the parallel heat flux and the second is the tensor-type moment similar to the parallel projection of the viscosity tensor. Both these higher-order moments enter into the Braginskii approximation. The nonlinear viscosity calculated in the scope of the 13-moment Grad approach is qualitatively the same as that found by Catto and Simakov. Its role is investigated for drift-Alfvén modes, driven by the combined effect of the dissipative part of perpendicular heat conductivity and the standard collisional viscosity, and it is shown to be essential for the radial transport of these modes. It is shown that the wave packet of drift-Alfvén modes, propagating in the diamagnetic drift direction and driven for reversed temperature gradient, is transported down the pressure gradient. In contrast to this, the wave packet propagating in the electron diamagnetic drift direction and driven for positive temperature gradient is transported up the pressure gradient. *© 2005 American Institute of Physics*. [DOI: 10.1063/1.2151169]

I. INTRODUCTION

The plasma temperature at the plasma edge in magnetic confinement systems can be sufficiently low for the Pfirsch-Schlüter regime to be valid in this region (see Ref. 1 and references therein). The plasma behavior in this regime can be described by transport equations derived from the Boltzmann kinetic equation by an expansion in the inverse of the collision frequency, taking into account the simplifying assumption that it is much smaller than the cyclotron frequency for all particle species. An example of the transport equations can be found in Refs. 2–4.

In the standard formalism, the transport equations are derived in the approximation that the collisional integral is linear with respect to deviation of the particle distribution function from a Maxwellian. One of the main results obtained in this approximation is the expression for viscosity tensor. Catto and Simakov[5] have recently suggested that, in deriving the expression for the viscosity tensor, the nonlinear part of the collisional integral related to the heat flux should be taken into account. With allowance for this part of the collisional integral, terms appear in the viscosity tensor that depend on the square of the heat flux, so that they can be referred to as the nonlinear plasma viscosity or, alternatively, the CS terms, in tribute to the original work of Catto and Simakov. Further analysis of the nature of the CS terms and their applications can be found in Ref. 6–8.

The standard viscosity, i.e., the linear one, was calculated in Refs. 2 and 3 by the moment approach generalizing the 13-moment Grad approach[9] for a larger number of moments. One of the goals of the present paper is to show that the nonlinear terms in the viscosity, i.e., the CS terms, also can be derived in the moment approach formulation. This seems to be relevant because it is not evident from Refs. 5–8 that the moment approach can be used for calculating the nonlinear part of viscosity. Thereby, we give a moment interpretation of the results of Refs. 5–8 and elucidate which moments should be taken into account for obtaining the CS terms.

In addition to the problem of deriving the nonlinear viscosity, it is also relevant to point out specified problems where this viscosity can be essential. One more goal of the present paper is to show that the nonlinear viscosity can be essential in the problem of drift-Alfvén modes.

In Sec. II we derive the starting equations for the nonlinear viscosity. In Sec. III we solve these equations and obtain the expression for the nonlinear viscosity.

The results presented in Sec. III are obtained for arbitrary magnetic field geometry. Their simplification for the particular case of magnetic field lines aligned along one of the axes, in a Cartesian coordinate system, is given in Sec. IV. In Secs. V and VI we study the instability of drift-Alfvén modes, driven by the standard collisional effects, and their radial transport sensitive to the nonlinear viscosity. The complete analyses of this problem require both the linear and nonlinear viscosities. The expressions for the linear viscosity tensor found in Ref. 2 (see also Refs. 3, 4, and 10) are given in the Appendix. The dispersion relation for the drift-Alfvén modes is derived in Sec. V. Its analysis is performed in Sec. VI.

Historically, the study of instabilities related to drift-Alfvén modes, driven by the combined effect of the perpendicular collisional heat conductivity and the collisional perpendicular viscosity, has been started in Refs. 2 and 10. In both these references the modes propagating across the equilibrium magnetic field, i.e., those with $k_z=0$ (see the definitions in Sec. V) were considered. Neglecting finite-beta effects, such modes reduce to the ion drift diamagnetic modes. Then, assuming that the equilibrium ion temperature is uniform, it was found in Ref. 2 that the growth/decay rate of these modes turns out to be vanishing. In contrast to Ref. 2, in Ref. 10, the temperature inhomogeneity was allowed for and it was found that the ion drift diamagnetic modes are unstable in the case of inverse temperature gradient, i.e., when the temperature and pressure gradients have opposite directions.

This result of Ref. 10 is confirmed in Sec. VI. In addition, in Sec. VI the modes with $k_z \neq 0$ are studied. Then, as is well known (see Refs. 11 and 4), besides the modes propagating in the ion diamagnetic drift direction, modes propagating in the electron diamagnetic drift direction appear. In Sec. VI we show that these modes are driven in the case of positive ratio of relative temperature and pressure gradients, if this ratio exceeds some threshold. The action of the nonlinear viscosity on the drift-Alfvén modes consists of contributing to their radial transport. The main results of this work are summarized in Sec. VII.

II. DERIVATION OF EQUATIONS FOR NONLINEAR VISCOSITY

A. General equations of moment approach

We take the distribution function of a particle species f in the form

$$f=f_0(1+\Phi). \tag{1}$$

Here, f_0 is the "shifted" Maxwellian,

$$f_0=n(M/2\pi T)^{3/2}\exp(-Mw^2/2T), \tag{2}$$

where $\mathbf{w}=\mathbf{v}-\mathbf{V}$ is the random velocity of particles, $\mathbf{v}$ is their total velocity, $\mathbf{V}$ is the macroscopic velocity, n is the plasma number density, and T is the temperature of the considered plasma species. The function Φ is a small additive, $\Phi \ll 1$.

1. Approximation for Φ

We take the function Φ in the form

$$\Phi = q_i\varphi^i + q_i^*\psi^i + \pi_{ij}\varphi^{ij} + \pi_{ij}^*\psi^{ij}. \tag{3}$$

The subscripts and superscripts mean the contravariant and covariant components, respectively.

The functions (φ^i,ψ^i), and (φ^{ij},ψ^{ij}) are defined by

$$(\varphi^i,\psi^i) = -\frac{2w^iM}{5nT^2}[L_1^{(3/2)}(x),L_2^{(3/2)}(x)] \tag{4}$$

and

$$(\varphi^{ij},\psi^{ij}) = \frac{M}{2nT^2}\left(w^iw^j-\frac{1}{3}w^2g^{ij}\right)[L_0^{(5/2)}(x),L_1^{(5/2)}(x)]. \tag{5}$$

Here, $x=(w/v_T)^2$, $v_T=(2T/M)^{1/2}$ is the thermal velocity, $L_l^{(m+1/2)}(x)$ are the Sonine-Laguerre polynomials, where m and l are integers, and g^{ij} is the metric tensor.

The values $\mathbf{q}$ and $\boldsymbol{\pi}$ are the heat flux and viscosity tensor, respectively. They are given by

$$q_i=\frac{M}{2}\int w_i\mathbf{w}^2fd\mathbf{w} \equiv \frac{5}{2}\frac{nT^3}{M}\int \varphi_i f d\mathbf{w}, \tag{6}$$

$$\pi_{ij}=M\int (w_iw_j-w^2g_{ij})fd\mathbf{w} \equiv 2nT^2\int \varphi_{ij}fd\mathbf{w}, \tag{7}$$

where g_{ij} is the metric tensor. The values $\mathbf{q}^*$ and $\boldsymbol{\pi}^*$ are related to the distribution function by [cf. Eqs. (6) and (7)]

$$q_i^*=-\frac{5}{2}\frac{nT^3}{M}\int \psi_i f d\mathbf{w}, \tag{8}$$

$$\pi_{ij}^*=2nT^2\int \psi_{ij}fd\mathbf{w}. \tag{9}$$

The function Φ, given by Eq. (3), is a particular case of that of Ref. 3. [Note that Eq. (2.5) of Ref. 3 for the function ψ^i contains a misprint; instead of 16/35 it should be 4/5.]

Although both ions and electrons can be included in the formulation, in the present paper we restrict ourselves only to ions and omit the indices characterizing the particle species.

2. Boltzmann kinetic equation

As in Refs. 2–4, we represent the Boltzmann kinetic equation in the form

$$\frac{df}{dt} + \mathbf{w}\cdot\nabla f + \frac{\partial f}{\partial \mathbf{w}}\cdot\{\mathbf{F} - \mathbf{w}\cdot\nabla\mathbf{V} + [\mathbf{w}\times\boldsymbol{\omega}_c]\} = C. \quad (10)$$

Here,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla, \quad \mathbf{F} = \frac{e\mathbf{E}}{M} + [\mathbf{V}\times\boldsymbol{\omega}_c] - \frac{d\mathbf{V}}{dt}, \quad (11)$$

where C is the Landau collision integral,[12] $\mathbf{E}$ is the electric field, e is the electric charge, $\boldsymbol{\omega}_c = e\mathbf{B}/(Mc)$ is the cyclotron frequency vector, $\mathbf{B}$ is the magnetic field, and c is the speed of light. Integrating Eq. (10) over velocity $\mathbf{w}$ with a weighting factor X leads to

$$\frac{d}{dt}\langle X\rangle - \left\langle \frac{dX}{dt}\right\rangle + \nabla\cdot\langle \mathbf{w}X\rangle - \langle \mathbf{w}\cdot\nabla X\rangle - \left\langle \mathbf{F}\cdot\frac{\partial X}{\partial \mathbf{w}}\right\rangle + \langle X\rangle \nabla\cdot\mathbf{V} - \left\langle ([\mathbf{w}\times\boldsymbol{\omega}_c] - \mathbf{w}\cdot\nabla\mathbf{V})\cdot\frac{\partial X}{\partial \mathbf{w}}\right\rangle = \langle CX\rangle, \quad (12)$$

where

$$\langle X\rangle = \int fX d\mathbf{w}. \quad (13)$$

Using Eq. (1) and the expression for the collision integral,[12] we find

$$\langle CX\rangle = \langle CX\rangle_L + \langle CX\rangle_{NL}, \quad (14)$$

where $\langle CX\rangle_L$ and $\langle CX\rangle_{NL}$ are the linear and nonlinear parts of the value $\langle CX\rangle$, respectively. The value $\langle CX\rangle_L$ is given by

$$\langle CX\rangle_L = \frac{2\pi\lambda e^4}{M^2}\int\int d\mathbf{w}d\mathbf{w}'\frac{\partial X}{\partial w_\beta}U_{\beta\gamma}\left(\Phi\frac{\partial f_0'}{\partial w_\gamma'} - \Phi'\frac{\partial f_0}{\partial w_\gamma}\right). \quad (15)$$

Here, λ is the Coulomb logarithm, f_0' and Φ' are functions of $\mathbf{w}'$, and

$$U_{\beta\gamma} = (u^2\delta_{\beta\gamma} - u_\beta u_\gamma)/u^3, \quad \mathbf{u} = \mathbf{w} - \mathbf{w}'. \quad (16)$$

Integrating by parts, the value $\langle CX\rangle_{NL}$ can be represented in the form

$$\langle CX\rangle_{NL} = \frac{2\pi\lambda e^4}{M^2}\int\int d\mathbf{w}d\mathbf{w}' f_0 f_0'\left(\frac{\partial X}{\partial w_\beta} - \frac{\partial X'}{\partial w_\beta'}\right)U_{\beta\gamma}\Phi\frac{\partial \Phi'}{\partial w_\gamma'}. \quad (17)$$

Note that, when describing plasma perturbations, both the shifted Maxwellian f_0 and the additive function Φ include not only the equilibrium terms but also the perturbed ones. For instance, the temperature T in Eq. (2) for the function f_0 is the total temperature of the respective plasma component, including both the equilibrium and perturbed parts, so that $T = T^{(0)} + \delta T$, where $T^{(0)}$ and δT are the equilibrium and perturbed temperatures, correspondingly. Therefore, if one uses Eq. (17) to investigate plasma perturbations, one should take $f_0 = f_0^{(0)} + \delta f_0$, $\Phi = \Phi^{(0)} + \delta\Phi$, where $f_0^{(0)}$ and $\Phi^{(0)}$ are the equilibrium parts, while δf_0 and $\delta\Phi$ are the perturbations. Then one obtains, for instance, the term $\Phi^{(0)}(\delta f_0)^2$ in the integrand of Eq. (17).

In accordance with the above explanations, the left-hand side of the kinetic equation, Eq. (10), includes both the equilibrium and perturbed temperature gradients. Therefore, our results can be used, in particular, in the problem of the ion temperature gradient (ITG) modes (see Ref. 11 and references therein).

3. Braginskii's approximation for $q_\parallel^*$

Using Eq. (4) and the above-given Boltzmann kinetic equation, one can calculate the function $q_\parallel^* \equiv \mathbf{h}\cdot\mathbf{q}^*$, where $\mathbf{h} = \mathbf{B}/B$ is the unit vector along the magnetic field. Then, one finds[2,3] that, in Braginskii's approximation,[12]

$$q_\parallel^* = -(4/15)q_\parallel, \quad (18)$$

where $q_\parallel \equiv \mathbf{h}\cdot\mathbf{q}$ is the parallel heat flux. The function $\mathbf{q}_\perp^* \equiv \mathbf{q}^* - \mathbf{h}(\mathbf{h}\cdot\mathbf{q}^*)$ is not important for our problem. Then, the vector Φ_i defined by Eq. (4), allowing for Eqs. (6) and (8), reduces to

$$\Phi_i = -\frac{2}{5}\frac{M}{nT^2}\left[q_i L_1^{(3/2)}(x) - \frac{4}{15}h_i q_\parallel L_2^{(3/2)}(x)\right]. \quad (19)$$

This expression is in accordance with Eq. (17) of Ref. 5. Therefore, it is clear that Ref. 5 has used the Braginskii's approximation for $q_\parallel^*$.

B. Starting equations for nonlinear viscosity

Let us choose

$$X = 2nT^2\varphi_{kl}, \quad (20)$$

and separate the viscosity tensor $\boldsymbol{\pi}$ into the linear and nonlinear parts, i.e.,

$$\boldsymbol{\pi} = \boldsymbol{\pi}^{(L)} + \boldsymbol{\pi}^{(NL)}. \quad (21)$$

Using the expression for Φ, given by Eq. (2.5) of Ref. 3, and Eqs. (12) and (15), we arrive at the expression for $\boldsymbol{\pi}^{(L)}$ given in Ref. 3. On the other hand, taking Φ in the form of Eq. (3) and allowing for Eq. (17), we obtain the following equation for the nonlinear viscosity:

$$\frac{6}{5}\nu_i\left(\pi_{kl}^{(NL)} + \frac{3}{4}\pi_{kl}^{*(NL)}\right) + \omega_{ci}\hat{\sigma}\pi_{kl}^{(NL)} = -nTW_{kl}^{(NL)}. \quad (22)$$

The quantity ν_i is the ion collision frequency defined in Ref. 12. The tensor $\boldsymbol{\pi}^{*(NL)}$ is the nonlinear part of the tensor $\boldsymbol{\pi}^*$. The operator $\hat{\sigma}$ is given by[3,4]

$$\hat{\sigma}a_{kl} = h_\tau\varepsilon^{\tau\sigma\lambda}(a_{\sigma k}g_{l\lambda} + a_{\sigma l}g_{k\lambda}) \equiv [\mathbf{h}\times\mathbf{a}_k]_l + [\mathbf{h}\times\mathbf{a}_l]_k, \quad (23)$$

where $\varepsilon^{\tau\sigma\lambda}$ is the antisymmetric tensor with the components $\pm 1/\sqrt{g} \equiv \pm 1/J$, where $J \equiv \sqrt{g}$ is the Jacobian of the coordinate transformation, and $\mathbf{a}_k$ is the vector with the covariant components a_{kj}, $j=(1,2,3)$. The tensor $W_{kl}^{(NL)}$ is determined by

$$W_{kl}^{(NL)} = -2T\langle C(\varphi_{kl})\rangle_{NL}. \tag{24}$$

Using the explicit form of the Sonine–Laguerre polynomials and calculating the velocity integral in Eq. (17), we arrive at

$$2nT^2\langle C\varphi_{kl}\rangle_{NL} = \frac{9}{25}\frac{\nu_i}{nMv_T^4}\left(\langle\langle\mathbf{qq}\rangle\rangle_{kl} + \frac{7}{4}\langle\langle\mathbf{qq}^*\rangle\rangle_{kl} + \frac{63}{64}\langle\langle\mathbf{q}^*\mathbf{q}^*\rangle\rangle_{kl}\right), \tag{25}$$

where the value $\langle\langle\mathbf{AB}\rangle\rangle_{kl}$ is determined by

$$\langle\langle\mathbf{AB}\rangle\rangle_{kl} = A_kB_l + A_lB_k - (2/3)g_{kl}\mathbf{A}\cdot\mathbf{B}. \tag{26}$$

The function $\mathbf{q}^*$ is given by Eq. (18) and $\mathbf{q}_\perp^*=0$.

Since Eq. (22) contains the tensor $\boldsymbol{\pi}^{*(NL)}$, we should complement it by an equation for this tensor. Then, we use Eq. (17) with X in the form

$$X = 2nT^2\psi_{kl}. \tag{27}$$

As a result, we find [cf. Eq. (22)]

$$\frac{12}{35}\nu_i\left(\frac{3}{4}\pi_{kl}^{(NL)} + \frac{205}{48}\pi_{kl}^{*(NL)}\right) + \omega_{ci}\hat{\sigma}\pi_{kl}^{*(NL)} = -\frac{2}{5}nTW_{kl}^{*(NL)}, \tag{28}$$

where [cf. Eq. (24)]

$$W_{kl}^{*(NL)} = -5T\langle C\psi_{kl}\rangle_{NL}. \tag{29}$$

Similarly to Eq. (25), one obtains

$$2nT^2\langle C\psi_{kl}\rangle_{NL} = \frac{267}{1750}\frac{\nu_i}{nMv_T^4}\left(\langle\langle\mathbf{qq}\rangle\rangle_{kl} + \frac{865}{356}\langle\langle\mathbf{qq}^*\rangle\rangle_{kl} + \frac{5915}{5696}\langle\langle\mathbf{q}^*\mathbf{q}^*\rangle\rangle_{kl}\right). \tag{30}$$

Equations (22) and (28) are the starting ones for calculating the nonlinear viscosity.

III. CALCULATION OF NONLINEAR VISCOSITY

A. Parallel nonlinear viscosity

1. General expression for $\pi_\parallel^{(NL)}$

Let us introduce the scalar of the parallel nonlinear viscosity $\pi_\parallel^{(NL)}$ determined by

$$\pi_\parallel^{(NL)} = h^kh^l\pi_{kl}^{(NL)}. \tag{31}$$

Multiplying Eqs. (22) and (28) by h^kh^l, we arrive at

$$\pi_\parallel^{(NL)} + \frac{3}{4}\pi_\parallel^{*(NL)} = -\frac{5}{6\nu_i}nTW_\parallel^{(NL)}, \tag{32}$$

$$\frac{3}{4}\pi_\parallel^{(NL)} + \frac{205}{48}\pi_\parallel^{*(NL)} = -\frac{7}{6\nu_i}nTW_\parallel^{*(NL)}. \tag{33}$$

Here, $\pi_\parallel^{*(NL)} \equiv h^kh^l\pi_{kl}^{*(NL)}$ and

$$W_\parallel^{(NL)} = \frac{6}{25}\frac{\nu_i}{n^2TMv_T^4}\left(\mathbf{q}^2 - 3q_\parallel^2 - \frac{7}{2}q_\parallel q_\parallel^* - \frac{63}{32}q_\parallel^{*2}\right), \tag{34}$$

$$W_\parallel^{*(NL)} = \frac{89}{350}\frac{\nu_i}{n^2TMv_T^4}\left(\mathbf{q}^2 - 3q_\parallel^2 - \frac{865}{178}q_\parallel q_\parallel^* - \frac{5915}{2848}q_\parallel^{*2}\right). \tag{35}$$

It follows from Eqs. (32) and (33) that

$$\pi_\parallel^{(NL)} = -\frac{nT}{89\nu_i}\left(\frac{1025}{12}W_\parallel^{(NL)} - 21W_\parallel^{*(NL)}\right). \tag{36}$$

Substituting here Eqs. (34) and (35), we obtain

$$\pi_\parallel^{(NL)} = -\frac{379}{8900}\frac{M}{pT}(\mathbf{q}^2 - c_\parallel q_\parallel^2), \tag{37}$$

where

$$c_\parallel = 3 + \frac{5}{379}\frac{q_\parallel^*}{q_\parallel}\left(229 + \frac{4683}{32}\frac{q_\parallel^*}{q_\parallel}\right). \tag{38}$$

2. The case of finite $q_\parallel^*$ calculated in Braginskii's approximation

For $q_\parallel^*$ given by Eq. (18), Eq. (38) reduces to

$$c_\parallel = \frac{8837}{3790} = 2.33. \tag{39}$$

Equation (37), with $c_\parallel$ given by Eq. (39), is in correspondence with that following from Ref. 5 (see the expression of Ref. 5 for $p_\parallel - p_\perp$ allowing for $p_\parallel - p_\perp = 3\pi_\parallel/2$).

3. The $q_\parallel^*=0$ approximation

For $q_\parallel^*=0$ Eq. (38) yields

$$c_\parallel = 3. \tag{40}$$

Comparing this expression with Eq. (39), one sees that the $q_\parallel^*=0$ approximation leads qualitatively to the same result as the Braginskii approximation.

4. The Grad approximation

In the Grad approximation, one should omit the term with $\pi_\parallel^{*(NL)}$ in the left-hand side of Eq. (32) and the term with $q_\parallel^*$ in Eq. (34). Then, one finds

$$\pi_\parallel^{(NL)} = -\frac{1}{20}\frac{M}{nT^2}(\mathbf{q}^2 - 3q_\parallel^2). \tag{41}$$

One can see that Eq. (41) differs from the corresponding result of Ref. 5 in two aspects. First, Eq. (39) for the coefficient $c_\parallel$ is changed by Eq. (40), which is a consequence of the $q_\parallel^*=0$ approximation, and second, the factor 379/8900 is changed as follows:

$$379/8900 \simeq 0.04 \to 1/20 = 0.05, \tag{42}$$

which is a consequence of the $\pi_\parallel^*=0$ approximation. Then, we conclude that the Grad approximation leads qualitatively to the same result as that obtained in Ref. 5.

B. Perpendicular nonlinear viscosity

1. Solution of equation for perpendicular nonlinear viscosity

Using Eq. (22), in accordance with Ref. 3, the perpendicular nonlinear viscosity tensor $\boldsymbol{\pi}^{(NL)\perp}$, for arbitrary tensor $W_{kl}^{(NL)}$, is given by

$$\pi_{ik}^{(NL)\perp} = \eta^{(3)} W_{(3)ik}^{(NL)} + \eta^{(4)} W_{(4)ik}^{(NL)}. \tag{43}$$

Here,

$$\eta^{(3)} = nT/(2\omega_c), \quad \eta^{(4)} = nT/\omega_c, \tag{44}$$

the tensors $W_{(3)ik}^{(NL)}$ and $W_{(4)ik}^{(NL)}$ are related to the tensor $W_{ik}^{(NL)}$ by

$$W_{(3)ik}^{(NL)} = (1/2)(\hat{g}_i^{\mu} g_{kt}\varepsilon^{t\gamma\nu} + \hat{g}_k^{\nu} g_{it}\varepsilon^{t\gamma\mu})h_\gamma W_{\mu\nu}^{(NL)}, \tag{45}$$

$$W_{(4)ik}^{(NL)} = (h_i h^{\mu} g_{kt}\varepsilon^{t\gamma\nu} + h_k h^{\nu} g_{it}\varepsilon^{t\gamma\mu})h_\gamma W_{\mu\nu}^{(NL)}, \tag{46}$$

and

$$\hat{g}_i^{\mu} = g_i^{\mu} - h_i h^{\mu}. \tag{47}$$

Note that Eq. (43) is in formal correspondence with Braginskii's expression for the oblique viscosity (gyroviscosity).[12] Therefore, the tensor $\pi_{ik}^{(NL)\perp}$ can be called the dissipative part of gyroviscosity.

2. Transformation of expression for perpendicular nonlinear viscosity

Equations (45) and (46) can be represented in the form

$$W_{(3)ik}^{(NL)} = (1/2)\{[\mathbf{h} \times (\mathbf{W}_i^{(NL)} - h_i \mathbf{h} \cdot \mathbf{W}^{(NL)})]_k + [\mathbf{h} \times (\mathbf{W}_k^{(NL)} - h_k \mathbf{h} \cdot \mathbf{W}^{(NL)})]_i\}, \tag{48}$$

$$W_{(4)ik}^{(NL)} = h_i[\mathbf{h} \times \mathbf{h} \cdot \mathbf{W}^{(NL)}]_k + h_k[\mathbf{h} \times \mathbf{h} \cdot \mathbf{W}^{(NL)}]_i. \tag{49}$$

Here, the value $\mathbf{W}_i^{(NL)}$ means the vector with covariant components $W_{i\alpha}^{(NL)}$, $\alpha=(1,2,3)$.

Using Eqs. (44), (48), and (49) and allowing for Eqs. (24) and (25), we transform Eq. (43) to

$$\pi_{ik}^{(NL)\perp} = -\frac{9}{200}\frac{\nu M}{nT^2\omega_c}\{(q_i + c_\perp q_\parallel h_i)[\mathbf{h} \times \mathbf{q}]_k + [\mathbf{h} \times \mathbf{q}]_i (q_k + c_\perp q_\parallel h_k)\}, \tag{50}$$

where

$$c_\perp = 3 + (7/2)q_\parallel^*/q_\parallel. \tag{51}$$

3. The case of finite $q_\parallel^*$ calculated in the Braginskii approximation

Substituting Eq. (18) into Eq. (51), we obtain

$$c_\perp = 31/15 = 2.07. \tag{52}$$

Equation (50) with $c_\perp$ given by Eq. (52) coincides with Eq. (61) of Ref. 5.

4. The Grad approach

Omitting the terms with $q_\parallel^*$, Eq. (51) reduces to

$$c_\perp = 3. \tag{53}$$

It is clear from comparison of Eqs. (52) and (53) that, as in the case of $\pi_\parallel^{(NL)}$, the Grad approach results qualitatively in the same $\pi_{ik}^{(NL)\perp}$ as in the Braginskii approximation.

IV. NONLINEAR VISCOSITY FOR THE SIMPLEST MAGNETIC FIELD GEOMETRY

Studying the ion dynamics, we can neglect perpendicular components of the magnetic field in the plasma viscosity. For better understanding the effects of the nonlinear viscosity and bearing in mind the application to the problem of drift-Alfvén modes, we consider a nonuniform plasma in plane geometry, with the equilibrium magnetic field $\mathbf{B}=\mathbf{e}_z B(x,y)$, in a Cartesian coordinate system (x,y,z); $\mathbf{e}_z$ is the unit vector along z.

One can find from Eq. (31) that

$$\pi_{zz}^{(NL)} = \pi_\parallel^{(NL)}, \tag{54}$$

where $\pi_\parallel^{(NL)}$ is defined by the right-hand side of Eq. (37) with $q_\parallel = q_z$. At the same time, the parallel nonlinear viscosity contributes to π_{xx} and π_{yy}, so that

$$\pi_{xx}^{(NL)\parallel} = \pi_{yy}^{(NL)\parallel} = -\pi_\parallel^{(NL)}/2, \tag{55}$$

where the superscript $\parallel$ means the corresponding contributions.

On the other hand, it follows from Eq. (50) that

$$\begin{pmatrix} \pi_{xx}^{(NL)\perp} & \pi_{xy}^{(NL)\perp} \\ \pi_{yx}^{(NL)\perp} & \pi_{yy}^{(NL)\perp} \end{pmatrix} = \frac{9}{200}\frac{\nu_i M}{nT^2\omega_c}\begin{pmatrix} 2q_x q_y & q_y^2 - q_x^2 \\ q_y^2 - q_x^2 & -2q_x q_y \end{pmatrix}, \tag{56}$$

$$\begin{pmatrix} \pi_{xz}^{(NL)\perp} & \pi_{yz}^{(NL)\perp} \\ \pi_{zx}^{(NL)\perp} & \pi_{zy}^{(NL)\perp} \end{pmatrix} = \frac{9}{200}(1 + c_\perp)\frac{\nu_i M q_z}{nT^2\omega_c}\begin{pmatrix} q_y & -q_x \\ q_y & -q_x \end{pmatrix}, \tag{57}$$

$$\pi_{zz}^{(NL)\perp} = 0. \tag{58}$$

Note that Eqs. (56) do not contain the coefficients $c_\parallel$ and $c_\perp$. Therefore, they can be derived in the scope of the 13-moment Grad approach.

V. DERIVATION OF LOCAL DISPERSION RELATION FOR DRIFT-ALFVÉN MODES

According to Ref. 4, the drift-Alfvén modes in the presence of viscosity are described by the mode equation of the form

$$\left[\nabla \times \left(Mn\frac{d\mathbf{V}_{i\perp}}{dt} + \nabla \cdot \pi - \frac{1}{c}\mathbf{j} \times \mathbf{B}\right)\right]_z = 0, \tag{59}$$

where $\mathbf{V}_{i\perp}$ is the perpendicular ion velocity and $\mathbf{j}$ is the electric current density. For simplicity, we consider Eq. (59) in the local approximation, assuming the plasma inhomogeneity to be directed along x and taking the mode dependence on time and coordinates in the eikonal form $\exp[-i\omega t + i(\int k_x dx + k_y y + k_z z)]$. Then, using the results of the Appendix, similar to Eq. (10.21) of Ref. 4, Eq. (59) reduces to

$$E_y\left[\omega - \frac{k_z^2 v_A^2}{\omega} + \frac{9}{40} i\alpha\left(1 - i\frac{\kappa_p}{2k_x}\right)\right] - \frac{ik_y\tilde{p}_i}{en_0}\left[\omega + i\alpha\left(\frac{93}{160} - i\frac{6}{5}\frac{\kappa_T}{k_x} - i\frac{81}{80}\frac{\kappa_p}{k_x}\right)\right] - \frac{\alpha k_y T_0 \tilde{n}}{en_0}\left(\frac{57}{160} - i\frac{117}{80}\frac{\kappa_p}{k_x}\right) - \frac{B_0 k_y}{cMn_0k_x^2}Q = 0. \quad (60)$$

Here, E_y is the y projection of the perturbed electric field, $\tilde{p}_i$ is the perturbed ion pressure, $\tilde{n}$ and n_0 are the perturbed and equilibrium plasma number density, respectively, T_0 is the equilibrium temperature, v_A is the Alfvén velocity, B_0 is the equilibrium magnetic field, $\kappa_p = \partial p_{0i}/\partial x$, $\kappa_T = \partial T_0/\partial x$. The parameter α means

$$\alpha = (4/3)\nu_i k_x^2 \rho_i^2, \quad (61)$$

where $\rho_i = (T_0/M)^{1/2}/\omega_{ci}$ is the ion Larmor radius. We restrict ourselves only to modes with $k_x \gg k_y$ since, according to Ref. 11, they prove to be "most dangerous" if one allows for shear.

The value Q allows for the perpendicular nonlinear viscosity and is defined by

$$Q = [\nabla \times \nabla \cdot \{\pi^{(NL)}\}^{\perp}]_z. \quad (62)$$

Using the results of Sec. IV, we calculate

$$Q = -i\frac{27}{64}\alpha\frac{k_x\kappa_T n_0}{\omega_{ci}}\tilde{T}_i. \quad (63)$$

Here, $\tilde{T}_i$ is the perturbed ion temperature related to $\tilde{p}_i$ and $\tilde{n}$ by

$$\tilde{T}_i = \tilde{p}_i/n_0 - T_0\tilde{n}/n_0. \quad (64)$$

In addition to the variable E_y, Eq. (60) also contains the variables $\tilde{n}$ and $\tilde{p}_i$. In order to express these variables in terms of E_y, we use the electron continuity equation and the ion heat balance equation. These equations are the following versions of Eqs. (10.1) and (10.20) of Ref. 4, respectively:

$$-i\omega\frac{\tilde{n}}{n_0} + \frac{cE_y}{B_0}\kappa_n = 0, \quad (65)$$

$$-i\frac{\tilde{p}_i}{n_0T_0}\left[\omega + i\alpha\left(1 - i\frac{\kappa_p + \kappa_T}{k_x}\right)\right] + \alpha\frac{\tilde{n}}{n_0}\left(1 - i\frac{\kappa_p}{k_x}\right) + \frac{cE_y}{B_0}\kappa_p = 0, \quad (66)$$

where $\kappa_n = \partial \ln n_0/\partial x$. Using Eqs. (60) and (63)–(66), we arrive at the dispersion relation,

$$\omega(\omega - \omega_{pi}^*) - k_z^2 v_A^2 + i\alpha\left[\frac{9}{40}(\omega - \omega_{pi}^*) + \frac{103}{160}\omega_{Ti}\right] + \frac{\alpha}{k_x}\left[\frac{27}{64}\kappa_T\omega_{Ti} + \frac{9}{20}\kappa_p\left(\omega + 29/9\,\omega_{pi}^* - \frac{223}{36}\omega_{Ti} + \frac{775}{144}\frac{\kappa_T}{\kappa_p}\omega_{Ti}\right)\right] = 0. \quad (67)$$

Here, ω_{pi}^* and ω_{Ti} are the ion diamagnetic drift frequencies determined by the pressure and temperature gradients, so that

$$\omega_{pi}^* = k_y c T_0 \kappa_p/(eB_0), \quad \omega_{Ti} = (\kappa_T/\kappa_p)\omega_{pi}^*. \quad (68)$$

The first term in the first square brackets in Eq. (67) corresponds to the dissipative effects due to the perpendicular ion heat conductivity and the collisional part of the standard perpendicular viscosity. The last term in the left-hand side of this equation is due to the same effects and the effect of nonlinear viscosity. Since we use the local approximation, it contains the formally small parameters κ_T/k_x and κ_p/k_x compared with the standard dissipative term. Nevertheless, the last term is physically important by two reasons. First, if one goes beyond the local approximation, i.e., considers the eigenvalue problem, it turns out to be of the same order of magnitude as the standard dissipative term. Second, the last term is real, in contrast to the standard dissipative term, which is imaginary. Therefore, the nonlinear viscosity does not influence growth or damping rate of the modes but contributes to their radial group velocity or, in other words, into their radial transport (see in detail below).

VI. INSTABILITY OF DRIFT-ALFVÉN MODES DUE TO STANDARD COLLISIONAL EFFECTS AND THEIR COLLISIONAL RADIAL TRANSPORT

A. Drift-Alfvén modes in neglecting their radial transport

In this subsection we consider the drift-Alfvén modes, neglecting their radial transport. Then, Eq. (67) reduces to

$$\omega(\omega - \omega_{pi}^*) - k_z^2 v_A^2 + i\alpha\left[\frac{9}{40}(\omega - \omega_{pi}^*) + \frac{103}{160}\omega_{Ti}\right] = 0. \quad (69)$$

Since the parameter α is small compared with the mode frequency, $\alpha \ll \omega$, Eq. (69) can be analyzed by the method of successive approximations, using a series expansion in the ratio α/ω.

1. Modes propagating in the ion diamagnetic drift direction

In the present section we analyze modes propagating in the ion diamagnetic drift direction, $\mathrm{Re}\,\omega/\omega_{pi}^* > 0$, for the simplest case $k_z \to 0$. Then, Eq. (69) is transformed to

$$\omega - \omega_{pi}^* + i\alpha\left[\frac{9}{40}\left(1 - \frac{\omega_{pi}^*}{\omega}\right) + \frac{103}{160}\frac{\omega_{Ti}}{\omega}\right] = 0. \quad (70)$$

Following the analysis of this dispersion relation presented in Ref. 10, we find from Eq. (70)

$$\mathrm{Re}\,\omega = \omega_{pi}^* \quad (71)$$

and

$$\mathrm{Im}\,\omega = -\frac{103}{160}\frac{\kappa_T}{\kappa_p}\alpha, \quad (72)$$

so that these modes become unstable for reversed temperature gradient,

$$\kappa_T/\kappa_p < 0. \tag{73}$$

Turning to Eq. (61) for α, we find that their growth rate is equal to

$$\mathrm{Im}\ \omega = -\frac{103}{120}\frac{\kappa_T}{\kappa_p}\nu_i k_x^2 \rho_i^2, \tag{74}$$

in coincidence with that obtained in Ref. 10.

2. Modes propagating in the electron diamagnetic drift direction

For simplicity, in studying the modes propagating in the electron diamagnetic drift direction, $\mathrm{Re}\ \omega/\omega_{pi}^* < 0$, we take $\omega_{pi}^* \gg k_z v_A$. Then, Eq. (69) reduces to

$$\omega + \frac{k_z^2 v_A^2}{\omega_{pi}^*} - i\alpha\left(\frac{103}{160}\frac{\kappa_T}{\kappa_p} - \frac{9}{40}\right) = 0. \tag{75}$$

Hence, we find

$$\mathrm{Re}\ \omega = -k_z^2 v_A^2/\omega_{pi}^* \tag{76}$$

and

$$\mathrm{Im}\ \omega = \frac{103}{160}\alpha\left(\frac{\kappa_T}{\kappa_p} - \frac{36}{103}\right). \tag{77}$$

In contrast to the modes with $\mathrm{Re}\ \omega/\omega_{pi}^* > 0$, these modes are unstable for positive ratio of the relative temperature and pressure gradients, κ_T/κ_p [cf. Eq. (73)],

$$\kappa_T/\kappa_p > 36/103. \tag{78}$$

In terms of κ_T/κ_n (and positive κ_T/κ_n), this inequality means $\kappa_T/\kappa_n > 36/67$.

B. Radial transport of the modes

1. Modes propagating in the ion diamagnetic drift direction

Allowing for the last term on the left-hand side of Eq. (67), we find, instead of Eq. (71),

$$\mathrm{Re}\ \omega = \omega_{pi}^* - \frac{1}{5}\nu_i k_x \rho_i^2 \kappa_p \mu, \tag{79}$$

where

$$\mu = \frac{38}{3} - \frac{293}{12}\frac{\kappa_T}{\kappa_p} + \frac{775}{46}\frac{\kappa_T^2}{\kappa_p^2}. \tag{80}$$

A remarkable feature of Eq. (79) is the fact that the nonlinear viscosity contributes to the radial group velocity v_g of the modes. This radial group velocity is equal to

$$v_g \equiv \frac{\partial\, \mathrm{Re}\ \omega}{\partial k_x} = -\frac{1}{5}\nu_i \rho_i^2 \kappa_p \mu. \tag{81}$$

Since $\mu > 0$, for a standard decreasing ion pressure profile, $\kappa_p < 0$, it follows from Eq. (81) that $v_g > 0$. This means that the wave packet of the drift-Alfvén modes, propagating in the ion diamagnetic drift direction, is transported in the direction of larger x, i.e., down the pressure gradient.

2. Modes propagating in the electron diamagnetic drift direction

Using Eq. (67), we find the following modification of Eq. (79):

$$\mathrm{Re}\ \omega = -\frac{k_z^2 v_A^2}{\omega_{pi}^*} + \frac{1}{5}\nu_i k_x \rho_i^2 \kappa_p \mu. \tag{82}$$

It hence follows that

$$v_g = \frac{1}{5}\nu_i \rho_i^2 \kappa_p \mu. \tag{83}$$

The sign of v_g is opposite to that of the radial group velocity determined by Eq. (81). Therefore, the wave packet of the drift-Alfvén modes, propagating in the electron diamagnetic drift direction, is transported in the direction of smaller x, i.e., up the pressure gradient.

VII. DISCUSSION

Following the moment approach, we have confirmed the expressions for the nonlinear viscosity derived in Ref. 5 and elucidated which moments should be taken into account to obtain these expressions. Then, we have shown that, in addition to the 13 moments entering the Grad approach (density, temperature, velocity, heat flux, and the viscosity tensor), one should allow for two higher-order moments. One of them is the vector-type moment, similar to the parallel heat flux, and the second is the tensor-type moment, similar to the parallel component of the viscosity tensor. The additional moments are characterized by the terms with q_i^* and π_{ij}^* in Eq. (3), assuming that the vector $\mathbf{q}^*$ contains only the parallel component $q_\parallel^*$ and the tensor π_{ij}^* consists only of the parallel component $\pi_\parallel^*$ determined similarly to Eq. (31). Both these moments enter the two-polynomial Braginskii hydrodynamics.[12] The rigid attachment of Ref. 5 to the moment $q_\parallel^*$ is demonstrated by the concordance of Eq. (19) with Eq. (17) of Ref. 5.

It is then clear that the numerical coefficients in the expressions of Ref. 5 for the nonlinear viscosity are not universal; they depend on number of higher-order moments taken into account in their derivation. This fact can be seen, in particular from Eqs. (37) and (50), containing the coefficients $c_\parallel$ and $c_\perp$ given by Eqs. (38) and (51), respectively, which are functions of the ratio $q_\parallel^*/q_\parallel$. Evidently, allowing for additional higher-order vector-type moments should result in changing these coefficients. At the same time, turning to Eqs. (32) and (33), one can see that Eq. (37) for $\pi_\parallel^{(NL)}$ is also not universal, since it is obtained only when the single higher-order tensor-type moment is allowed in the calculations.

Therefore, a reasonable question is, what is the minimal number of moments necessary to derive analytically the correct nonlinear viscosity? The answer is the total Grad 13 moments. Using it, one arrives at the expressions for the parallel nonlinear viscosity $\pi_\parallel^{(NL)}$ given by Eq. (41) and the perpendicular nonlinear viscosity $\pi_{ik}^{(NL)\perp}$ defined by Eq. (50) with $c_\perp$ given by Eq. (53).

The only universal coefficient in the nonlinear viscosity is the factor $9/200$ in Eq. (50) for the tensor $\pi_{ik}^{(NL)\perp}$. Such a universality can be understood if one allows for the known universality of the standard gyroviscosity and the fact that, according to Eqs. (43)–(47), this tensor can be treated as the dissipative part of the gyroviscosity.

We have explained that separation of the collision operator into nonlinear and linear parts is not the same as separation of a value into the linear and nonlinear parts of the perturbation amplitude in the nonlinear plasma theory; see in detail the discussion after Eq. (17). Respectively, the expressions for the nonlinear viscosity derived in the paper can be used in both the linear and nonlinear plasma theories as well as in the theory of plasma equilibrium.

We have analyzed the problem of linear drift-Alfvén modes in a collisional plasma using the transport equations including the nonlinear and linear viscosities and the perpendicular heat conductivity. Then, we have considered specifically the case of small-scale modes only, so that the local approximation could be used; see in detail Sec. V. In this approximation, the nonlinear viscosity contributes to the radial transport of the modes, i.e., to their radial group velocity, while the growth/decay rate is determined by the linear viscosity and perpendicular heat conductivity.

Note that our analysis of the drift-Alfvén modes generalizes the analysis of Ref. 10, addressed to the ion diamagnetic drift modes, by inclusion of the effects of finite parallel wave vector and the nonlinear viscosity. In addition to the modes unstable for negative ratio of temperature and pressure gradients, studied in Ref. 10, we have brought up the modes unstable for positive ratio of these gradients. We have shown that the wave packet of drift-Alfvén modes, propagating in the diamagnetic drift direction and driven for reversed temperature gradient, is transported down the pressure gradient. In contrast to this, the wave packet propagating in the electron diamagnetic drift direction and driven for positive temperature gradient is transported up the pressure gradient. We think that the instability studied can be of interest for the problem of interpretation of the edge localized modes (ELMs) in tokamaks.[13]

The recent nonlinear tokamak plasma theory[14] includes a rather wide trend of the ITG modes going back to Refs. 15–17 and many other works cited in Ref. 11. We have performed preliminary analysis of whether the nonlinear viscosity contributes to the problem of linear ITG modes. However, we have not revealed significant contributions.

Though we have applied the nonlinear viscosity in Secs. V and VI only for the case of slab geometry, there are problems that require knowing the viscosity tensor in general toroidal geometry. An example of such problems is the problem of plasma rotation in toroidal systems (see Refs. 18 and 19 and works cited therein). The role of nonlinear viscosity in this problem can be elucidated using the general-geometry results given in Sec. III.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, Grant No. 03-02-16294, by the Russian Federal Program on Support of Leading Scientific School Researches, Grant No. 2024.2003.2, by the Department of Atomic Science and Technology of the Russian Agency of Atomic Industry, by the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union, Grant BRHE REC-011, by the National Council of Scientific and Technological Development (CNPq), and by the State of São Paulo Research Foundation (FAPESP), Brazil.

APPENDIX: PERPENDICULAR LINEAR VISCOSITY

In correspondence with Eq. (21), in describing the viscosity effects, we should allow for, in addition to the nonlinear viscosity $\boldsymbol{\pi}^{(NL)}$, also the linear viscosity $\boldsymbol{\pi}^{(L)}$. For the problem of drift-Alfvén modes in the simplest magnetic field geometry, considered in Sec. V, we need the tensor components $\pi_{\alpha\beta}^{(L)}$, where $(\alpha,\beta)=(x,y)$. Similarly to Refs. 2–4 and 10, we separate this tensor into two parts, the oblique viscosity (gyroviscosity), denoted $\boldsymbol{\pi}^{(L)\wedge}$, independent of collisions, and the collisional perpendicular viscosity, denoted $\boldsymbol{\pi}^{(L)\perp}$, so that

$$\boldsymbol{\pi}^{(L)} = \boldsymbol{\pi}^{(L)\wedge} + \boldsymbol{\pi}^{(L)\perp}. \tag{A1}$$

The gyroviscosity tensor is given by

$$\pi_{xx}^{(L)\wedge} = -\pi_{yy}^{(L)\wedge} = -\frac{p_i}{2\omega_{ci}} W_{xy}^{(1)}, \tag{A2}$$

$$\pi_{xy}^{(L)\wedge} = \pi_{yx}^{(L)\wedge} = \frac{p_i}{4\omega_{ci}} (W_{xx}^{(1)} - W_{yy}^{(1)}), \tag{A3}$$

where $p_i = nT_i$,

$$W_{\lambda\mu}^{(1)} = \langle\langle \boldsymbol{\nabla}\mathbf{V}_{\perp i}\rangle\rangle_{\lambda\mu} + \frac{2}{5p_i}\langle\langle \boldsymbol{\nabla}\mathbf{q}_{\perp i}\rangle\rangle_{\lambda\mu}. \tag{A4}$$

The collisional part of the linear viscosity tensor is as follows:

$$\pi_{xx}^{(L)\perp} = -\pi_{yy}^{(L)\perp} = -\frac{3}{20}\frac{p_i\nu_i}{\omega_{ci}^2}\left[W_{xx}^{(1)} - W_{yy}^{(1)} + \frac{3}{10}(W_{xx}^{(2)} - W_{yy}^{(2)})\right], \tag{A5}$$

$$\pi_{xy}^{(L)\perp} = \pi_{yx}^{(L)\perp} = \frac{3}{10}\frac{p_i\nu_i}{\omega_{ci}^2}\left(W_{xy}^{(1)} + \frac{3}{10} W_{xy}^{(2)}\right), \tag{A6}$$

where

$$W_{\lambda\mu}^{(2)} = \frac{1}{p_i}\left\langle\left\langle \frac{\boldsymbol{\nabla}p_i}{p_i}\mathbf{q}_{\perp i} - \boldsymbol{\nabla}\mathbf{q}_{\perp i}\right\rangle\right\rangle_{\lambda\mu}. \tag{A7}$$

In contrast to Secs. II–IV, here we have restored the ion indices. The subscript $\perp$ at the vectors $\mathbf{V}_i$ and $\mathbf{q}_i$ means the components of these vectors perpendicular to the magnetic field. It is implied that, in substituting these vectors into the viscosity, they are taken in the form $\mathbf{V}_{\perp i} = \mathbf{V}_{\perp i}^{(0)}$, $\mathbf{q}_{\perp i} = \mathbf{q}_{\perp i}^{(0)} + \mathbf{q}_{\perp i}^{(1)}$, where

$$\mathbf{V}_{\perp i}^{(0)} = \frac{1}{n\omega_{ci}M_i}[\mathbf{h} \times \boldsymbol{\nabla} p_i] + \frac{c}{B}[\mathbf{E} \times \mathbf{h}], \tag{A8}$$

$$\mathbf{q}_{\perp i}^{(0)} = \frac{5}{2}\frac{p_i}{\omega_{ci}M_i}[\mathbf{h} \times \boldsymbol{\nabla} T_i], \tag{A9}$$

$$\mathbf{q}_{\perp i}^{(1)} = -\frac{2p_i\nu_i}{\omega_{ci}^2 M_i}\nabla_\perp T_i. \tag{A10}$$

Equations (A1)–(A10) are used in Sec. V in deriving Eq. (60). In addition, Eq. (A9) for $\mathbf{q}_{\perp i}^{(0)}$ is used for transition from Eq. (62) to Eq. (63) for Q by means of Eq. (56).

[1] ITER Physics Expert Groups on Confinement and Transport Modelling and Database *et al.*, Nucl. Fusion **39**, 2175 (1999).
[2] A. B. Mikhailovskii and V. S. Tsypin, Plasma Phys. **13**, 785 (1971).
[3] A. B. Mikhailovskii and V. S. Tsypin, Beitr. Plasmaphys. **24**, 335 (1984).
[4] A. B. Mikhailovskii, *Electromagnetic Instabilities in an Inhomogeneous Plasma* (Institute of Physics, Bristol, 1992).
[5] P. J. Catto and A. N. Simakov, Phys. Plasmas **11**, 90 (2004).
[6] A. N. Simakov and P. J. Catto, Contrib. Plasma Phys. **44**, 83 (2004).
[7] A. N. Simakov and P. J. Catto, Phys. Plasmas **12**, 012105 (2005).
[8] P. J. Catto and A. N. Simakov, Phys. Plasmas **12**, 012501 (2005).
[9] H. Grad, Commun. Pure Appl. Math. **2**, 311 (1949).
[10] V. S. Tsypin, Sov. Phys. Tech. Phys. **17**, 543 (1972).
[11] A. B. Mikhailovskii, *Theory of Plasma Instabilities*, Vol. 2, Instabilities of an Inhomogeneous Plasma (Consultants Bureau, New York, 1974).
[12] S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[13] J. W. Connor, Plasma Phys. Controlled Fusion **40**, 191 (1998).
[14] B. Coppi, Nucl. Fusion **42**, 1 (2002).
[15] L. I. Rudakov and R. Z. Sagdeev, Sov. Phys. Dokl. **6**, 415 (1961).
[16] A. B. Mikhailovskii, Sov. Phys. JETP **25**, 831 (1967).
[17] B. Coppi, M. N. Rosenbluth, and R. Z. Sagdeev, Phys. Fluids **10**, 582 (1967).
[18] A. B. Mikhailovskii and V. S. Tsypin, Sov. J. Plasma Phys. **10**, 142 (1984).
[19] V. S. Tsypin, A. B. Mikhailovskii, R. M. O. Galvão, I. C. Nascimento, M. Tendler, C. A. de Azevedo, and A. S. de Asis, Phys. Plasmas **5**, 3358 (1998).