Unified theory of Mercier-ballooning and Alfvén eigenmodes in positive-shear tokamaks with large-orbit energetic ions

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A unified theory of the Mercier-ballooning and the compensating-electron Alfvén eigenmodes (CEAEs) in positive-shear tokamaks with large-orbit energetic ions is developed. It is shown that the cross-field drift effect of electrons compensating the electric charge of energetic ions (the compensating-electron effect) leads to rotation of the Mercier-ballooning modes. If the Mercier stability criterion is satisfied, the ballooning modes rotate in the direction of compensating-electron frequency, while in violation of this stability criterion the unstable modes rotate against this frequency. The compensating-electron effect also results in decreasing the growth rate of ballooning modes, though their instability condition is unchanged. The Mercier and ballooning effects influence both rotation and decay rate of the CEAEs, the ballooning effect being smaller than the Mercier effect. As a result, rotation and damping of CEAEs increases/decreases in the case of magnetic well/hill. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1877519]

I. INTRODUCTION

Large-orbit energetic ions influence rather effectively the Alfvén eigenmodes in tokamak discharges.[1–4] In the case of reversed shear they lead to the reversed-shear Alfvén eigenmodes (RSAEs) (Refs. 1–3) whereas in the case of positive shear, the compensating-electron Alfvén eigenmodes (CEAEs) (Ref. 4) appear. At the same time, it is known that Mercier and ballooning modes can exist in tokamak plasmas when the effect of large-orbit energetic ions is not relevant (see, e.g., Chap. 8 of Ref. 5). The goal of the present paper is to develop a unified theory of Mercier-ballooning and Alfvén eigenmodes in positive-shear tokamaks with large-orbit energetic ions.

The standard theory to describe the effect of energetic particles on magnetohydrodynamics modes, which we refer to as energetic-particle modes[6,7] theory, is rather well known and widely employed. However, in a recent paper it has been shown that this theory is not self-consistent and should be replaced by the CEAE theory.[4] For the sake of completeness, additional argumentation on the basic error of Refs. 6 and 7 is presented in the Appendix, which also helps on understanding the motivation for appearance of the compensating-electron effect in the basic equations.

The paper is organized as follows. In Sec. II we present the starting equations. Section III is addressed to derivation of the dispersion relations. In Sec. IV we study the compensating-electron effect on ballooning and Mercier modes using the analytical theory of these modes with finite growth rates initially developed in Ref. 8 and then summarized in Ref. 5. Section V treats the Mercier and ballooning effects on CEAEs. Discussion of the results and conclusions are given in Sec. VI.

II. STARTING EQUATIONS

We work in the weak-ballooning approximation described in Sec. 7.5 of Ref. 5. Then, following Ref. 9, we introduce the ballooning variable y by formally replacing $\theta \to y$, where θ is the poloidal angle, while y, in contrast to θ, runs over an infinite range. Formally, in the weak-ballooning approximation, the ballooning representation looks like the radial Fourier representation with the radial wave number k_x connected with the ballooning variable by

$$k_x = nq' y, \tag{1}$$

where n is the toroidal mode number, q is the safety factor, and the prime is the radial derivative.

According to Sec. 8.5 of Ref. 5, neglecting the energetic ions and finiteness of the mode frequency, the current continuity equation, allowing for the Mercier-ballooning effects in the weak-ballooning approximation, is of the form

$$\frac{d}{dt}\left[(1+t^2)\frac{d\phi}{dt}\right] - \left(U_0 + \frac{U_1}{1+t^2}\right)\phi = 0. \tag{2}$$

Here $t=k_x/k_y$, $k_y=nq/r$, the r is the radial coordinate, ϕ is the electrostatic potential averaged over the metric oscillations (see, in detail, Sec. 7.5 of Ref. 5), U_0 and U_1 are the parameters characterizing the Mercier and ballooning effects, respectively. In the case of weak shear and parabolic profile of the thermal plasma pressure, the values U_0 and U_1 can be taken in the form [see Eqs. (8.17), (8.33), and (8.43) of Ref. 5]

$$U_0 = -\frac{4\beta_p r^2}{s^2 q^2 R^2}\left[1 - q^2\left(1 + \frac{3}{2}\beta_p^3\frac{r^2}{R^2} - \frac{e^2}{2}\beta_p + 6e\tau R\right)\right] \quad (3)$$

and

$$U_1 = \frac{8\beta_p^2 r^2}{s^2 R^2}\left(3\beta_p^2\frac{r^2}{R^2} - 2s\right). \quad (4)$$

Here β_p is the poloidal beta, $s=rq'/q$ is the shear, R is the major torus radius, e and τ are the parameters characterizing the ellipticity and triangularity of the magnetic surfaces (their definitions can be found in Sec. 2.5.1 of Ref. 5).

In addition to Eq. (1), we can take from Ref. 2 an equation for the Alfvén eigenmodes neglecting the Mercier-ballooning effects and allowing for the compensating-electron effect and finiteness of the mode frequency. In the coordinate representation, this mode equation has the form [see Eq. (55) of Ref. 2]

$$\frac{d}{dr}\left[\left(\frac{\omega^2}{v_A^2} - k_\parallel^2\right)\frac{d\phi}{dr}\right] - k_y^2\left(\frac{\omega^2}{v_A^2} - k_\parallel^2\right)\phi + \frac{4\pi e_h\omega k_y}{cB_0}\frac{dn_h}{dr}\phi = 0. \quad (5)$$

Here ω is the mode frequency [the time-dependence of the perturbation is taken in the form $\exp(-i\omega t)$, where t means time], $v_A^2 = B_0^2/(4\pi M_c n_c)$ is the squared Alfvén velocity, B_0 is the equilibrium magnetic field averaged over the poloidal oscillations, n_c is the number density of the thermal ions, M_c is the thermal ions mass, $k_\parallel = (m/q - n)/R$, m is the poloidal mode number, e_h is the electric charge of energetic ion, n_h is the number density of energetic ions, and c is the speed of light. For a monotonic q profile, Eq. (5) reduces to

$$\frac{d}{d\hat{x}}\left[\left(\frac{\omega^2}{\omega_A^2} - \hat{x}^2\right)\frac{d\phi}{d\hat{x}}\right] - \left(\frac{\omega^2}{\omega_A^2} - \hat{x}^2\right)\phi - Q_h\phi = 0. \quad (6)$$

Here $\hat{x} = xk_y$, x is the radial deviation from the corresponding singular magnetic surface, $\omega_A^2 = (sv_A/qR)^2$ is the squared characteristic Alfvén frequency, and Q_h describes the compensating-electron effect and is given by

$$Q_h = \omega/(4\omega_{CE}), \quad (7)$$

where ω_{CE} is the characteristic compensating-electron frequency defined by

$$\omega_{CE} = -\frac{1}{4}\frac{k_y\omega_A^2}{\Omega_h\kappa_h}\frac{n_c M_c}{n_h M_h}, \quad (8)$$

where $\kappa_h = d\ln n_h/dr$, Ω_h is the cyclotron frequency of energetic ions, and M_h is the energetic ion mass.

In the Fourier representation, Eq. (6) yields (see also comments in the Appendix)

$$\frac{d}{dt}\left[(1+t^2)\frac{d\phi}{dt}\right] + \frac{\omega^2}{\omega_A^2}(1+t^2)\phi + Q_h\phi = 0. \quad (9)$$

Combining Eqs. (2) and (9), we arrive at the mode equation of the form

$$\frac{d}{dt}\left[(1+t^2)\frac{d\phi}{dt}\right] + \frac{\omega^2}{\omega_A^2}(1+t^2)\phi + Q_h\phi - \left(U_0 + \frac{U_1}{1+t^2}\right)\phi = 0. \quad (10)$$

This is our starting mode equation in the ballooning (Fourier) representation. Using the explanations of Sec. 8.5.3 of Ref. 5, it can also be written in the coordinate representation. Then the term with U_1 looks as a nonlocal one.

III. DERIVATION OF DISPERSION RELATIONS

A. Solution of the mode equation in the inertialless region

Neglecting the inertial effect, i.e., the term with ω^2/ω_A^2, Eq. (10) reduces formally to Eq. (8.44) of Ref. 5, i.e., to

$$\frac{d}{dt}\left[(1+t^2)\frac{d\phi}{dt}\right] - \left[\nu(\nu+1) - \frac{b^2}{1+t^2}\right] = 0. \quad (11)$$

Here

$$\nu = -1/2 + i\alpha, \quad (12)$$

$$\alpha = (Q_h - U_0 - 1/4)^{1/2}, \quad (13)$$

$$b = (-U_1)^{1/2}. \quad (14)$$

According to Sec. 8.5 of Ref. 5, the even and odd solutions of Eq. (11), $\phi = \phi_\pm$, are given by

$$\phi_+ = (1+t^2)^{-b/2}F\left(-\frac{\nu+b}{2}, \frac{1+\nu-b}{2}; \frac{1}{2}; -t^2\right), \quad (15)$$

$$\phi_- = (1+t^2)^{-b/2}F\left(\frac{1-\nu-b}{2}, 1+\frac{\nu-b}{2}; \frac{3}{2}; -t^2\right), \quad (16)$$

where F is the hypergeometrical function. Similarly to Eqs. (8.48)–(8.50) of Ref. 5, taking $t \gg 1$ in Eqs. (15) and (16), we find that the short-wavelength inertialless asymptotic of the solutions $\phi_\pm$ in the Fourier space is of the form

$$\phi \sim t^\nu(1 + t^{-(2\nu+1)}\Delta_\pm), \quad (17)$$

where

$$\Delta_\pm = f_\pm(\nu,b)/f(\nu), \quad (18)$$

$$f_+(\nu,b) = \Gamma\left(\frac{1+\nu-b}{2}\right)\Gamma\left(\frac{1+\nu+b}{2}\right) \Big/ \left[\Gamma\left(-\frac{\nu+b}{2}\right)\Gamma\left(\frac{-\nu+b}{2}\right)\right], \quad (19)$$

$$f_-(\nu,b) = \Gamma\left(1+\frac{\nu-b}{2}\right)\Gamma\left(1+\frac{\nu+b}{2}\right) \Big/ \left[\Gamma\left(\frac{1-\nu-b}{2}\right)\Gamma\left(\frac{1-\nu+b}{2}\right)\right], \quad (20)$$

$$f(\nu) = \Gamma(\nu+1/2)/\Gamma(-\nu-1/2), \quad (21)$$

Γ is the gamma function.

B. Dispersion relation for the Mercier-ballooning modes modified by the compensating-electron effect

For Im $\omega > 0$ Eq. (10) in the inertial region ($t \gg 1$) takes the form

$$\frac{d}{dt}\left(t^2 \frac{d\phi}{dt}\right) - [\nu(\nu+1) + \lambda^2 t^2]\phi = 0, \tag{22}$$

where

$$\lambda^2 = -\omega^2/\omega_A^2. \tag{23}$$

According to Sec. 5.1.4 of Ref. 5, solution of Eq. (22) finite at $t \to \infty$ is of the form

$$\phi \sim t^{-1/2} K_{i\alpha}(\lambda t), \tag{24}$$

where $K_{i\alpha}$ is the Bessel function of the second kind of imaginary argument. We assume Re $\lambda \sim$ Im ω, so that the solution given by Eq. (24) decreases for $t \to \infty$.

The asymptotic of Eq. (24) for $\lambda t \ll 1$ (the long-wavelength inertial asymptotic) is

$$\phi \sim t^{\nu}[1 + (\lambda t/2)^{-(2\nu+1)} f(\nu)]. \tag{25}$$

Matching Eqs. (17) and (25) leads to the dispersion relation which coincides formally with Eq. (10.36) of Ref. 5,

$$(\lambda/2)^{2\nu+1} = f^2(\nu)/f_{\pm}(\nu,b). \tag{26}$$

This dispersion relation describes both the ballooning and Mercier modes.

C. Dispersion relation for the CEAEs modified by the Mercier-ballooning effects

1. Short-wavelength inertialless asymptotic in the coordinate space

As in the case Im $\omega > 0$, in order to derive dispersion relation for the modes with Im $\omega < 0$, we should match the short-wavelength inertialless asymptotic solution with the long-wavelength inertial one. In Ref. 4 the last asymptotic solution was obtained in the coordinate representation. Therefore, following the approach of Ref. 4, we transit in Eq. (17) from the Fourier space to the coordinate space. Such a transition can be performed using explanations of Sec. 6.3 of Ref. 5. As a result, we arrive at the following expression for the short-wavelength inertialless asymptotic in the coordinate space:

$$\phi_{\pm} \sim \hat{x}^{\nu}[\Delta_{\pm}^{\mathrm{b}} + \hat{x}^{-(2\nu+1)}]. \tag{27}$$

Here the parameters $\Delta_{\pm}^{b}$ are defined by

$$\Delta_{\pm}^{b} = \Delta \frac{f_{\pm}(\nu,b)}{f_{\pm}(\nu,0)}, \tag{28}$$

$$\Delta = 2^{-(2\nu+1)}/f(\nu), \tag{29}$$

the values $f_{\pm}(\nu,b)$ are given by Eqs. (19) and (20), while $f_{\pm}(\nu,0)$ are these values for $b=0$.

2. Solution of the mode equation in the inertial region

Taking $t \gg 1$ in Eq. (10) and transiting to the coordinate representation, we arrive at

$$\frac{d}{dz}\left[(1-z^2)\frac{d\phi}{dz}\right] + \nu(\nu+1)\phi = 0, \tag{30}$$

where

$$z = \hat{x}\omega_A/\omega. \tag{31}$$

Equation (30) coincides with the similar mode equation of Ref. 4 for generalization of the parameter ν by including the Mercier effect U_0, see Eqs. (12) and (13). A peculiarity of this equation is that for Im $\omega = 0$ it has the singular points $z = \pm 1$.

Similarly to Ref. 4, we express solution of Eq. (30) in terms of the Legendre functions $P_{\nu}(z)$ and $Q_{\nu}(z)$, so that

$$\phi = C_P P_{\nu}(z) + C_Q Q_{\nu}(z). \tag{32}$$

Here $C_P = (C_P^+, C_P^0, C_P^-)$, $C_Q = (C_Q^+, C_Q^0, C_Q^-)$ are numerical coefficients and the superscripts $(+,0,-)$ mean the regions $z > 1$, $-1 \leq z \leq 1$, $z < -1$. Then, we find that for $|z| \gg 1$ the function ϕ has the asymptotic (details of calculations can be found in Ref. 4)

$$\phi \sim z^{\nu}\left[1 + \frac{2^{-(2\nu+1)}\Gamma(-i\alpha)\Gamma(1+\nu)}{\Gamma(i\alpha)\Gamma(\nu)} \times \left(1 - i\pi \operatorname{th} \pi\alpha \frac{C_Q^+}{C_P^+}\right) z^{-(2\nu+1)}\right], \tag{33}$$

where the ratio C_Q^+/C_P^+ satisfies the relation

$$i\pi \frac{C_Q^+}{C_P^+} = -\operatorname{sgn}\omega(1 \mp \cos\pi\nu) \mp i\sin\pi\nu. \tag{34}$$

In obtaining Eq. (34), it is necessary to allow for the Landau bypass rule of the points $z = \pm 1$. The upper/lower signs in the right-hand side of Eq. (34) correspond to the even/odd solutions.

Requiring that Eqs. (27) and (33) are the same, we arrive at the dispersion relation

$$\left(\frac{\omega}{4\omega_A}\right)^{-2i\alpha} = \frac{f_{\pm}(\nu,b)}{f_{\pm}(\nu,0)} \frac{\Gamma^2(-i\alpha)\Gamma(1/2 + i\alpha)}{\Gamma^2(i\alpha)\Gamma(1/2 - i\alpha)} \times \left(1 - i\pi \operatorname{th}\pi\alpha \frac{C_Q^+}{C_P^+}\right). \tag{35}$$

This dispersion relation generalizes a similar dispersion relation of Ref. 4 by including the finiteness of the parameters U_0 and U_1.

As in Ref. 4, we are interested in the case $\alpha \ll 1$. Then Eq. (35) for the even modes reduces to

$$\frac{\omega}{\omega_A} = 16\exp\left\{-\frac{\pi l}{\alpha} - \gamma_E - \frac{1}{2}\left[\psi\left(\frac{1}{4} - \frac{b}{2}\right) + \psi\left(\frac{1}{4} + \frac{b}{2}\right) - 2\psi\left(\frac{1}{4}\right)\right] + \frac{\pi}{2} + \frac{i\pi}{2}\operatorname{sgn}\omega\right\}, \tag{36}$$

where γ_E is the Euler constant, $\psi = \Gamma'/\Gamma$ is logarithmic derivation of the gamma function, $l = 1,2,3,\ldots$. The dispersion relation of the odd modes for $\alpha \ll 1$ has a similar form with substitutions $\psi(1/4 \mp b/2) \to \psi(3/4 \mp b/2)$, $\psi(1/4) \to \psi(3/4)$, $\pi/2 \to -\pi/2$.

IV. COMPENSATING-ELECTRON EFFECT ON BALLOONING AND MERCIER MODES

A. Ballooning modes far from the Mercier stability boundary

According to Sec. 10.3.1 of Ref. 5 (see also Ref. 8), for ν not close to $-1/2$ and small $b-\nu-1$, Eq. (26) reduces to

$$(\lambda/2)^{2\nu+1} = (b-\nu-1)c_B, \tag{37}$$

where

$$c_B = \frac{\nu+1/2}{2\pi^{1/2}\Gamma(\nu+1)}\sin\left[\pi\left(\nu+\frac{1}{2}\right)\right]\Gamma^3\left(\nu+\frac{1}{2}\right). \tag{38}$$

For $U_0=0$, $Q_h=0$ and

$$b>1 \tag{39}$$

this dispersion relation describes aperiodically unstable ballooning modes (Re $\omega=0$) with the growth rate

$$\mathrm{Im}\,\omega \equiv \omega_I = \frac{\pi}{2}(b-1)\omega_A. \tag{40}$$

Let us analyze the compensating-electron effect on the modes considered. Then we assume Q_h to be small but nonzero, $Q_h \ll 1$, and, as before, take $U_0=0$. As a result, it follows from Eq. (37) that

$$\omega_I = \frac{\pi}{2}(b-1)\frac{\omega_A}{1+\pi^2\omega_A^2/(64\omega_{CE}^2)}, \tag{41}$$

$$\mathrm{Re}\,\omega \equiv \omega_R = \frac{\pi^2}{16}(b-1)\frac{\omega_A^2/\omega_{CE}}{1+\pi^2\omega_A^2/(64\omega_{CE}^2)}. \tag{42}$$

Hence it can be seen that the stability boundary of ballooning modes remains unchanged in the presence of compensating-electron effect. The consequences of this effect are, first, decreasing the growth rate, and second, appearance of a real part of the mode frequency. These consequences are essential for, qualitatively,

$$\omega_{CE} < \omega_A. \tag{43}$$

According to Eq. (42), the modes rotate in the direction of compensating-electron frequency ω_{CE}.

For $\omega_{CE}/\omega_A \ll 1$, Eqs. (41) and (42) reduce to

$$\omega_I = \frac{32}{\pi}(b-1)\frac{\omega_{CE}^2}{\omega_A}, \tag{44}$$

$$\omega_R = 4(b-1)\omega_{CE}. \tag{45}$$

We see that in this limiting case the growth rate is small compared with the real part of mode frequency as $(\omega_{CE}/\omega_A)^2$.

B. Ballooning modes near the Mercier stability boundary

Similar to Sec. 10.3.2 of Ref. 5, we now take

$$b = 1/2 + \delta, \tag{46}$$

$$\nu = -1/2 + \beta, \tag{47}$$

where δ is a small positive number, $0<\delta\ll 1$, while β is defined by

$$\beta = \left(\frac{1}{4}+U_0-Q_h\right)^{1/2}. \tag{48}$$

It is assumed that for $Q_h \to 0$ the parameter β is real, so that the Mercier stability criterion is satisfied.

As in Ref. 5, we transform Eq. (26) to the form similar to Eq. (10.49) of Ref. 5:

$$\left(\frac{\lambda}{2}\right)^{2\beta} = \left\{1+\beta\left[3\psi(1)-\psi\left(\frac{1}{2}\right)\right]\right\}\frac{1-\beta/\delta}{1+\beta/\delta}. \tag{49}$$

As in Ref. 5, we are interested in the modes with small ratio β/δ. However, in contrast to Ref. 5, allowing Q_h we can not transit to the limit $\beta \to 0$. Then we introduce

$$\beta_0 = \left(\frac{1}{4}+U_0\right)^{1/2} \tag{50}$$

and, assuming Im $\omega \gg$ Re ω and $|Q_h| \ll \beta_0^2$, expand β in a series in Q_h:

$$\beta = \beta_0 + i\beta_I, \tag{51}$$

where

$$\beta_I = -\frac{\omega_I}{8\omega_{CE}\beta_0}. \tag{52}$$

As a result, we arrive at the same expression for ω_I as Eq. (10.50) of Ref. 5,

$$\omega_I = 4\omega_A \exp\left(-\frac{1}{\delta}-\gamma_E\right), \tag{53}$$

and following expression for ω_R:

$$\omega_R = \frac{\omega_I^2}{16\delta\beta_0^2\omega_{CE}}. \tag{54}$$

We see from Eq. (54) that, as in Sec. IV A, the modes rotate in the direction of ω_{CE}. The value ω_{CE} should be sufficiently large compared with ω_A since otherwise the condition $\omega_R/\omega_I \ll 1$ would be invalid.

C. Mercier modes neglecting the ballooning effects

Now we assume the ballooning effect to be negligible, $b=0$, and the Mercier stability criterion to be violated, $1/4+U_0<0$, so that the Mercier modes are unstable. This means that, in neglecting the compensating-electron effect, $Q_h \to 0$, the parameter α is real. We designate this limiting value of the parameter α as α_0, so that

$$\alpha_0 = (-U_0-1/4)^{1/2}. \tag{55}$$

Assuming $|\alpha| \ll 1$, we reduce Eq. (26) for the even modes to the form of Eq. (5.50) of Ref. 5, i.e., to

$$\left(\frac{\lambda}{2}\right)^{2i\alpha} = 1 + 2i\alpha\left[\psi(1) - \psi\left(\frac{1}{4}\right)\right]. \quad (56)$$

Similar to expansion of the parameter β in Sec. IV B, we expand the parameter α in the series in Q_h assuming $\omega_I \gg \omega_R$. Then we have

$$\alpha = \alpha_0 + \frac{i}{8}\frac{\omega_I}{\omega_{CE}}. \quad (57)$$

We transform the left-hand side of Eq. (56) to

$$\left(\frac{\lambda}{2}\right)^{2i\alpha} = \left(\frac{\omega_I}{2\omega_A}\right)^{2i\alpha_0}\left[1 + \frac{2\alpha_0\omega_R}{\omega_I} - \frac{\omega_I}{4\omega_{CE}}\ln\left(\frac{\omega_I}{2\omega_A}\right)\right]. \quad (58)$$

Substituting Eq. (58) into Eq. (56) and following the procedure of Sec. 5.1.3 of Ref. 5, we obtain that the growth rate of the modes is given by Eq. (5.36) of Ref. 5, i.e.,

$$\omega_I = 16\omega_A \exp\left(-\frac{\pi l}{\alpha_0} - \gamma_E + \frac{\pi}{2}\right), \quad (59)$$

while the real part of the mode frequency is equal to

$$\omega_R = -\frac{\pi l \omega_I^2}{8\alpha_0^2\omega_{CE}}. \quad (60)$$

Note that Eq. (60) can be obtained from the condition that the second and third terms in the square brackets on the right-hand side of Eq. (58) are mutually canceled.

It follows from Eq. (60) that, in contrast to the ballooning modes (see Secs. IV A and IV B), the Mercier modes rotate in the direction against ω_{CE}. Similar to Eq. (54), Eq. (60) is valid only for sufficiently large ω_{CE}/ω_A (the requirement $\omega_R/\omega_I \ll 1$ should be satisfied).

V. MERCIER AND BALLOONING EFFECTS ON CEAES

Now we assume that both the ballooning and Mercier modes are stable and consider the role of the finite parameters U_0 and U_1 in the problem of CEAEs. We analyze Eq. (36) similarly to Ref. 4. Then we take $\omega=\omega_R+i\omega_I$, $\alpha=\alpha_R+i\alpha_I$, where ω_R and ω_I were introduced in Sec. IV A, while α_R and α_I are real. According to Eqs. (7), (8), and (13), the parameter α_R is related to ω_R by

$$\alpha_R = \frac{1}{2}\left(\frac{\omega_R}{\omega_{CE}} - 1 - 4U_0\right)^{1/2}. \quad (61)$$

It follows from Eq. (36) that, approximately,

$$\frac{\omega_R}{\omega_A} = 16\exp\left\{-\frac{\pi l}{\alpha_R} - \gamma_E + \frac{\pi}{2} - \frac{1}{2}\left[\psi\left(\frac{1}{4} - \frac{b}{2}\right) + \psi\left(\frac{1}{4} + \frac{b}{2}\right) - 2\psi\left(\frac{1}{4}\right)\right]\right\}, \quad (62)$$

$$\omega_I = -\frac{4\alpha_R^3}{l}|\omega_{CE}|. \quad (63)$$

In obtaining Eq. (62) we have used that contribution of the ballooning effect in this equation is real even for imaginary b. Since this contribution is small compared with $1/\alpha_R$, we neglect it below.

Assuming the ratio ω_{CE}/ω_A to be small parameter, we can use the method of successive approximations in the small parameter $[\ln(\omega_A/\omega_{CE})]^{-1}$. Then we find

$$\omega_R = \omega_R^{(0)} + \omega_R^{(1)}, \quad (64)$$

$$\alpha_R = \alpha_R^{(1)}, \quad (65)$$

where

$$\omega_R^{(0)} = (1 + 4U_0)\omega_{CE}, \quad (66)$$

$$\omega_R^{(1)} = \frac{4\pi^2 l^2 \omega_{CE}}{[\ln(16\omega_A/\omega_R^{(0)})]^2}, \quad (67)$$

$$\alpha_R^{(1)} = \frac{\pi l}{\ln(16\omega_A/\omega_R^{(0)})}. \quad (68)$$

Since $1/4+U_0>0$ it follows from Eq. (66) that the sign of the real part of the mode frequency for $U_0\neq 0$ is the same as for $U_0=0$. This part of the mode frequency increases for $U_0>0$ (the case of stabilizing curvature or, in other words, the case of magnetic well) and decreases for $U_0<0$ (the case of destabilizing curvature or the case of magnetic hill).

Using Eqs. (64)–(68), Eq. (63) is transformed to

$$\omega_I = -\frac{4\pi^3 l^2}{[\ln(16\omega_A/\omega_R^{(0)})]^3}|\omega_{CE}|. \quad (69)$$

According to Eq. (69), the damping of the CEAEs increases in the case of magnetic well, $U_0>0$, and decreases in the case of magnetic hill, $U_0<0$. This result is in agreement with the general notion of the stabilizing/destabilizing role of the magnetic well/hill.

VI. DISCUSSIONS AND CONCLUSIONS

The subject of our study was eigenmodes in a thermal plasma in positive-shear tokamaks in the presence of energetic ions under the assumption that the characteristic Larmor radius of these ions ρ_h is large compared with the characteristic perpendicular wavelength of the modes, $k_\perp\rho_h \gg 1$, so that direct effects of energetic ions can be neglected. Nevertheless, their presence is crucial for our analysis because of the presence of electrons compensating the equilibrium electric charge of energetic ions. These electrons contribute into the mode equation due to their cross-field drift effect called in our presentation as the compensating-electron effect. This effects causes appearance of a family of the CEAEs pointed out in Ref. 4, which are damped eigenmodes rotating in the direction of the compensating electron frequency ω_{CE}.

In allowing for the Mercier-ballooning effects, i.e., phenomena due to combined action of the equilibrium magnetic field curvature and the thermal plasma pressure gradient, we deal with two families of eigenmodes: the Mercier-ballooning ones and the CEAEs. It is known that, in the absence of energetic ions and in neglecting the diamagnetic drift effect of the thermal plasma, the Mercier-ballooning eigenmodes are aperiodically unstable perturbations. We have developed an analytical theory allowing one to yield benchmarks for problems what is influence of compensating-electron effect on the Mercier-ballooning modes and what is modification of the CEAEs in the presence of the Mercier-ballooning effects.

According to our analysis, the compensating-electron effect leads to the appearance of the real part of the frequency of the ballooning-Mercier modes, i.e., to rotation of these unstable modes. It has been shown that, if the Mercier stability criterion is satisfied, the ballooning modes rotate in the direction of the compensating-electron frequency ω_{CE}, while in violation of this stability criterion the unstable modes rotate against ω_{CE}. The compensating-electron effect results also in decreasing the growth rate of ballooning modes, though the instability condition of them remains unchanged. As for the CEAEs, the Mercier and ballooning effects influence both real part of their frequency and decay rate. The ballooning effects on the CEAEs proves to be small compared with the Mercier effect. The last leads to increasing/decreasing the rotation frequency in the cases of magnetic well/hill. Similarly, the magnetic well/hill lead to increasing/decreasing of the CEAEs. In principle, the described analytical regularities of the Mercier-ballooning modes and CEAEs can be studied numerically by means of the MISHKA-H code.[1]

As known (see, e.g., Chap. 21 of Ref. 5), the diamagnetic drift effect also leads to rotation of the Mercier-ballooning modes. Therefore, it seems reasonable to incorporate the diamagnetic drift effect into the theory described above. One more interesting problem is the generalization of our theory for arbitrary $k_\perp \rho_h$. In the scope of this problem, allowing for direct contribution of energetic ions into the mode equation seems to be necessary. Evidently, a similar program can also be of interest for reversed-shear tokamaks. Thus, we believe that our paper, together with Refs. 1–4, can be considered as the first links in the chain of subsequent studies of Alfvén eigenmodes in tokamaks in the presence of energetic ions.

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APPENDIX: EXPLANATION OF THE ESSENCE OF ERROR OF REFS. 6 AND 7

The compensating-electron effect is the cylindrical effect, i.e., that revealed in the cylindrical approximation. In this approximation and neglecting the Mercier effect, Eq. (2) takes the form of Eq. (9). Meanwhile, turning to Eq. (1) of Ref. 6 or to Eq. (2) of Ref. 7, we see that these equations in the same approximation are of the form

$$\left(\frac{d}{d\theta}f\frac{d}{d\theta}+\Omega^2 f\right)\delta\phi=0. \tag{A1}$$

Here $\delta\phi$ is the same as our ϕ, θ is the same as our ballooning variable y, $f=1+s^2\theta^2$, i.e., in our definitions,

$$f=1+t^2, \tag{A2}$$

$\Omega^2=\omega^2/\omega_A^2$ (in neglecting the diamagnetic drift frequency of the thermal plasma). Thus, substituting $\delta\phi\to\phi$, Eq. (A1) can be rewritten in the form

$$\frac{d}{dt}\left[(1+t^2)\frac{d\phi}{dt}\right]+\frac{\omega^2}{\omega_A^2}(1+t^2)\phi=0. \tag{A3}$$

We see that the only difference between our Eq. (9) and Eq. (A3) used by Refs. 6 and 7 is the term with Q_h in Eq. (9). Therefore, we should explain why Eq. (A3) does not contain this term.

Now we elucidate which starting equation has been used in Refs. 6 and 7 for obtaining Eq. (A1). Then we conclude that such a starting equation is Eq. (1) of Ref. 10. Neglecting the toroidicity and the temperature effects of the thermal plasma, this equation has the form

$$-\omega^2\rho_m\boldsymbol{\xi}=c^{-1}(\delta\mathbf{j}\times\mathbf{B}+\mathbf{j}\times\delta\mathbf{B}). \tag{A4}$$

Here $\boldsymbol{\xi}$ is the fluid displacement vector related to the perturbed perpendicular electric field $\delta\mathbf{E}_\perp=i\omega\boldsymbol{\xi}\times\mathbf{B}/c$, $\mathbf{B}$ is the equilibrium magnetic field, ρ_m is the mass density of the thermal plasma, $\mathbf{j}$ is the equilibrium electric current, $\delta\mathbf{j}$ and $\delta\mathbf{B}$ are the perturbed electric current and the perturbed magnetic field, respectively.

One can justify oneself that using Eq. (A4) leads to Eq. (A3). Therefore, we should elucidate whether Eq. (A4) is valid for the description of the modes considered. Then we allow for the following explanation of Ref. 6.

"Our analysis contains effects due to finite-size orbits of both circulating and trapped energetic particles, such as Larmor radii, magnetic drift orbits, and banana widths, which render energetic-particle dynamics effective only in the ideal region instead of the extended inertial region as in the case of negligible orbit sizes."

In this connection, our question can be reformulated as follows: whether Eq. (A4) is valid for description of the effects of finite-size orbits of energetic particles. Now we allow for the explanation of Ref. 10 that Eq. (A4) has been obtained by summing of the equations of motion for each species. Then we note that the equation of motion of energetic ions for finite Larmor radius in the cylindrical approximation is qualitatively of the form

$$0=e_h n_h J_0^2(k_\perp\rho_h)\delta\mathbf{E}_\perp+\cdots, \tag{A5}$$

where J_0 is the Bessel function, ρ_h is an effective Larmor radius of energetic ions, and the ellipses means other terms of this equation unimportant for our discussion. In addition, we should allow for that there are groups of electrons compensating the equilibrium charge of energetic ions. Their equation of motion can be taken in the form

$$0 = - e_h n_h \delta \mathbf{E}_\perp + \cdots , \tag{A6}$$

where the ellipses means the terms unimportant for our discussion. After summing Eqs. (A5) and (A6), and the equation of motion of the thermal plasma we obtain, instead of Eq. (A4)

$$- \omega^2 \rho_m \boldsymbol{\xi} = - e_h n_h (1 - J_0^2(k_\perp \rho_h)) \delta \mathbf{E}_\perp + c^{-1} (\delta \mathbf{j} \times \mathbf{B} + \mathbf{j} \times \delta \mathbf{B}). \tag{A7}$$

In accordance with Refs. 6 and 7, we are interested in the modes with $k_\perp \rho_h \gg 1$. Then Eq. (A7) reduces to

$$- \omega^2 \rho_m \boldsymbol{\xi} = - e_h n_h \delta \mathbf{E}_\perp + c^{-1} (\delta \mathbf{j} \times \mathbf{B} \times \mathbf{j} \times \delta \mathbf{B}). \tag{A8}$$

Thus, we see that, in contrast to Eq. (A4), Eq. (A8) contains an additional term with $\delta \mathbf{E}_\perp$. It is the term that leads to the term with Q_h in Eq. (9).

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