Nonextensive statistical mechanics and central limit theorems I - Convolution of independent random variables and q-product

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Abstract.
In this article we review the standard versions of the Central and of the Lévy-Gnedenko Limit Theorems, and illustrate their application to the convolution of independent random variables associated with the distribution $\mathscr{G}_q(X) \equiv \mathscr{A}_q \left[1+(q-1)\mathscr{B}_q(X-\bar{\mu}_q)^2\right]^{\frac{1}{1-q}}$ $(\mathscr{A}_q > 0;\, \mathscr{B}_q > 0;\, q < 3)$, known as q-Gaussian. This distribution emerges upon extremisation of the nonadditive entropy $S_q \equiv k\left(1-\int [p(X)]^q\, dX\right)/(1-q)$, basis of nonextensive statistical mechanics. It has a finite variance for $q < 5/3$, and an infinite one for $q \geq 5/3$. We exhibit that, in the case of (standard) independence, the q-Gaussian has either the Gaussian (if $q < \frac{5}{3}$) or the α-stable Lévy distributions (if $q > \frac{5}{3}$) as its attractor in probability space. Moreover, we review a generalisation of the product, the q-product, which plays a central role in the approach of the specially correlated variables emerging within the nonextensive theory.

Keywords: central limit theorem, independence, nonextensive statistical mechanics
PACS: 02.30.-f, 02.50.-r, 05.40.-a

INTRODUCTION

Science, whether in its pure or applied form, is frequently related to the description of the behaviour exhibited by systems when some quantity approaches a particular value. We might consider the effect on the state of a particle as we furnish it a certain energy, E, when this quantity tends to some critical value, E_c, or the response of a system when the number of elements goes to infinity, like it usually occurs in thermodynamics and statistical mechanics, i.e., the *thermodynamic limit*. In the latter case, we may focus on the outcome of the addition, or arithmetic average, of a large (infinite) sequence of random variables associated with a certain observable. This constitutes the basis of the celebrated central limit theorem (CLT), which is at the core of the theory of probabilities and mathematical statistics. The CLT has its origin at the weak law of large numbers of JACOB BERNOULLI [1]. For independent random variables, it had its first version introduced by ABRAHAM DE MOIVRE in 1733 [2], who used the normal distribution to approximate the functional form of a binomial distribution for a large number of events. In 1812, his result was later extended by PIERRE-SIMON LAPLACE, who formulated the now called *Theorem of de Moivre-Laplace* [3]. Laplace also used the normal distribution in the analysis of errors in experiments, but it was CARL FRIEDRICH GAUSS who first proved, in 1809, the connection between error in measurement and the

CP965, *Complexity, Metastability and Nonextensivity –CTNEXT 07*,
edited by S. Abe, H. Herrmann, P. Quarati, A. Rapisarda, and C. Tsallis

normal distribution. It is due to this relation that the normal distribution is widely called in Physics as *Gaussian distribution*. Although discovered more than once by different people, the fact is that only in 1901, mathematician ALEKSANDR LYAPUNOV defined the CLT in general terms and proved it in a precisely mathematical fashion [4, 5].

After the establishment of a central limit theorem for the addition of independent random variables with finite second-order moment, other versions have appeared, namely, the *Lévy-Gnedenko extension* for the sum of independent random variables with diverging second-order moment [6, 7], the *m-dependent central limit theorem*, *martingale central limit theorem* [8], the *central limit theorem for mixing processes* among others [9–11, 14–20].

In this article, we review the fundamental properties of nonadditive entropy, S_q, its optimising distribution (known as q-Gaussian), and the q-product, a generalisation of the product [21, 22] formulated within nonextensive statistical mechanics. We analyse, both analytically and numerically, the sum of conventional independent random variables and show that, in this case, the attractor in probability space is the Gaussian distribution if random variables have a finite second-order moment, or the α-stable Lévy distributions otherwise.

NONADDITIVE ENTROPY S_q

Statistical mechanics, *i.e.*, the application of statistics to large populations whose state is governed by some Hamiltonian functional, is strongly attached to the concept of entropy originally introduced by RUDOLF JULIUS EMMANUEL CLAUSIUS in 1865 [23]. The relation between entropy and the number of allowed microscopic states was firstly established by LUDWIG EDUARD BOLTZMANN in 1877 when he was studying the approach to equilibrium of an ideal gas [24]. Mathematically, this relation is,

$$S = k \ln W, \tag{1}$$

where k is a positive constant and W the number of microstates compatible with the macroscopic state. This equation is known as *Boltzmann principle*.

When a system is not isolated, but instead in contact with some kind of reservoir, it is possible to derive, from Eq. (1) under some assumptions, the Boltzmann-Gibbs entropy, $S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i$, where p_i is the probability of microscopic configuration i [25]. Boltzmann-Gibbs statistical mechanics is based on the molecular chaos [24] and ergodic [26] hypotheses [27]. It has been very successful in the treatment of systems in which *short* spatio/temporal interactions dominate. In this case, ergodicity and independence are justified and Khinchin's approach to S_{BG} is valid [26]. Therefore, it appears as entirely plausible that physical entropies other than the Boltzmann-Gibbs one, can be defined in order to treat anomalous systems, for which ergodicity and/or independence are not verified.

Inspired by this kind of systems it was proposed in 1988 [28] the entropy $S_q \equiv k\left(1 - \sum_{i=1}^{W} p_i^q\right) / (1-q)$ $(q \in \Re;\ \lim_{q\to 1} S_q = S_{BG})$ as the basis of a possible extension of Boltzmann-Gibbs statistical mechanics [29, 30] where the *entropic index* q should be determined *a priori* from microscopic dynamics. Just like S_{BG}, S_q is *nonnegative*, *concave* ($\forall q > 0$), *experimentally robust* (or *Lesche-stable* [31]) ($\forall q > 0$), composable, and leads to a *finite entropy production per unit time* [32, 33]. Moreover, it has been shown that it is also *extensive* [15, 34, 35, 38], hence in compliance with Clausius concept on macroscopic entropy and thermodynamics, for a special class of *correlated* systems. More precisely, systems whose phase-space is occupied in a (asymptotically) scale-invariant manner. It is upon this kind of correlations that the q-generalised Central Limit Theorems are constructed.

At this stage let us emphasize the difference between the *additivity* and *extensivity* concepts for entropy [1]. An entropy is said to be *additive* [36] if for two *probabilistically independent* systems, let us say A and B, the total entropy equals the sum of the entropies for the two independent systems, *i.e.*, $S(A+B) = S(A) + S(B)$. According to this definition, Boltzmann-Gibbs entropy, S_{BG}, and Rényi entropy, S_α^R [37], $S_\alpha^R = \frac{1}{1-\alpha} \ln [\sum_{i=1}^n p_i^\alpha]$, are *additive*, while S_q ($q \neq 1$), among others [11], is *nonadditive*. Despite the fact of being nonadditive, S_q, just as additive entropies S_{BG} and S_α^R, is *composable*, as already mentioned. By this we mean that, for a system composed by two independent subsystems, A and B, if we know the entropy of each sub-system, then we are able to evaluate the entropy of the entire system. Composability for S_{BG} and S_α^R is a consequence of its additivity, whereas for S_q it results from the fact that, considering independent subsystems, the total entropy satisfies

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\,\frac{S_q(A)}{k}\frac{S_q(B)}{k}\,. \tag{2}$$

On the other hand, an entropy is defined as *extensive* whenever the condition,

$$\lim_{N\to\infty} \frac{S(N)}{N} = s \in (0,\infty), \tag{3}$$

is verified (N represents the number of elements of the system). In this definition, the correlation between elements of the system is arbitrary, *i.e.*, *it is not important to state whether they are independent or not*. If the elements of the system are independent (e.g., the ideal gas, the ideal paramagnet), then the additive entropies $S_1 = S_1^R = S_{BG}$ and S_α^R ($\forall \alpha$) are extensive, whereas the nonadditive entropy S_q ($q \neq 1$) is nonextensive. In such case we have

$$s = S(1)\,, \tag{4}$$

where $S(1)$ is the entropy of one element when it is considered as isolated. Furthermore, for short-range interacting systems, *i.e.*, whose elements are only asymptotically independent (*e.g.*, air molecules at normal conditions), i.e., strict independence is now violated, S_{BG} and S_α^R still are extensive, whereas S_q ($q \neq 1$) still is nonextensive. For such systems, $s_{BG} \equiv \lim_{N\to\infty} S_{BG}(N)/N \neq S(1)$. Conversely, for subsystems of systems exhibiting long-range correlations [12], S_{BG} and S_α^R are *nonextensive*, *whereas* S_q *can be*

[1] In this discussion we will treat elements of a system as strictly identical and distinguishable.

extensive for an appropriate value of the entropic index $q \neq 1$. This class of systems has been coined as *q-describable* [13].

Optimising S_q

Let us consider the continuous version of the nonadditive entropy S_q, *i.e.*,

$$S_q = k \frac{1 - \int [p(X)]^q \, dX}{1-q}. \tag{5}$$

The natural constraints in the maximisation of (5) are (hereinafter $k = 1$), $\int p(X) \, dX = 1$, corresponding to normalisation, and

$$\int X \frac{[p(X)]^q}{\int [p(X)]^q dX} \, dX \equiv \langle X \rangle_q = \bar{\mu}_q, \tag{6}$$

$$\int (X - \bar{\mu}_q)^2 \frac{[p(X)]^q}{\int [p(X)]^q dX} \, dX \equiv \left\langle (X - \bar{\mu}_q)^2 \right\rangle_q = \bar{\sigma}_q^2, \tag{7}$$

corresponding to the *q-generalised* mean and variance of X, respectively.

From the variational problem we obtain

$$\mathscr{G}_q(X) = \mathscr{A}_q \left[1 + (q-1)\, \mathscr{B}_q \left(X - \bar{\mu}_q\right)^2\right]^{\frac{1}{1-q}}, \qquad (q < 3), \tag{8}$$

(if the quantity within brackets is nonnegative, and zero otherwise) where,

$$\mathscr{A}_q = \begin{cases} \dfrac{\Gamma\left[\frac{5-3q}{2-2q}\right]}{\Gamma\left[\frac{2-q}{1-q}\right]} \sqrt{\frac{1-q}{\pi} \mathscr{B}_q} & \Leftarrow \quad q < 1 \\ \sqrt{\frac{\mathscr{B}_q}{\pi}} & \Leftarrow \quad q = 1 \\ \dfrac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2q-2}\right]} \sqrt{\frac{q-1}{\pi} \mathscr{B}_q} & \Leftarrow \quad q > 1 \end{cases}, \tag{9}$$

and $\mathscr{B}_q = \left[(3-q)\, \bar{\sigma}_q^2\right]^{-1}$. Standard and generalised variances, $\bar{\sigma}_q^2$ and $\bar{\sigma}^2$ are related through $\bar{\sigma}_q^2 = \bar{\sigma}^2 \frac{5-3q}{3-q}$, for $q < \frac{5}{3}$.

Defining the *q-exponential* function [2] as

$$e_q^x \equiv [1 + (1-q)\, x]^{\frac{1}{1-q}} \qquad (e_1^x \equiv e^x), \tag{10}$$

($e_q^x = 0$ if $1 + (1-q)x \leq 0$) we can rewrite PDF (8) as

$$\mathscr{G}_q(x) \equiv \mathscr{A}_q \, e_q^{-\mathscr{B}_q (x - \bar{\mu}_q)^2}, \tag{11}$$

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[2] Other generalisations for the exponential function can be found at Ref. [39].

hereon referred to as *q-Gaussian*. The inverse function of the q-exponential, the q-logarithm, is $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$ $(x>0)$.

For $q = \frac{3+m}{1+m}$, the q-Gaussian recovers Student's t-distribution with m degrees of freedom ($m = 1,2,3,\ldots$) and finite moment up to order m. So, for $q>1$, PDF (11) presents an asymptotic *power-law* behaviour. Complementarily, if $q = \frac{n-4}{n-2}$ with $n = 3,4,5,\ldots$, $p(x)$ recovers the r-distribution with n degrees of freedom. Consistently, for $q<1$, $p(x)$ has a *compact support* defined by the condition $|x-\bar{\mu}_q| \le \sqrt{\frac{3-q}{1-q}\,\bar{\sigma}_q^2}$.

q-calculus

The nonadditivity property of S_q, assuming *for independent systems A* and B,

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k}, \tag{12}$$

has inspired the introduction of a new algebra [21, 22] composed by *q-sum*, $x \oplus_q y \equiv x+y+(1-q)xy$, and the *q-product*

$$x \otimes_q y \equiv \left[x^{1-q}+y^{1-q}-1\right]^{\frac{1}{1-q}}. \tag{13}$$

The corresponding inverse operations are the *q-difference*, $x \ominus_q y$, and the *q-division*, $x \oslash_q y$, such that, $(x \otimes_q y) \oslash_q y = x$. The q-product can be written by using the basic function of nonextensive formalism, the q-exponential, and its inverse, the q-logarithm. Hence, $x \otimes_q y \equiv \exp_q\left[\ln_q x + \ln_q y\right]$, which for $q \to 1$, recovers the usual property $\ln(x \times y) = \ln x + \ln y$ $(x,y>0)$, where $x \times y \equiv x \otimes_1 y$. Since $\exp_q[x]$ is a non-negative function, the q-product must be restricted to values of x and y that respect condition

$$|x|^{1-q} + |y|^{1-q} - 1 \ge 0 \tag{14}$$

We can enlarge the domain of the q-product to negative values of x and y by writing it as

$$x \otimes_q y \equiv \text{ sign}(xy)\exp_q\left[\ln_q|x| + \ln_q|y|\right]. \tag{15}$$

We list now a set of properties of the q-product:

1. $x \otimes_1 y = x\,y$;
2. $x \otimes_q y = y \otimes_q x$;
3. $(x \otimes_q y) \otimes_q z = x \otimes_q (y \otimes_q z) = x \otimes_q y \otimes_q z = \left[x^{1-q}+y^{1-q}+z^{1-q}-3\right]^{\frac{1}{1-q}}$;
4. $(x \otimes_q 1) = x$;
5. $\ln_q\left[x \otimes_q y\right] \equiv \ln_q x + \ln_q y$;
6. $\ln_q(xy) = \ln_q(x) + \ln_q(y) + (1-q)\ln_q(x)\ln_q(y)$;
7. $(x \otimes_q y)^{-1} = x^{-1} \otimes_{2-q} y^{-1}$;

8. $\left(x \otimes_q 0\right) = \begin{cases} 0 & \text{if } (q \geq 1 \text{ and } x \geq 0) \text{ or if } (q < 1 \text{ and } 0 \leq x \leq 1)\,, \\ \left(x^{1-q} - 1\right)^{\frac{1}{1-q}} & \text{if q} < 1\,\text{and}\,\text{x} > 1. \end{cases}$

For special values of q, *e.g.*, $q = 1/2$, the argument of the q-product can attain nonpositive values, specifically at points for which $|x|^{1-q} + |y|^{1-q} - 1 < 0$. In these cases, and consistently with the cut-off for the q-exponential we have set $x \otimes_q y = 0$. With regard to the q-product domain, and restricting our analysis of Eq. (14) to $x, y > 0$, we observe that for $q \to -\infty$ the region $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ leads to a vanishing q-product. As the value of q increases, the forbidden region decreases its area, and when $q = 0$ we have the limiting line given by $x + y = 1$, for which $x \otimes_0 y = 0$. Only for $q = 1$, the whole set of real values of x and y has a defined value for the q-product. For $q > 1$, condition (14) yields a curve, $|x|^{1-q} + |y|^{1-q} = 1$, at which the q-product diverges. This undefined region increases as q goes to infinity. At the $q \to \infty$ limit, the q-product is only defined in $\{x > 1, y \leq 1\} \cup \{0 \leq x \leq 1, 0 \leq y \leq 1\} \cup \{x \leq 1, y > 1\}$. This entire scenario is depicted on the panels of Fig. 1. The profiles presented by $x \otimes_\infty y$ and $x \otimes_{-\infty} y$ illustrate the above property (8). To illustrate the q-product in another simple form, we show, in Fig. 2, a representation of $x \otimes_q x$ for typical values of q.

LÉVY DISTRIBUTIONS

In the context of the CLT for independent variables, apart from the Gaussian distribution, $\mathscr{G}(X)$, another stable distribution plays a key role, the α-Lévy distribution, $L_\alpha^B(X)$ $(0 < \alpha < 2)$. If $\mathscr{G}(X)$ is characterised by its fast decay, $L_\alpha^B(X)$ is characterised by its 'fat' tails, since it allows both small and large values of X to be effectively measurable. Widely applied in several areas, $L_\alpha^B(X)$ is defined through its Fourier Transform [6],

$$\hat{L}_\alpha^B(k) \;=\; \exp\left[-a\,|k|^\alpha \left\{1 + i B \tan\left(\alpha \tfrac{\pi}{2}\right) \tfrac{k}{|k|}\right\}\right], \qquad (\alpha \neq 1)\,, \tag{16}$$

(where α is the *Lévy exponent,* and B represents the *asymmetry parameter*). By this we mean that Lévy distributions have no analytical form in X, excepting for special values of $\alpha = 1$. For $B = 0$, $L_\alpha^0 \equiv L_\alpha$, the distribution is symmetric. Regarding α values, one can verify that $L_{\frac{1}{2}}(x)$ corresponds to the *Lévy-Smirnov* distribution, and that $L_1(X)$ is the *Cauchy* or *Lorentz* distribution ($L_1(X)$ coincides with the $\mathscr{G}_2(X)$ distribution). For $\alpha = 2$, $\hat{L}_\alpha(k)$ has a Gaussian form, thus the corresponding distribution is a Gaussian.

Carrying out the inverse Fourier Transform on $\tilde{L}_\alpha(k)$, $L_\alpha(X) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikX}\,\hat{L}_\alpha(k)\,dk$, we can straightforwardly evaluate the limit for small X, $L_\alpha(X) \approx \left(\pi\,\alpha\,a^{a/\alpha}\right)^{-1}\Gamma\left[\frac{1}{\alpha}\right]$, and the limit for large X,

$$L_\alpha(X) \sim \frac{a\,\alpha}{\pi}\frac{\Gamma[\alpha]\sin\left[\frac{\pi\alpha}{2}\right]}{|X|^{1+\alpha}}, \qquad X \to \infty. \tag{17}$$

From condition, $0 < \alpha < 2$, it is easy to prove that variables associated with a Lévy distribution do not have a finite second-order moment, just like $\mathscr{G}_q(X)$ with $\frac{5}{3} \leq q < 3$

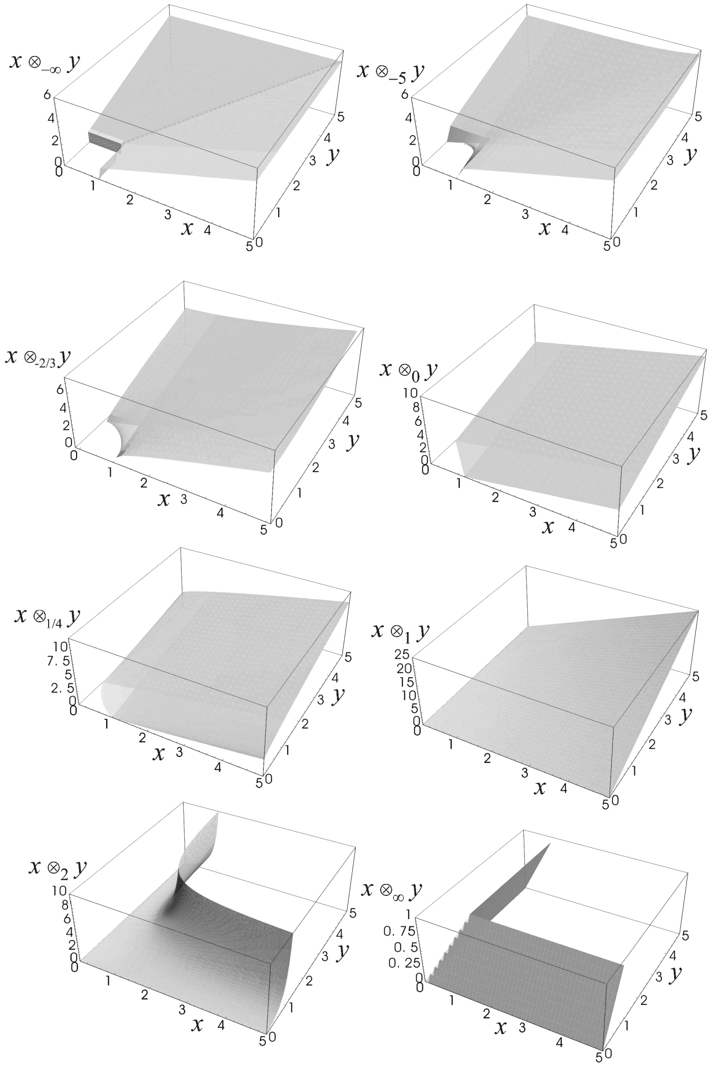

FIGURE 1. Representation of the q-product, Eq. (15), for $q = -\infty$, -5, $-2/3$, 0, $1/4$, 1, 2, ∞. As it is visible, the squared region $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ is gradually integrated into the nontrivial domain as q increases up to $q = 1$. From this value on, a new prohibited region appears, but this time coming from large values of $(|x|, |y|)$. This region reaches its maximum when $q = \infty$. In this case, the domain is composed by a horizontal and vertical strip of width 1.

(see [40] and references therein). In spite of the fact that we can write two distributions, $\mathscr{G}_q(X)$ and $L_\alpha(X)$, which present the same asymptotic power-law decay (with $\alpha = \frac{3-q}{q-1}$), there are interesting differences between them. The first one is that, as we shall see later

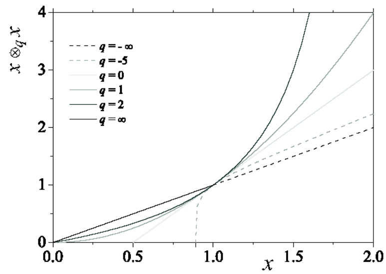

FIGURE 2. Representation of the q-product, $x \otimes_q x$ for $q = -\infty, -5, 0, 1, 2, \infty$. Excluding $q = 1$, there is a special value $x^* = 2^{1/(q-1)}$, for which $q < 1$ represents the lower bound [in figure $x^*(q = -5) = 2^{-1/6} \simeq 0.89089$ and $x^*(q = 0) = 1/2$], and for $q > 1$ the upper bound [in figure $x^*(q = 2) = 2$]. For $q = \pm\infty$, $x \otimes_q x$ lies on the diagonal of bisection, but following the lower and upper limits mentioned above.

on, $L_\alpha(X)$ together with $\mathscr{G}(X)$ are the only two stable functional forms whenever we convolute *independent* variables [20]. Therefore, distribution $\mathscr{G}_q(X)$ $(q \neq 1, 2)$ is not stable for this case. The other point concerns their representation in a log − log scale. Contrarily to what happens in log − log representations of $\mathscr{G}_q(X)$, an inflexion point exists at X_I for $L_\alpha(X)$ if $1 < \alpha < 2$, see Fig. 3. Since in many cases the numerical adjustment of several experimental/computational probability density functions for either a Lévy or a q-Gaussian distribution seems to be plausible, the presence of an inflexion point might be used as an extra criterion to conclude which one is the most adequate. This has clear implications on phenomena modelling.

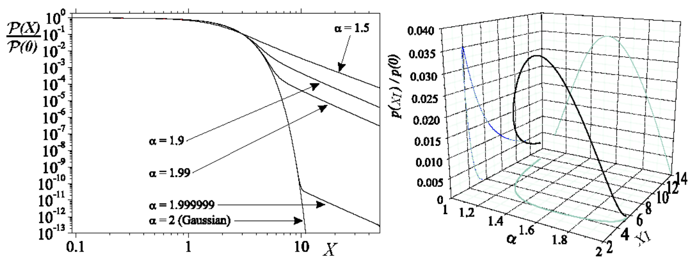

FIGURE 3. *Left panel*: Gaussian and α-stable Lévy distributions for α approaching 2 in Eq. (16) with $a = 1$ and $B = 0$. As referred in the text for values of α closer to 2, Lévy distribution becomes almost equal to a Gaussian up to some critical value for which the power law behaviour emerges. *Right panel*: Locus of the inflexion point of the α-stable Lévy distributions, Eq. (16), with $a = 1$ and $B = 0$. Contrarily to what happens with $\mathscr{G}_q(X)$, when Lévy distributions are represented in a log-log scale, they exhibit an inflexion point which goes to infinity as $\alpha \to 1$ (Cauchy-Lorentz distribution $\mathscr{G}_2(X)$) and $\alpha \to 2$ (Gaussian distribution $\mathscr{G}(X)$) too. We also show the projections onto the planes $\frac{p(X_I)}{p(0)} - X_I$, $\frac{p(X_I)}{p(0)} - \alpha$, and $\alpha - X_I$.

CENTRAL LIMIT THEOREMS FOR INDEPENDENT VARIABLES

Variables with finite variance

Let us consider a sequence $X_1, X_2, \ldots, X_N$ of random variables which are defined on the same probability space, share the same probability density function, $p(X)$, with $\sigma < \infty$, and are independent in the sense that the joint probability density function of any two X_i and X_j, $P\left(X_i, X_j\right)$, is just $p(X_i)\, p\left(X_j\right)$.

Hence [41, 42], a new variable

$$Y = \frac{X_1 + X_2 + \ldots + X_N}{N}, \tag{18}$$

with raw moments, $\langle Y^n \rangle \equiv \left\langle \frac{1}{N^n} \left(\sum_{j=1}^{N} X_j \right)^n \right\rangle$, has its probability density function given by the convolution of N probability density functions, or, since variables are independent, by the Fourier Transform,

$$\begin{aligned} \mathscr{F}\left[P(Y)\right](k) &= \frac{1}{2\pi} \left\{ \int e^{i k \frac{X}{N}} p(X)\, dX \right\}^N = \frac{1}{2\pi} \left\{ \int \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \frac{\langle X^n \rangle}{N} dX \right\}^N \\ &= \frac{1}{2\pi} \exp\left[N \ln \left[1 + ik \frac{\langle X \rangle}{N} - \frac{1}{2} k^2 \frac{\langle X^2 \rangle}{N^2} + O\left(N^{-3}\right) \right] \right]. \end{aligned} \tag{19}$$

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$, expanding up to second order in X we asymptotically have, in the limit $N \to \infty$,

$$\mathscr{F}\left[P(Y)\right](k) \approx \frac{1}{2\pi} \exp\left[N \left\{ + i k \frac{\langle X \rangle}{N} - \frac{k^2}{2N^2} \left(\left\langle X^2 \right\rangle - \langle X \rangle^2 \right) \right\} \right]. \tag{20}$$

Defining $\mu_X \equiv \langle X \rangle$, as the mean value, and $\sigma_X^2 \equiv \left\langle X^2 \right\rangle - \langle X \rangle^2$, as the standard deviation we have

$$\mathscr{F}\left[P(Y)\right](k) \approx \frac{1}{2\pi} \exp\left[-i k \mu_X - \frac{k^2}{2N} \sigma_X^2 \right] . \tag{21}$$

Performing the Inverse Fourier Transform for $\mathscr{F}\left[P(Y)\right](k)$, we obtain the distribution $P(Y)$,

$$P(Y) = \int e^{-ikY} \mathscr{F}\left[P(Y)\right](k)\, dk , \tag{22}$$

which yields,

$$P(Y) \approx \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma_X^2}{N}}} e^{-(Y-\mu_X)^2/(2\frac{\sigma_X^2}{N})} \tag{23}$$

Remembering that, from Eq. (18) $\mu_Y = \mu_X$ and $\sigma_Y = \sigma_X N^{-1/2}$ we finally get

$$P(Y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-(Y-\mu_Y)^2/(2\sigma_Y^2)} , \tag{24}$$

which is a Gaussian. It is also easy to verify that $P\left(Z = \frac{Y}{\sqrt{N}}\right) = \sqrt{N} P(Y)$, *i.e.*, the Gaussian distribution scales as $N^{-\frac{1}{2}}$.

Example: The convolution of independent q-variables

Let us assume that the probability density function $p(X)$ is a q-Gaussian distribution, $\mathscr{G}_q(X)$, which, as a simple illustration, has $q = \frac{3}{2}$, and $\sigma = 1 < \infty$ [3]. In this case, the Fourier Transform for $\mathscr{G}_{\frac{3}{2}}(X)$ is $\mathscr{F}\left[\mathscr{G}_{\frac{3}{2}}(X)\right](k) = (1+|k|)\exp[-|k|]$. The distribution, $\mathscr{P}(Y)$, where $Y = X_1 + X_2 + \ldots + X_N$, is given by

$$\mathscr{P}(Y) = \frac{1}{2\pi}\int \exp[-ikY]\left\{\mathscr{F}\left[\mathscr{G}_{\frac{3}{2}}(X)\right](k)\right\}^N dk \; . \tag{25}$$

Expanding $\left\{\mathscr{F}\left[\mathscr{G}_{\frac{3}{2}}(X)\right](k)\right\}^N$ around $k = 0$ we obtain, $\left\{\mathscr{F}\left[\mathscr{G}_{\frac{3}{2}}(X)\right](k)\right\}^N \simeq 1 - \frac{1}{2}Nk^2 + \frac{1}{3}N|k|^3$. From the CLT, we observe that distribution $\mathscr{P}(Y)$, for large N, is well described by a Gaussian, $\mathscr{G}(Y) \approx \frac{1}{\sqrt{2\pi N}}\exp\left[-\frac{Y^2}{2N}\right]$, at its central region (for N large). Because of singularity $|k|^3$, associated with the divergence in n-order statistical moments $(n \geq 3)$, we have the remaining distribution described by a power-law, $\mathscr{P}(Y) \sim \frac{2N}{\pi}Y^4$, for large Y. This specific behaviour is depicted in Fig. 4. Therein we observe a crossover from Gaussian to power law at $YN^{-1/2}$ of order $\sqrt{\ln N}$, which tends to infinity.

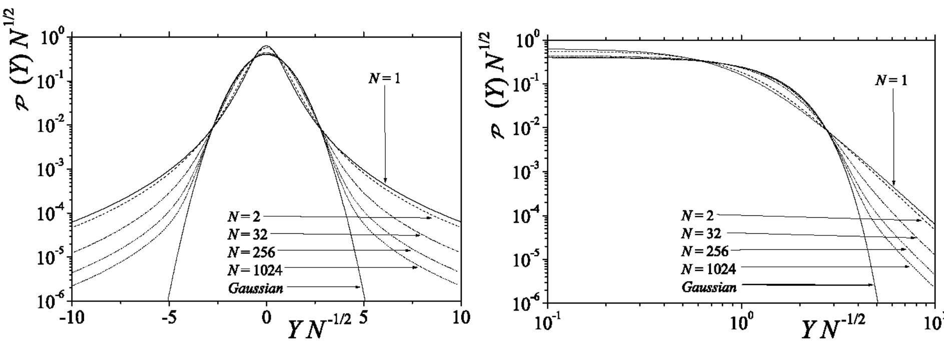

FIGURE 4. Both panels represent probability density function $\mathscr{P}(Y)$ *vs.* Y (properly scaled) in log-linear (left) and log-log (right) scales, where Y represents the sum of N independent variables X having a $\mathscr{G}_{\frac{3}{2}}(X)$ distribution. Since variables are independent and its variance is finite, $\mathscr{P}(Y)$ converges to a Gaussian as it is visible. It is also visible in the log-linear representation that, although the central part of the distribution approaches a Gaussian, the power-law decay subsists even for large N as it is depicted in log-log representation.

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[3] The value $q = \frac{3}{2}$ appears in a wide range of phenomena which goes from long-range hamiltonian systems [43–45] to economical systems [46].

Variables with infinite variance

Figure 4 is similar to what is presented in Fig. 3 (left panel). As it is visible there, as α goes to 2, the distribution $L_\alpha(X)$ nearly collapses onto the Gaussian distribution up to some critical value X^*. Beyond that point, the asymptotic power-law character emerges and the distribution falls as $|X|^{-\alpha-1}$.

If we consider the sum, $Y = X_1 + X_2 + \ldots + X_N$, of N random variables, $X_1, X_2, \ldots, X_N$, which share the same Lévy distribution, $L_\alpha(X)$. The distribution $\mathscr{P}(Y)$ is then given by,

$$\begin{aligned} \mathscr{P}(Y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikY}\left\{\tilde{L}_\alpha(k)\right\}^N dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[-ikY - aN|k|^\alpha\right] dk. \end{aligned} \tag{26}$$

Introducing a new variable, $\omega = kN^{1/\alpha}$, we get $\mathscr{P}(Y) = N^{-1/\alpha} L_\alpha\left(\frac{Y}{N^{1/\alpha}}\right)$. Hence, for Lévy distributions we also have the scaling property, but with exponent $1/\alpha$. This generalised version of the Central Limit Theorem, originally due to Gnedenko-Kolmogorov [7], is applied to any distribution $p(x)$ which, in the limit $x \to \infty$, behaves as $|x|^{-\mu-1}$ ($\mu < 2$). In other words, *the probability density function of the sum of N variables, each one with the same distribution* $p(x) \sim |x|^{-\alpha-1}$ $(0 < \mu < 2)$, *converges, in the* $N \to \infty$ *limit,* to a *α-stable Lévy distribution*.

FINAL REMARKS

In this article we have reviewed central limit theorems for the sum of independent random variables. As we have illustrated, in the absence of correlations, the convolution of identical probability density functions which maximise nonadditive entropy S_q, behave in the same way as any other distribution. In other words, when the entropic index $q < \frac{5}{3}$ the variance of $\mathscr{G}_q(X)$ is finite, hence the convolution leads to a Gaussian distribution. On the other hand, *i.e.*, when $q \geq \frac{5}{3}$ the variance diverges. As a consequence the convolution leads to a α-stable Lévy distribution with the same asymptotic power-law decay of $\mathscr{G}_q(X)$ (the marginal case $q = 5/3$ yields a Gaussian distribution, but with a logarithmic correction on the standard $x^2 \propto t$ scaling, i.e., it yields anomalous diffusion). We have also exhibited that there is an important difference between q-Gaussian and α-stable Lévy distributions, namely the emergence of an inflexion point on the latter type when they are represented in a log-log scale. In the subsequent paper (Part II) we will show that the strong violation of the independence condition introduces a drastic change in the probability space attractor. Specifically, for a special class of correlations (*q-independence*), it is the q-Gaussian distribution which is a stable one.

ACKNOWLEDGMENTS

We are deeply thankful to E.P. Borges and R. Hilfer for fruitful discussions related to both Part I and Part II. Partial financial support from Pronex, CNPq, Faperj (Brazilian agencies) and FCT/MCES (Portuguese agency) is acknowledged as well.

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