PHYSICS OF PLASMAS **15**, 052103 (2008)

Nonaxisymmetric magnetorotational instability in ideal and viscous plasmas

A. B. Mikhailovskii,[1] J. G. Lominadze,[2] R. M. O. Galvão,[3,4] A. P. Churikov,[5] N. N. Erokhin,[1] A. I. Smolyakov,[1,6] and V. S. Tsypin[4]

[1]*Institute of Nuclear Fusion, Russian Research Centre Kurchatov Institute, 1, Kurchatov Sq., Moscow 123182, Russia*
[2]*Kharadze Abastumani National Astrophysical Observatory, 2a, Kazbegi Ave., Tbilisi 0160, Georgia*
[3]*Physics Institute, University of São Paulo, Cidade Universitaria, 05508-900, São Paulo, Brazil*
[4]*Brazilian Center for Physics Research, Rua Xavier Sigaud, 150, 22290-180, Rio de Janeiro, Brazil*
[5]*Syzran Branch of Samara Technical University, 45, Sovetskaya Str., Syzran, Samara Region 446001, Russia*
[6]*University of Saskatchewan, 116 Science Place, Saskatoon, Saskatchewan S7N 5E2, Canada*

(Received 2 October 2007; accepted 20 March 2008; published online 2 May 2008)

The excitation of magnetorotational instability (MRI) in rotating laboratory plasmas is investigated. In contrast to astrophysical plasmas, in which gravitation plays an important role, in laboratory plasmas it can be neglected and the plasma rotation is equilibrated by the pressure gradient. The analysis is restricted to the simple model of a magnetic confinement configuration with cylindrical symmetry, in which nonaxisymmetric perturbations are investigated using the local approximation. Starting from the simplest case of an ideal plasma, the corresponding dispersion relations are derived for more complicated models including the physical effects of parallel and perpendicular viscosities. The Friemann–Rotenberg approach used for ideal plasmas is generalized for the viscous model and an analytical expression for the instability boundary is obtained. It is shown that, in addition to the standard effect of radial derivative of the rotation frequency (the Velikhov effect), which can be destabilizing or stabilizing depending on the sign of this derivative in the ideal plasma, there is a destabilizing effect proportional to the fourth power of the rotation frequency, or, what is the same, to the square of the plasma pressure gradient, and to the square of the azimuthal mode number of the perturbations. It is shown that the instability boundary also depends on the product of the plasma pressure and density gradients, which has a destabilizing effect when it is negative. In the case of parallel viscosity, the MRI looks like an ideal instability independent of viscosity, while, in the case of strong perpendicular viscosity, it is a dissipative instability with the growth rate inversely proportional to the characteristic viscous decay rate. We point out, however, that the modes of the continuous range of the magnetohydrodynamics spectrum are not taken into account in this paper, and they can be more dangerous than those that are considered. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2907788]

I. INTRODUCTION

Over several decades, magnetorotational instability (MRI)[1,2] was investigated exclusively as a magnetohydrodynamic (MHD) phenomenon. Then, beginning with Refs. 3 and 4, it started to be looked at as an astrophysical phenomenon directly related to the physics of accretion disks, and currently it is being gradually incorporated into plasma physics, thereby becoming a plasma physical phenomenon (see, e.g., Refs. 5–8 and references therein). This is a welcome trend because the standard books and reviews on the general theory of plasma instabilities[9–19] do not contain the MRI.

The axisymmetric MRI was considered in Refs. 1–3. It corresponds to perturbations with $m=0$, where m is the azimuthal mode number, assuming the equilibrium geometry to be cylindrical with radial, azimuthal, and longitudinal coordinates (r,θ,z), respectively. The nonaxisymmetric MRI was initially investigated in Ref. 20, stimulating many other works (see, e.g., Ref. 21 and references therein). Nevertheless, the results achieved in these studies are definitely less impressive than those for the axisymmetric MRI, justifying further work in the development of the theory of nonaxisymmetric MRI. This is the goal of the present paper.

We are interested in experimental magnetic confinement configurations, so that the gravitational force, allowed for in the papers of the astrophysical trend,[3] is assumed to vanish. In this case, the equilibrium Coriolis force is balanced by the plasma pressure gradient.

In Sec. II, we discuss the equilibrium and preliminary questions regarding the description of perturbations. Section III studies the nonaxisymmetric MRI in an ideal plasma. In Sec. IV, we allow for parallel viscosity, while Sec. V addresses the perpendicular viscosity effects. Discussions of the results are given in Sec. VI.

We note that the inclusion of the perpendicular viscosity in our analysis was stimulated by the work of Refs. 22–25, based on numerical studies of nonaxisymmetric MRI. The parallel viscosity, on the other hand, was studied in Ref. 26 for the case of axisymmetric MRI, and we note that physically this effect is the gyrorelaxation effect discovered by Budker[27] and Schluter;[28] see also Ref. 29. Both the parallel and perpendicular viscosity effects were studied in detail in the theory of drift instabilities of collision-dominated plasmas (see Refs. 13 and 30).

We make use of the cylindrical approximation as the simplest geometry allowing the study of the effect of a nonuniform rotation frequency profile. This approximation seems to be useful for designing experiments aiming at the investigation of MRI as well as for predicting the behavior of a rotating plasma in axisymmetric magnetic confinement configurations relevant for fusion research. In the last case, the analysis of the nonaxisymmetric perturbations should be extended to allow for magnetic shear and toroidicity effects.

The local approximation in the MRI theory goes back to the classical Ref. 3 and is widely used in the theory of drift instabilities[9,11] for obtaining analytical results. Nevertheless, it is known that, when the local approach predicts any instability, the results have to be verified by a global treatment taking into account proper boundary conditions.

In the local theory of axisymmetric MRI developed in Ref. 3, the radial dependence of the perturbed functions was characterized by $\exp(ik_r r)$, where k_r is the radial projection of the wave vector, neglecting the weak radial dependence of the amplitudes of these functions. In contrast to this, the local theory of nonaxisymmetric MRI requires that this radial dependence should also be taken into account, making its mathematical treatment more complex than the axisymmetric case.

The most advanced approach to studying the perturbations of a rotating plasma is based on the equations derived by Frieman and Rotenberg.[31] Continuing the approach of Ref. 31 (the FR approach), Refs. 32 and 5 have derived a pair of first-order differential equations [the Hameiri–Bondeson–Iacono–Bhattacharjee (HBIB) equations] for the perturbed radial plasma displacement and the Frieman–Rotenberg variable (the FR variable) denoting the sum of the perturbed plasma pressure and magnetic pressure; see also Ref. 6. The remarkable feature of these equations is the fact that their cross matrix elements are antisymmetric (see in detail Refs. 32, 5, and 6). This fact is crucial when one excludes the FR variable and thereby derives the second-order differential equation for the perturbed plasma displacement; such an equation proves to be self-adjoint. In other words, dealing with the above pair of equations with antisymmetric cross matrix elements is an additional guarantee of the correctness of the respective analytical procedure. On the other hand, analytical results should be considered as doubtful if one arrives at a similar pair of equations without this antisymmetry.

The FR approach has been elaborated for the ideal plasma, so that it is applicable only to the problem considered in Sec. III. Therefore, we cannot use this approach *a priori* in Secs. IV and V where dissipative effects are allowed for. Nevertheless, it is evident that introducing the FR variable is possible also in the cases of dissipative plasmas. This is the reasoning used in the present paper. In Sec. VI, the mechanism of influence of the parallel and perpendicular viscosities on nonaxisymmetric MRI is commented upon. Discussions are given in Sec. VII.

We restrict ourselves to only the eigenmodes of rotating plasma. However, according to Refs. 5 and 6 (see also Refs. 33–35), in addition to the eigenmodes, one should study also the continuous ranges in the MHD spectrum, which can be more dangerous than the eigenmodes. In other words, our instability criteria may not be necessary and sufficient and the investigation of the continuous range of the MHD spectrum should be continued.

Note also that an important issue of our topic is the role of kinetic effects in the MRI. This issue has been studied in Ref. 36. The Landau damping allowed for in this reference is similar to the parallel viscosity considered here, but a complete investigation has to be pursued further.

II. EQUILIBRIUM AND DESCRIPTION OF PERTURBATIONS

A. Equilibrium without perpendicular viscosity

We consider an axisymmetric plasma cylinder placed in a magnetic field $\mathbf{B}_0$ directed along the axis, $\mathbf{B}_0=(0,0,B_0)$, and suppose that the plasma rotates in the azimuthal direction, so that its equilibrium velocity $\mathbf{V}_0$ is given by $\mathbf{V}_0=(0,V_0,0)$, where $V_0=r\Omega$ and Ω is the rotation frequency dependent on the radial coordinate, $\Omega=\Omega(r)$.

We assume the equilibrium magnetic field to be uniform, $dB_0/dr=0$, so that the equilibrium equation is given by

$$\rho_0 r\Omega^2 = p_0', \tag{1}$$

where ρ_0 is the equilibrium mass density, p_0 is the equilibrium plasma pressure, and the prime is the radial derivative.

B. Perturbations

We take time and spatial dependence of the perturbations in the form

$$\tilde{F}(\mathbf{r},t) = \tilde{F}(r)\exp(-i\omega t + im\theta + ik_z z). \tag{2}$$

Here $\tilde{F}(\mathbf{r},t)$ is any perturbed value, ω is the oscillation frequency, k_z is the parallel projection of the wave vector, and m is the azimuthal mode number. In addition to m, we will use the azimuthal projection of the wave vector k_y defined by $k_y=m/r$.

For an ideally conducting plasma, the freezing condition is given by

$$\partial\mathbf{B}/\partial t - \nabla\times[\mathbf{V}\times\mathbf{B}] = 0, \tag{3}$$

where $\mathbf{B}$ and $\mathbf{V}$ are the total magnetic field and velocity, respectively, given by $\mathbf{B}=\mathbf{B}_0+\tilde{\mathbf{B}}$, $\mathbf{V}=\mathbf{V}_0+\tilde{\mathbf{V}}$, where $\tilde{\mathbf{B}}=(\tilde{B}_r,\tilde{B}_\theta,\tilde{B}_z)$ and $\tilde{\mathbf{V}}=(\tilde{V}_r,\tilde{V}_\theta,\tilde{V}_z)$ are the perturbed magnetic field and perturbed velocity, respectively. It follows from Eq. (3) that

$$\tilde{V}_r = -\frac{\tilde{\omega}}{k_z B_0}\tilde{B}_r, \tag{4}$$

$$\tilde{V}_\theta = -\frac{\tilde{\omega}}{k_z B_0}\left(\tilde{B}_\theta - \frac{i}{\tilde{\omega}}\frac{d\Omega}{d\ln r}\tilde{B}_r\right). \tag{5}$$

Here $\tilde{\omega}$ is the Doppler-shifted oscillation frequency defined by

$$\tilde{\omega} \equiv \omega - m\Omega. \tag{6}$$

We express $\tilde{B}_z$ in terms of $\tilde{B}_r$ and $\tilde{B}_\theta$ using the Maxwell equation

$$\nabla \cdot \tilde{\mathbf{B}} = 0. \tag{7}$$

Then we find

$$\tilde{B}_z = \frac{1}{k_z}(i\tau_B - k_y\tilde{B}_\theta), \tag{8}$$

where

$$\tau_B = \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_r). \tag{9}$$

The perturbed mass density $\tilde{\rho}$ is given by the continuity equation

$$-i\tilde{\omega}\tilde{\rho} + \tilde{V}_r\rho_0' + \rho_0\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{V}_r) + ik_y\tilde{V}_\theta + ik_z\tilde{V}_z\right] = 0. \tag{10}$$

Then, using Eqs. (4) and (5), we arrive at

$$\frac{\tilde{\rho}}{\rho_0} = \frac{1}{k_zB_0}\left[i\left(\tau_B + \frac{d\ln\rho_0}{dr}\tilde{B}_r\right) - k_y\tilde{B}_\theta\right] + \frac{k_z\tilde{V}_z}{\tilde{\omega}}. \tag{11}$$

C. Equilibrium in the presence of perpendicular viscosity

In the presence of perpendicular viscosity, it is necessary to allow for the condition of azimuthal plasma equilibrium. According to Ref. 37, this condition is of the form

$$\frac{d^2(r\Omega)}{dr^2} + \frac{1}{r}\frac{d(r\Omega)}{dr} - \frac{\Omega}{r} = 0, \tag{12}$$

which is satisfied by the Velikhov rotation frequency profile

$$\Omega(r) = a^V + e^V/r^2. \tag{13}$$

Here a^V and e^V are the constants, and the superscript "V" stands for the first letter in the name of the author of Ref. 1.

III. IDEAL PLASMA

A. Starting equations

In the simplest case of ideal plasma, we use the motion equation

$$\left(\rho\frac{d\mathbf{V}}{dt}\right)^{\sim} = -\nabla\tilde{p} + \frac{1}{4\pi}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\frac{\mathbf{B}^2}{2}\right]^{\sim}. \tag{14}$$

Here $d/dt = \partial/\partial t + \mathbf{V}\cdot\nabla$ and the tilde stands for the perturbed part of the corresponding value. The total mass density is given by $\rho = \rho_0 + \tilde{\rho}$ and $\tilde{p}$ is the perturbed plasma pressure. In the case of ideal plasma, we use the adiabatic condition

$$\frac{d}{dt}\left(\frac{p}{\rho^\Gamma}\right) = 0, \tag{15}$$

where Γ is the adiabatic exponent. Then we arrive at

$$\tilde{p} = \frac{\rho_0c_s^2}{k_zB_0}\left(i\tau_B - k_y\tilde{B}_\theta + \frac{k_z^2B_0}{\tilde{\omega}}\tilde{V}_z\right) + \frac{i\rho_0r\Omega^2\tilde{B}_r}{k_zB_0}, \tag{16}$$

where $c_s^2 = \Gamma p_0/\rho_0$ is the sound velocity square. In order to find $\tilde{V}_z$, we turn to the longitudinal plasma equation of motion,

$$-i\tilde{\omega}\tilde{V}_z = -ik_z\tilde{p}/\rho_0. \tag{17}$$

Combining Eqs. (16) and (17), we obtain

$$\tilde{p} = \frac{\rho_0}{k_zB_0\alpha_s}[c_s^2(i\tau_B - k_y\tilde{B}_\theta) + ir\Omega^2\tilde{B}_r], \tag{18}$$

$$\tilde{V}_z = \frac{1}{B_0\tilde{\omega}\alpha_s}[c_s^2(i\tau_B - k_y\tilde{B}_\theta) + ir\Omega^2\tilde{B}_r], \tag{19}$$

where

$$\alpha_s = 1 - k_z^2c_s^2/\tilde{\omega}^2. \tag{20}$$

Substituting Eq. (19) into Eq. (11) leads to

$$\frac{\tilde{\rho}}{\rho_0} = \frac{1}{k_zB_0\alpha_s}\left\{i\left[\tau_B + \left(\alpha_s\frac{d\ln\rho_0}{dr} + \frac{k_z^2\Omega^2r}{\tilde{\omega}^2}\right)\tilde{B}_r\right] - k_y\tilde{B}_\theta\right\}. \tag{21}$$

The (r,θ)-th projections of Eq. (14) take the form

$$\rho_0(-i\tilde{\omega}\tilde{V}_r - 2\Omega\tilde{V}_\theta) - \tilde{\rho}\Omega^2r = -\frac{\partial}{\partial r}\tilde{p} + \frac{B_0}{4\pi}\left(ik_z\tilde{B}_r - \frac{\partial\tilde{B}_z}{\partial r}\right), \tag{22}$$

$$\rho_0\left(-i\tilde{\omega}\tilde{V}_\theta + \frac{\kappa^2}{2\Omega}\tilde{V}_r\right) = -ik_y\tilde{p} + \frac{iB_0}{4\pi}(k_z\tilde{B}_\theta - k_y\tilde{B}_z). \tag{23}$$

Here

$$\kappa^2 = \frac{2\Omega}{r}\frac{d(r^2\Omega)}{dr} \equiv 4\Omega^2 + \frac{d\Omega^2}{d\ln r} \tag{24}$$

and we introduced the Frieman–Rotenberg variable p_* defined by

$$p_* = \tilde{p} + B_0\tilde{B}_z/(4\pi). \tag{25}$$

B. The HBIB equations

By means of Eqs. (8) and (18), Eq. (25) is represented in the form

$$p_* = \frac{B_0}{4\pi k_z}\left[\left(1 + \frac{\beta}{\alpha_s}\right)(i\tau_B - k_y\tilde{B}_\theta) + \frac{ir\Omega^2}{\alpha_sv_A^2}\tilde{B}_r\right], \tag{26}$$

where $\beta = c_s^2/v_A^2$. Using Eqs. (4) and (5), Eq. (23) yields

$$\tilde{B}_\theta = -(4\pi v_A^2k_yk_zp_*/B_0 + i2\Omega\tilde{\omega}\tilde{B}_r)/D_0. \tag{27}$$

Here

$$D_0 = \tilde{\omega}^2\alpha_A, \tag{28}$$

$$\alpha_A = 1 - k_z^2 v_A^2/\tilde{\omega}^2, \tag{29}$$

where $v_A^2 = B_0^2/(4\pi\rho_0)$ is the Alfvén velocity square. By means of Eq. (27), Eq. (26) results in

$$D\tau_B = C_1 \tilde{B}_r - i4\pi k_z B_0 C_2 p_*. \tag{30}$$

Here

$$D = D_0(1+\beta/\alpha_s), \tag{31}$$

$$C_1 = -\Omega\left[2k_y\tilde{\omega}\left(1+\frac{\beta}{\alpha_s}\right)+\frac{D_0 r\Omega}{\alpha_s v_A^2}\right], \tag{32}$$

$$C_2 = \frac{1}{B_0^2}\left[D_0 - k_y^2 v_A^2\left(1+\frac{\beta}{\alpha_s}\right)\right]. \tag{33}$$

Using the variable p_*, we represent Eq. (22) in the form

$$-ip_*' = \frac{\rho_0}{k_z B_0}\left[\Omega^2 r\frac{\tau_B}{\alpha_s} - \left(D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\tilde{\omega}^2\alpha_s}\right)\tilde{B}_r + i\Omega\left(2\tilde{\omega}+\frac{m\Omega}{\alpha_s}\right)\tilde{B}_\theta\right]. \tag{34}$$

Substituting here Eq. (27), we arrive at

$$-ip_*' = -i\Omega\left(2\tilde{\omega}+\frac{m\Omega}{\alpha_s}\right)\frac{k_y}{D_0}p_* + \frac{\rho_0}{k_z B_0}\left\{\Omega^2 r\frac{\tau_B}{\alpha_s} - \left[D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\alpha_s\tilde{\omega}^2} - \frac{2\Omega^2}{D_0}\tilde{\omega}\left(2\tilde{\omega}+\frac{m\Omega}{\alpha_s}\right)\right]\tilde{B}_r\right\}. \tag{35}$$

We exclude the value τ_B from Eq. (35) using Eq. (30). Then we obtain

$$i4\pi k_z B_0 D p_*' = -i4\pi k_z B_0 \bar{C}_1 p_* + C_3\tilde{B}_r. \tag{36}$$

Here

$$\bar{C}_1 = -\Omega\left[\left(2\tilde{\omega}+\frac{m\Omega}{\alpha_s}\right)k_y\left(1+\frac{\beta}{\alpha_s}\right)+\frac{4\pi\rho_0 r\Omega}{\alpha_s}C_2\right], \tag{37}$$

$$C_3 = 4\pi\rho_0\left\{D\left[D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\alpha_s\tilde{\omega}^2} - \frac{2\Omega^2}{D_0}\tilde{\omega}\left(2\tilde{\omega}+\frac{m\Omega}{\alpha_s}\right)\right] - \frac{r\Omega^2}{\alpha_s}C_1\right\}. \tag{38}$$

Substituting Eqs. (33) and (32) into Eqs. (37) and (38), we have

$$\bar{C}_1 = C_1 \tag{39}$$

$$C_3 = 4\pi\rho_0\left\{D_0\left[\left(1+\frac{\beta}{\alpha_s}\right)\left(D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{\Omega^4 r^2 D_0}{\tilde{\omega}^2\alpha_s v_A^2}\right] - 4\Omega^2\tilde{\omega}^2\left(1+\frac{\beta}{\alpha_s}\right)\right\}. \tag{40}$$

C. The mode equation and dispersion relation

In order to exclude the value p_* from our problem, we find from Eq. (30)

$$i4\pi k_z B_0 p_* = (C_1\tilde{B}_r - D\tau_B)/C_2. \tag{41}$$

Then Eq. (36) takes the form

$$D(D\tau_B/C_2)' + \Lambda\tilde{B}_r = 0. \tag{42}$$

Here

$$\Lambda = a + b, \tag{43}$$

$$a = C_3 - C_1^2/C_2, \tag{44}$$

$$b = -Dr[C_1/(rC_2)]'. \tag{45}$$

In the case of small-scale modes, one can take

$$\tilde{B}_r = \tilde{B}_r^{(0)}(r)\exp(ik_r r), \tag{46}$$

where $\tilde{B}_r^{(0)}(r)$ is a slowly varying amplitude. As a result, Eq. (42) leads to the dispersion relation

$$-k_r^2 D^2/C_2 + \Lambda = 0. \tag{47}$$

In the case of nonaxisymmetric modes, there is the small parameter

$$\alpha_A\tilde{\omega}^2 r^2/(m^2 v_A^2) \ll 1. \tag{48}$$

Using Eq. (33), the value $1/C_2$ is approximated as

$$\frac{1}{C_2} = -\frac{4\pi\rho_0 r^2}{m^2}\frac{1}{1+\beta/\alpha_s}\left[1+\frac{D_0}{k_y^2 v_A^2(1+\beta/\alpha_s)}\right]. \tag{49}$$

Then

$$a = 4\pi\rho_0 D_0\left[\left(1+\frac{\beta}{\alpha_s}\right)\left(D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{\Omega^4 r^2 D_0}{\alpha_s\tilde{\omega}^2 v_A^2} + \frac{4\tilde{\omega}^2\Omega^2 r^2}{m^2 v_A^2} + \frac{4r^2}{m}\frac{\tilde{\omega}\Omega^3}{\alpha_s v_A^2}\right]. \tag{50}$$

$$b = b^{(0)} + b^{(1)}. \tag{51}$$

Here

$$b^{(0)} = -\frac{4\pi D_0}{m^2}\left(1+\frac{\beta}{\alpha_s}\right)r\frac{d}{dr}\times\left\{\rho_0\Omega\left[2m\tilde{\omega}+\frac{D_0 r^2\Omega}{\alpha_s v_A^2(1+\beta/\alpha_s)}\right]\right\}, \tag{52}$$

$$b^{(1)} = -\frac{4\pi D_0}{m^3 v_A^2}\left(1+\frac{\beta}{\alpha_s}\right)r\frac{d}{dr}\left(2\rho_0\Omega\tilde{\omega}\frac{D_0 r^2}{1+\beta/\alpha_s}\right). \tag{53}$$

Calculating $b^{(0)}$, we use the chain of identities

$$2\frac{d}{dr}(\rho_0\Omega\tilde{\omega}) = 2\omega\frac{d}{dr}(\rho_0\Omega) - 2m\frac{d}{dr}(\rho_0\Omega^2)$$
$$= 2\tilde{\omega}\frac{d}{dr}(\rho_0\Omega) + 2m\Omega\frac{d}{dr}(\rho_0\Omega) - 2m\frac{d}{dr}(\rho_0\Omega^2)$$
$$= 2\tilde{\omega}\frac{d}{dr}(\rho_0\Omega) - m\rho_0\frac{d\Omega^2}{dr}. \quad (54)$$

At the same time, in the second term in the square brackets of Eq. (52) we differentiate only Ω^2. Then we arrive at

$$b^{(0)} = 4\pi\rho_0 D_0\left(1+\frac{\beta}{\alpha_s}\right)\left(\frac{d\Omega^2}{d\ln r} - \frac{2\tilde{\omega}}{m}\frac{d\Omega}{d\ln r}\right). \quad (55)$$

The derivative of $2\Omega\tilde{\omega}$ in Eq. (53) is calculated similarly to Eq. (54). Then one has

$$b^{(1)} = \frac{4\pi\rho_0 D_0^2 r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r}. \quad (56)$$

Substituting Eqs. (50), (51), (55), and (56) into Eq. (43), we find

$$\Lambda = 4\pi\rho_0 D_0\left[\left(1+\frac{\beta}{\alpha_s}\right)\left(D_0 - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{D_0 r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r} + \frac{\Omega^4 r^2 D_0}{v_A^2\tilde{\omega}^2\alpha_s} + \frac{4\tilde{\omega}^2\Omega^2 r^2}{m^2 v_A^2} - \frac{2\tilde{\omega}}{m}\left(1+\frac{\beta}{\alpha_s}\right)\frac{d\Omega}{d\ln r}\right]. \quad (57)$$

Here we have allowed for the following remarkable fact: mutual cancellation of the terms with $d\Omega^2/d\ln r$ in a and b to major order in the expansion.

Equation (57) shows that there is a hierarchy in contributions of the derivatives of the rotation frequency and plasma density; the first enters with the small weight, while the second with the large one. Equations (55) and (56) have been simplified using this hierarchy.

D. Analysis of dispersion relation

Using Eq. (49), the dispersion relation (47) can be represented in the form

$$k_r^2 D_0^2 + \frac{m^2}{4\pi\rho_0 r^2(1+\beta/\alpha_s)}\Lambda = 0. \quad (58)$$

Substituting here Eq. (57), we arrive at

$$D_0\left[\tilde{\omega}^2(\alpha_s+\beta) + \frac{1}{k_\perp^2 v_A^2}\left(\alpha_s\tilde{\omega}^2\frac{d\Omega^2}{d\ln r} + m^2\Omega^4\right)\right] - \frac{k_y^2}{k_\perp^2}\tilde{\omega}^2(\alpha_s+\beta)\Omega^2\frac{d\ln\rho_0}{d\ln r} + \frac{4\alpha_s\tilde{\omega}^4\Omega^2}{k_\perp^2 v_A^2} = 0, \quad (59)$$

where $k_\perp^2 = k_r^2 + m^2/r^2$. Note that we have neglected here the terms as small as $1/(k_\perp^2 r^2)$. In the explicit form, Eq. (59) means

$$Q_0 \equiv \tilde{\omega}^4\left(1+\beta+\frac{\kappa^2}{k_\perp^2 v_A^2}\right) - \tilde{\omega}^2 k_z^2 v_A^2 g_2 + k_z^4 c_s^2 v_A^2 g_0 = 0. \quad (60)$$

Here

$$g_0 = 1 + \frac{1}{k_\perp^2 v_A^2}\left(\frac{d\Omega^2}{d\ln r} - \frac{m^2\Omega^4}{v_A^2\beta^2 k_z^2} + \frac{m^2\Omega^2}{r^2 k_z^2}\frac{d\ln\rho_0}{d\ln r}\right), \quad (61)$$

$$g_2 = 1 + 2\beta + \frac{1}{k_\perp^2 v_A^2}\left[(1+\beta)\frac{d\Omega^2}{d\ln r} - \frac{m^2\Omega^4}{v_A^2 k_z^2} + \frac{k_y^2}{k_z^2}(1+\beta)\Omega^2\frac{d\ln\rho_0}{d\ln r} + 4\beta\Omega^2\right]. \quad (62)$$

For small $\tilde{\omega}$ it follows from Eq. (60) that

$$Q_0 = Q_0^{(0)} \equiv -k_z^2 v_A^2(g_2\tilde{\omega}^2 - k_z^2 c_s^2 g_0) = 0. \quad (63)$$

Taking here $\tilde{\omega}=0$, we obtain the expression for the instability boundary

$$g_0 = 0, \quad (64)$$

or, in the explicit form,

$$\frac{d\Omega^2}{d\ln r} + k_\perp^2 v_A^2 - \frac{m^2\Omega^4}{c_s^2 k_z^2} + \frac{m^2\Omega^2}{r^2 k_z^2}\frac{d\ln\rho_0}{d\ln r} = 0. \quad (65)$$

For $\beta\to 0$, the second term on the left-hand side of this equation does not lead to infinity since, according to the equilibrium condition (1), the value Ω^2 is small as β.

Note that allowing for Eq. (1), Eq. (65) is represented as follows:

$$\frac{d\Omega^2}{d\ln r} + k_\perp^2 v_A^2\left[1 - \frac{k_y^2}{k_\perp^2 r^2 k_z^2}\frac{\beta}{\Gamma^2}\left(\frac{d\ln p_0}{d\ln r}\right)^2 + \frac{k_y^2}{k_\perp^2 r^2 k_z^2}\frac{\beta}{\Gamma}\frac{d\ln p_0}{d\ln r}\frac{d\ln\rho_0}{d\ln r}\right] = 0. \quad (66)$$

According to Eq. (66), the perturbations can be unstable not only due to negative $d\Omega^2/d\ln r$, as studied in Refs. 1 and 2, but also due to the square of the equilibrium plasma pressure gradient and the equilibrium density gradient. Since $d\ln p_0/d\ln r > 0$ in our model, the equilibrium density gradient proves destabilizing for negative density gradient,

$$d\ln\rho_0/d\ln r < 0. \quad (67)$$

Near the instability boundary, i.e., for $g_0\approx 0$, Eq. (62) yields

$$g_2 = \beta + \frac{\Omega^2}{k_\perp^2 v_A^2}\left(4\beta + \frac{m^2\Omega^2}{c_s^2 k_z^2}\right). \quad (68)$$

For $\beta \gg 1$ it follows from Eq. (62) that

$$\tilde{\omega}^2 - k_z^2 v_A^2 g_0 = 0. \quad (69)$$

In the opposite case $\beta\to 0$, Eq. (63) reduces to

$$\left[1 + \frac{\Omega^2}{k_\perp^2 c_s^2}\left(4 + \frac{m^2\Omega^2}{v_A^2\beta^2 k_z^2}\right)\right]\tilde{\omega}^2 - k_z^2 v_A^2 g_0 = 0. \quad (70)$$

IV. EFFECTS OF PARALLEL VISCOSITY

Let us consider the parallel viscosity effects. Then, in accordance with Ref. 14, we should substitute into the perpendicular projections of Eq. (14),

$$\tilde{p} \to \tilde{p}_\perp, \tag{71}$$

where $\tilde{p}_\perp$ is the perturbed perpendicular plasma pressure, and this value can be expressed in terms of $\tilde{p}$ and the parallel viscosity scalar $\tilde{\pi}_\parallel$ by

$$\tilde{p}_\perp = \tilde{p} - \tilde{\pi}_\parallel/2. \tag{72}$$

Thus, instead of the perturbed version of the perpendicular projections of Eq. (14), one has

$$\left(\rho\frac{d\mathbf{V}}{dt}\right)^\sim_\perp = -\boldsymbol{\nabla}_\perp\left(\tilde{p} - \frac{\tilde{\pi}_\parallel}{2}\right) + \frac{1}{4\pi}\left\{(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} - \frac{1}{2}\boldsymbol{\nabla}\,\mathbf{B}^2\right\}^\sim_\perp. \tag{73}$$

Thereby, Eqs. (22) and (23) are modified as follows:

$$-i\tilde{\omega}\tilde{V}_r - 2\Omega\tilde{V}_\theta + \frac{1}{\rho_0}\frac{\partial}{\partial r}\left(\tilde{p} - \frac{\tilde{\pi}_\parallel}{2}\right) - \Omega^2 r\tilde{\rho} + \frac{v_A^2}{B_0}\left(\frac{\partial\tilde{B}_z}{\partial r} - ik_z\tilde{B}_r\right) = 0, \tag{74}$$

$$-i\tilde{\omega}\tilde{V}_\theta + \frac{\kappa^2}{2\Omega}\tilde{V}_r + \frac{ik_y}{\rho_0}\left(\tilde{p} - \frac{\tilde{\pi}_\parallel}{2}\right) + \frac{iv_A^2}{B_0}(k_y\tilde{B}_z - k_z\tilde{B}_\theta) = 0. \tag{75}$$

The parallel viscosity modifies also the parallel projection motion equation (14) by the substitution[14]

$$\tilde{p} \to \tilde{p} + \tilde{\pi}_\parallel. \tag{76}$$

Then, instead of Eq. (17), we have

$$-i\tilde{\omega}\tilde{V}_z = -\frac{ik_z}{\rho_0}(\tilde{p} + \tilde{\pi}_\parallel). \tag{77}$$

In the Braginskii approximation,[29]

$$\tilde{\pi}_\parallel = -i\Delta_\parallel\rho_0\tilde{V}_z/k_z, \tag{78}$$

where $\Delta_\parallel$ is the characteristic parallel-viscosity-induced decay rate given by

$$\Delta_\parallel = \frac{2}{3}\cdot 0.96\frac{Tk_z^2}{M\nu_i}. \tag{79}$$

Here ν_i is the ion collision frequency, T is the equilibrium plasma temperature, and M is the ion mass. Using Eq. (78), Eq. (77) takes the form

$$-i(\tilde{\omega} + i\Delta_\parallel)\tilde{V}_z = -ik_z\tilde{p}/\rho_0. \tag{80}$$

It then follows from Eqs. (16) and (80) that

$$\tilde{p} = \frac{\rho_0}{\bar{\alpha}_s k_z B_0}[c_s^2(i\tau_B - k_y\tilde{B}_\theta) + ir\Omega^2\tilde{B}_r], \tag{81}$$

where

$$\bar{\alpha}_s = 1 - \frac{k_z^2c_s^2}{\tilde{\omega}(\tilde{\omega} + i\Delta_\parallel)}. \tag{82}$$

Allowing for Eq. (81), Eq. (82) yields

$$\tilde{V}_z = \frac{1}{\bar{\alpha}_s B_0}\frac{c_s^2(i\tau_B - k_y\tilde{B}_\theta) + ir\Omega^2k_z^2\tilde{B}_r}{\tilde{\omega} + i\Delta_\parallel}. \tag{83}$$

Then Eq. (11) reduces to [cf. Eq. (21)]

$$\frac{\tilde{\rho}}{\rho_0} = \frac{1}{k_zB_0\bar{\alpha}_s}\left\{i\tau_B - k_y\tilde{B}_\theta + i\left[\bar{\alpha}_s\frac{d\ln\rho_0}{dr} + \frac{k_z^2r\Omega^2}{\tilde{\omega}(\tilde{\omega} + i\Delta_\parallel)}\right]\tilde{B}_r\right\}. \tag{84}$$

Instead of Eq. (25), it is reasonable to introduce the FR variable p_* by

$$p_* = \tilde{p} - \tilde{\pi}_\parallel/2 + B_0\tilde{B}_z/(4\pi). \tag{85}$$

Using Eqs. (8) and (79)–(81), we represent Eq. (85) in the form

$$p_* = \frac{B_0}{4\pi k_z}\left[\left(1 + \frac{\beta}{\bar{\alpha}_s}\right)(i\tau_B - k_y\tilde{B}_\theta) + i\frac{r\Omega^2}{v_A^2\bar{\alpha}_s}\tilde{B}_r\right]. \tag{86}$$

In terms of p_*, Eqs. (74) and (75) reduce to

$$-i\tilde{\omega}\tilde{V}_r - 2\Omega\tilde{V}_\theta + \frac{1}{\rho_0}p_*' - \Omega^2 r\frac{\tilde{\rho}}{\rho_0} - \frac{iv_A^2}{B_0}k_z\tilde{B}_r = 0, \tag{87}$$

$$-i\tilde{\omega}\tilde{V}_\theta + \frac{\kappa^2}{2\Omega}\tilde{V}_r + \frac{ik_y}{\rho_0}p_* - \frac{iv_A^2}{B_0}k_z\tilde{B}_\theta = 0. \tag{88}$$

Substituting Eqs. (4) and (5) into Eq. (88), we find that Eq. (27) for $\tilde{B}_\theta$ in terms of p_* and $\tilde{B}_r$ remains in force. Then Eq. (86) takes the form of Eq. (30) with [cf. Eqs. (31)–(33)]

$$D = D_0(1 + \beta/\bar{\alpha}_s), \tag{89}$$

$$C_1 = -\Omega\left[2\tilde{\omega}k_y\left(1 + \frac{\beta}{\bar{\alpha}_s}\right) + \frac{D_0r\Omega}{\bar{\alpha}_sv_A^2}\right], \tag{90}$$

$$C_2 = \frac{1}{B_0^2}\left[D_0 - k_y^2v_A^2\left(1 + \frac{\beta}{\bar{\alpha}_s}\right)\right]. \tag{91}$$

Substituting Eqs. (4), (5), and (84) into Eq. (87), we find [cf. Eq. (34)]

$$-ip_*' = \frac{\rho_0}{k_zB_0}\left[-\left(D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4r^2k_z^2}{\bar{\alpha}_s\tilde{\omega}(\tilde{\omega} + i\Delta_\parallel)}\right)\tilde{B}_r + i\Omega\left(2\tilde{\omega} + \frac{m\Omega}{\bar{\alpha}_s}\right)\tilde{B}_\theta + \frac{\Omega^2r}{\bar{\alpha}_s}\tau_B\right]. \tag{92}$$

Using Eq. (88), this equation reduces to [cf. Eq. (35)]

$$-ip'_* = -i\Omega\left(2\tilde{\omega} + \frac{m\Omega}{\bar{\alpha}_s}\right)\frac{k_y}{D_0}p_* + \frac{\rho_0}{k_z B_0}\left\{\frac{\Omega^2 r}{\bar{\alpha}_s}\tau_B - \left[D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\bar{\alpha}_s\tilde{\omega}(\tilde{\omega}+i\Delta_\parallel)} - \frac{2\Omega^2}{D_0}\tilde{\omega}\left(2\tilde{\omega}+\frac{m\Omega}{\bar{\alpha}_s}\right)\right]\tilde{B}_r\right\}. \quad (93)$$

Excluding here τ_B by means of Eq. (30) leads to Eq. (36) with

$$\bar{C}_1 = -\Omega\left[\left(2\tilde{\omega} + \frac{m\Omega}{\bar{\alpha}_s}\right)k_y\left(1+\frac{\beta}{\bar{\alpha}_s}\right) + \frac{4\pi\rho_0 r\Omega}{\bar{\alpha}_s}C_2\right], \quad (94)$$

$$C_3 = 4\pi\rho_0\left\{D\left[D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\bar{\alpha}_s\tilde{\omega}(\tilde{\omega}+i\Delta_\parallel)} - \frac{2\Omega^2}{D_0}\tilde{\omega}(2\tilde{\omega}+m\Omega)\right] - \frac{r\Omega^2}{\bar{\alpha}_s}C_1\right\}. \quad (95)$$

For C_1 and C_2 given by Eqs. (90) and (91), Eq. (94) leads to Eq. (39), while Eq. (94) yields [cf. Eq. (40)]

$$C_3 = 4\pi\rho_0\left[D_0\left(1+\frac{\beta}{\bar{\alpha}_s}\right)\left(D_0 - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) - 4\Omega^2\tilde{\omega}^2\left(1+\frac{\beta}{\bar{\alpha}_s}\right) + \frac{\Omega^4 r^2 D_0^2}{v_A^2\bar{\alpha}_s\tilde{\omega}(\tilde{\omega}+i\Delta_\parallel)}\left(1+\frac{i\Delta_\parallel\tilde{\omega}}{D_0}\right)\right]. \quad (96)$$

It is clear that Eqs. (42)–(45) as well as the dispersion relation, Eq. (47), remain in force. We use the approximation for $1/C_2$ given by Eq. (49) with the substitution $\alpha_s \to \bar{\alpha}_s$. Then we arrive at Eqs. (50)–(53) and (55) with the same substitution and Eq. (56). As a result, Eq. (57) for Λ is modified as follows:

$$\Lambda = \Lambda_\parallel \equiv 4\pi\rho_0 D_0\left[\left(1+\frac{\beta}{\bar{\alpha}_s}\right)\left(D_0 - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{D_0 r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r} + \frac{\Omega^4 r^2 D_0}{v_A^2\tilde{\omega}(\tilde{\omega}+i\Delta_\parallel)\bar{\alpha}_s}\left(1+\frac{i\Delta_\parallel\tilde{\omega}}{D_0}\right) + \frac{4\tilde{\omega}^2\Omega^2 r^2}{m^2 v_A^2}\right]. \quad (97)$$

Correspondingly, the dispersion relation proves to be the following generalization of Eq. (58):

$$k_r^2 D_0^2 + \frac{m^2}{4\pi\rho_0 r^2(1+\beta/\bar{\alpha}_s)}\Lambda = 0. \quad (98)$$

Similarly to Eq. (59), Eq. (98) reduces to

$$D_0\left[\tilde{\omega}^2(\bar{\alpha}_s+\beta) + \frac{1}{k_\perp^2 v_A^2}\left(\bar{\alpha}_s\tilde{\omega}^2\frac{d\Omega^2}{d\ln r} + \frac{\tilde{\omega}}{\tilde{\omega}+i\Delta_\parallel}m^2\Omega^4\left(1+\frac{i\Delta_\parallel\tilde{\omega}}{D_0}\right)\right)\right] - \frac{k_y^2}{k_\perp^2}\tilde{\omega}^2(\bar{\alpha}_s+\beta)\Omega^2\frac{d\ln\rho_0}{d\ln r} + \frac{4\bar{\alpha}_s\tilde{\omega}^4\Omega^2}{k_\perp^2 v_A^2} = 0. \quad (99)$$

We represent

$$\tilde{\omega}^2\bar{\alpha}_s \equiv \tilde{\omega}^2\alpha_s + \frac{i\Delta_\parallel k_z^2 c_s^2}{\tilde{\omega}+i\Delta_\parallel}. \quad (100)$$

Let us consider the case of strong parallel viscosity, $\Delta_\parallel \gg \tilde{\omega}$. Then Eq. (100) leads to

$$\bar{\alpha}_s = 1, \quad (101)$$

$$\tilde{\omega}^2 g_2^\parallel - k_z^2 v_A^2 g_0^\parallel = 0. \quad (102)$$

Here

$$g_0^\parallel = 1 + \beta + \frac{1}{k_\perp^2 v_A^2}\left[\frac{d\Omega^2}{d\ln r} - \frac{m^2\Omega^4}{k_z^2 v_A^2} + \frac{m^2}{k_z^2 r^2}(1+\beta)\Omega^2\frac{d\ln\rho_0}{d\ln r}\right], \quad (103)$$

$$g_2^\parallel = 1 + \beta + \kappa^2/(k_\perp^2 v_A^2). \quad (104)$$

Then, similarly to Eq. (64), we arrive at the expression for the instability boundary,

$$g_0^\parallel = 0, \quad (105)$$

or, in the explicit form [cf. Eq. (65)],

$$\frac{d\Omega^2}{d\ln r} + (1+\beta)\left[k_\perp^2 v_A^2 - \frac{m^2\Omega^4}{k_z^2 v_A^2} + \frac{m^2}{k_z^2 r^2}\Omega^2\frac{d\ln\rho_0}{d\ln r}\right]. \quad (106)$$

It follows from Eq. (102) that

$$\tilde{\omega}^2 = k_z^2 v_A^2 g_0^\parallel/g_2^\parallel. \quad (107)$$

V. EFFECTS OF PERPENDICULAR VISCOSITY

Instead of Eq. (14), now we start with the model motion equation

$$\left(\rho\frac{d\mathbf{V}}{dt}\right)^{\sim} = -\nabla\tilde{p} + \frac{1}{4\pi}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\frac{\mathbf{B}^2}{2}\right]^{\sim} + (\rho\nu\nabla^2\mathbf{V})^{\sim}, \quad (108)$$

where ν is the kinematic viscosity coefficient assumed to be a constant. Physically, the term with ν describes the perpendicular viscosity. Then we arrive at the following equations for the amplitudes of the perturbed functions:

$$-i(\tilde{\omega}+i\Delta_\perp)\tilde{V}_r - 2\Omega\tilde{V}_\theta - \frac{\tilde{\rho}}{\rho_0}\Omega^2 r + \frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial r} + \frac{v_A^2}{B_0}\left(\frac{\partial}{\partial r}\tilde{B}_z - ik_z\tilde{B}_r\right) = 0, \quad (109)$$

$$-i(\tilde{\omega}+i\Delta_\perp)\tilde{V}_\theta + \frac{\kappa^2}{2\Omega}\tilde{V}_r + \frac{ik_y}{\rho_0}\tilde{p} + \frac{iv_A^2}{B_0}(k_y\tilde{B}_z - k_z\tilde{B}_\theta) = 0, \quad (110)$$

$$-i(\tilde{\omega}+i\Delta_\perp)\tilde{V}_z = -ik_z\tilde{p}/\rho_0, \quad (111)$$

where, similarly to $\Delta_\parallel$, see Eq. (79), $\Delta_\perp$ is the characteristic perpendicular-viscosity-induced decay rate given by

$$\Delta_\perp = \nu(k_r^2 + k_y^2 + k_z^2). \tag{112}$$

In general, the value k_r should be considered as the operator $-i\partial/\partial r$.

Substituting Eq. (107) into Eq. (16) yields [cf. Eq. (18)]

$$\tilde{p} = \frac{\rho_0}{k_z B_0 \hat{\alpha}_s}[c_s^2(i\tau_B - k_y \tilde{B}_\theta) + ir\Omega^2 \tilde{B}_r], \tag{113}$$

where

$$\hat{\alpha}_s = 1 - \frac{k_z^2 c_s^2}{\tilde{\omega}(\tilde{\omega} + i\Delta_\perp)}. \tag{114}$$

Using Eq. (113), Eq. (111) leads to [cf. Eq. (19)]

$$\tilde{V}_z = \frac{1}{(\tilde{\omega} + i\Delta_\perp) B_0 \hat{\alpha}_s}[c_s^2(i\tau_B - k_y \tilde{B}_\theta) + ir\Omega^2 \tilde{B}_r]. \tag{115}$$

Respectively, using Eq. (11), Eq. (21) is modified as follows [cf. Eq. (84)]:

$$\frac{\tilde{\rho}}{\rho_0} = \frac{1}{k_z B_0 \hat{\alpha}_s}\left\{ i\left[\tau_B + \left(\hat{\alpha}_s \frac{d\ln\rho_0}{dr} + \frac{k_z^2\Omega^2 r}{\tilde{\omega}(\tilde{\omega}+i\Delta_\perp)}\right)\tilde{B}_r\right] - k_y \tilde{B}_\theta \right\}. \tag{116}$$

Equations (22) and (23) are substituted by

$$\rho_0[-i(\tilde{\omega}+i\Delta_\perp)\tilde{V}_r - 2\Omega\tilde{V}_\theta] - \tilde{\rho}\Omega^2 r = -\frac{\partial}{\partial r}\tilde{p} + \frac{B_0}{4\pi}\left(ik_z\tilde{B}_r - \frac{\partial \tilde{B}_z}{\partial r}\right), \tag{117}$$

$$\rho_0\left[-i(\tilde{\omega}+i\Delta_\perp)\tilde{V}_\theta + \frac{\kappa^2}{2\Omega}\tilde{V}_r \right] = -ik_y\tilde{p} + \frac{iB_0}{4\pi}(k_z\tilde{B}_\theta - k_y\tilde{B}_z). \tag{118}$$

Introducing the variable p_* by means of Eq. (25) and using Eq. (118), we find, instead of Eq. (27),

$$\tilde{B}_\theta = -[4\pi v_A^2 k_y k_z p_*/B_0 + i\Omega \times (2\tilde{\omega} - i\Delta_\perp d\ln\Omega/d\ln r)\tilde{B}_r]/D_\nu, \tag{119}$$

where

$$D_\nu = \tilde{\omega}(\tilde{\omega}+i\Delta_\perp) - k_z^2 v_A^2. \tag{120}$$

Similarly to Eq. (26), one has [cf. Eq. (86)]

$$p_* = \frac{B_0}{4\pi k_z}\left[\left(1+\frac{\beta}{\hat{\alpha}_s}\right)(i\tau_B - k_y\tilde{B}_\theta) + \frac{ir\Omega^2}{\hat{\alpha}_s v_A^2}\tilde{B}_r \right]. \tag{121}$$

Substituting here Eq. (119) leads to Eq. (30) with the following modification of Eqs. (31)–(33):

$$D = D_\nu(1+\beta/\hat{\alpha}_s), \tag{122}$$

$$C_1 = -\Omega\left[k_y\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\left(1+\frac{\beta}{\hat{\alpha}_s}\right) + \frac{D_\nu}{\hat{\alpha}_s v_A^2}r\Omega \right], \tag{123}$$

$$C_2 = \frac{1}{B_0^2}\left[D_\nu - k_y^2 v_A^2\left(1+\frac{\beta}{\hat{\alpha}_s}\right)\right]. \tag{124}$$

We transform Eq. (117) similarly to Eq. (34),

$$-ip_*' = \frac{\rho_0}{k_z B_0}\left\{ \frac{\Omega^2 r}{\hat{\alpha}_s}\tau_B - \left[D_\nu - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\tilde{\omega}(\tilde{\omega}+i\Delta_\perp)\hat{\alpha}_s}\right]\tilde{B}_r + i\Omega\left(2\tilde{\omega}+\frac{m\Omega}{\hat{\alpha}_s}\right)\tilde{B}_\theta \right\}. \tag{125}$$

Using Eq. (119), this equation leads to the following modification of Eq. (35) [cf. Eq. (93)]:

$$-ip_*' = -i\Omega\left(2\tilde{\omega}+\frac{m\Omega}{\hat{\alpha}_s}\right)\frac{k_y}{D_\nu}p_* + \frac{\rho_0}{k_z B_0}\Omega^2 r\frac{\tau_B}{\hat{\alpha}_s} - \frac{\rho_0}{k_z B_0} \times\left[D_\nu - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{\Omega^4 k_z^2 r^2}{\hat{\alpha}_s\tilde{\omega}(\tilde{\omega}+i\Delta_\perp)} - \frac{\Omega^2}{D_\nu}\left(2\tilde{\omega}+\frac{m\Omega}{\hat{\alpha}_s}\right)\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\right]\tilde{B}_r. \tag{126}$$

Substituting here the value τ_B from Eq. (30) with C_1 and C_2 given by Eqs. (123) and (124), we arrive at Eq. (36) with

$$\bar{C}_1 = -\Omega\left[\left(2\tilde{\omega}+\frac{m\Omega}{\hat{\alpha}_s}\right)k_y\left(1+\frac{\beta}{\hat{\alpha}_s}\right) + \frac{4\pi\rho_0 r\Omega}{\hat{\alpha}_s}C_2 \right], \tag{127}$$

$$C_3 = 4\pi\rho_0\left\{ D\left[D_\nu - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r} - \frac{k_z^2}{\hat{\alpha}_s}\frac{\Omega^4 r^2}{\tilde{\omega}(\tilde{\omega}+i\Delta_\perp)} - \frac{\Omega^2}{D_\nu}\left(2\tilde{\omega}+\frac{m\Omega}{\hat{\alpha}_s}\right)\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\right] - \frac{C_1}{\hat{\alpha}_s}r\Omega^2 \right\}. \tag{128}$$

Substituting here Eqs. (123) and (124), one has [cf. Eqs. (39) and (96)]

$$\bar{C}_1 = -\Omega\left[2k_y\tilde{\omega}\left(1+\frac{\beta}{\hat{\alpha}_s}\right) + \frac{r\Omega D_\nu}{\hat{\alpha}_s v_A^2}\right], \tag{129}$$

$$C_3 = 4\pi\rho_0\left[D_\nu\left(1+\frac{\beta}{\hat{\alpha}_s}\right)\left(D_\nu - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) - + \frac{r^2\Omega^4 D_\nu{}^2}{\tilde{\omega}(\tilde{\omega}+i\Delta_\perp)\hat{\alpha}_s v_A^2} - 2\Omega^2\tilde{\omega}\left(1+\frac{\beta}{\hat{\alpha}_s}\right) \times\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\right]. \tag{130}$$

Comparing Eq. (129) with Eq. (123) leads to the conclusion that, due to the dissipative term with $\Delta_\perp d\ln\Omega/d\ln r$ in Eq. (123), in the case considered,

$$\bar{C}_1 \neq C_1. \tag{131}$$

For the condition (131), instead of Eq. (42), one arrives at

$$D\left[\left(\frac{D\tau_B}{C_2}\right)' + \frac{\bar{C}_1 - C_1}{C_2}\tau_B\right] + \Lambda\tilde{B}_r = 0. \quad (132)$$

Here the coefficient Λ is given by Eq. (43) with

$$a = C_3 - C_1\bar{C}_1/C_2. \quad (133)$$

When C_1 and $\bar{C}_1$ are given by Eqs. (123) and (129), one has

$$\delta C_1 \equiv \bar{C}_1 - C_1 = -\frac{ik_y\Delta_\perp}{2\Omega v_A^2}\left(1 + \frac{\beta}{\hat{\alpha}_s}\right)\frac{d\Omega^2}{d\ln r}. \quad (134)$$

Equation (132) yields the dispersion relation given by Eq. (47). We have neglected here the term with δC_1 which is small as $1/(k_r r)$.

The value $1/C_2$ is approximated by Eq. (49) with the substitution $\alpha_s \to \hat{\alpha}_s$ and $D_0 \to D_\nu$. Then, instead of Eq. (50), one has

$$a = 4\pi\rho_0 D_\nu\left\{\left(1 + \frac{\beta}{\hat{\alpha}_s}\right)\left(D_\nu - \frac{d\Omega^2}{d\ln r} - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{\Omega^4 r^2 D_\nu}{\tilde{\omega}(\tilde{\omega} + i\Delta_\perp)\hat{\alpha}_s v_A^2} + 2\tilde{\omega}\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\frac{r^2\Omega^2}{m^2 v_A^2}\right\}. \quad (135)$$

The value b is represented in the form of Eq. (51). Similarly to Eqs. (55) and (56), we obtain

$$b^{(0)} = 4\pi\rho_0 D_\nu\left(1 + \frac{\beta}{\hat{\alpha}_s}\right)\left[\frac{d\Omega^2}{d\ln r} - \frac{1}{m}\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\frac{d\Omega}{d\ln r}\right], \quad (136)$$

$$b^{(1)} = \frac{4\pi\rho_0 D_\nu^2 r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r}. \quad (137)$$

As a result, similarly to Eq. (57), we arrive at

$$\Lambda = \Lambda_\perp \equiv 4\pi\rho_0 D_\nu\left[\left(1 + \frac{\beta}{\hat{\alpha}_s}\right)\left(D_\nu - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{D_\nu r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r} + 2\tilde{\omega}\left(2\tilde{\omega} - i\Delta_\perp\frac{d\ln\Omega}{d\ln r}\right)\frac{r^2\Omega^2}{m^2 v_A^2} + \frac{\Omega^4 r^2}{\tilde{\omega}(\tilde{\omega} + i\Delta_\perp)\hat{\alpha}_s v_A^2}D_\nu\right]. \quad (138)$$

Using Eq. (138), the dispersion relation (47) takes the form [cf. Eqs. (59) and (99)]

$$\frac{k_\perp^2}{k_y^2}D_\nu(\hat{\alpha}_s + \beta) - (\hat{\alpha}_s + \beta)\Omega^2\frac{d\ln\rho_0}{d\ln r} + \frac{\hat{\alpha}_s}{k_y^2 v_A^2}(\tilde{\omega}^2 - k_z^2 v_A^2)\frac{d\Omega^2}{d\ln r} + \frac{4\tilde{\omega}^2\hat{\alpha}_s}{k_y^2 v_A^2}\Omega^2 + \frac{\Omega^4 r^2 D_\nu}{\tilde{\omega}(\tilde{\omega} + i\Delta_\perp)v_A^2} = 0. \quad (139)$$

Let us consider Eq. (139) for strong perpendicular viscosity, $\Delta_\perp \gg \tilde{\omega}$. Then, similarly to Eq. (101), one has

$$\hat{\alpha}_s = 1. \quad (140)$$

In this limit, it follows from Eq. (139) that

$$\tilde{\omega} = -\frac{ik_z^2 v_A^2}{\Delta_\perp(1 + \beta)}g_0^\perp, \quad (141)$$

where $g_0^\perp$ is given by

$$g_0^\perp = 1 + \beta + \frac{1}{k_\perp^2 v_A^2}\left[\frac{d\Omega^2}{d\ln r} + \frac{m^2}{r^2 k_z^2}(1 + \beta)\frac{d\ln\rho_0}{d\ln r} - \frac{m^2\Omega^4}{k_z^2 v_A^2}\right]. \quad (142)$$

VI. COMMENTS ON THE MECHANISM OF INFLUENCE OF PARALLEL AND PERPENDICULAR VISCOSITIES ON NONAXISYMMETRIC MRI

It follows from Eq. (97) that for $\Delta_\parallel \to \infty$,

$$\Lambda_{\parallel(\Delta_\parallel\to\infty)} = 4\pi\rho_0 D_0\left[(1 + \beta)\left(D_0 - \Omega^2\frac{d\ln\rho_0}{d\ln r}\right) + \frac{D_0 r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r} + \frac{\Omega^4 r^2}{v_A^2} + \frac{4\tilde{\omega}^2\Omega^2}{m^2 v_A^2}\right]. \quad (143)$$

Comparing Eq. (143) with Eq. (57), one can see that the parameter α_s does not enter Eq. (143). Physically this parameter describes the engagement of the perpendicular plasma motion with the parallel one. Therefore, it is reasonable to conclude that the mechanism of influence of the parallel viscosity on the MRI is modification of this engagement.

Turning to Eq. (138), one can find that for $\Delta_\perp \gg (k_z c_s, k_z v_A)$,

$$\Lambda_\perp = 4\pi\rho_0 D_\nu\left\{i\Delta_\perp\tilde{\omega}(1 + \beta) - k_z^2 v_A^2\left[(1 + \beta)\times\left(1 + \frac{\Omega^2}{k_z^2 v_A^2}\frac{d\ln\rho_0}{d\ln r}\right) + \frac{r^2}{m^2 v_A^2}\frac{d\Omega^2}{d\ln r} - \frac{\Omega^4 r^2}{k_z^2 v_A^2}\right]\right\}. \quad (144)$$

Comparing Eq. (144) with Eq. (138), one can see that, as in the case of strong parallel viscosity, the strong perpendicular viscosity switches off the engagement between the parallel and perpendicular motion. In addition, according to Eqs. (109) and (110), for $\Delta_\perp \gg \tilde{\omega}$ it increases the perturbed magnetic field induced for given perturbed velocities.

VII. DISCUSSIONS

We have analyzed the nonaxisymmetric perturbations in a cylindrical rotating plasma aiming at the conditions of laboratory experiments. We have allowed for a positive plasma pressure gradient, $p_0' > 0$, balancing the fluid rotation, in contrast to the astrophysics scenario, where the plasma rotation is balanced by the gravitational force.

Having allowed for the equilibrium plasma pressure, we have been forced to take into account the effects of the perturbed plasma pressure and density, which were not considered in the classical works.[1,2] Correspondingly, in addition to the effects studied in Refs. 1 and 2, our equations contain the terms related to the equilibrium plasma pressure and density gradients. This fact is documented by the expression for the instability boundary in ideal plasma given by Eqs. (65) and (66).

Thereby, our analysis predicts the three driving mechanisms of the ideal MRI: the classical one due to the negative $d\Omega^2/d\ln r$ (the Velikhov effect) and two new mechanisms, one of which is related to the square of the plasma pressure gradient and the second one to the cross effect of pressure and density gradients.

We have shown that for strong viscosities, the parallel-viscosity-driven MRI looks like an ideal instability independent of the viscosity, see Eq. (107), while the perpendicular-viscosity-driven one looks like a dissipative one with the growth rate inversely proportional to the characteristic perpendicular-viscosity-induced decay rate; see Eq. (141).

To obtain the above physical results, we have elaborated a mathematical apparatus based on using the Frieman–Rotenberg variable p_*, introduced by Eqs. (25) and (85). Similarly to Refs. 32 and 5, we have derived the pair of first-order differential equations for p_* and the radial perturbed magnetic field $\tilde{B}_r$. These equations have the form of Eqs. (30) and (36) with the coefficients D, C_1, $\bar{C}_1$, C_2, and C_3. In the case of ideal plasma, these coefficients are given by Eqs. (31)–(33) and (39). Instead of this, in the presence of parallel viscosity one has Eqs. (89)–(91) and (96). In both of these cases, $\bar{C}_1 = C_1$. In the presence of perpendicular viscosity, the above coefficients are determined by Eqs. (122)–(124), (129), and (130). Then $\bar{C}_1 \neq C_1$; see Eq. (131).

Using the above pair of the first-order differential equations, we have derived second-order differential equation for $\tilde{B}_r$; see Eq. (42). By means of this differential equation, we have derived the local dispersion relation given by Eq. (47).

This dispersion relation contains two contributions of different mathematical structure: the local and differential ones. We have explained that, for correctly allowing for the effect of $d\Omega^2/d\ln r$, one should carefully calculate the differential contribution; see Eqs. (52)–(55). Then one can see the following remarkable fact: the local and differential contributions are mutually canceled to major order of the expansion in a series in $k_z^2 r^2$; see Eq. (57). In other words, this effect looks like a small additive term in this expansion, which is, nevertheless, just the same as the one that enters in the classical instability criterion;[1–3] see Eq. (65).

ACKNOWLEDGMENTS

We are grateful to I. V. Khal'zov, E. A. Pashitskii, and K. N. Stepanov for useful discussions.

This work was supported by the Russian Foundation of Basic Researches (RFBR), Grant No. 06-02-16767, the International Science and Technology Center (ISTC), Grant No. G-1217, the NSERC (Canada) and NATO Collaborative Linkage Grant, the Research Foundation of the State of São Paulo (FAPESP), the National Council of Scientific and Technological Development (CNPq), and the Ministry of Science and Technology, Brazil.

[1] E. P. Velikhov, Sov. Phys. JETP **9**, 995 (1959).
[2] S. Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A. **46**, 253 (1960).
[3] S. A. Balbus and J. F. Hawley, Astrophys. J. **376**, 214 (1991).
[4] S. A. Balbus and J. F. Hawley, Rev. Mod. Phys. **70**, 1 (1998).
[5] A. Bondeson, R. Iacono, and A. Bhattacharjee, Phys. Fluids **30**, 2167 (1987).
[6] C. Wang, J. W. S. Blokland, R. Keppens, and J. P. Goedbloed, J. Plasma Phys. **70**, 651 (2004).
[7] Y. M. Huang and A. B. Hassam, Phys. Plasmas **10**, 204 (2003).
[8] A. Y. Aydemir, Phys. Plasmas **11**, 5065 (2004).
[9] A. B. Mikhailovskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1967), Vol. 3, p. 158.
[10] A. B. Mikhailovskii, *Theory of Plasma Instabilities*, Instabilities of a Homogeneous Plasma (Consultants Bureau, New York, 1974), Vol. 1.
[11] A. B. Mikhailovskii, *Theory of Plasma Instabilities*, Instabilities of an Inhomogeneous Plasma (Consultants Bureau, New York, 1974), Vol. 2.
[12] A. B. Mikhailovskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1975), Vol. 6, p. 77.
[13] A. B. Mikhailovskii, *Electromagnetic Instabilities in an Inhomogeneous Plasma* (Institute of Physics, Bristol, 1992).
[14] A. B. Mikhailovskii, *Instabilities in a Confined Plasma* (Institute of Physics, Bristol, 1998).
[15] A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Plasma Electrodynamics* (Pergamon, Oxford, 1975), Vol. 1.
[16] A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Plasma Electrodynamics* (Pergamon, Oxford, 1975), Vol. 2.
[17] J. G. Lominadze, *Cyclotron Waves in Plasmas* (Pergamon, New York, 1981).
[18] R. C. Davidson, in *Handbook of Plasma Physics*, edited by M. N. Rosenbluth and R. Z. Sagdeev (North-Holland, Amsterdam, 1983), Vol. 1, p. 519.
[19] A. B. Mikhailovskii, in *Handbook of Plasma Physics*, edited by M. N. Rosenbluth and R. Z. Sagdeev (North-Holland, Amsterdam, 1983), Vol. 1, p. 587.
[20] S. A. Balbus and J. F. Hawley, Astrophys. J. **400**, 610 (1992).
[21] I. V. Khal'zov, V. I. Ilgisonis, A. I. Smolyakov, and E. P. Velikhov, Phys. Fluids **18**, 124107 (2006).
[22] L. L. Kitchatinov and G. Rudiger, Mon. Not. R. Astron. Soc. **286**, 757 (1997).
[23] G. Rudiger, L. Primavera, R. Arlt, and D. Elsner, Mon. Not. R. Astron. Soc. **306**, 913 (1999).
[24] D. A. Shalybkov, G. Rudiger, and M. Schultz, Astron. Astrophys. **395**, 339 (2002).
[25] G. Rudiger, M. Schultz, and D. Shalybkov, Phys. Rev. E **67**, 046312 (2003).
[26] S. A. Balbus, Astrophys. J. **616**, 857 (2004).
[27] G. I. Budker, in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, edited by M. A. Leontovich (Pergamon, Oxford, 1961), Vol. 1, p. 145.
[28] A. Schluter, Z. Naturforsch. A **12A**, 822 (1957).
[29] S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[30] V. S. Tsypin, Sov. Phys. Tech. Phys. **17**, 543 (1972).
[31] E. Frieman and N. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).
[32] E. Hameiri, J. Math. Phys. **22**, 2080 (1981).
[33] R. J. Nijboer and J. P. Goedbloed, J. Plasma Phys. **61**, 241 (1999).
[34] R. Keppens, F. Casse, and J. P. Goedbloed, Astrophys. J. Lett. **569**, L121 (2002).
[35] H. Goedbloed and R. Keppens, Space Sci. Rev. **121**, 55 (2005).
[36] A. Bondeson and R. Iacono, Phys. Fluids B **1**, 1431 (1989).
[37] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).