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A generalized nonlinear Schrödinger equation: Classical field-theoretic approach

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Abstract – An exact classical field theory, for a recently proposed nonlinear generalization of the Schrödinger equation, is presented. In this generalization, a nonlinearity depending on an index q appears in the kinetic term, such that the free-particle linear Schrödinger equation is recovered in the limit $q \to 1$. It is shown that besides the usual $\Psi(\vec{x},t)$, a new field $\Phi(\vec{x},t)$ must be introduced, which becomes $\Psi^*(\vec{x},t)$ only when $q \to 1$. In analogy to the linear case, $\rho(\vec{x},t) = \frac{1}{2V}[\Psi(\vec{x},t)\Phi(\vec{x},t) + \Psi^*(\vec{x},t)\Phi^*(\vec{x},t)]$ is interpreted as the probability density for finding the particle at time t, in a given position $\vec{x}$ inside an arbitrary finite volume V, for any q. The possible physical consequences are discussed, and, in particular, the important property that the fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ do not interact with light.

Nonlinear (NL) equations have become an important subject of physics in the latest years, due to the possibility of explaining several complex behaviors in nature [1,2]. In many cases, NL equations cannot be solved exactly and one is obliged to make use of approximative analytical methods, or numerical procedures; in the last case, the recent improvements in computer technology have contributed to considerable progresses in their study. Many areas of physics have benefited from the advances in the study of NL equations, like nonlinear optics, superconductivity, plasma physics, and nonequilibrium statistical mechanics, since many physical situations in these areas are described in terms of these equations.

In most cases, NL equations are proposed as generalizations of linear ones, so that the latter may be recovered in certain limits. Essentially, two ways have been mostly used in the literature for constructing such generalizations: i) the addition of new NL terms to a linear equation; ii) the modification of the exponent of existing linear terms. As examples of the procedures above, one could mention the NL Schrödinger equations (NLSEs) [3] and Fokker-Planck ones [4]. In the most common formulation of the NLSE one uses the first procedure by introducing a new cubic term in the wave function; this term, for some particular type of solution, is responsible for the modulation of the wave function. On the other hand, the second procedure has been mostly employed for the NL Fokker-Planck equations; in particular, the introduction of a power $(2-q)$ in the probability of the diffusion term [5,6] has been used within the framework of nonextensive statistical mechanics. This type of equation has led to the possibility of explaining many interesting physical phenomena related to anomalous diffusion [7] (for which $q \neq 1$).

In the same way that the linear Fokker-Planck equation is associated to normal diffusion and to the Boltzmann-Gibbs entropy, its NL counterparts may be also related to anomalous-diffusion phenomena and to generalized entropies [8,9], like the nonadditive one of nonextensive statistical mechanics [10,11]. Such a theory emerged from a generalization of the Boltzmann-Gibbs entropy, through the introduction of a real parameter q, in such a way that the former is recovered in the limit $q \to 1$ [12]. Since then, a lot of progress occurred in this area, leading to many generalized functions, distributions, and important equations of physics. In particular, the q-Gaussian distribution, which represents a generalization of the standard Gaussian, appears naturally from an extremization procedure of the entropy [12], or from the solution

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of the corresponding nonlinear Fokker-Planck equation [5,6], or even from generalized forms of the Central Limit Theorem [13,14]. This distribution has been very useful for experiments in many real systems [10,11]; among many others, one could highlight: i) the velocities of cold atoms in dissipative optical lattices [15]; ii) the velocities of particles in quasi two-dimensional dusty plasma [16]; iii) single ions in radio frequency traps interacting with a classical buffer gas [17]; iv) the relaxation curves of RKKY spin glasses, like CuMn and AuFe [18]; v) transverse momenta distributions at LHC experiments [19]; vi) the overdamped motion of interacting vortices in type II superconductors [20].

Recently, generalized Schrödinger equations direclty related to nonextensive statistical mechanics were proposed in the literature [21,22]. The equation introduced in ref. [22] is essentially linear, but characterized by position-dependent masses; this effect comes naturally from a nonadditive translation operator, which can be identified with a q-exponential. On the other hand, the NLSE of ref. [21] consists of a generalization similar to the one introduced in refs. [5,6] for the NL Fokker-Planck equation. Considering a particle of mass m in a three-dimensional space, the NLSE is given by

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\phi(\vec{x},t)}{\phi_0}\right] = -\frac{1}{2-q}\,\frac{\hbar^2}{2m}\nabla^2\left[\frac{\phi(\vec{x},t)}{\phi_0}\right]^{2-q}, \qquad (1)$$

where the scaling of the wave function by ϕ_0 guarantees the correct physical dimensionalities for all terms. Such a scaling becomes irrelevant only for linear equations (*e.g.*, in the particular case $q=1$ of eq. (1)). Considering a free particle normalized to an arbitrary finite volume V, one has that $\phi_0 = 1/\sqrt{V}$. In this way, one may define the dimensionless wave function $\Psi(\vec{x},t)$ through $\phi(\vec{x},t) = (1/\sqrt{V})\Psi(\vec{x},t)$, leading to

$$i\hbar\frac{\partial\Psi(\vec{x},t)}{\partial t} = -\frac{1}{2-q}\,\frac{\hbar^2}{2m}\nabla^2\left[\Psi(\vec{x},t)\right]^{2-q}. \qquad (2)$$

As proposed in ref. [21], a solution of the equation above is the q-plane wave,

$$\Psi(\vec{x},t) = \exp_q[i(\vec{k}\cdot\vec{x}-\omega t)], \qquad (3)$$

expressed in terms of the q-exponential function $\exp_q(u)$ that emerges in nonextensive statistical mechanics [10]. This function generalizes the standard exponential, and for a pure imaginary iu, it is defined as

$$\exp_q(iu) = [1+(1-q)iu]^{\frac{1}{1-q}}; \quad [\exp_1(iu) \equiv \exp(iu)], \qquad (4)$$

where we used $\lim_{\epsilon\to 0}(1+\epsilon)^{1/\epsilon} = e$. For $q \neq 1$ it exhibits an oscillatory behavior characterized by an amplitude $\rho_q \neq 1$ [23–25]; in particular, for $q>1$ this amplitude decreases for increasing arguments, since

$$\exp_q(\pm iu) = \cos_q(u) \pm i\sin_q(u), \qquad (5)$$

$$\cos_q(u) = \rho_q(u)\cos\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \qquad (6)$$

$$\sin_q(u) = \rho_q(u)\sin\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\}, \qquad (7)$$

$$\rho_q(u) = \left[1+(1-q)^2u^2\right]^{1/[2(1-q)]}. \qquad (8)$$

Moreover, $\exp_q(iu)$ presents other peculiar properties,

$$[\exp_q(iu)]^* = \exp_q(-iu) = [1-(1-q)iu]^{\frac{1}{1-q}}, \qquad (9)$$

$$\exp_q(iu)[\exp_q(iu)]^* = \rho_q^2(u) = \left[1+(1-q)^2u^2\right]^{\frac{1}{1-q}}, \qquad (10)$$

$$\exp_q(iu_1)\exp_q(iu_2) = \exp_q[iu_1+iu_2-(1-q)u_1u_2], \qquad (11)$$

$$\left\{[\exp_q(iu)]^\alpha\right\}^* = \left\{[\exp_q(iu)]^*\right\}^\alpha = [\exp_q(-iu)]^\alpha, \qquad (12)$$

for any α real. By integrating eq. (10) from $-\infty$ to $+\infty$, one obtains [24]

$$\mathcal{I}_q = \int_{-\infty}^{\infty} \mathrm{d}u\,[\rho_q(u)]^2 = \left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{3-q}{2(q-1)}\right)}{(q-1)\Gamma\left(\frac{1}{q-1}\right)}\right]^{1/2}, \qquad (13)$$

from which one readily verifies the physically important property of square integrability for $1<q<3$; as typical examples, one has $\mathcal{I}_{3/2} = \mathcal{I}_2 = \sqrt{\pi}$. Notice that this integral diverges in both limits $q\to 1$ and $q\to 3$, as well as for any $q<1$.

For $1<q<3$, due to the property that the amplitude of the q-plane wave decreases when its argument $(\vec{k}\cdot\vec{x}-\omega t)$ increases, this new type of solution describes a typical nonlinear oscillatory phenomenon. A given physical system may be characterized by a single value of q; by specifying the rate of decay of the q-plane wave amplitude (which should be a characteristic of a given physical system), *i.e.,* the modulation of eq. (8), we may determine the appropriate value of q. Therefore, this type of solution can be considered as a good candidate for describing nonlinear physical phenomena characterized by oscillatory motion with modulation in both space and time, like those appearing in superconductivity, plasma physics, nonlinear optics, and lattice dynamics of solids.

We will now develop an exact classical field theory associated with eq. (2) and its complex conjugate. For reasons that will become clear later, we introduce also an additional dimensionless field, $\Phi(\vec{x},t)$. The equations of motion for classical fields can be derived by means of the principle of stationary action, through the definition of a Lagrangian density $\mathcal{L}$, which will depend on the dimensionless fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$, as well as on its spatial and time derivatives,

$$\mathcal{L} \equiv \mathcal{L}\Big(\Psi, \vec{\nabla}\Psi, \dot{\Psi}, \Phi, \vec{\nabla}\Phi, \dot{\Phi}, \Psi^*, \vec{\nabla}\Psi^*, \dot{\Psi}^*, \Phi^*, \vec{\nabla}\Phi^*, \dot{\Phi}^*\Big). \qquad (14)$$

Now we consider heuristically the Lagrangian density (whose justification will emerge later on),

$$\begin{aligned}\mathcal{L} = A\Big\{ & i\hbar\Phi(\vec{x},t)\dot{\Psi}(\vec{x},t) \\ & - \frac{\hbar^2}{2m}\,[\Psi(\vec{x},t)]^{1-q}\,[\vec{\nabla}\Phi(\vec{x},t)]\cdot[\vec{\nabla}\Psi(\vec{x},t)] \\ & - i\hbar\Phi^*(\vec{x},t)\dot{\Psi}^*(\vec{x},t) \\ & - \frac{\hbar^2}{2m}\,[\Psi^*(\vec{x},t)]^{1-q}\,[\vec{\nabla}\Phi^*(\vec{x},t)]\cdot[\vec{\nabla}\Psi^*(\vec{x},t)]\Big\}, \end{aligned} \quad (15)$$

where $A \equiv 1/(2qV)$ is a multiplicative constant. As will be shown below, a simple Lagrangian density constructed only in terms of $\Psi(\vec{x},t)$ and $\Psi^*(\vec{x},t)$ (as usually done in the case $q=1$ [26,27]) does not lead to the NLSE of eqs. (2) and its complex conjugate; for this reason, the second field, $\Phi(\vec{x},t)$, was introduced.

From the above Lagrangian density one may construct a classical action, which may be extremized to yield the Euler-Lagrange equations for each field [26,27]. The Euler-Lagrange equation for the field Φ,

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \vec{\nabla}\cdot\left[\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\Phi)}\right] - \frac{\partial}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = 0, \quad (16)$$

leads to

$$\begin{aligned} & i\hbar\frac{\partial\Psi(\vec{x},t)}{\partial t} + \frac{\hbar^2}{2m}(1-q)[\Psi(\vec{x},t)]^{-q}\,[\nabla\Psi(\vec{x},t)]^2 \\ & + \frac{\hbar^2}{2m}[\Psi(\vec{x},t)]^{1-q}\,\nabla^2\Psi(\vec{x},t) = 0, \end{aligned} \quad (17)$$

which corresponds to the NLSE of eq. (2). Carrying the same procedure for the field $\Psi(\vec{x},t)$ one obtains

$$i\hbar\frac{\partial\Phi(\vec{x},t)}{\partial t} = \frac{\hbar^2}{2m}[\Psi(\vec{x},t)]^{1-q}\,\nabla^2\Phi(\vec{x},t), \quad (18)$$

with corresponding equations holding for the complex-conjugate fields, $\Psi^*(\vec{x},t)$ and $\Phi^*(\vec{x},t)$.

It is important to notice that eq. (18) becomes the complex conjugate of eq. (2) only for $q=1$, in which case $\Phi(\vec{x},t) = \Psi^*(\vec{x},t)$. For all $q \neq 1$, one has that $\Phi(\vec{x},t)$ is distinct from $\Psi^*(\vec{x},t)$, with the fields $\Phi(\vec{x},t)$ and $\Psi(\vec{x},t)$ being related by eq. (18).

Now, if one substitutes the solution of eq. (2) (*i.e.*, the q-exponential of eq. (3)) in eq. (18), one finds

$$\Phi(\vec{x},t) = \{\exp_q[i(\vec{k}\cdot\vec{x} - \omega t)]\}^{-q} = [\Psi(\vec{x},t)]^{-q}. \quad (19)$$

The canonical conjugate fields to those above are

$$\begin{aligned} & \Pi_\Psi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = i\hbar A\Phi; \quad \Pi_{\Psi^*} = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}^*} = -i\hbar A\Phi^*; \\ & \Pi_\Phi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = 0; \quad \Pi_{\Phi^*} = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}^*} = 0, \end{aligned} \quad (20)$$

which yield the Hamiltonian density,

$$\begin{aligned}\mathcal{H} & = \Pi_\Psi\dot{\Psi} + \Pi_{\Psi^*}\dot{\Psi}^* + \Pi_\Phi\dot{\Phi} + \Pi_{\Phi^*}\dot{\Phi}^* - \mathcal{L} \\ & = A\,\frac{\hbar^2}{2m}\Big\{[\Psi(\vec{x},t)]^{1-q}\,[\vec{\nabla}\Phi(\vec{x},t)]\cdot[\vec{\nabla}\Psi(\vec{x},t)] \\ & \quad + [\Psi^*(\vec{x},t)]^{1-q}\,[\vec{\nabla}\Phi^*(\vec{x},t)]\cdot[\vec{\nabla}\Psi^*(\vec{x},t)]\Big\}. \end{aligned} \quad (21)$$

This Hamiltonian density, for the solutions presented in eqs. (3) and (19), leads to an important property, namely, that each of the two terms above is real separately, *i.e.,* for all times,

$$\begin{aligned} & [\Psi(\vec{x},t)]^{1-q}\,[\vec{\nabla}\Phi(\vec{x},t)]\cdot[\vec{\nabla}\Psi(\vec{x},t)] = \\ & [\Psi^*(\vec{x},t)]^{1-q}\,[\vec{\nabla}\Phi^*(\vec{x},t)]\cdot[\vec{\nabla}\Psi^*(\vec{x},t)] = \\ & q(k_x^2 + k_y^2 + k_z^2) = qk^2. \end{aligned} \quad (22)$$

Then, the total energy of the system becomes

$$E = \int \mathrm{d}^3x\;\mathcal{H} = \frac{\hbar^2k^2}{2m}, \quad (23)$$

preserving the energy spectrum of the free particle, for all values of q, in agreement with ref. [21].

Now we define the momentum density,

$$\begin{aligned}\vec{\mathcal{P}} & = \frac{1}{2V}\Big\{-i\hbar[\vec{\nabla}\Psi(\vec{x},t)]\Phi(\vec{x},t) + i\hbar[\vec{\nabla}\Psi^*(\vec{x},t)]\Phi^*(\vec{x},t)\Big\} \\ & = -q\Big\{\Pi_\Psi\vec{\nabla}\Psi(\vec{x},t) + \Pi_{\Psi^*}\vec{\nabla}\Psi^*(\vec{x},t)\Big\}, \end{aligned} \quad (24)$$

which presents a property similar to the Hamiltonian density, *i.e.*, using the solutions above both terms are real,

$$-q\Pi_\Psi\vec{\nabla}\Psi(\vec{x},t) = -q\Pi_{\Psi^*}\vec{\nabla}\Psi^*(\vec{x},t) = \frac{\hbar\vec{k}}{2V}, \quad (25)$$

leading to the total momentum,

$$\vec{p} = \int \mathrm{d}^3x\;\vec{\mathcal{P}} = \hbar\vec{k}. \quad (26)$$

Consistently with the above, we define the probability density for finding a particle at time t, in a given position $\vec{x}$ inside an arbitrary finite volume V, as

$$\rho(\vec{x},t) = \frac{1}{2V}[\Psi(\vec{x},t)\Phi(\vec{x},t) + \Psi^*(\vec{x},t)\Phi^*(\vec{x},t)], \quad (27)$$

for any value of q. Its time derivative is given by

$$\begin{aligned} i\hbar\frac{\partial\rho(\vec{x},t)}{\partial t} = \frac{i\hbar}{2V}\Big[& \frac{\partial\Psi(\vec{x},t)}{\partial t}\Phi(\vec{x},t) + \Psi(\vec{x},t)\frac{\partial\Phi(\vec{x},t)}{\partial t} \\ & + \frac{\partial\Psi^*(\vec{x},t)}{\partial t}\Phi^*(\vec{x},t) + \Psi^*(\vec{x},t)\frac{\partial\Phi^*(\vec{x},t)}{\partial t}\Big]. \end{aligned} \quad (28)$$

Now, using eqs. (2), (18), and their complex conjugates, one may write the equation above as

$$\begin{aligned} & i\hbar\frac{\partial\rho}{\partial t} + \frac{\hbar^2}{4mV}\;\vec{\nabla}\cdot\Big\{\Psi^{1-q}(\vec{\nabla}\Psi)\Phi - (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\Phi^* \\ & - \Psi^{2-q}(\vec{\nabla}\Phi) + (\Psi^*)^{2-q}(\vec{\nabla}\Phi^*)\Big\} + \frac{\hbar^2}{4mV}(1-q) \\ & \times\Big\{\Psi^{1-q}(\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Phi) - (\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\cdot(\vec{\nabla}\Phi^*)\Big\}. \end{aligned} \quad (29)$$

It is important to stress that the equation above may be readily written in the form of a continuity equation only for $q=1$. Indeed, an anomalous term appears for

$q \neq 1$, in such a way that one should impose an additional condition for the solutions. Therefore, within the present frame, solutions must satisfy

$$[\Psi(\vec{x},t)]^{1-q}[\vec{\nabla}\Psi(\vec{x},t)]\cdot[\vec{\nabla}\Phi(\vec{x},t)] \;\in\; \Re, \qquad (30)$$

for the preservation of probability. Since herein we are dealing with nonlinear equations, which usually present more than one solution, some possible solutions may not satisfy the above requirement; such an additional requirement is trivially satisfied for linear equations. One important point is that condition (30) is satisfied for the classical solutions of eqs. (3) and (19), as already seen in eq. (22), *i.e.*, $\Psi^{1-q}(\vec{\nabla}\Psi)\cdot(\vec{\nabla}\Phi)=(\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\cdot(\vec{\nabla}\Phi^*)= qk^2$. Therefore, one may write eq. (29) in the form of a continuity equation,

$$\frac{\partial\rho(\vec{x},t)}{\partial t}+\vec{\nabla}\cdot\vec{j}(\vec{x},t)=0, \qquad (31)$$

where the probability density follows eq. (27) and the current density is given by

$$\vec{j}(\vec{x},t) \;=\; \frac{\hbar}{4miV}\Big\{\Psi^{1-q}(\vec{\nabla}\Psi)\Phi-(\Psi^*)^{1-q}(\vec{\nabla}\Psi^*)\Phi^* - \Psi^{2-q}(\vec{\nabla}\Phi)+(\Psi^*)^{2-q}(\vec{\nabla}\Phi^*)\Big\}. \qquad (32)$$

One readily sees that the above definitions for $\rho(\vec{x},t)$ and $\vec{j}(\vec{x},t)$ recover those of the particular case $q=1$ [28], *i.e.*,

$$\vec{j}(\vec{x},t)=\frac{\hbar}{2miV}\Big(\Psi^*\vec{\nabla}\Psi-\Psi\vec{\nabla}\Psi^*\Big). \qquad (33)$$

The results above show that, in spite of the fact that the q-plane wave of eq. (3) presents a modulation for $q>1$, leading to a finite integral of the product $\Psi(\vec{x},t)\Psi^*(\vec{x},t)$ over space, one cannot plainly associate such a property to a localized particle. In fact, we have shown that the important quantity to be considered should be the probability density $\rho(\vec{x},t)$ of eq. (27), defined in terms of both fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$. For the solutions presented herein, one notices that a free particle is characterized by $\rho(\vec{x},t)=1/V$, and similarly to what is done for the case $q=1$, one needs to consider a normalization with respect to a finite volume V.

Another important property of the field theory presented above concerns the fact that the global gauge invariance [26] does not hold for the equations of the present formalism. Considering the form of the solution in eq. (19), such transformations are given by

$$\begin{aligned}\Psi(\vec{x},t)&\to \mathrm{e}^{i\alpha}\Psi(\vec{x},t); \qquad \Psi^*(\vec{x},t)\to \mathrm{e}^{-i\alpha}\Psi^*(\vec{x},t),\\ \Phi(\vec{x},t)&\to \mathrm{e}^{-iq\alpha}\Phi(\vec{x},t); \qquad \Phi^*(\vec{x},t)\to \mathrm{e}^{iq\alpha}\Phi^*(\vec{x},t).\end{aligned} \qquad (34)$$

Since the Lagrangian density in eq. (15) does not present this symmetry for $q\neq 1$, all equations derived from it also violate this symmetry. In standard field theory, the invariance of the Lagrangian under the transformations above is directly related to the property that the corresponding fields may be associated to charged particles that interact with light. Due to the violation of this symmetry, the fields introduced herein, $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$, cannot interact with light and so, they might be relevant for physical phenomena that are not detectable through light.

To conclude, we have developed an exact classical field theory for a recently proposed nonlinear generalization of the Schrödinger equation, along a path similar to nonextensive statistical mechanics. We have verified that a simple Lagrangian density defined solely in terms of the standard fields, $\Psi(\vec{x},t)$ and $\Psi^*(\vec{x},t)$, does not reproduce the nonlinear Schrödinger equation and its corresponding complex conjugate. Therefore, besides the usual $\Psi(\vec{x},t)$, an extra field $\Phi(\vec{x},t)$ was introduced, with $\Phi(\vec{x},t)\to\Psi^*(\vec{x},t)$ only when $q\to 1$; for $q\neq 1$ these two fields are related by means of an additional nonlinear Schrödinger-like equation. Based on that, a Hamiltonian density was constructed, preserving the usual energy spectrum of the free particle, $E=(\hbar^2k^2)/2m$, as well as the momentum, $\vec{p}=\hbar\vec{k}$, for all q. The appropriate probability density was defined as $\rho(\vec{x},t)=\frac{1}{2V}[\Psi(\vec{x},t)\Phi(\vec{x},t)+\Psi^*(\vec{x},t)\Phi^*(\vec{x},t)]$, which, for the solutions presented, is associated to an extended state, similarly to what happens in the case $q=1$. The conservation of probability, ensured by a continuity equation, does not follow generically from the equations for the two fields (and their corresponding complex conjugates) as happens in the linear case; for that, an additional condition is required for the solutions. It was argued that although the present solutions satisfy this condition, it is possible that other (mathematically admissible) solutions of these nonlinear equations may not fulfill a continuity equation.

Exploratory efforts in possible applications to nonlinear physical phenomena, like those emerging in plasma physics, superconductivity, and nonlinear optics, would be welcome. As natural extensions of this work, the quantization and renormalization of the present classical field theory appear to be nontrivial and constitute goals for future investigations.

A further interesting aspect concerns the fact the Lagrangian density proposed is not invariant under global gauge transformations, implying that the fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ cannot interact with light. In other words, for $q\neq 1$, we can have free massive particles of a nonlinear nature that appear to be unable to couple with light. Consequently, further investigations could render the present mechanism as capable to describe one of the most intriguing features of our Universe: the presence of dark matter.

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