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Has the Hamming Distance a Conjugate Field?

by

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Through Monte Carlo simulations of a standard cellular automaton (Domany-Kinzel), we introduce a new type of external field h defined as the frequency at which differ the random numbers used in the updating of the two replicas involved in the damage spread method. We show, for the first time, that h is the field conjugate to the Hamming distance Ψ (chaotic order parameter).

Key-words: Cellular automata; Hamming distance; Chaos; Conjugate field.

Cellular automata (CA) are totally discrete dynamical systems (discrete space, discrete time and discrete number of states) which provide simple models for a great number of problems in science. CA are frequently used to model chemical reactions, crystal growth models, turbulence, neural networks, biological systems and other nonlinear processes far from thermal equilibrium.[1,2]

In the CA context, the discrete space is represented by a regular lattice and with each site i of the lattice one associates a variable σ_i which can take k different values $\sigma_i = 0,\ 1, \cdots, k-1$. The CA time evolution is defined, at each time step, by local rules where the value σ_i at time t depends, in a deterministic or probabilistic way, on the state of the system at time $t-1$. All the sites are simultaneously updated at each time step. Since its dynamics is not restricted to the usual Boltzmann weight and detailed balance, CA do not necessarily evolve (in the long time asymptotic limit) towards the standard thermal equilibrium.

Although d-dimensional (probabilistic) CA describe processes that might be far from equilibrium, they can frequently be mapped onto $(d+1)$-dimensional statistical-mechanics models.[3] The corresponding spin model is, in general, anisotropic and involves multispin interactions and fields, with coupling constants related to the parameters (conditional probabilities) which specify the evolution rule of the CA. Therefore it is not surprising that even one-dimensional CA exhibit continuous phase transitions with universal critical exponents and scaling laws. In particular, critical frontiers between chaotic and non-chaotic phases do exist and are usually characterized by a vanishing Hamming distance, which plays the role of an order parameter. Its conjugate field is, up to now, an open question which we focus herein.

One of the most studied one-dimensional probabilistic CA is that considered by Domany and Kinzel.[4] It consists of a linear chain of N lattice sites $(i = 1, 2, \cdots, N)$, with periodic boundary conditions. Each site has two possible states $\sigma_i = 0, 1$. The state of the system at time t is specified by the set $\{\sigma_i\}$. At the next time step, the state of the i-th site is $\sigma_i(t+1)$ which equals 0 or 1 according to the conditional probabilities $\{p(\sigma_{i-1}(t), \sigma_i(t)/\sigma_i(t+1))\}$,i.e. $p(00/1)$, $p(10/1)$, $p(01/1)$ and $p(11/1)$. (For instance, $p(00/1)$ denotes the probability of the i-th site to be at state 1 once we know that its "parents" were both at state 0.) Naturally $p(\sigma_{i-1}, \sigma_i/0) = 1 - p(\sigma_{i-1}, \sigma_i/1)$. A possible application of this CA is to model catalysis in chemical reactions.[5] Also, the Domany-Kinzel CA contains, as special cases, the problem of directed percolation and directed compact percolation[6,7] on the square lattice. For simplicity, we shall discuss here the isotropic case $p(01/1) = p(10/1) \equiv p_1$, $p(11/1) \equiv p_2$ and $p(00/1) \equiv p_0$.

The complete phase diagram for the isotropic and "legal" ($p_0 = 0$) Domany-Kinzel CA is depicted[8] in Fig.1. The order parameters characterizing the three phases (*frozen, active* and *chaotic*) are the "magnetization" M, defined as the fraction of sites with value 1, and the normalized Hamming distance Ψ , defined as the fraction of sites that differ in two replicas that started whith *different* initial configurations and evolve under the *same* noise (i.e., the same random numbers sequence). In the chaotic phase, the automaton is sensitive to the initial conditions and an "initial damage", created by flipping randomly a fraction p of the sites, spreads through the entire CA. In the frozen phase we have $M = 0$ and $\Psi = 0$; in the active phase $M \neq 0$ and $\Psi = 0$; and in the chaotic phase $M \neq 0$ and $\Psi \neq 0$.

On general thermodynamic grounds, an external field can be the *conjugate* of a given order parameter. In that case it satisfies: (i) if it is different from zero, the long time asymptotic value of the order parameter cannot vanish (hence the phase transition is destroyed); (ii) the vanishing field susceptibility of the order parameter diverges at the (continuous) phase transition. The question we address here is whether this structure is preserved even for dynamical transitions such as those appearing in CA.

For example, if one considers $p_0 \equiv p(00/1) \neq 0$ and the system is at say configuration zero (i.e., all the sites are at state 0) at time t, it will acquire a nonvanishing fraction of sites with states different from zero ($M_\infty \neq 0$), at all subsequent time steps. The probability p_0 is, consequently, analog to an external field in ferromagnets, which destroys the phase transition. So, if one considers $p_0 \neq 0$, the frozen-active critical surface disappears but the active-chaotic one remains,[9] as it is shown in Fig.2. We can define the associated susceptibility

$$\chi_M \equiv \left.\frac{\partial M}{\partial p_0}\right|_{p_0=0} \quad . \tag{1}$$

It diverges at the frozen-active critical surface as shown[9] in Fig.3. Consequently p_0 can be legitimately considered as the conjugate parameter of M.

Let us now address the central aim of the present paper, namely *the* *existence of a field conjugate to the active-chaotic order parameter* Ψ. This external field h must satisfy two standard properties: h must destroy the active-chaotic phase transition, and the associated linear response function

$$\chi_\Psi \equiv \left.\frac{\partial\Psi}{\partial h}\right|_{h=0} \tag{2}$$

must diverges at the active-chaotic critical surface.

In order to propose the appropriate conjugate field h we are looking for, let us observe that the chaotic phase is detected through the damage spread method. In this method we make (typically after that the CA attains equilibrium) a replica of the system containing an initial damage (a fraction p of the sites flipped at random) and study how this damage behaves as time evolves using the *same* sequence of random numbers for the two systems. So, the basic idea is simply that, say in the active phase ($\Psi_\infty \equiv \lim_{t\to\infty}\Psi(t) = 0$) where the CA normally annihilate the initial damage, if we use (from time to time) different random numbers to update the replica system, it will acquire a nonvanishing fraction of sites which differ from their counterparts in the original CA, at subsequent time steps. Therefore, we shall propose the conjugate field h to be *proportional to the frequency at which distinct random numbers are used to update the replica system.*

As it is shown in Fig.4, there are at least four ways to implement the external field h. The first one (*random individual*; Fig.4(a)) is to apply (with frequency h_1) *discrepant* random numbers for the two replicas at any site at any time. Another way (*regular individual*; Fig.4(b)) is, at any time, to use different random numbers (with frequency h_2) to update the same regularly located sites. Similarly we define the *random collective* (see Fig.4(c)) and the *regular collective* (see Fig.4(d)) cases, respectively associated with h_3 and h_4. In our simulations we implement all four cases with $N = 3200$ and $N = 6400$ sites. For each set of parameters defining the CA we take a quite large number (typically up to 100) of random starting configurations where all states were equally probable. In these simulations the transient time was 10000 time steps and the Hamming distance was averaged over another 30000 time steps. The corresponding Hamming distance as function of the external field in the neighborhood of the active-chaotic critical point is illustrated in Fig.5. The main features of the field dependence of Ψ are the same for h_1, h_2, h_3 and h_4.

The results exhibited in Figs.5 (order parameter) and 6 (susceptibility) clearly show, on one hand, that h destroys the active-chaotic phase transition and, on the other, that the corresponding susceptibility tends to diverge at the $N \to \infty$ limit. Consequently the proposed definition of h (frequency at which discrepant random numbers are applied to both replicas, herein implemented through four slightly different manners) can legitimatelly be considered as the conjugate parameter of the Hamming distance. The robustness of the present definition has been recently checked[10] within a mean-field-like approximation, and full confirmation has been obtained.

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FIGURE CAPTIONS

Figure 1. $p_0 \equiv p(00/1) = 0$ and $p(10/1) = p(01/1)$ (isotropic) phase diagram. The data correspond to simulations with 3200 sites; transients of 10000 (3000) time steps were used for the frozen-active (active-chaotic) phase transitions. The damage was averaged over another 30000 time steps.

Figure 2. Phase diagram ,for the isotropic case ($p(10/1) = p(01/1)$) and arbitrary $p_0 \equiv p(00/1)$.

Figure 3. Susceptibility $\chi_M = \frac{\partial M}{\partial p_0}\Big|_{p_0=0}$ obtained numerically for $p_2 = 0.1$. The data used to take the numerical derivative of M correspond to $p_0 \simeq 10^{-4}$ ($\square$) and $p_0 \simeq 10^{-5}$ ($*$). The system consists of 3200 sites.

Figure 4. Four different ways to implement the external field $h = 1/2$. The black circles represent the sites updated in both replicas with *discrepant* random numbers. τ is the time t at which discrepancies are first introduced. *Random-individual*: at *every* time step, we *randomly* choose a concentration h_1 of sites and update them with discrepant random numbers. *Regular-individual*: at *every* time step, we update with discrepant random numbers a concentration h_2 of *regularly* spaced sites. *Random-collective*: at a concentration h_3 of *randomly* chosen time steps, we update *all* the sites with discrepant random numbers. *Regular-collective*: at a concentration h_4 of *regularly* spaced time steps, we update *all* the sites with discrepant random numbers.

Figure 5. Hamming distance Ψ as a function of the external field h_2 for $p_1 = 0.85$ and $N = 6400$. The data correspond to simulations with transients of 10000 time steps and Ψ was averaged over another 30000 time steps. For $p_1 = 0.85$ the active-chaotic critical point is $p_2 \simeq 0.154$.

Figure 6. Dynamical susceptibility $\chi_\Psi \simeq (\langle\Psi\rangle_{h\neq 0} - \langle\Psi\rangle_{h=0})/h$ as a function of p_2 for (a) $h_1 = 1/640$ (*), $1/576$ ($\times$) and $1/512$ (+); (b) $h_2 = 1/320$ (*), $1/256$ ($\times$) and $1/192$ (+); (c) $h_3 = 1/640$ (*), $1/576$ ($\times$) and $1/512$ (+) and (d) $h_4 = 1/2500$ (*), $1/2000$ ($\times$) and $1/1500$ (+). The data correspond to simulations with $N = 6400$ sites and fixed $p_1 = 0.85$; transients of 10000 time steps and Ψ was averaged over another 30000 time steps. For $p_1 = 0.85$ the active-chaotic critical point is $p_2 \simeq 0.154$

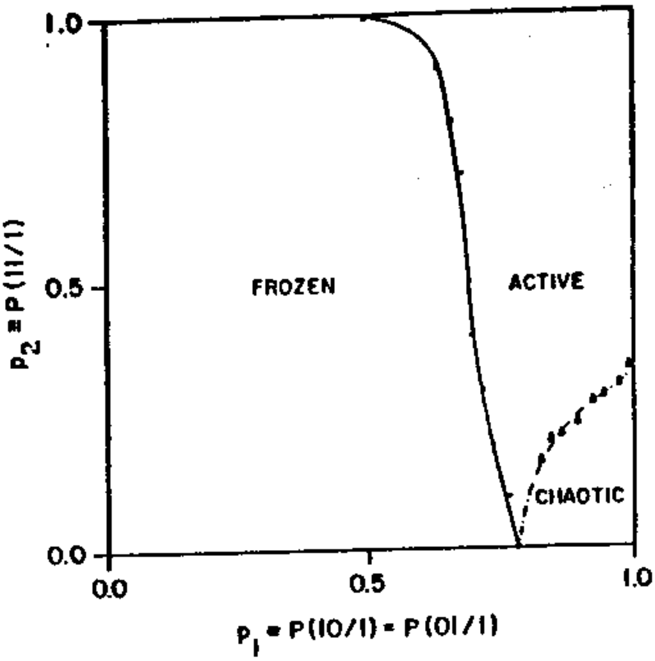
1.0
0.5
0.0
0.0
0.5
1.0
$p_2 = P(11/1)$
$p_1 = P(10/1) = P(01/1)$
FROZEN
ACTIVE
CHAOTIC

Fig. 1

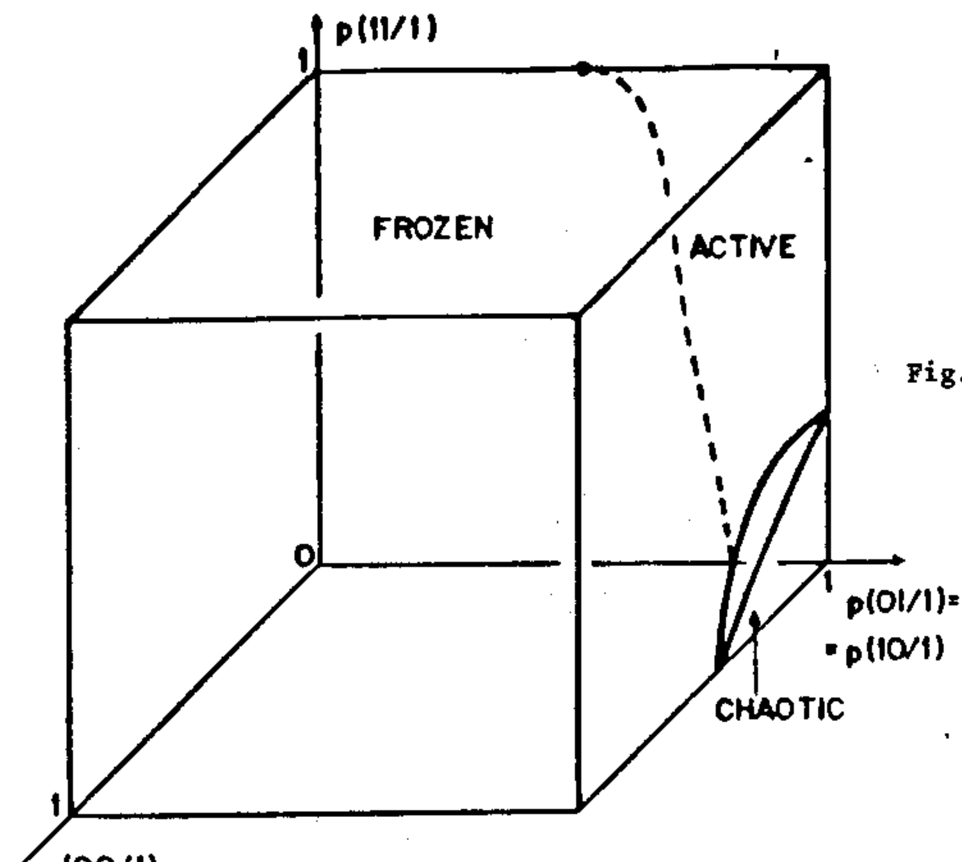
p(11/1)
1
0
1
1
FROZEN
ACTIVE
CHAOTIC
p(01/1)=
= p(10/1)
p(00/1)

Fig. 2

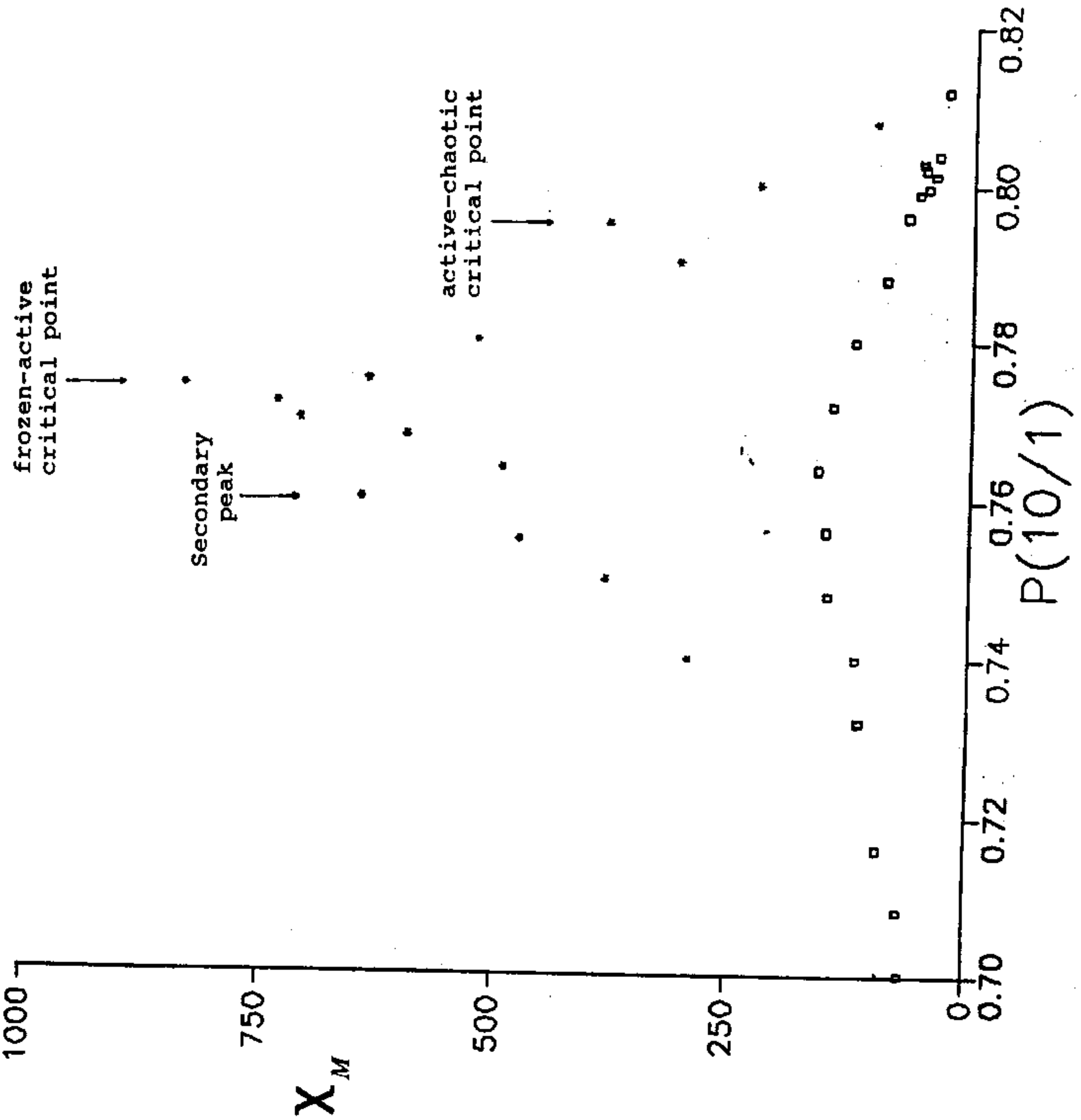
frozen-active
critical point
Secondary
peak
active-chaotic
critical point
X_M
1000
750
500
250
0
0.70
0.72
0.74
0.76
0.78
0.80
0.82
P(10/1)

Fig. 3

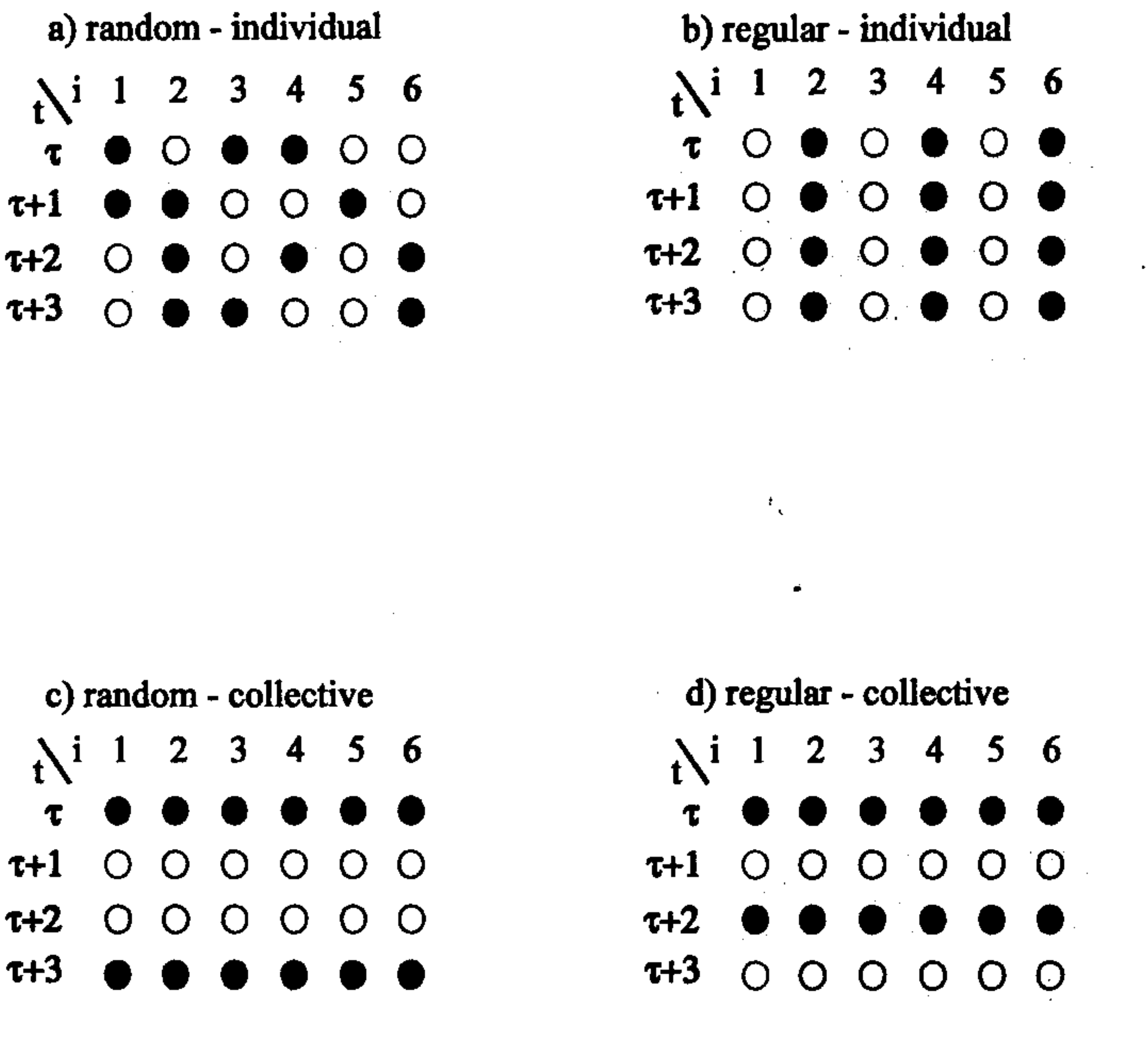
a) random - individual
b) regular - individual
c) random - collective
d) regular - collective
t\i 1 2 3 4 5 6
τ
τ+1
τ+2
τ+3

Fig. 4

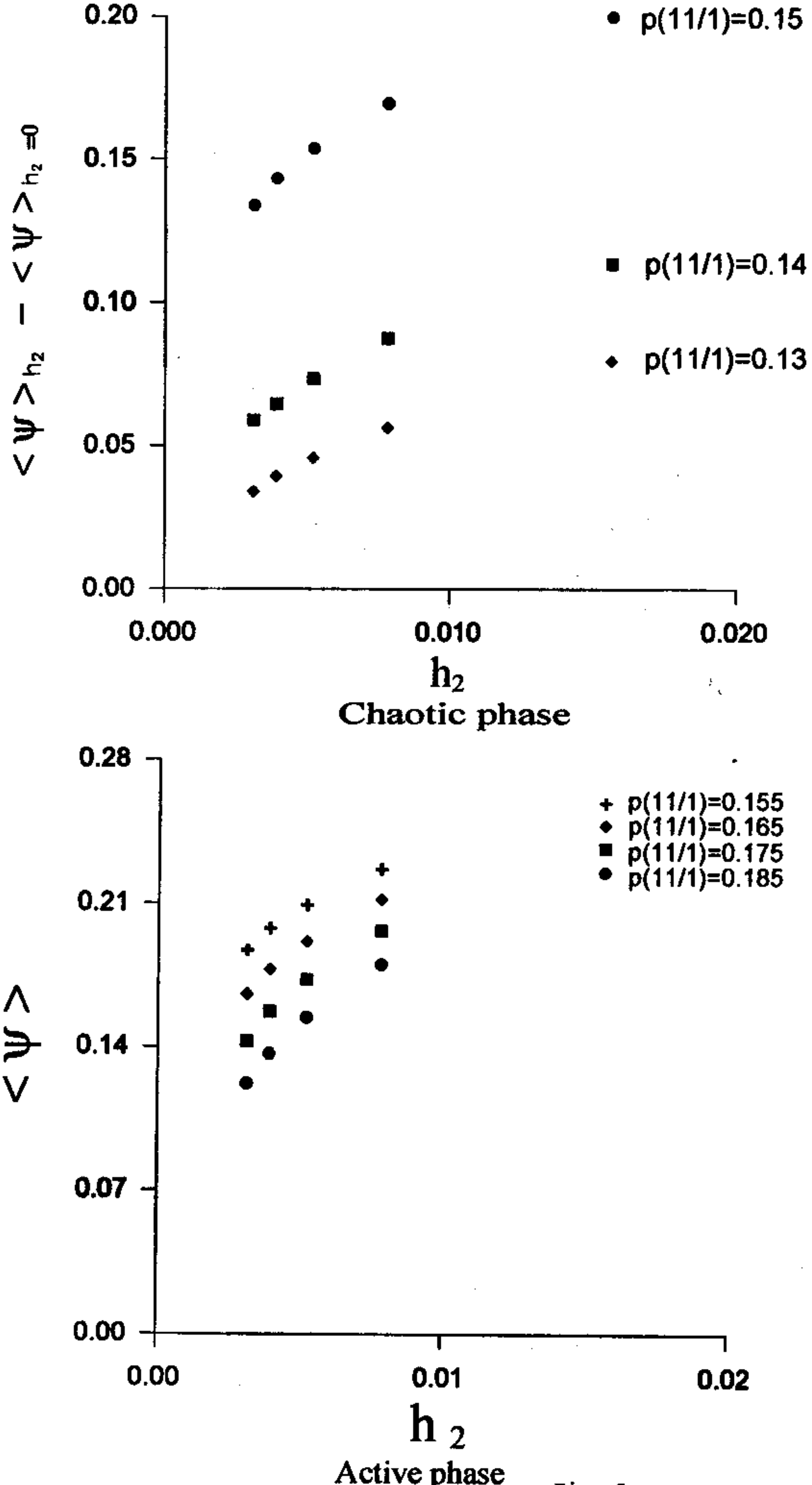
0.20
0.15
0.10
0.05
0.00
0.000
0.010
0.020
h2
Chaotic phase
p(11/1)=0.15
p(11/1)=0.14
p(11/1)=0.13
0.28
0.21
0.14
0.07
0.00
0.00
0.01
0.02
h 2
Active phase
p(11/1)=0.155
p(11/1)=0.165
p(11/1)=0.175
p(11/1)=0.185

Fig. 5

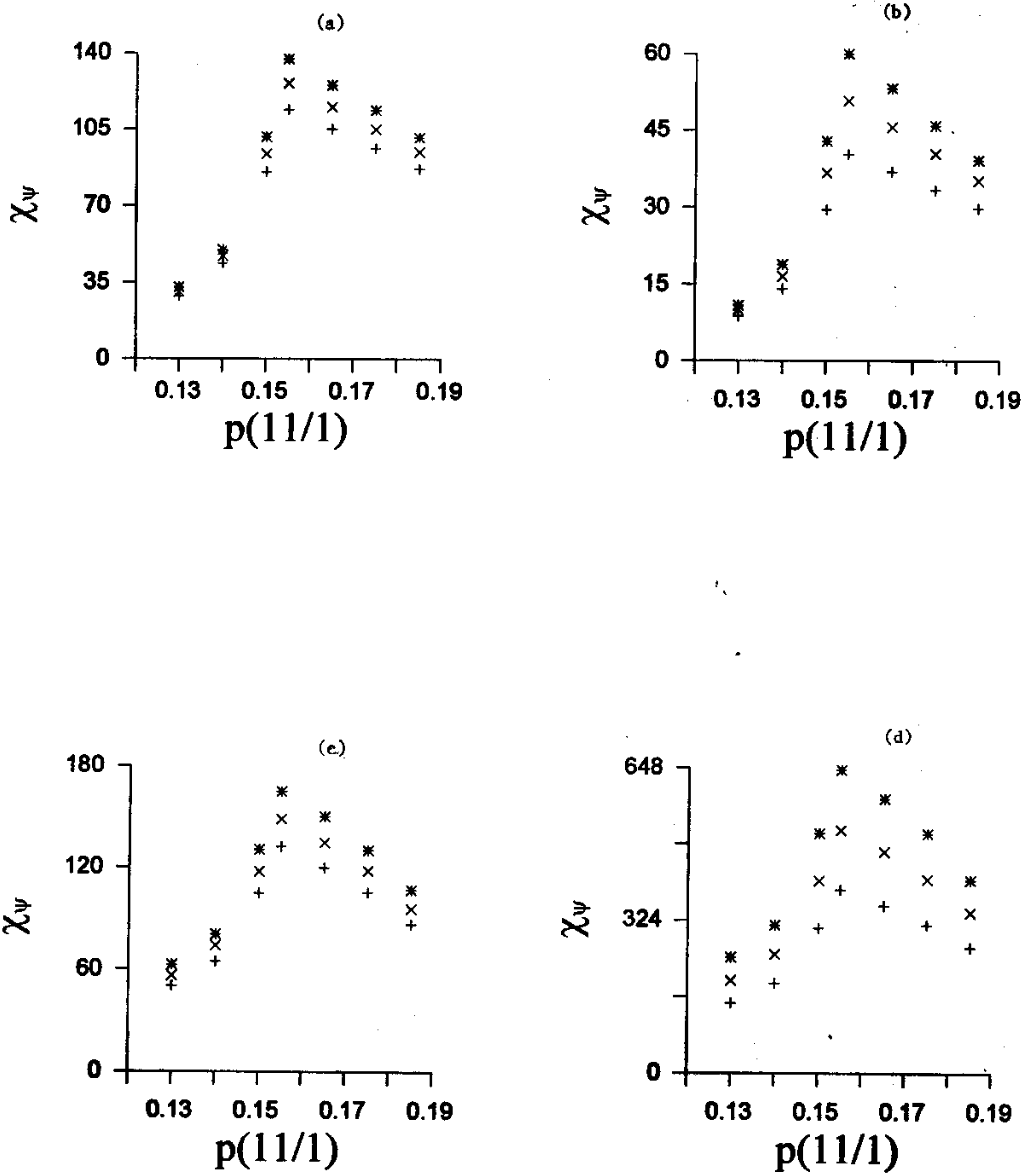
(a)
140
105
70
35
0
χ_ψ
0.13
0.15
0.17
0.19
p(11/1)
(b)
60
45
30
15
0
χ_ψ
0.13
0.15
0.17
0.19
p(11/1)
(c)
180
120
60
0
χ_ψ
0.13
0.15
0.17
0.19
p(11/1)
(d)
648
324
0
χ_ψ
0.13
0.15
0.17
0.19
p(11/1)

Fig. 6

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