# Percolation and nucleation approaches to nuclear fragmentation: criticality in very small systems

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**Abstract.** Different criteria for criticality in very small systems are discussed in the context of percolation and nucleation approaches to nuclear fragmentation. It is shown that the probability threshold in percolation and interaction radius threshold in nucleation are very strongly dependent upon the adopted criterion. By using Monte Carlo method, similarities and dissimilarities between nucleation and percolation pictures are also appointed out.

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Hot nuclear systems, resulting from nucleus-nucleus collisions at intermediate bombarding energies, are unstable against the breakup into many pieces [1,2,3]. The main features of the experimental data seem to be satisfactorily described by simple statistical models, which rely on probabilistic laws, such as percolation and nucleation models. Also, it has been seen even that the ALADIN data [3] are better reproduced by lattice percolation model than by more sophisticated models, like the statistical multifragmentation model [4] and the sequential evaporation model [5]. Generally speaking, site [6], bond [7] or site-bond percolations [8] and the so-called nucleation model [9] have been successful in reproducing, at least qualitatively, the fragment mass distribution, the intermediate mass fragment (IMF) multiplicity and the fragment kinetic energy spectra. It is believed that the importance of such models can not be underestimated, in spite of its simplicity.

More than a decade ago, it was found that the fragment mass distribution obtained from very high energetic proton-induced reactions follows a power law [10], in the same way than the cluster size distribution in both the Fisher's gas condensation model and in percolation theory at the respective critical points. This behavior has been interpreted as a possible signature of a critical phenomenon (a liquid-gas-type phase transition [10] or a percolation-type transition [7]), reinforcing the expectation that one should be able to extract important information about the nuclear equation of state from nuclear fragmentation events. After the report of the Purdue's group data, several other data have been published, involving also nucleus-nucleus collisions at intermediate energies, whose charge distributions have been observed to obey also a power law. Based on the above mentioned interpretation, many works on nuclear liquid-gas phase transition have been performed by assuming a van der Waals-type equation of state [11]. On the other hand, the hypothesis of a percolation transition has been boosted by Campi [12], which introduced the event-by-event analyses, through the study of the cluster moment correlations. This brief report addresses only this latter hypotesis.

In percolation theory, where infinite lattices are assumed, the critical probabilities are not a universal parameter but depend upon the lattice type, the dimension of the space and the percolation type, *i.e.*, site or bond percolation. Except for few cases, there is no exact manner to calculate these critical probabilities. So, there are standard approximations in order to compute them, namely, series expansion, renormalization group and Monte Carlo methods [13]. On the other hand, the critical phenomena are usually characterized by a small number of critical exponents, which do not depend upon the specific type of lattice, but only upon the dimension of the space, and they have the same value in both site and bond percolations. However, this universality feature of the critical exponents might not be obeyed in the case of finite systems, like nuclear systems.

Actually, nuclear systems have necessarily very small number of constituents

(usually less than few hundreds). This introduces serious difficulties, such as finite size effects which are expected to be far from negligible. As a matter of fact, there is no unique way to define the percolation threshold in finite systems which is to be the analogous of the critical concentration in infinite lattices. And, what is more embarassing, different definitions give different values for the percolation threshold, sometimes even very apart one from another, such as shown below.

Furthermore, similarities have been appointed out between percolation and nucleation approaches. The most striking one is that the physics of the whole process is assumed by both models to be effectively described by a unique parameter, namely, the concentration probability (in percolation) or the interaction radius (in nucleation). In fact, it has been even suggested in Ref.[14] that nucleation may be regarded simply as percolation without lattice, *i.e.*, a type of percolation in the continuum, since nucleation picture shares many qualitatively features with lattice percolation, such as, *e.g.*, the ability to take into account critical phenomena and the same definition for clusters.

The main goal of this report is the following: By using Monte Carlo method, we compare the results for three different criteria for criticality in very small systems and we compare lattice percolation and nucleation calculations in the context of nuclear fragmentation events. In particular, critical exponents are extracted in both models. Although the close paralelism between percolation and nucleation models, we find important dissimilarities, as it will be shown below.

If the nuclear fragmentation is indeed a critical phenomenon, then the problem is how to estimate the "critical" probability and the "critical" interaction radius in very small systems. Let us consider in the following only the percolation case, since the nucleation case can be obtained by analogy. The threshold probability (the analogous to the critical probability of infinite systems) has been suggested as 1) the position of the maximum of the second moment of the cluster distribution [12], 2) the position of the minimum of the critical exponent of the cluster distribution [7], both regarded as function of the concentration probability or 3) by assuming the definition of percolating cluster in finite systems as the cluster with size equal to or greater than half of total of constituents, the probability threshold is regarded as the value of the probability in which the percolating cluster appears first [8]. However, it was appointed out in Ref.[13] that the multiplicity of clusters with size equal to or greater than 4 is correlated with the second moment and the critical exponent of the cluster distribution. As a matter of fact, this multiplicity, which we hereafter denote by $M_4$, is nothing but the cluster multiplicity $M_0 = \sum_s n_s$ excluded the number of cluster with $A = 1, 2$ and 3, and it is shown also in Ref. [13] that similar exclusion leads to better agreement with the scaling law. Furthermore $M_4$ is equal to zero in the limit of very small concentration (in this case, the system is build up only by isolated constituents), and equal to 1 in the opposite limit of very high

concentration. In between, $M_4$ assumes a maximum value, which occurs in almost the same position than the maximum of the second moment or the minimum of the critical exponent, when plotted against the concentration. Hence, the probability threshold may be given by the criterion 4), *i.e.*, the position of the maximum of $M_4$. In fact, this criterion has been adopted in previous simulations of nuclear fragmentation events giving encouraging results [15]. Of course, the extraction of the critical exponents in very small systems depends heavily upon the percolation threshold.

Before going on, we summarize few fundamental results from percolation theory (thermodynamical limit), concerning to the description of the critical phenomena, which are assumed to hold also in very small systems [12,16]. The cluster size distribution obeys a scaling law, *i.e.*,

$$n(s,\epsilon) \propto s^{-\tau} f[\epsilon s^{\sigma}] \quad \epsilon \to 0,\ s \to \infty \tag{1}$$

where $s$ is the number of cluster constituents, $\epsilon = p - p_c$ the distance from the critical point, and $f[\epsilon s^{\sigma}]$ the scaling function, restrained to $f(0) = 1$. $\tau$ e $\sigma$ are critical exponents. For $p \to p_c$, the scaling law yields the potential law,

$$n_s \propto s^{-\tau}. \tag{2}$$

The critical exponents can be evaluated by means of the moments of the distribution $n_s$, defined by [13]:

$$M_k = \sum_s{}' s^k n_s \tag{3}$$

where the prime means that the percolating cluster is excluded from the sum. In finite systems, the percolating cluster is assumed to be the greatest cluster [12]. It should be noted, such as mentioned above, that for $k = 0$, $M_0$ is the total cluster multiplicity.

As a matter of fact, for $p \to p_c$ and $A_0 \to \infty$, the singular part of the first moments behaves as:

$$M_0 \propto |p - p_c|^{2-\alpha} \tag{4}$$

$$M_1 \propto (p - p_c)^{\beta} \tag{5}$$

and

$$M_2 \propto |p - p_c|^{-\gamma} \tag{6}$$

where $\alpha$, $\beta$ and $\gamma$ are critical exponents and $M_1$ and $M_2$ are the percolating cluster strength and the mean cluster size, respectively.

If $\tau$ and $\beta$ are known, then $\sigma$ and $\gamma$ can be calculated by [13]

$$\sigma = \frac{\tau - 2}{\beta} \tag{7}$$

and

$$\gamma = \frac{3 - \tau}{\sigma}. \tag{8}$$

Hence, it is enough to extract independently the two critical exponents $\tau$ and $\beta$.

In what follows, we first discuss how the probability threshold depends upon the criterion of criticality and the system size. We have performed calculations for systems with $A_0 = 81$ (full triangles), 203 (open circles) and 350 (full squares) sites, with a statistics of 3000 runs. In Fig.1, we display $M_2/M_1$ (instead $M_2$, such as suggested by Campi [6]), $\tau$ and $M_4$ against site occupation probability $P$ (left panels), bonding probability $P_b$ (central panels) and effective interaction radius $R_{int}$ (right panels). It is clearly seen that $M_2/M_1$ and $M_4$ follow a bell-shaped behavior, with broader width for smaller systems, and $\tau$ is described by an inverted bell-shaped curve, also with broader width for smaller systems. It is seen from the curves of $\tau$ that the apparent exponent rapidly estabilizes its value as function of the system size. In our case, for instance, $\tau$ is almost the same for $A_0 = 203$ and 350. In table 1, we display the probability thresholds and interaction radius threshold, taken from Fig.1, by using the criteria 1), 2) and 4), such as mentioned above. From table 1, it is clear that the three criteria give different values for the probability and interaction radius thresholds. Furthermore, it is seen also that the finite size effects in the case of criterion 4) are significantly different from the case of criteria 1) and 2). In what very small systems are concerned the probability threshold calculated by using criterion 4) decreases from above $p_c$ - the critical probability in infinite lattices - ($p_c \approx 0.25$ for bond percolation and $p_c \approx 0.31$ for site percolation) to bellow $p_c$ when $A_0$ is increased from 81 to 350, while it is always above $p_c$ if the criteria 1) and 2) are used.

Different thresholds are expected to cause strong effects on the size distribution, fragment multiplicity and energy spectra. To show these effects explicitly in the case of size distribution, we display in Fig.2 the fragment size distribution for three values of $A_0$, by using the three above-mentioned criteria in the context of the bond percolation (full squares for criterion 1), open circles for 2) and full triangles for 4)). The experimental data from Ref.[1,2] are also displayed (crosses). It is apparent that the criterion 4), given by the position of the maximum of $M_4$, reproduces

better the experimental power law. It was checked that the qualitative results are the same also in the case of site percolation and nucleation.

**Table 1.** Probability thresholds and the "critical" interaction radius by using the criteria 1), 2) and 4).

| $A_0$ | | (1) $M2/M1$ (max) | (2) $\tau$ (min) | (4) $M_4$ (max) |
|---|---|---|---|---|
| 81 | site | 0.50 | 0.44 | 0.40 |
| | bond | 0.37 | 0.32 | 0.25 |
| | nucl | 1.66 | 1.57 | 1.32 |
| 203 | site | 0.41 | 0.35 | 0.32 |
| | bond | 0.31 | 0.28 | 0.22 |
| | nucl | 1.54 | 1.45 | 1.26 |
| 350 | site | 0.40 | 0.32 | 0.30 |
| | bond | 0.30 | 0.26 | 0.21 |
| | nucl | 1.52 | 1.39 | 1.24 |

In order to see how the critical exponents depend upon the criticality condition, we display in table 2 the exponents $\beta$ and $\tau$, along with $\lambda_{32}$ – the slope of *ln* $M_3$ *versus ln* $M_2$ – which is shown, in Ref. [12], to be $\lambda_{32} = 1 + \frac{1}{\sigma\gamma}$. The critical exponent $\tau$ is obtained directly from the charge (mass) distribution and $\beta$ is calculated by looking for the slope of the plot of *ln* $S_{max}$ versus $ln|p - p_c|$, instead of using Eq.(5), following procedure suggested by Elliot *et al.* [16]. $S_{max}$ denotes the size of the largest cluster. With the estimated values of $\tau$ and $\beta$, then the exponents $\sigma$ and $\gamma$ can be calculated, by using the scaling equations (7-8).

In table 2, the figures result from calculations by using site percolation, bond percolation and nucleation and adopting each one of the three above-mentioned criteria. The system considered is $A_0 = 203$. For comparison, the values extracted from *Au* fragmentation experiments [2,17] are also shown. It is seen that the criterion given by $M_4$ yields critical exponents with closer values to the experimental ones, except in the bond calculation of $\tau$, in which case it is overestimated in the $M_4$ criterion. Hence, since the $M_4$ criterion allows better agreement with the experimental charge (mass) distribution and which other critical exponents, it seems

that the $M_4$ criterion is more appropriate if simulation in very small systems is concerned.

**Table 2.** Critical exponents by using the criteria 1), 2) and 4).

| Exponent | | (1) $M_2/M_1$ (max) | (2) $\tau$ (min) | (4) $M_4$ (max) | $Au$ fragmentation (exp) |
|---|---|---|---|---|---|
| $\tau$ | site | 1.78 | 1.70 | 1.82 | 2.14±0.06 |
| | bond | 2.25 | 2.13 | 2.39 | |
| | nucl | 1.89 | 1.85 | 2.20 | |
| $\beta$ | site | 0.17 | 0.23 | 0.27 | 0.29±0.02 |
| | bond | 0.21 | 0.29 | 0.29 | |
| | nucl | 0.10 | 0.14 | 0.16 | |
| $\lambda_{32}$ | site | 1.68 | 1.72 | 2.00 | 2.22± 0.10 |
| | bond | 1.93 | 2.01 | 2.23 | |
| | nucl | 1.85 | 1.96 | 2.32 | |

Now, let us address the question of similarities and dissimilarities between percolation and nucleation pictures.

In Fig.3, it is plotted the second largest fragment against the largest fragment, both normalized to 1. This correlation gives information about the impact parameter and/or the violence of the collision since the cluster sizes are very dependent upon these two initial conditions. It is clear that the bond percolation is much closer to nucleation than the site percolation. For comparison, we display also the experimental data of Ref.[2], whose polinomial fitting (solid curve) follows closely to the bond percolation and nucleation curves. In spite of the great similarity between bond percolation and nucleation (Fig.1 and Fig.3), it is found also important dissimilarity. This is better shown in Fig.4, where it is displayed the *rms* cluster radius $< R_F >$ against the fragment size $A_F$, for $A_0 = 350$. It is seen that the nucleation (full squares) gives sistematically smaller values, when compared with bond percolation (crosses) and site percolation (full triangles). This can be easily understood, since in the nucleation calculation the interaction spheres associated with the nucleons are allowed to interpenetrate, *i.e.*, the interdistance between two closest neighbors may be lesser or equal to 2 $R_{int}$, while in percolation calculations, this interdistance is constant and equal to the lattice size. As consequence of this

interpenetration, the clusters are more compact in the nucleation model than in the percolation models. It is seen that the bond percolation results almost coincide with the curve $\alpha r_0 A_F^{1/3}$, which represents the spherical cluster radii in the case of expanded system. It is also seen that for $A_F > 10$, the site percolation begins to deviate more and more from the bond percolation points. This means that for large fragments, the site percolation produces very ramified and/or noncompact (swiss cheese-type) clusters. For completeness, it is also displayed the curve $r_0 A_F^{1/3}$, which represents the normal density cluster radii.

Finally, it is worthwhile to clarify somewhat better how close the nucleation is to the bond percolation. In order to do that, let us consider the mentioned fixed points, such as the maximum $M_2/M_1$, the minimum $\tau$ and the maximum $M_4$ for few different system sizes (*cf* table 1). Hence, we are able to obtain useful correlations, by plotting in Fig.5 the site occupation probability $P$ (panel (a)) and the bonding probability $P_b$ (panel (b)) as function of $R_{int}$. It is clearly seen that $P_b$ is linearly correlated to $R_{int}$. This behavior is not well followed by $P$, which presents some points quite deviated from the straight line. This result stresses the isomorphism between the nucleation and the bond percolation, meaning that one approach is essentially the same than another.

In summary, we have shown that for very small systems the criticality criterion is not unique and the percolation and interaction radius thresholds depend significantly upon the adopted criterion, *i.e.* the percolation and interation radius threshold are located at different positions for different criteria. As a matter of fact, we have performed Monte Carlo simulations and shown that the so-called maximum $M_4$ criterion is more appropriate for nuclear fragmentation events. Furthermore, it is seen that the nucleation approach is linearly correlated to the bond percolation. This and other results suggest an isomorphic relationship between nucleation and bond percolation approaches.

## Acknowledgment

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## FIGURE CAPTIONS

**Figure 1.** The ratio $M_2/M_1$, the apparent exponent $\tau$ and the multiplicity $M_4$ against concentration probabilities and interaction radius for $A_0 = 81$ (full triangles), 203 (open circles) and 350 (full squares).

**Figure 2.** The cluster size distribution (normalized at $A_F = 1$ for $A_0 = 81$, and the charge distribution (normalized at $Z_F = 2$) for $A_0 = 203$ and 350. The full squares, open circles, full triangles denote $M_2/M_1$, $\tau$ and $M_4$ criteria, respectively and the crosses, the experimental data taken from Ref.[1,2,10].

**Figure 3.** The second largest fragment *versus* the largest fragment both normalized to 1, in site (full triangles) and bond (crosses) percolation and in nucleation (full squares) calculations. The full curve is a fitting of the experimental results of Ref.[2].

**Figure 4.** The *rms* cluster radius $< R_F >$ *versus* the fragment size $A_F$ in site (full triangles) and bond (crosses) percolation and in nucleation (full squares) calculations. The full (broken) curve denotes the corresponding spherical cluster radius in normal (expanded) nuclear density.

**Figure 5.** The site occupation probability $P$ (panel (a)) and the bonding probability $P_b$ (panel (b)) *versus* the interaction radius $R_{int}$. The linear regression is also displayed.

**Figure 1.**

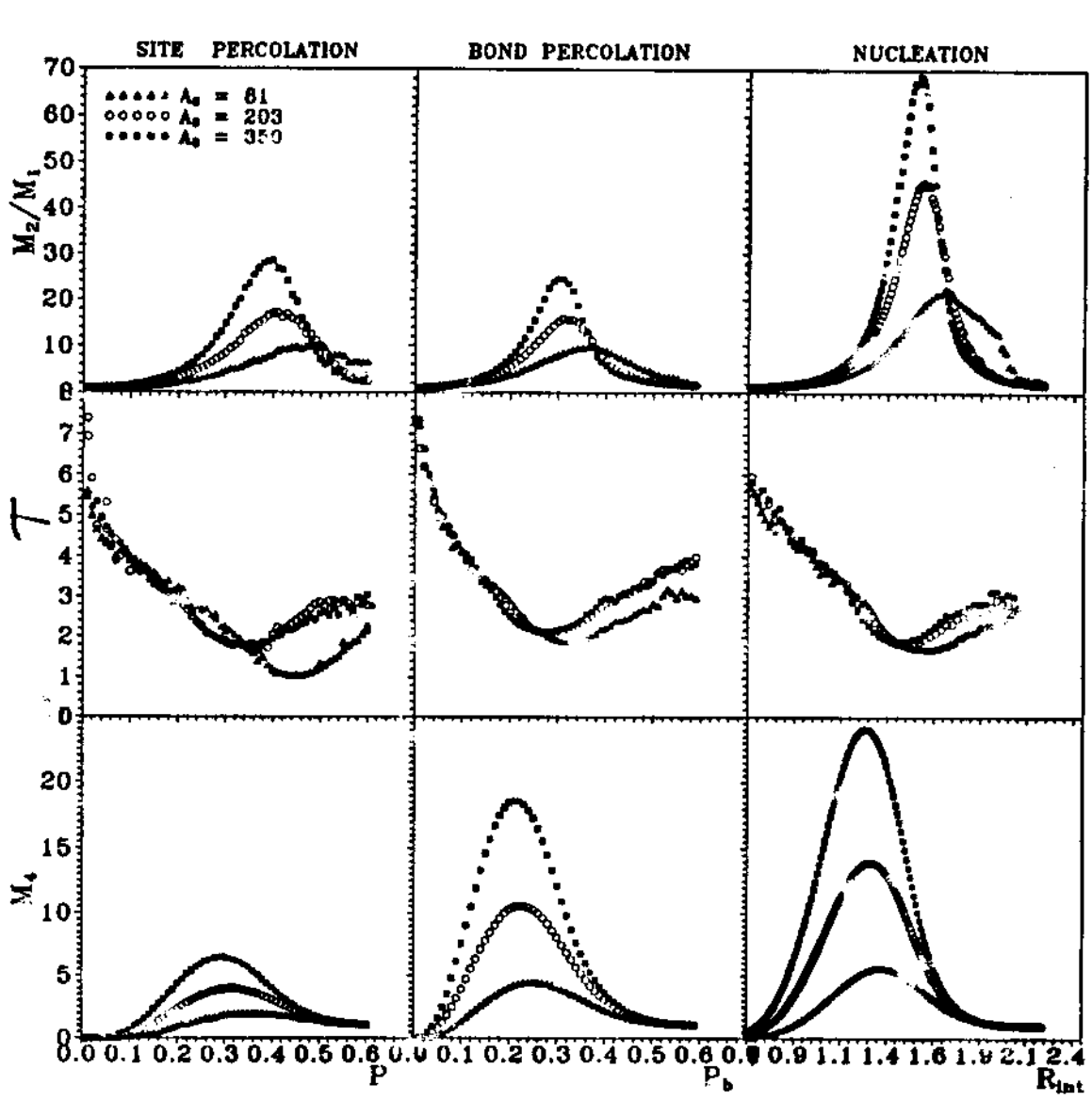

SITE PERCOLATION
BOND PERCOLATION
NUCLEATION
$A_0$ = 81
$A_0$ = 203
$A_0$ = 359
$M_2/M_1$
T
$M_4$
P
$P_b$
$R_{int}$

**Figure 2.**

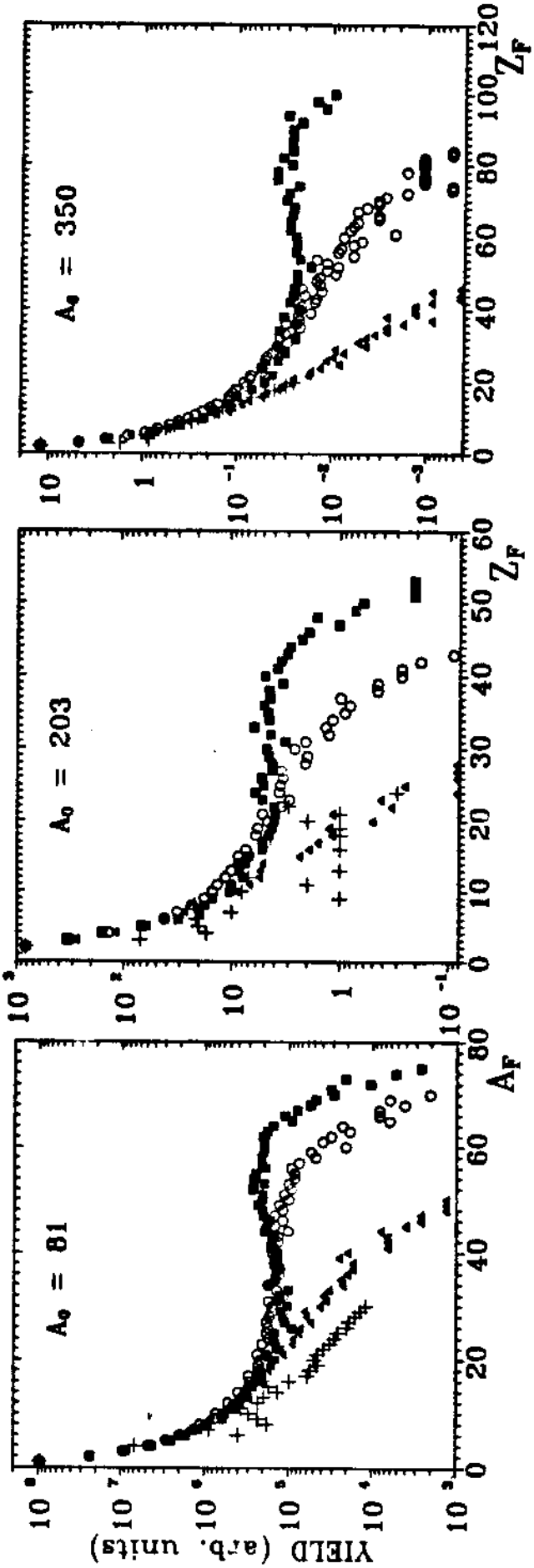


**Figure 3.**

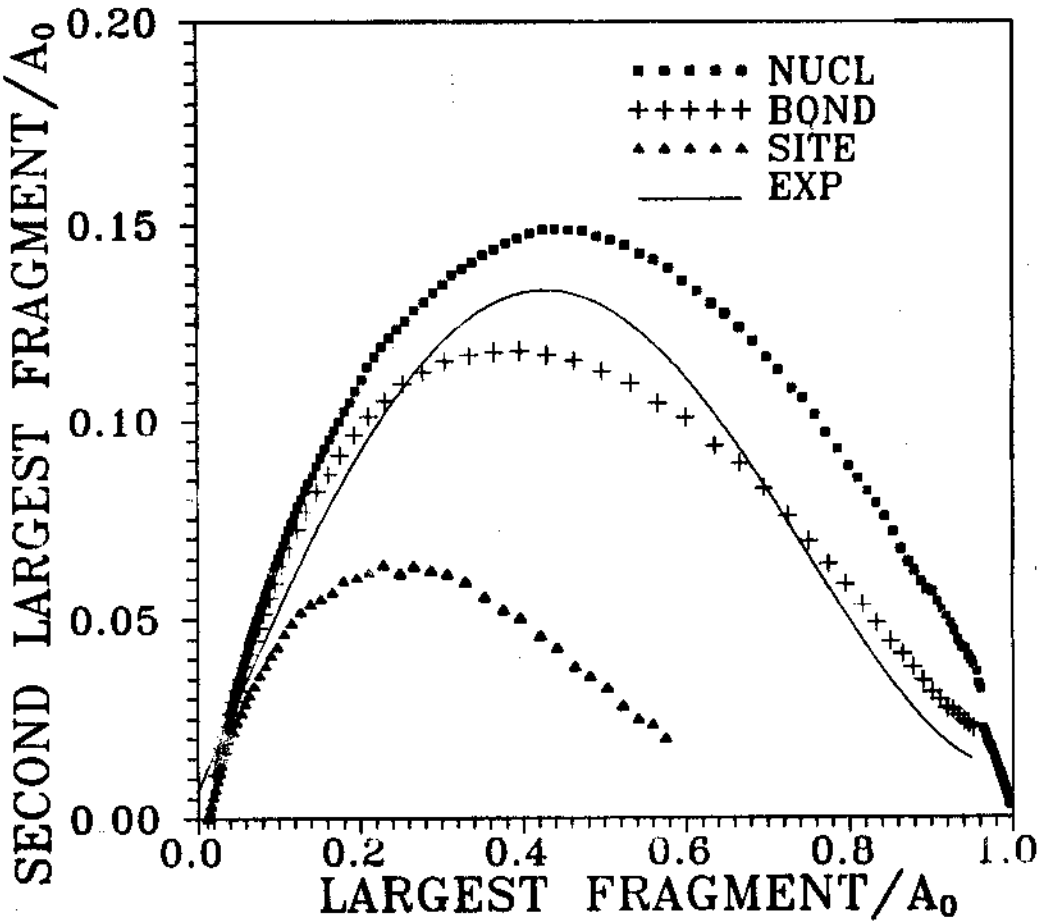

NUCL
BOND
SITE
EXP
SECOND LARGEST FRAGMENT/$A_0$
LARGEST FRAGMENT/$A_0$
0.00
0.05
0.10
0.15
0.20
0.0
0.2
0.4
0.6
0.8
1.0

**Figure 4.**

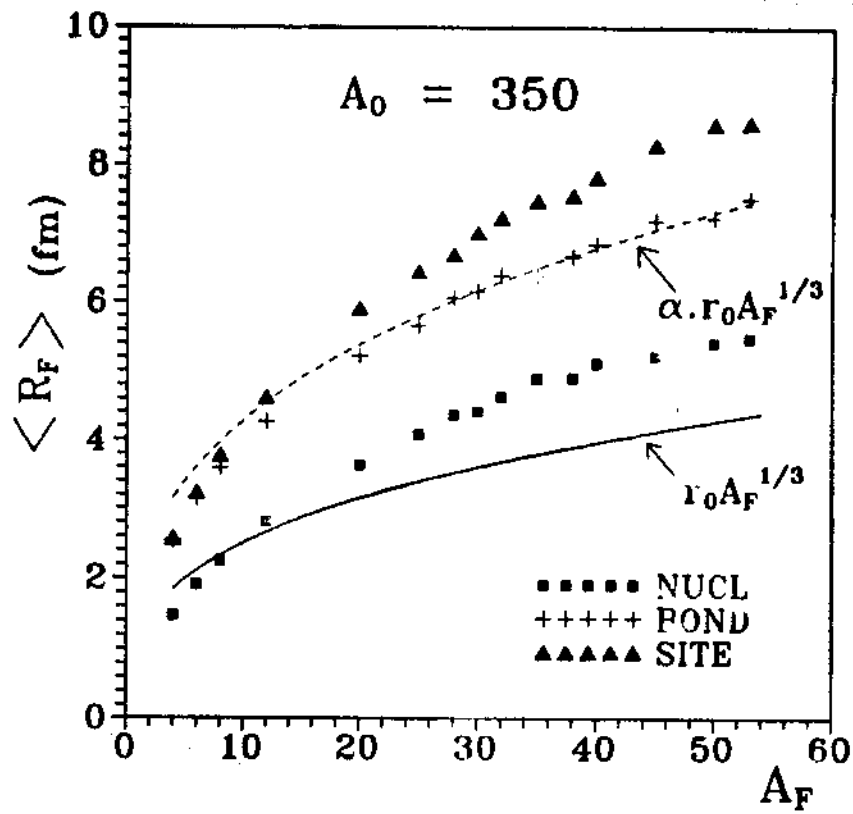
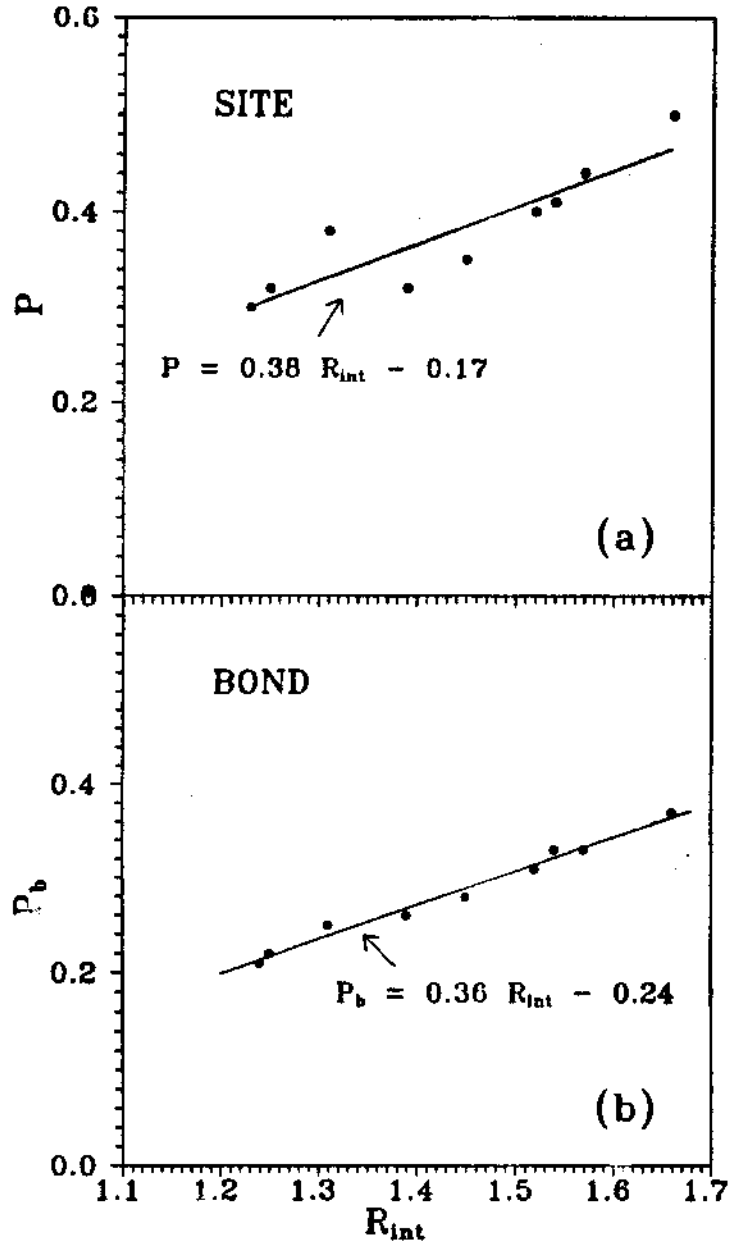

$A_0 = 350$
$\alpha . r_0 A_F^{1/3}$
$r_0 A_F^{1/3}$
NUCL
POND
SITE
$\langle R_F \rangle$ (fm)
$A_F$
0
2
4
6
8
10
0
10
20
30
40
50
60

**Figure 5.**


SITE
P = 0.38 R_int - 0.17
(a)
BOND
P_b = 0.36 R_int - 0.24
(b)
P
P_b
R_int
0.0
0.2
0.4
0.6
1.1
1.2
1.3
1.4
1.5
1.6
1.7

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