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NON-MINIMAL COUPLING AND QUANTUM COSMOLOGY

by

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Abstract

We consider a minisuperspace cosmological model generated by coupling non-minimally a vector field with the gravitational field. The classical solutions are divided in three sets: singular solutions, eternal universes with a expansion phase and flat spacetime. We apply quantum cosmological arguments to investigate which of them is the most probable classical solution. The semi-classical cosmological wave functions satisfy the Correspondence Principle and examples are shown in which the three set of classical solutions can be predicted from different classes of WKB wave functions. As a particular example, we have considered the no-boundary wave function and shown that it predicts flat spacetime which is however classically unstable. The unique expansion solution that arises from this instability is an Eternal Universe.

Key-words: Minimal coupling; Quantum cosmology.

I Introduction

One of the most fundamental problems of Cosmology concerns the existence of singularities in cosmological solutions of Einstein's equation that may describe the Universe we liven in. At the singular point physical quantities, such as the curvature of the Universe and the energy density of matter, diverges and, consequently, no laws of physics can be there applied. There has been some proposals to avoid such undesirable behaviour either classically (through the effects of classically non-minimally coupled fields to gravity[1]; viscous processes[2]; modifications of Einsteins's equations through spin dominance mechanisms[3]; and so on), or by quantum gravitational effects[4].

Concerning the coupling of classical fields to the curvature of spacetime, in the context of a theory of non-minimally interaction between the electromagnetic field and gravity[5,6], it has been shown that there exists homogeneous and isotropic non-stationary classical solutions which are free from singularities, having a very condensate epoch between a contracting and an expansion phase, without any particle horizon, which may thus be fitted with cosmological observations (e.g., they admit the existence of a cosmic microwave back-ground radiation, a hot era). Many interesting features of these solutions have been studied elsewhere[7,8] and the theory from which they where obtained has been used to provide a possible mechanism for the origin of the primordial mangnetic field that pervades our universe[9]. However, it is also possible to find classical solutions within this theory which have singularities. Then, a natural question appears: what is the relative probability between singular and non-singular solutions? The aim of this paper is to try to answer this question by recurring to the ideas recently developped in quantum cosmology[10] and apply them to this model in order to select the most probable classical solutions.

In the next section we will summarize some of the classical aspects of the theory with non-minimal coupling between the electromagnetic and the gravitational fields. We will deal with a reduced configuration space in order to construct a minisuperspace model which contains both the singular and the non-singular classical solutions we are interested in. The action and the Hamiltonian in this minisuperspace model will be obtained in the standard way. In section III the quantum minisuperspace model will be developped, the semi-classical wave functions evaluated and predictions about which of the classical solutions are the most probable will be made for some of these wave functions including the no-boundary and tunneling ones. A summary and comments about our results will be made on section IV.

II The Classical Model

The minisuperspace model we shall discuss in this paper was developed in the context of a theory in which gravity is non-minimally coupled to electromagnetism[5,6], the Lagrangian being given by[5] a combination of Einstein's and Maxwell's theory plus an interacting (non-minimal) term:

$$\mathscr{L} = \sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{k}R + \sigma R W_\mu W_\nu g^{\mu\nu}\right], \tag{2.1}$$

where g is the determinant of the space-time metric $g_{\mu\nu}$, R is the respective curvature (four-) scalar, $k = 16\pi G$ (G is Newton's constant), σ is a dimensionless coupling constant, W_μ is the electromagnetic field and $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$.

The field equations of this theory are:

$$\begin{cases} (1+\sigma W^2)G_{\mu\nu} = -E_{\mu\nu} + \sigma\Box(W^2)g_{\mu\nu} - \sigma R W_\mu W_\nu - \sigma(W^2)_{,\mu;\nu} & (2.2) \\ F^{\mu\nu}{}_{;\nu} = -\sigma R W^\mu \quad , & (2.3) \end{cases}$$

where $W^2 = g^{\mu\nu}W_\mu W_\nu$, $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ ($R_{\mu\nu}$ is the Ricci tensor), $\Box$ is the covariant Laplacian operator, the semi-colon denotes covariant derivative with respect to the four-metric $g_{\mu\nu}$, and $E_{\mu\nu} := F_{\mu\alpha}F^{\alpha}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha k}F^{\alpha k}$. (We have put k = 1).

Our minisuperspace model is characterized by the following ansatz:

$$\begin{cases} ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2 \\ W_\mu = (\psi(t),0,0,0) \end{cases} \tag{2.4}$$

The four-metric is of Robertson-Walker form, where $d\Omega_3^2$ is the metric on the spatial sections with constant positive or negative curvature ε ($\varepsilon = +1$ or $\varepsilon = -1$ respectively). The topology of these sections is considered to be closed.

With these assumptions it follows that $F_{\mu\nu} = 0 = E_{\mu\nu}$ and

$$W^2 = g^{\mu\nu} W_\mu W_\nu = -\psi^2/N^2 =: -\phi^2 \tag{2.5}$$

The field equations (2.2) and (2.3) reduce to:

$$\begin{cases} (1-\sigma\phi^2)R_{\mu\nu} + \sigma(\phi^2)_{,\mu;\nu} = 0 & \text{(2.6a)} \\ \Box(\phi^2) = 0 & \text{(2.6b)} \end{cases}$$

From (2.6b) and taking the trace of (2.6a) it follows that the curvature scalar R vanishes.

Defining $\beta := (1 - \sigma\phi^2)$, and substituting β, ϕ and N into (2.6) we obtain:

$$\begin{cases} a\ddot{a} + \dot{a}^2 + \varepsilon N^2 - a\dot{a}\dfrac{\dot{N}}{N} = 0 \\ \dfrac{\beta}{\beta} + 3\left(\dfrac{\ddot{a}}{a} = \dfrac{\dot{a}}{a}\dfrac{\dot{N}}{N} - \dfrac{\dot{\beta}}{\beta}\dfrac{\dot{N}}{N}\right) = 0 \\ a\ddot{a} + 2\dot{a}^2 + 2\varepsilon N^2 - a\dot{a}\dfrac{N}{N} + a\dot{a}\dfrac{\dot{\beta}}{\beta} = 0 \end{cases} \tag{2.7}$$

We define the associated minisuperspace action substituting the restrictions (2.4) directly into the Lagrangian (2.1), yielding the corresponding action:

$$S := \int dt\ d^3x\ \mathcal{L}(a,\beta,N) \tag{2.8}$$

The equations of motion obtained from the action (2.8) form a system which is equivalent to (2.7). This result validates the interpretation of our model as a minisuperspace model.

The calculation of the Hamiltonian from the action (2.8) yields:

$$H = N\left[-\frac{\Pi_a \Pi_\beta}{a^2} + \beta\frac{\Pi_\beta^2}{a^3} + \beta a\right] =: N\mathcal{H} \tag{2.9}$$

where Π_a and Π_β are respectively the momenta associated to the variables a and β and N plays the role of a Lagrange multiplier (its variation yields the super-Hamiltonian constraint which makes $H \approx 0$ in Dirac's notation[11]).

We can notice that, if we proceed through the Dirac quantization of our model using variables (a,β), the Hamiltonian (2.9) leads to factor-ordering problems. In order, to circumvent this difficulty, we shall now introduce a new equivalent set of coordinates (x,y) in terms of which the Hamiltonian will present no ordering ambiguities. We set

$$\begin{cases} x := \beta^{a} \\ y := \frac{a^2}{a} \end{cases} \tag{2.10}$$

This method is not specific of the present problem but it appears in many others situations. For instance, when dealing with a scalar field coupled minimally to gravity one faces this factor ordering problem as it has been noticed by many authors. One of the possible solutions for this is the Laplacian factor ordering[12,13] in the Wheeler-DeWitt equation. However, when the coupling of the scalar field to gravity is non-minimal there is a direct way to obtain a quantum version without recurring to such arbitrainess.

Indeed, consider the model of a scalar field $\phi(x)$ interacting conformally to gravity through the standard action

$$S = \int \sqrt{-g}\ d\phi x \left[R + \partial_{\mu}\phi \partial_{\nu}\phi g^{\mu\nu} - \frac{1}{6} R\phi^2 - V(\phi) \right]$$

We consider the ansatz

$$ds^2 = dt^2 - a^2(t) d\Omega_3^2$$

and limit ϕ to be spatially homogeneous $\phi = \phi(t)$.

We then make a change of variable for ϕ to $\psi = \phi a$ and consider not the global time t but the conformal time $\eta = \int a(t)dt$.

A direct calculation yields the Hamiltonian:

$$H = \frac{1}{2}\,\Pi_y^2 - \frac{1}{2}\,\Pi_x^2 - \frac{\varepsilon}{2}\,x^2 + \frac{\varepsilon}{2}\,y^2 - \frac{1}{12}\,x^4 V$$

in which we have redefined new variables x,y by setting $x = \sqrt{12}a$ and $y = \sqrt{2}\psi$ and which is free from factor ordering problems.

The action (2.8) is then given by

$$S = + \int dt \left[\varepsilon N x - \frac{1}{N}\,\dot{x}\,\dot{y}\right] \tag{2.11}$$

The general solutions to the equations of motion are:

$$y = -\frac{\varepsilon t^2}{2} + \Sigma \tag{2.12a}$$

$$x = ct \tag{2.12b}$$

which can be expressed as

$$y = -\varepsilon\,\frac{x^2}{2c^2} + \Sigma \tag{2.13}$$

where c and Σ are integration constants.

As can be seen from (2.12a) we may have the following possible classical solutions:

a) For $\varepsilon = 1$.

If $\Sigma > 0$ there is a singularity on $t = -\sqrt{\Sigma}$ when the Universe is created, it expands till maximum size at $t = 0$ and then recolapse at $t = \sqrt{\Sigma}$.

If $\Sigma \leq 0$ there is no classical solution.

b) For $\varepsilon = -1$

If $\Sigma > 0$ the universe is flat at $t \to -\infty$, contracts to its minimum size at $t = 0$ and then expands to become flat again at $t \to \infty$. No singularities are present: it is an eternal universe.

If $\Sigma = 0$ it is just the flat spacetime in Milne coordinates.

If $\Sigma < 0$ we may have a universe that contracts from flat spacetime till singularity or an expanding universe coming from a singularity and going to flat spacetime.

Thus, for $\varepsilon = -1$, there is the possibility of having eternal or singular universes depending on the value of the constant of integration Σ.

The Hamiltonian of the system (2.11) is given by

$$H = -N(\Pi_x \Pi_y + \varepsilon x) \tag{2.14}$$

which yields the supper-Hamiltonian constraint

$$\mathcal{H} := -(\Pi_x \Pi_y + \varepsilon x) \approx 0 \tag{2.15}$$

where $\Pi_x = -\frac{\dot{y}}{N}$ and $\Pi_y = -\frac{\dot{x}}{N}$.

This super-Hamiltonian will be the starting point for the quantization of the model.

III The Quantum Model

Dirac's method for the quantization of a parametrized theory[7] leads to the Wheeler-De Witt equation, which governs the dynamics of the quantum state $\psi(x,y)$. For our model this equation is given by:

$$\hat{\mathcal{H}}\left(x, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right)\psi(x,y) = 0 \tag{3.1}$$

where $\hat{\mathcal{H}}$ is the operator version of (2.15). The explicit form of (3.1) is:

$$-\frac{\partial^2\psi}{\partial x\partial y} + \varepsilon x\psi = 0 \tag{3.2}$$

A solution of (3.2) is given by

$$\psi(x,y) = \psi_0 \exp\left[\sqrt{-\varepsilon}\left(py - \frac{x^2}{2p}\right)\right] \tag{3.3}$$

where ψ_0 and p are arbitrary complex constants.

Note that ψ is an eigenfunction of the momentum operator Π_y with eigenvalue $-i\sqrt{-\varepsilon}\,p$. In order for this eigenvalue to be real, p must be real for $\varepsilon = 1$ and pure imaginary for $\varepsilon = -1$. Thus, we may write (3.3) as:

$$\psi(x,y) = \psi_0 \exp\left[-i\left(cy - \varepsilon\,\frac{x^2}{2c}\right)\right] \tag{3.4}$$

where c is a real constant. This wave function is also a semi-classical wave function because the argument in the exponential of (3.4)

$$S := -\,cy + \varepsilon\,\frac{x^2}{2c} \tag{3.5}$$

is the complete solution of the Hamilton-Jacobi equation of the model

$$\frac{\partial S}{\partial x}\frac{\partial S}{\partial y} + \varepsilon x = 0 \tag{3.6}$$

The general solution of (3.2) constructed from (3.4) is:

$$\psi(x,y) = \int dc G(c) \exp\left[-i(cy - \varepsilon\,\frac{x^2}{2c})\right] = \int dc F(c) \exp\left[-i\left(cy - \varepsilon\,\frac{x^2}{2c} + \beta(c)\right)\right] \tag{3.7}$$

where $G(c) = |G(c)|e^{-i\beta(c)} = F(c)e^{-i\beta(c)}$ is an arbitrary complex function.

We are interested in the possibility of the wave function of the universe predicting a classical universe. In particular we want to known which of the possibilities of having an eternal or singular classical universe is more probable. In that case, the occurence of eternal or singular solutions of eqs. (2.12) depends on the sign of the constant ε, as discussed in Section II. As it is impossible in the above model to obtain a classical eternal universe with $\varepsilon = 1$ we will, from now on, limit our discussion to the case $\varepsilon = -1$.

The idea that a quantum solution predicts a classical universe is meaningful, of course, only in the semi-classical limit which will be identified here with the behaviour of the wave function in the region where the scale factor is very large. Both quantities x^2 and y are proportional to a^2 (a is the scale factor) and so is the term $\left(cy + \frac{x^2}{2c}\right)$ in the phase of (3.7). Hence, when $a \to \infty$, this phase varies rather rapidly, enabling us to approximate $\psi(x,y)$, in the semi-classical limit, employing the stationary phase method. The stationary phase condition for (3.7) with $\varepsilon = -1$ is

$$\frac{d\beta(c)}{dc} + y - \frac{x^2}{2c^2} = 0 \tag{3.8}$$

Suppose (3.8) have N solutions, $C_n(x,y), n = 1,\ldots,N$. In the semi-classical limit $\psi(x,y)$ can be written as

$$\psi_{sc}(x,y) = \sum_{n=1}^{N} F\left(C_n(x,y)\right)\exp\left(-iS_n(x,y)\right) \tag{3.9}$$

with S_n defined by

$$S_n(x,y) := -\left[\beta\left(C_n(x,y)\right) + yC_n(x,y) + \frac{x^2}{2C_n(x,y)}\right] \tag{3.10}$$

It is easy to show that the $S_n(x,y)$ given by (3.10), are solutions of (3.6) with $\varepsilon = -1$:

$$\frac{\partial S}{\partial x}\frac{\partial S}{\partial y} - x = 0 \tag{3.11}$$

In the theory of the nonlinear partial differential equations of first order[10] they are called the general integral of (3.11). This type of equation admits two other sets of solutions: the complete and the singularintegral. The complete integral of (3.11) is given by:

$$S_c(x,y) = -cy - \frac{x^2}{2c} \tag{3.12}$$

which is (3.5) for $\varepsilon = -1$. There is no singular integral of (3.11).

The functions $S_c(x,y)$ given by (3.12) can be used to construct another set of semi-classical wave functions by the WKB approximation method. These WKB wave functions are approximate solutions in first order of $\hbar$ of eq. (3.2) (with $\varepsilon = -1$) and have the form:

$$\psi_w(x,y) = A(x,y)\exp[iS(x,y)] \tag{3.13}$$

For the particular case of eq. (3.8), the functions $A(x,y)$ and $S(x,y)$ must satisfy, besides the Hamilton-Jacobi equation (3.11), the following equation

$$\frac{\partial A}{\partial x}\frac{\partial S}{\partial y} + \frac{\partial A}{\partial y}\frac{\partial S}{\partial x} = 0 \tag{3.14}$$

where we have assumed that

$$\left|A\frac{\partial^2 S}{\partial x\partial y}\right| << \left|\frac{\partial A}{\partial x}\frac{\partial S}{\partial y} + \frac{\partial A}{\partial y}\frac{\partial S}{\partial S}\right| \tag{3.15}$$

Using $S_c(x,y)$ given by (3.12), which satisfies (3.11) and (3.15) because $\frac{\partial^2 S}{\partial x\partial y} = 0$, we obtain for (3.13)

$$\psi_w(x,y) = A(x,y)\exp\left[-i\left(cy + \frac{x^2}{2c}\right)\right] \tag{3.16}$$

Inserting $S_c(x,y)$ into (3.14) and solving it by the separation of variables method ($A(x,y) = X(x)Y(y)$) we obtain for the pre-factor $A(x,y)$

$$A(x,y) = B\exp\left[W\left(-cy + \frac{x^2}{2c}\right)\right] \tag{3.17}$$

where B and W are real constants.

Thus, we may obtain for (3.13):

$$\psi_w(x,y) = B\exp\left[W\left(-cy + \frac{x^2}{2c}\right) + i\left(cy + \frac{x^2}{2c}\right)\right] \tag{3.18}$$

Note that each term of the semi-classical wave funtions $\psi_{sc}(x,y)$ given by (3.9) satisfies the WKB equations because the $S_n(x,y)$ satisfy (3.11), the pre-factors $f\left(C_n(x,y)\right)$ satisfy (3.14) (this can be shown by using the fact that the $C_n(x,y)$ are solutions of (3.8)) and they satisfy (3.15) if f(c) is assumed to be a rapidly varying function of C or if $\frac{\partial^2 S}{\partial x \partial y}$ goes to as $\underline{a}$ goes to infinity, which is the case of the examples that follow.

The wave functions ψ_{sc} and ψ_w given by (3.9) and (3.16) respectively are the most general forms of WKB solutions in the form $\psi \sim e^{is}$ where S is a solution of the Hamilton-Jacobi. The $\psi_w(x,y)$ given by (3.18) is a particular case of (3.16).

In order to make predictions from (3.9) and (3.16) we will use the fact that these wave functions have a strong peak about the correlations[11]

$$p_i = \frac{\partial S}{\partial q_i} \tag{3.19}$$

where the q^i are the minisuperspace variables and the the p_i the canonical momenta.

The correlations (3.12) are in fact first integrals of the classical equations of motion[11]. This implies that WKB wave functions of the form e^{is} are peaked over an n-parameter subset of all classical trajectories. Besides, the pre-factors will provide a measure over the ensemble of classical trajectories about which the wave function is peaked. If the WKB solutions are of the form $\psi = A(q)e^{is(q)}$ (which is the case of (3.9) and (3.16)) the conserved probability measure will be:

$$dp = \vec{J}.d\vec{\sigma} \tag{3.20}$$

where

$$\vec{J} = A(q)^2\vec{\nabla}S(q) \tag{3.21}$$

and $d\vec{\sigma}$ is the "area element" of a suitably chosen hyper-surface, so that all the trajectories of the ensemble cross it only once.

In quantum cosmology, a peak in the wave function is usually interpreted as a prediction[11] and we will follow this idea here. The probability measure (3.20) can be used to calculate conditional probabilities, a prediction being made when the result is close to zero or one. We will also assume that interference between terms of a superposition of WKB wave functions that describe ensembles of classical universes (which is the case of (3.9)) can be neglected[15].

Let us now apply these concepts to the wave-functions (3.9) and (3.16). We will begin by the wave function (3.18) which is a particular example of the WKB function (3.16) and satisfies the WKB equations (3.11), (3.14) and (3.15). Applying (3.19) to our model we get:

$$\Pi_x = \frac{\partial S_c}{\partial x} = - \frac{x}{c} \tag{3.22}$$

$$\Pi_y = \frac{\partial S_c}{\partial y} = - c \tag{3.23}$$

where S_c is given by (3.12).

In the gauge N = 1, $\Pi_x = - \dot{y}$ and $\Pi_y = - \dot{x}$. Thus equations (3.22) and (3.23) yields.

$$\frac{dy}{dx} = \frac{x}{C^2} \Rightarrow y = \frac{x^2}{2C^2} + \Sigma \tag{3.24}$$

which is just equation (2.13) for $\varepsilon = - 1$.

The sign of the constant Σ is still unknown. In order to determine this sign we have to apply equations (3.20) and (3.21) to our model. We define

$$\eta = y - \frac{x^2}{2C^2} \tag{3.25}$$

$$\xi = - y - \frac{x^2}{2C^2} \tag{3.26}$$

We then obtain for $\vec{J}$ given in (3.21)

$$\vec{J} = \exp(2CW\eta)\ \nabla(\vec{C}\xi) \tag{3.27}$$

In the plane (ξ,η), the surfaces ξ = const. (ξ is essentially S_c given in

(3.12)) are orthogonal to the classical trajectories (3.24) (by virtue of (3.22) and (3.23)) which have $\eta = \Sigma =$ const. Thus $\vec{J}$ points largely in the ξ direction. Choosing the surfaces σ in (3.20) to be the surfaces of constant ξ it will be guaranteed that $\vec{J}$ crosses them only once. This choice yields for (3.20) the following equation:

$$dP = \vec{J}.d\vec{\sigma} \approx \exp(2CW\eta)d\eta$$

The conditional probability of having $\eta \approx \Sigma > 0$ will then be given by:

$$P(\eta \approx \Sigma > 0 | -\infty < n < \infty) = \frac{\int_0^{\infty} \exp(2CW\eta)d\eta}{\int_{-\infty}^{\infty} \exp(2CW\eta)d\eta} \tag{3.28}$$

If CW > 0 then P = 1 and if CW < 0, P = 0. Thus, we can make a definite prediction about the sign of Σ if we know the sign of CW. To known the sign of CW we must have boundary conditions that may select just one among the many solutions of (3.2) or of (3.11), (3.14) and (3.15).

We will now turn to the semi-classical wave function of the type given in (3.9). Let us take some particular examples of $\beta(c)$ and see what kind of classical universes they may predict.

i) $\beta(c) = \alpha = cte$

In that case, equation (3.8) gives:

$$C = \pm \frac{x}{\sqrt{2y}}$$

which, for $S_n(x,y)$ given by (3.10), yields

$$S_\pm := \pm S_r := \pm x\sqrt{2y}$$

The wave function (3.9) is

$$\psi_{sc}(x,y) = F\left(\frac{x}{\sqrt{2y}}\right)\exp(iS_r) + F\left(-\frac{x}{\sqrt{2y}}\right)\exp(-iS_r) \qquad (3.29)$$

The correlations (3.19) applied to this case gives, for N = 1.

$$\Pi_x = -\dot{y} = \frac{\partial S_\pm}{\partial x} = \pm\sqrt{2y}$$

$$\Pi_y = -\dot{x} = \frac{\partial S_\pm}{\partial y} = \pm\frac{x}{\sqrt{2y}}$$

which implies

$$\frac{dy}{dx} = \frac{2y}{x} \Rightarrow y = \frac{x^2}{2c^2} \qquad (3.30)$$

This is equation (2.13) with $\Sigma = 0$. We conclude that the wave-function (3.29) predicts Minkowski spacetime.

ii) $\beta(c) = -\alpha c$ and $\beta(c) = \frac{\alpha}{c}$, $\alpha = \text{const} \neq 0$.

In these cases the wave function predicts an eternal universe for $\alpha > 0$ and a singular universe for $\alpha < 0$. It is not possible to obtain Minkowski spacetime.

Note that in both cases the knowledge of the form of $S(x,y)$ was enough to make exact predictions about the most probable classical solutions. In fact, the pre-factors $F_n(C_0(x,y))$ wouldn't give much information about the singular nature of the most probable classical solutions because, near the classical trajectories, the functions $C_0(x,y)$ would be almost equal to the constant C of (2.13) and, if we apply a similar reasoning with the one used in (3.18), we would only get information about the most probable values of C, which are irrelevant to answer the question we are addressing.

To make definite predictions out of (3.9) we need the exact form of the function $\beta(c)$. Again, boundary conditions on the general wave function (3.7) are important in order to find a particular solution with a definite $\beta(c)$. Thus we will now see what predictions can be made if we use the no-boundary[16] and tunelling[17] boundary conditions.

The no-boundary semi-classical wave function can be calculated by evaluating the Euclidean action that comes from (2.11) with $\varepsilon = -1$.

$$I = -\int d\tau \left(\frac{\dot{x}\dot{y}}{N} - Nx\right)$$

in the classical solutions of the Euclidean field equations

$$\frac{\ddot{y}}{N} + N = 0$$

$$\frac{\ddot{x}}{N} = 0$$

$$\frac{\dot{x}\dot{y}}{N^2} + x = 0$$

In order for the four-geometry to close-off in a regular way, $a(\tau)$ must be a pure imaginary function of τ (consequently, $x(\tau)$ mut also be a pure imaginary function and $y(\tau)$ negative definite) subjected to the boundary condition that $a(0) = y(0) = 0$. Also $x(0) = 0$ in order to preserve regularity of the solutions at $\tau = 0$. The solution is

$$\psi_{NB} = \exp(-I_+) + \exp(-I_-) = \cos(x\sqrt{2y}) \tag{3.31}$$

which is a particular case of example i) discussed before which predicts

Minkowski spacetime.

For the $\varepsilon = +1$ case the solution is

$$\psi_{NB} = \exp(x\sqrt{2y}) \tag{3.22}$$

As it is not oscilatory, it cannot describe a classical universe (it can be viewed as generating solutions with $\varepsilon = 0$ in (2.12a) which are not allowed for $\varepsilon = +1$).

We do not know how to obtain a unique solution using the tunneling boundary conditions. Many solutions of the form of (3.9) may satisfy these boundary conditions by setting appropriate choices of F(c) and $\beta(c)$.

IV Conclusion

The minisuperspace quantum model we have studied generates a Wheeler - De Witt equation that can be exactly solved. Its general solution yields semi-classical wave functions which are in the most general WKB form and can be divided into two sets: one given by eq. (3.9) and the other by (3.16) (the latter can be obtained from the general solution by setting F(c) in (3.7) to be a sharply peaked function of C around a fixed value $\bar{C}$). All these semi-classical wave functions correspond to classical solutions of the equations of motion described on Section II and so they satisfy the Correspondence Principle. For wave functions of the type of eq. (3.9) it seems that it is enough the knowledge of the functions $\beta(C)$ and, consequently, of $S_n(x,y)$ given in (3.10) in order to decide which are in the most probable classical solutions. Examples were given in Section III where flat space-time, singular and eternal universes were obtained. In the case of wave functions given by (3.16), the knowledge of the pre-factor is important to decide about the most probable classical solution. Examples which show the way the pre-factor provide predictions were also given.

We could construct a measure on the space of all possible solutions of the Wheeler - De Witt equation (see, e.g., ref. 14) which is spanned by the function $F(c)$ and $\beta(c)$ (see eq. (3.7)). However we still do not know what could be the relation of the specific forms of $F(c)$ and $\beta(c)$ with the singular and non-singular nature of the solutions.

We conclude that, in order to make definite predictions about the probability of having eternal universes, it is necessary to have a unique wave function that could be obtained by imposing suitable boundary conditions. Imposing the no-boundary condition, the semi-classical wave function selects flat spacetime as the most probable classical solution. This is in accordance with the general belief that the no-boundary condition selects a kind of "ground-state" wave function. The tunneling boundary condition seems not to be restrictive enough in order to select a unique semi-classical wave function.

Let us remark that the flat spacetime selected by the no-boundary wave function is classically unstable[7]. It can evolve either to a contracting singular universe or to an eternal universe. As we are now surely living in an expanding one, we conclude that, in the context of the no-boundary proposal applied to our model, our universe must be eternal.

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