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POTTS MODEL ON INFINITELY RAMIFIED SIERPINSKI GASKET-TYPE FRACTALS AND ALGEBRAIC ORDER AT ANTIFERROMAGNETIC PHASES

by

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Abstract

We propose families of infinitely ramified fractals, which we call the m–sheet Sierpinski gasket with side b ($(mSG)_b$), on which the q-state Potts model can be exactly solvable through a Real Space Renormalisation Group (RSRG) technique and which presents phase transitions at finite temperatures for $m > 1$. We also propose, within a cell-to-cell RSRG sheme, a criterion for a suitable choice of cells in the study of antiferromagnetic (AF) classical spin models defined on (or approximated by) multi-rooted hierarchical lattices, and apply it for the AF Potts model on some $(mSG)_b$ fractals. Concerning the Ising model on the $(mSG)_2$ family, we obtained the *exact* para (P) – ferromagnetic (F) critical temperature as a function of m; we verified that, for $m = 1$ and 2, there is no AF order (not even at zero temperature). We calculated the *exact* P–F and possibly exact P–AF critical frontiers and the corresponding correlation length critical exponents for $q = 2, 3$ and 4-state Potts model on the $(mSG)_4$. The AF $q = 2$ and 4 cases have highly degenerate ground states and each one presents, above a certain critical fractal dimension $D_f^c(q, m)$, an unusual low–temperature phase whose attractor occurs at a non-null temperature. We proved, for $q = 2$ and $D_f(2, m) \geq 5.1$, that the correlations have a power law decay with distance along this entire phase.

Key Words: Potts model, Fractals, Real Space Renormalisation Group, Criticality.

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I - Introduction

Gefen and coworkers [1] presented the first systematic study of critical phenomena on fractals. Since then, much attention has been paid to the study of spin models on fractals [2–12] and, in particular, on hierarchical lattices (HL) [13–17]. Despite the fact that different models defined on several fractal lattices have been exactly solved (see, for example, [2, 3, 10]), as far as we know, there is no one with finite and short range interactions which exhibits phase transition at a non-null temperature (except on bond hierarchical lattices). The existence of such a case would certainly contribute to a better undertanding of critical phenomena on these scale (but not translationally) invariant lattices.

In this paper we propose families of deterministic fractals (called by us the m-sheet Sierpinski gasket with side b) on which the q-state Potts model (see [18]) can be exactly solvable and which presents phase transitions at finite temperature for $m > 1$. These fractals constitute hierarchical lattices on which the aggregated objects are triangles generating, thus, three-rooted HL (differently from the bond ones which have only two roots). Due to their hierarchical character, the Real Space Renormalisation Group (RSRG) employed here provides the exact para (P) - ferromagnetic (F) critical frontiers and their corresponding correlation length critical exponent $\nu_T^F(m)$.

Concerning the antiferromagnetic (AF) Potts model, one has to be very cautious when a negative coupling constant changes sign under the first scaling leading, thus, to renormalised ferromagnetic couplings in all subsequent iterations. Our interpretation of this fact is that the symmetries of the antiferromagnetic ground state are not being preserved under renormalisation, and that we should choose sufficiently large cells (which appear in two subsequent steps of construction of the fractal) which conserve these symmetries. This point has been neglected on Migdal-Kadanoff-like HL with even chemical distance [19, 20] and, for $q = 2$, on the Sierpinski gasket ($b = 2$, $m = 1$) [9, 11, 12]. Herein we discuss, within a cell-to-cell RSRG sheme where the spin states on the roots of the cells are fixed under renormalisation, this point and propose a criterion for a suitable choice of cells in the study of antiferromagnetic classical spin models defined on HL (or on Bravais lattices which are approximated by these ones). Applying this criterion, we obtained the P-AF phase boundaries and their respective critical exponent $\nu_T^{AF}(m)$ for Potts antiferromagnets on the m-sheet Sierpinski gasket families (which we shall refer hereafter to as $(mSG)_b$) which we expect to be the exact ones.

Another interesting feature of the fractals $(mSG)_b$ on which we define the Potts antiferromagnet is the appearance, above a certain critical fractal dimension $D_f^c(q,m)$, of a unusual low-temperature phase in which the correlations have a power law decay with distance. Such systems have a highly degenerate ground state which generates a non-zero entropy per site at zero temperature, violating thus the third law of Thermodynamics. This residual entropy appears also, for example, in AF Potts models on systems like: bipartite lattices (for $q \geq 3$), FCC lattices (for $q \geq 2$ [21]), decorated square lattice [22], and some fractals [6, 9, 11, 12]. If one applies to such a systems an argument similar to that of Wannier [23], one would expect no long-range order of the usual type. But, after Berker and Kadanoff [24] suggested that such systems may present a distintive low-temperature phase with algebraic decay of correlations, much work has been done [25, 21, 26, 27, 18, 28, 6, 29, 20] looking for this phase. Although, in some cases, there are indications that such a phase exists (see, for instance, [6, 20, 27, 29]), there is no proof, as far as we know, that the correlations decay algebraically along this unusual phase. Herein we prove this for the Ising antiferromagnet on the $(mSG)_4$ for $m \geq 116$ (and hence for fractal dimensions $D_f \geq 5.1$). The existence of an attractor at finite temperature ($T \neq 0$) for the $q = 4$ AF Potts model with two and three spin interactions on the $(mSG)_4$ with $m \geq 17$ ($D_f \geq 3.7$) indicates that such a phase appears also in this case.

The outline of this paper is as follows. In Sec.II, we define the fractal families $(mSG)_b$ and the q-state Potts model with two and three spin interations. In Sec.III, we present the two-parameter RSRG formalism (valid for $q \neq 2$), as well as that with one parameter suitable for treating the Ising model without three spin interations. In Sec.IV, we propose a criterion for a convenient choice of cells in the study of antiferromagnetic classical spin models defined on (or approximated by) HL. This criterion is a generalisation of the one derived in this section from a mathematical analysis at $T = 0$ of the most general form that the mentioned one-parameter RSRG can have. The appearance of an unusual attractor at a finite temperature emerges naturally from such analysis. The application of this criterion led to the results reported in Sec.V for the Ising model on the $(mSG)_2$ and for the $q = 2, 3$ and 4-state Potts model on the $(mSG)_4$. Finally, in Sec.VI, the conclusions are given.

II - Model

The m-sheet Sierpinski gasket $(mSG)_b$ is a generalisation of the two-dimensional case of the Sierpinski gasket family proposed by Hilfer and Blumen [30]. Different problems have been studied in the $(1SG)_b$ family, e.g., spectral dimension [30, 31], moments of a voltage distribution in resistor networks [32], criticality of self-avoiding walks [33], residual entropy [9, 34] and other thermodynamical properties [9] of the Ising model. Each member of the $m = 1$ family has a generator $G(b)$ (b is an integer) constitued by an equilateral triangle of side length b which contains $b(b+1)/2$ upward oriented triangles of unit side. The $(mSG)_b$ fractal (where b and m are fixed) has a generator $G(b, m)$ (see Fig. 1) which consists of m structures topologically similar to $G(b)$ connected only at three external sites A, B and C (hereafter called roots). $G(b, m)$ constitues the $n = 1$ stage of construction of the $(mSG)_b$ fractal obtained in the $n \to \infty$ limit. Any stage is obtained from the previous one by replacing each upward-oriented triangle of each sheet by the respective generator, and leaving the downward triangles empty (see, for $m = 2$, Fig. 1(a)). In each step $mb(b+1)/2$ new units are generated leading, thus, to a fractal dimension D_f given by

$$D_f = \frac{\ln[b(b+1)m/2]}{\ln b} \tag{1}$$

One can easily show that in each stage (n) the order of ramification [35] R_n satisfies the recursive equation $R_n(m) = mR_{n-1}$, where $R_1(m) > 1$ for all m. Therefore, unlike the $m = 1$ case which is finitely ramified [30], the $(mSG)_b$ has an infinite order of ramification for $m > 1$. The particular case $(2SG)_2$ was introduced [13] as an example of an HL in which the aggregated objects are more complex than bonds.

At each site of the $(mSG)_b$ fractal with fixed b and m, we associate a Potts spin variable $\sigma_i = 1, 2, \cdots, q$ and consider the q-state Potts model with two (J_2) and three (J_3) spin interactions described by the following dimensionless Hamiltonian:

$$\beta\mathcal{H} = -K_2 \sum_{<i,j>} \delta(\sigma_i, \sigma_j) \quad - K_3 \sum_{<i,j,l>} \delta(\sigma_i, \sigma_j, \sigma_l) \tag{2}$$

where $\beta = 1/K_bT$, $K_i = J_i\beta$ ($i = 2, 3$) and $\delta(\sigma_i, \cdots, \sigma_l) = 1(0)$, if $\sigma_i = \cdots = \sigma_l$ (otherwise). The first sum is over all NN pairs of spins and the second one is over the spins on all the upward-pointing triangles. We consider here either positive or negative values for both coupling constants.

III - Formalism

Let us now define our renormalisation group. For this, we perform a scale transformation from one cell of side b to another of smaller side b' together with a renormalisation of the parameters K_2 and K_3. The last step consists in summing over the spin states of the b and b'−cells, with the restriction that the spins (σ_A, σ_B and σ_C) on the roots are held in fixed states α, β and γ, namely

$$W_m(\alpha,\beta,\gamma) = D\, W'(\alpha,\beta,\gamma) \tag{3}$$

where

$$\begin{aligned} W_m(\alpha,\beta,\gamma) \equiv & \sum_{\{\sigma_i\}} \delta(\sigma_A,\alpha)\delta(\sigma_B,\beta)\delta(\sigma_C,\gamma)\ e^{-\beta\mathcal{H}} = \\ & = Z\ \langle\, \delta(\sigma_A,\alpha)\delta(\sigma_B,\beta)\delta(\sigma_C,\gamma)\,\rangle_{b-cell} = \\ & = Z\ p(\sigma_A=\alpha,\sigma_B=\beta,\sigma_C=\gamma) \end{aligned} \tag{4}$$

$$\begin{aligned} W'(\alpha,\beta,\gamma) \equiv & \sum_{\{\sigma_i'\}} \delta(\sigma_A',\alpha)\delta(\sigma_B',\beta)\delta(\sigma_C',\gamma)\ e^{-\beta\mathcal{H}'} = \\ & = Z'\ \langle\, \delta(\sigma_A',\alpha)\delta(\sigma_B',\beta)\delta(\sigma_C',\gamma)\,\rangle_{b'-cell} = \\ & = Z'\ p'(\sigma_A'=\alpha,\sigma_B'=\beta,\sigma_C'=\gamma) \end{aligned} \tag{5}$$

where $\langle\cdots\rangle$ represents the standart thermal average, and D is a constant due to the renormalisation of the zero energy. Z and Z' are the respective partition functions of the non-renormalised (b-cell) and the renormalised (b') one. $p(\sigma_A = \alpha, \sigma_B = \beta, \sigma_C = \gamma)$ is the probability that the rooted spins σ_A, σ_B and σ_C of the b-cell are in the respective states α, β and γ; a similar definition follows for $p'(\sigma_A' = \alpha, \sigma_B' = \beta, \sigma_C' = \gamma)$ in the renormalised cell. Due to the symmetry of the considered cell, it appears only three different "constrained" partition functions $W_m(\alpha,\beta,\gamma)$: $W_F^{(m)} \equiv W_m(\alpha,\alpha,\alpha)$, $W_I^{(m)} \equiv W_m(\alpha,\alpha,\gamma)$, and $W_{AF}^{(m)} \equiv W_m(\alpha,\beta,\gamma)$; where $(\alpha,\beta,\gamma = 1,2,\cdots,q)$ and $\alpha \neq \beta \neq \gamma$, $\alpha \neq \gamma$. A similar notation W_l' ($l = F, I$ and AF) is used for the b'-cell, where the superscript has been suppressed since $m = 1$ for this cell. Eliminating D from Eq. (3), we obtain that

$$\frac{W_F^{(m)}}{W_{AF}^{(m)}} = \frac{W_F'}{W_{AF}'} \tag{6}$$

$$\frac{W_I^{(m)}}{W_{AF}^{(m)}} = \frac{W_I'}{W_{AF}'} \tag{7}$$

Therefore our RG preserves the ratios p_F/p_{AF} and p_I/p_{AF} of the above probabilities. Since the m-sheets are connected only at the roots, the "constrained" partition functions factorise trivially as

$$W_m(\alpha,\beta,\gamma) = (W_1(\alpha,\beta,\gamma))^m \tag{8}$$

For the Ising case ($q = 2$) obviously only $W_F^{(m)}$ and $W_I^{(m)}$ are defined, and the RG equation (with $K_3 = 0$) is given by

$$\frac{W_F^{(m)}}{W_I^{(m)}} = \frac{W_F'}{W_I'} \quad (q = 2) \tag{9}$$

In this case, the K_2 parameter space is closed by renormalisation (i.e., if $K_3 = 0$ then $K_3' = 0$ for the considered triangular cell types).

Notice that Eq. (9) is equivalent to the preservation of the correlation function between any pair of the rooted spins of the $(mSG)_b$, namely:

$$\frac{W_F^{(m)}}{W_I^{(m)}} = \frac{W_F'}{W_I'} \quad \Leftrightarrow \quad \Gamma_{ij}^{(1)}(G^{(1)}, K_2) = \Gamma_{ij}^{(0)}(G^{(0)}, K_2') \qquad (10)$$

$$(i, j = A, B, C)$$

where $\Gamma_{ij}^{(1)}(G^{(1)}, K_2)$ and $\Gamma_{ij}^{(0)}(G^{(0)}, K_2')$ denote the correlation functions between any two rooted spins at the respective stages $n = 1$ and $n = 0$ (see Fig. 1(a)) with NN couplings constants K_2 and K_2' respectively.

Eq. (10) follows from the isotropic case ($\Gamma_{AB} = \Gamma_{AC} = \Gamma_{BC}$) of the following relations (which are easily derivable from Eqs. (4) and (5)):

$$W_F^{(m)} = W_m(1,1,1) = \frac{Z}{8}\left[\Gamma_{AB} + \Gamma_{AC} + \Gamma_{BC} + 1\right] \qquad (11)$$

$$W_I^{(m)} = W_m(1,1,-1) = \frac{Z}{8}\left[\Gamma_{AB} - \Gamma_{AC} - \Gamma_{BC} + 1\right] \qquad (12)$$

where

$$\Gamma_{ij} \equiv \langle \alpha_i \alpha_j \rangle \quad (\alpha_i, \alpha_j = \pm 1; \;\; i, j = A, B, C) \qquad (13)$$

IV - A criterion for the choice of cells in an AF spin model

It is well know that the choice of cells in a cell-to-cell RSRG is an important factor for the reliability of its results. One should cells that reproduce the geometrical properties of the whole lattices as well as the symmetries of the ground state configurations of the ordered phases. Although this condition has been taken into account in most RSRG calculations of spin models on Bravais lattices (see, for example, [38] and references therein), this has been neglected [9, 11, 12, 19, 20] in the case of antiferromagnetic models defined on HL. As we will see below, when the symmetries of the ground state are not preserved under renormalisation, the RG generates unphysical disconnected basins of attraction. In this section, we propose a criterion for a suitable choice of cells which avoids, for example, this kind of problem. First we shall consider the one-parameter RG described by Eq. (9) and derive such a criterion for either Ising antiferromagnets defined on (or approximated by) three-rooted HL or AF Potts model on two-rooted ones. Afterwards we formulate, inspired in the above results, a general criterion which we expect to be valid for antiferromagnetic classical spin models defined on (or approximated by) HL with an arbitrary number of roots.

Although Eq. (9) was mentioned in the context of the AF Ising model on the $(mSG)_b$, this equation is valid for systems described by a one-parameter Hamiltonian defined on cells which have only two different constrained partition functions. Let us, thus, consider the q-state Potts model on two-rooted graphs (or the Ising model on three-rooted ones) with only two-spin interactions whose Hamiltonian is given by Eq. (2) with $K_3 = 0$. Each spin pair (σ_i, σ_j) where $\sigma_i = \sigma_j \quad (\sigma_i \neq \sigma_j)$ contributes with a term $e^{-\beta \mathcal{H}_{ij}} = e^{K_2} \quad (e^{-\beta \mathcal{H}_{ij}} = 1)$ to the constrained partition functions. Consequently, these functions can be written, for the non-renormalised cell, as

$$W_l(X) = \sum_{i=0}^{N_b} g_l^{(i)}\, X^i \quad (l = F, I) \;\; (X \equiv e^{K_2}) \tag{14}$$

where N_b is the number of bonds of the cell and $g_l^{(i)}$ is the degeneracy of the state with energy $\beta\varepsilon_i = -K_2\, i$.

Obviously, the first nonzero term $\left(g_l^{(j_l)}\, X^{j_l}\right)$ of the lowest order in X^i of Eq. (14) represents the dominant term of W_l for the antiferromagnetic case ($J_2 < 0$) of this model at $T = 0$. We shall denote each such term simply by

$$g_F^{(j_F)}\, X^{j_F} \equiv g_F\, e^{-\beta\varepsilon_F}$$

$$g_I^{(j_I)}\, X^{j_I} \equiv g_I\, e^{-\beta\varepsilon_I}$$

where ε_F (ε_I) is the lowest energy configuration of the AF Potts model on the non-renormalised cell with the restriction that the spins on the roots are all (are not all) in the same state. Therefore, ε_F (ε_I) refers to the configuration F (I) where neighbour spins are in different states and: (i) either $\sigma_A = \sigma_B$ ($\sigma_A \neq \sigma_B$) in the case of two-rooted cells, (ii) or $\sigma_A = \sigma_B = \sigma_C$ ($\sigma_A = \sigma_B \neq \sigma_C$) in the case of three-rooted ones. Observe that ε_F and ε_I constitute the two lowest energies of the energy spectrum associated with the considered cell and, therefore, the smallest of them is the model's ground state energy for the cell. When $\varepsilon_F < \varepsilon_I$ ($\varepsilon_F > \varepsilon_I$) we shall say that the ground state of the cell is of type F (type I). When $\varepsilon_F = \varepsilon_I$, indeed, the type of the ground state is given by the one with higher degeneracy.

Notice that $\frac{W_F(X)}{W_I(X)}$ satisfies the following properties:

$$\frac{W_F(1)}{W_I(1)} = 1 \tag{15}$$

and

$$\lim_{X\to\infty} \frac{W_F(X)}{W_I(X)} \to \infty \tag{16}$$

So we can rewrite Eq. (9) as:

$$\frac{g_F\ e^{-\beta\varepsilon_F}\ \ [1+f_F(X)]}{g_I\ e^{-\beta\varepsilon_I}\ \ [1+f_I(X)]} = A(X') \tag{17}$$

with

$$A(X') = \frac{g'_F\ e^{-\beta\varepsilon'_F}\ \ \left[1+f'_F(X')\right]}{g'_I\ e^{-\beta\varepsilon'_F}\ \ \left[1+f'_I(X')\right]} \tag{18}$$

where the superscript " ' ", refers to the renormalised b'-cell and where we suppressed the superscript (m) since we are now considering systems which, in general, differ from the $(mSG)_b$ fractal.

The function $f_l(X)$ $(l = F, I)$ (and similarly $f'_l(X')$) defined by

$$f_l(X) = \sum_{i>j_l}^{N_b} \frac{g_l^{(i)}}{g_l^{(j_l)}}\ X^{(i-j_l)} \qquad (l = F, I) \tag{19}$$

is a monotonously increasing function with the following properties:

$$f_l(0) = 0 \tag{20}$$

and

$$\lim_{X \to \infty} f_l(X) = \infty \tag{21}$$

Let us now analyse Eq. (17) in the X=0 limit (i.e., $T \to 0$ in the antiferromagnetic ($J_2 < 0$) case). Using the general properties of $f_l(X)$ ($l = F, I$) (Eqs. (20) and (21)) and relations (15) − −(16), we obtain (see table I) the possible solutions of Eq. (17) in this limit for different relationships between ε_F, ε_I and ε_F', ε_I'. Three main situations arise from an analysis of this table, namely

1) $\varepsilon_F < \varepsilon_I$ and $\varepsilon_F' > \varepsilon_I'$ (case $B1$). In this case the point $X = 0$ ($K_2 \to -\infty$) is renormalised into $X' \to \infty$ ($K_2' \to \infty$). Since $K_2^* \to \infty$ is the attractor of the ferromagnetic phase, we conclude that the RG transformation leads to a ferromagnetic solution for the AF case at $T = 0$. This situation arises, for example, in the Ising model on bond HL with even chemical distance b (such as the linear chain, diamond HL (see Fig. (1) of [13]), Wheatstone-brigde HL (see Fig. (2) of [13])) where we renormalise a b−cell into a single bond ($b' = 1$). Their typical flow and phase diagrams in the transmissivity variable [37] $t \equiv \tanh(K_2/2)$ are shown in Fig. 2. Notice that, when the basin of attraction of the ferromagnetic phase contain more than one point (Fig. 2(b)), this choice of cells generates an unphysical disconnected ferromagnetic phase.

2) The inequality between ε_F and ε_I is preserved under renormalisation (cases $A1$ and $B2$). For these cases $X' = 0$ is a solution of Eq. (17) for $X = 0$, which means that the RG transformation presents an antiferromagnetic solution, at least, for $T = 0$. This situation can be found, for example, in the Ising model on bond HL with odd chemical distance such as the linear chain, Migdal-Kadanoff-type HL (see Fig. $1(a) - (f)$ of [20]) and also on bond HL with even chemical distances where we renormalise a $b-$cell into a $b'-$one with $b' = b/2 \neq 1$. In Fig. 3 we plotted the two branches $t'_+(t)$ and $t'_-(t)$ of the renormalised transmissivity ($t'_+(t)$ is always positive and $t'_-(t)$ is always negative) which appear in the latter case for $b = 4$ and $b' = 2$ in the Ising model on the linear chain (Fig. $3(a)$), and on the diamond HL (Fig. $3(b)$). The union of $t'_+(t)$ (which is equal to the solution obtained for the renormalisation of the $b = 2$ cell into a bond) for $0 \leq t \leq 1$ with $t'_-(t)$ for $-1 \leq t \leq 0$ provides a solution (which we denote by $t'_{ph}(t)$) which is physically meaninful. On the linear chain this solution leads to the exact known phase diagram, while on the diamond HL it yields to the expected antiferromagnetic order at zero temperature for the Ising model with $J_2 < 0$ (notice that in this case there is no frustration in the thermodynamic limit). Concerning the Ising model on HL with odd chemical distances the solution $t'(t)$ obtained for $b' = 1$ are qualitatively similar to $t'_{ph}(t)$, not requiring therefore the use of bigger cells.. Differently from situation (1), we note that points with $K_2 < 0$ $(-1 < t < 0)$ are renormalised into $K'_2 < 0$, causing no disconectedness in the basins of attraction of the phases.

3) $\varepsilon_F = \varepsilon_I$ (cases $C1$, $C2$ and $C3$). In this situation the RG transformation provides a finite and non-null solution for X'. Furthemore, a solution $X' < 1$ can appear when the type of the ground state is preserved under renormalisation. On the other hand, when there is no such preservation we do not know any example in which $X' < 1$ is a solution. When $X' > 1$ the behaviour is qualitatively similar to situation (1): any $K_2 < 0$ renormalises after one RG step to $K_2' > 0$. One example of this situation will be given in Sec. V-A (AF Ising model on the $(mSG)_2$ for $b = 2$ and $b' = 1$). When $X' < 1$ it can happen two possibilities:

3a) Sucessive iterations of any $K_2' < 0$ converges to the paramagnetic attractor $t_P^* = 0$ through negative values of K_2. Such behaviour occurs, for instance, in the AF Ising model on the $(2SG)_2$ for $b = 4$ and $b' = 2$ (see Sec. V-A).

3b) If $|K_2'|$ is large enough it can lead to an antiferromagnetic attractor at a finite temperature, i.e., $t_{AF}^* \neq -1$ as illustrated on Fig. 4. We will see one example of this situation in the AF Ising model on the $(mSG)_4$ for $m \geq 116$ (see Sec. V-B.1). Using a rescaling argument, Berker and Kadanoff [24] showed how this behaviour can arise, in the RSRG scheme, on systems whose residual entropy per particle is non-zero, such as the antiferromagnetic q-state Potts model on hypercubic lattices [24], on Migdal-Kadanoff-type HL [20], on Sierpinski Carpet [6, 36] and on Sierpinski Pastry Shell [6]; they also suggested that this unusual phase is characterised by a power law decay of correlations.

We conclude from the above analysis that when $\varepsilon_F \neq \varepsilon_I$ and $\varepsilon_F' \neq \varepsilon_I'$ one should choose cells whose ground state are from the same type, otherwise it can appear unphysical disconnected phases. When $\varepsilon_F = \varepsilon_I$ (which is the case of the Ising model on all the fractal families $(mSG)_b$) we showed that $K_2 \to -\infty$ is not a fixed point, or in other words, the zero temperature character of the antiferromagnet is not preserved under the RG, since the rooted spins after renormalisation become nearest neighbours and have a non-zero probability of being all in the same state. In the latter case if there is preservation of the type of the cellular ground state, then it can appear an unusual phase characterised by an attractor at a non-null temperature.

We also verified in some examples of a 2−parameter RG (see, for example, Sec. V-B.2 and Sec. V-B.3) that a necessary (but not sufficient) condition for obtaining reliable results is to choose cells which preserve the type of the ground state under renormalisation. In fact we believe that this criterion can be generalised to an n−parameter RG ($n > 1$) for AF classical spin models defined on (or approximated by) HL with many roots and such that there are $(n+1)$ different restricted partitions functions. The RG recursive relations are, then, constructed by preserving n different ratios of these restricted partition functions. In this case there will be $(n+1)$ configurations (with respective energies $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}$ and degeneracies $g_1, g_2, \cdots, g_{n+1}$) where the rooted spins are at frozen states and the remaining spins are such that two nearest neighbours are at different states. If ε_i is the smallest energy (or in the case of equalities among the energies, if g_i is the biggest degeneracy) then we say that the ground state of the cell is of type

i. The generalisation of the above criterion would then be:

- *a necessary, but not sufficient condition for the above n−parameter RG to describe well AF classical spin models defined on (or approximated by) multi-rooted HL is to use cells which preserve the type of the ground state under renormalisation.*

V - Results

In this section we consider the Ising model on the $(mSG)_2$ and the $q = 2, 3$ and 4−state Potts model on the $(mSG)_4$ family.

V-A Ising model on the $(mSG)_2$

In this case, renormalising the $b = 2$ cell with m sheets (Fig. 1(a), $n = 1$) into the $b' = 1$ cell with one-sheet (Fig. 1(a), $n = 0$), Eq. (17) becomes:

$$\left[\frac{4\,X^3\,(1 + f_F(X))}{3\,X^3\,(1 + f_I(X))}\right]^m = \frac{X'^3}{X'} = X'^2 \tag{22}$$

with

$$f_F(X) = (3/4)\,X^2 + (1/4)\,X^6 \tag{23}$$

and

$$f_I(X) = (4/3)\,X^2 + (1/3)\,X^4 \tag{24}$$

which agrees, for $m = 1$, with Eq. (1) of [1].

Rewriting Eq. (22) in terms of the finite valued $t \equiv \tanh(K_2/2)$ variable, we obtained the $t'(t)$ and the corresponding phase diagram shown in Fig. 5. The exact ferromagnetic critical temperature $K_B\, T_c^F\, / J_2$ is plotted as a function of m in Fig. 6. It should be noted that, for $m = 2$, points with $-1 \leq t < 0$ converge to the paramagnetic attractor (P) through positive (instead of negative) values of K_2'. Moreover, for $m \geq 3$, points with $-1 \leq t < t_0$ ($t'(t_0) = t_c^F$) converge to the ferromagnetic attractor (F), generating, thus, a disconnected unphysical ferromagnetic phase. This behaviour can be understood in terms of the analysis of the energies ($\varepsilon_F, \varepsilon_I$) and degeneracies ($g_F, g_I$) introduced in the previous section given by:

$$\varepsilon_F = -3\, J_2\, m \quad g_F = 4^m \qquad \varepsilon_F' = -3\, J_2' \quad g_F' = 1$$

$$\varepsilon_I = -3\, J_2\, m \quad g_I = 3^m \qquad \varepsilon_I' = -\ J_2' \quad g_I' = 1$$

So, the ground state for the antiferromagnetic (AF) case of this model on the non-renormalised cell ($b = 2$, m sheets) is of type F (because, in spite of $\varepsilon_F = \varepsilon_I$, $g_F > g_I$) while it is of type I on the renormalised one ($b' = 1$) (because $\varepsilon_I' < \varepsilon_F'$). In order to avoid the unphysical behaviour above caused by the lack of preservation of the type of the ground state, we should choose other cells that reproduce exactly the same fractal, for instance the $b = 4$ with m sheets (see Fig. 1(a), $n = 2$) and the renormalised cell $b' = 2$ with m sheets (see Fig. 1(a), $n = 1$).

The RG transformation (Eq. (17)) becomes, with this new choice of cells:

$$\frac{g_F(m)\,X^{9m}\,\left[1+f_F^{(m)}(X)\right]}{g_I(m)\,X^{9m}\,\left[1+f_I^{(m)}(X)\right]}=\frac{4\,X'^3\,\left[1+f_F'(X')\right]}{3\,X'^3\,\left[1+f_I'(X')\right]} \tag{25}$$

where the dependence on m of the $g_F(m)$ and $g_I(m)$ are not any more powers of $g_F(1)$ and $g_I(1)$, for example: $g_F(1)=280$, $g_I(1)=273$, $g_F(2)=10900$, $g_I(2)=9675$, $g_F(3)=480844$, $g_I(3)=356265$. $f_F^{(m)}$ and $f_I^{(m)}$ are smooth polynomials of respective degrees $18m$ and $16m$, $f_F'(X')$ and $f_I'(X')$ have the same functional forms as those of Eqs. (23) and (24), respectively. It should be noted that, for any value of m, the ground state for the AF case of this model is of type F for both cells since, in spite of $\varepsilon_F=\varepsilon_I=-9\,J_2\,m$ and $\varepsilon_F'=\varepsilon_I'=-3\,J_2'\,m$, $g_F(m)>g_I(m)$ for all m and $g_F'>g_I'$. On the other hand, one can easily show that Eq. (25) presents solutions with $X'<1$ at $X=0$ only for $g_F(m)/g_i(m)<\frac{4}{3}$. As we can see from the above degeneracies, this condition holds exclusively for $m=1$ and 2. In Fig. 7 it is plotted, for $m=2$, the real solutions on the $t-$variable which can have any physical meaning. The positive solution $t_+'(t)$ is exactly the one obtained in the previous renormalisation ($b=2\rightarrow b'=1$). Similarly to the examples of case (2) of Sec. IV, the physical solution is obtained by the union of $t_+'(t)$ for $0\leq t\leq 1$ with the negative one t_-' for $-1\leq t\leq 0$. Similarly to the Ising antiferromagnet on the triangular lattice which due to its full frustration, is paramagnetic even at $T=0$ [23], there is no AF order (not even at $T=0$) for this model on the fully frustrated fractal $(2SG)_2$. For $m\geq 3$ the negative branch becomes complex near $t=-1$ and, therefore, $t'(t)$ is given by the positive branch $t'_+(t)$ which generates a disconnected ferromagnetic phase. Despite of the preservation, for $m > 3$, of the type of the celular ground state, the considered RG leads to unphysical results — this example illustrates the insufficiency of the criterion stated in Sec. IV. We believe that, for a given $m \geq 3$, the convenient choice of cells which leads to the exact results for the AF Ising model on the $(mSG)_2$ should have sides $b = 2^n$ and $b' = 2^{n-1}$ where $n(m)$ is such that each basin of attraction of a phase is connected.

V-B Potts Model on the $(mSG)_4$

Let us consider now the $q = 2, 3$ and 4−state Potts model on a different fractal family, namely, on the $(mSG)_4$. In this case, we renormalise the $b = 4$ cell with m sheets (see Fig. 1(c)) into the $b' = 1$ cell with one sheet (see Fig. 1(a), $n = 0$).

V-B.1 $q = 2$ case (Ising model)

In this case the RG transformation (Eq. (17)) is written as

$$\frac{(168)^m\ X^{10m}\ \ [1 + f_F(q = 2, X)]^m}{(175)^m\ X^{10m}\ \ [1 + f_I(q = 2, X)]^m} = \frac{X'^3}{X'} \tag{26}$$

where

$$f_F(q = 2, X) = (1/168)[847\,X^2 + 1200\,X^4 + 975\,X^6 + 595\,X^8 + 213\,X^{10} + 75\,X^{12} + 13\,X^{14} + 9\,X^{16} + X^{20}] \tag{27}$$

and

$$f_I(q=2,X) = (1/175)[812\,X^2 + 1243\,X^4 + 991\,X^6 + 545\,X^8$$
$$+223\,X^{10} + 81\,X^{12} + 21\,X^{14} + 4\,X^{16} + X^{18}] \qquad (28)$$

Notice that the above expressions lead, for $m = 1$ and $K \gg 1$, to $e^{-2K'} \cong e^{-2K} + 4\ e^{-4K} + O(e^{-6K})$ which agrees with Eq. (9) of [9] valid for all b and $m = 1$. In Fig. 8 is showed the plot of Eq. (26) on the t−variable and the respective phase diagrams for different values of m. The critical temperatures for the P-F and P-AF transitions are plotted in Fig. 9.

It should be noted that this is an example of the case ($C1$) of table I where the ground state for the AF case of this model is of type I on both $b = 4$ and $b' = 1$ cells and where the suitable negative solution for negative values of K_2 ($-1 < t < 0$) appears. We want to stress that it does not appear a new negative solution for $-1 < t < 0$ when we increase the sizes of the cells to $b = 16$ and $b' = 4$ since for any fixed value of $\frac{W'_F}{W'_I}|_{b'=4}$ there corresponds a unique value of t' where $-1 \leq t' \leq 0$. This is an indication that our solution might be the exact one.

From Fig. 8 we see that an unusual AF phase appears only for $m > m_c \cong 115.57$ ($D_f^c \cong 5.1$). This occurs only when, for $T = 0$, the probability $p_I(m)$ of the configuration (I) of the non-renormalised cell becomes much bigger than the probability $p_F(m)$ of the configuration (F) ($p_I(m_c)/p_F(m_c) \cong 112$). We obtained, for $J_2 < 0$, the same RG behavior with increasing D_f as the one proposed by

Berker and Kadanoff [24] for $d-$dimensional systems with residual entropy per site with increasing d. In Fig. 9 the para-ferro and para-antiferromagnetic critical temperatures for this case are showed.

Let us now calculate the correlation function Γ_{AB} on the $(mSG)_4$ for a fixed $m > m_c$ along the unusual phase with attractor $X^*_{AF} \neq 0$.

Using the $X-$variable ($X \equiv e^{K_2}$) and interacting n times Eq. (10) we obtain that

$$\Gamma^{(n)}_{AB}(G^{(n)}, X) = \Gamma^{(0)}_{AB}(G^{(0)}, X^{(n)} = (T_m)^n) \tag{29}$$

where

$$X' = T_m(X) \tag{30}$$

$T_m(X)$ is given explicitly by the square root of the left hand side of Eq. (26). $G^{(n)}$ is the $n^{th}-$stage of the $(mSG)_4$ and $X^{(n)}$ is the coupling constant obtained after n iterations of the recursive equation (30).

Through linearisation of T_m around the AF attractor fixed point, X^*_{AF}, we obtain

$$X^{(1)} - X^*_{AF} = \lambda_m \, (X - X^*_{AF}) + O(X^2) \tag{31}$$

where

$$\lambda_m = (\partial T_m / \partial X) \Big|_{X = X^*_{AF}} < 1 \tag{32}$$

Iterating Eq. (31) n times leads to

$$X^{(n)} - X^*_{AF} \cong (\lambda_m)^n \, (X - X^*_{AF}) \tag{33}$$

Combining Eqs. (29), (33) with the expression of $\Gamma^{(0)}_{AB}(G^{(0)}, X)$ given by:

$$\Gamma^{(0)}_{AB}(G^{(0)}, X) = 2\,\frac{(X^2+1)}{(X^2+3)} - 1 \tag{34}$$

we finally arrive, in the fractal limit ($n \to \infty$), at:

$$\Gamma_{AB}(G, X) \equiv \lim_{n\to\infty} \left[\Gamma^{(n)}_{AB}(G^{(n)}, X) - \Gamma^{(\infty)}_{AB}(G^{(\infty)}, X^*_{AF})\right] \cong$$

$$\cong \;\; B_m(X)\, r_{AB}^{-a(m)} \qquad (r_{AB} \to \infty) \tag{35}$$

where

$$B_m(X) = 4\,\frac{(X^{*2}_{AF}+1)}{(X^{*2}_{AF}+3)}\left[X^*_{AF}\left(\frac{1}{(1+X^{*2}_{AF})} - \frac{1}{(3+X^{*2}_{AF})}\right)\right]\;(X - X^*_{AF}) \tag{36}$$

r_{AB} is the chemical distance between the roots A and B at the n^{th}–stage of the $(mSG)_4$ given by:

$$r_{AB} = 4^n \tag{37}$$

and $a(m)$ is defined as:

$$a(m) = -\frac{\ln\lambda_m}{\ln 4} \tag{38}$$

Eq. (35) confirms, thus, the power law decay of correlations along this whole unusual phase as suggested by Berker and Kadanof [24].

Assuming that, similarly to the asymptotic behavior of $\Gamma(r \to \infty)$ in d-dimensional Bravais lattices

$$\Gamma_{AB}(G,X) \cong r_{AB}^{D_f(m)-2-\eta_{AF}(m)} \quad (r_{AB} \to \infty) \tag{39}$$

we obtained the critical exponent η_{AF} as a function of m shown in Fig. 10.

V-B.2 3-state Potts model

For $q = 3$ the RG transformation (Eqs. (6) and (7)) can be written as

$$\frac{(24)^m \; X^{2m} \; [1 + f_F(q=3,X,Y)]^m}{[1 + f_{AF}(q=3,X,Y)]^m} = \frac{X'^3 \, Y'}{1} \tag{40}$$

$$\frac{(4)^m \; X^m \; [1 + f_I(q=3,X,Y)]^m}{[1 + f_{AF}(q=3,X,Y)]^m} = \frac{X'}{1} \tag{41}$$

where $f_l(q = 3, X, Y)$ $(l = F, I, AF)$ are polynomials in X and Y with many terms whose first and last ones are given by:

$$f_F(q=3,X,Y) = (1/24)\left[204\, X + \cdots + X^{28}\, Y^{10}\right] \tag{42}$$

$$f_I(q=3,X,Y) = (1/4)\left[28\, X + \cdots + X^{27}\, Y^{9}\right] \tag{43}$$

and

$$f_{AF}(q=3,X,Y) = \left[48\, X^2 + \cdots + 3\, X^{26}\, Y^{8}\right] \tag{44}$$

Using the finite value transmissivity [37] variable t_2, $(-1/(q-1) \leq t_2 \leq 1)$ defined by

$$t_2 = \frac{e^{K_2} - 1}{e^{K_2} + (q-1)}$$

and a similar variable associated with three-spin interactions:

$$t_3 = \frac{e^{K_3} - 1}{e^{K_3} + (q-1)}$$

we obtained the flow and phase diagrams for $m = 2$ shown in Fig. 11(a) as well as the critical frontiers for different values of m (see Fig. 11(b)). Table II contains the semi-stable fixed points t_c^F and t_c^{AF} which govern the respective critical behaviours of the P-F and P-AF transitions.

It should be noted that the critical frontier P-F [P-AF] is tangent to the axis $t_3 = -1/2$ [$t_3 = 1$] and finishes at the respective attractor F $(K_2 \to \infty, K_3 \to -\infty)$ $[AF(K_2 \to -\infty, K_3 \to \infty)]$. This unusual behaviour, where the attractor is localized on a critical line was also obtained for the pair of attractors $(K_2 \to -\infty, K_{nn} \to -\infty)$ and $(K_2 \to \infty, K_{nn} \to -\infty)$ in the Ising model with nearest-neighbour (K_2) and $n.n.$−neighbour (K_{nn}) interations on the square lattice [38]. In both cases the attractors are characterised by infinite coupling constants and asymptotic behaviors which depend on the angle of approach.

For increasing values of m, the connectivity increases strenghening the correlations; consequently, the regions of the ordered phases become larger as shown in

Fig. 11(b). On the other hand, on the $m \to 1$ limit, where the order of ramification (R) becomes finite, the critical frontier P-AF [P-F] reduces to the $t_2 = -1/2$ [$t_2 = 1 \cup t_3 = 1$] axis bringing the critical temperature down to $T_c = 0$ in both cases, and the semi-stable fixed point t_c^{AF} [t_c^F] coincides with its respective attractor AF [F]. This is in agreement with the observed fact (see, for example, [3]) that short-range spin models on structures with finite order of ramification do not present phase transition at finite temperatures. It should be noted that, differently from the fully frustrated Ising model on the $(mSG)_4$ for $m \leq m_c$, the 3−state Potts model presents, for all values of m, an antiferromagnetic ordering at $T = 0$ with a non-frustrated ground state. Notice also that, in the AF case, the ground states of the used cells are both of type AF (considering $J_2 < 0$ and $J_3 < -3J_2$) and are non-degenerated. This case is a generalisation to 2 parameters of the case ($A1$) of table I, where $T = 0$ ($X \to 0, Y \to \infty$) is a fixed point.

V-B.3 4−state Potts model

For $q = 4$ the RG transformation (Eqs. (6) and (7)) are given by

$$\frac{(360)^m \quad [1 + f_F(q = 4, X, Y)]^m}{(616)^m \quad [1 + f_{AF}(q = 4, X, Y)]^m} = \frac{X'^3\, Y'}{1} \tag{45}$$

and

$$\frac{(456)^m \quad [1 + f_I(q = 4, X, Y)]^m}{(616)^m \quad [1 + f_{AF}(q = 4, X, Y)]^m} = \frac{X'\, Y'}{1} \tag{46}$$

where the first and last terms of $f_i(q = 4, X, Y)$ are:

$$f_F(q=4,X,Y) = (1/360)\left[8064\,X + \cdots + X^{30}\,Y^{10}\right] \tag{47}$$

$$f_I(q=4,X,Y) = (1/456)\left[8928\,X + \cdots + X^{28}\,Y^{9}\right] \tag{48}$$

and

$$f_{AF}(q=4,X,Y) = (1/616)\left[9168\,X + \cdots + 3\,X^{26}\,Y^{8}\right] \tag{49}$$

The AF-ground state of this model is of type AF on the non-renormalised cell (since, in spite of $\varepsilon_{AF} = \varepsilon_I = \varepsilon_F = 0$, $g_{AF} > g_I > g_F$) and also on the renormalised one (since $\varepsilon'_{AF} < \varepsilon'_I$ and $\varepsilon'_{AF} < \varepsilon'_F$, considering $J_2 < 0$ and $J_3 < -3J_2$). This is a generalisation to 2 parameters of the case $(C1)$ of table I where, due to the degeneracies g_I and g_F, $T = 0$ is not a fixed point for any value of m.

The phase diagram on the (t_2, t_3) variables for $m = 2$ and $m = 20$ are shown in Fig. 12. Similarly to the $q = 2$ case, it appears an unusual AF phase with attractor at $T \neq 0$ only for $D_f(m) \geq 3.7$ ($m \geq m_c \cong 17.63$), when the probability $p_{AF}(m)$ of the configuration (AF) of the non-renormalised cell at $T = 0$ becomes much bigger than those of the configurations (I) and (F) $\left(\frac{p_{AF}(m_c)}{p_I(m_c)} \cong 199\right.$ and $\left.\frac{p_{AF}(m_c)}{p_F(m_c)} \cong 12775\right)$. Table III contains the semi-stable fixed points t_c^F and t_c^{AF} governing the P-F and P-AF phase transitions respectively as well as the attractor t^*_{AF} of the unusual AF phase. Similarly to the Ising case, as m tends to m_c the fixed points t_c^{AF} and t^*_{AF} approach each other until they merge, for $m = m_c$, into a single marginal one where $\nu_T^{AF} \to \infty$. Notice also that, for $m \gg m_c$, the AF attractor converges to $T = 0$ since p_I and p_F become neglectable in comparison with p_{AF} $\left(\frac{p_I(m=45)}{p_{AF}(m=45)} \cong 10^{-6} \text{ and } \frac{p_F(m=45)}{p_{AF}(m=45)} \cong 10^{-11}\right)$. Although the above behaviour confirms the one suggested by Berker and Kadanoff [24], it remains to be proved that the correlations decay algebraically along this entire distinctive phase.

V-B.4 Critical exponents ν_T^F and ν_T^{AF} for $q = 2, 3$ and 4

The correlation length critical exponents ν_T^F and ν_T^{AF} for the respective P-F and P-AF transitions are given by

$$\nu_T^s = \ln(b/b')/\ln \lambda_>^s \qquad (s = F, AF) \tag{50}$$

where $\lambda_>^s$ is the greatest eigenvalue ($\lambda_>^s > 1$) of the Jacobian matrix obtained through the derivation of the RG transformation with respect to the parameters evaluated at the semi-stable fixed point t_c^s ($s = F, AF$).

The dependences of ν_T^F and ν_T^{AF} with $D_f(m)$ for the $q = 2, 3$ and 4$-$state Potts model on the $(mSG)_4$ are shown on Figs. 13(a) and 13(b). It should be noted that ν_T^F diverges for $m \to 1$ as expected [9]. Its asymptotic behavior is given by

$$\nu_T^F(q) \sim 1/(D_f(m) - D_f(1)) \qquad m \to 1 \tag{51}$$

which is similar to the result

$$\nu_T \sim 1/(d-1) \quad d \to 1$$

obtained for the Potts ferromagnet on hypercubic lattices [39]and Migdal-Kadanoff-like HL [40] (where $d = 1$ is the lower critical dimension for the P-F phase transition on these lattices). Similarly to [40] (with $q > 2$) the exponent ν_T^F presents, as a function of $D_f(m)$, a minimum at a value $D_f(m) = D_f^{min.}$ which increases for increasing values of q (see Fig. 13(a)). In the $D_f(m) \to \infty$ limit ν_T^F tends to 1 for all values of q, similarly to the result obtained for Migdal-Kadanoff-like HL [40] which differs form the behaviour found for hypercubic lattices [41] where $\nu_T^F \to \frac{1}{2}$ $(d \to \infty)$.

For the P-AF transition, ν_T^{AF} diverges for $m = m_c(q)$ as a power law, namely

$$\nu_T^{AF}(q) \sim D(q)(m - m_c(q))^{-\theta(q)} \quad m \to m_c(q) \tag{52}$$

where $\theta(2) \cong 0.494$, $\theta(3) = 1$, $\theta(4) \cong 0.511$, $D(2) \cong 7.00$, $D(3) = \ln 4$ and $D(4) \cong 2.20$.

Notice that the exact behaviour of $\nu_T^{AF}(3)$ has the same form as that of Eq. (51).

VI Conclusion

We have proposed families of deterministic fractals, the $m-$sheet Sierpinski gaskets with side b ($(mSG)_b$), on which the $q-$state Potts model can be exactly solvable and which presents phase transitions at finite temperatures for $m > 1$. Using a RSRG transformation which preserves under renormalisation the ratios of the probabilities that the rooted spins are in fixed states, we obtained the exact para (P) - ferromagnetic (F) critical frontiers and the thermal correlation length critical exponent, $\nu_T^F(m)$, as a function of m for the Ising model on the $(mSG)_2$ fractal family and for the $q = 2, 3$ and 4−state Potts model on the $(mSG)_4$.

The RG transformation for the antiferromagnetic (AF) Potts model provides undesirable results on some cells of the $(mSG)_b$ for cetain values of b like, for example, a renormalised ferromagnetic coupling in the first iteration which can lead to the appearance of a disconnected ferromagnetic phase. We have discussed this point and concluded that it occurs due to a failure in the preservation of the antiferromagnetic ground state of the chosen cells. Then, we proposed a criterion for a suitable choice of cells in the study of AF classical spin models defined on (or approximated by) multi-rooted hierarchical lattice. Applying this criterion we verified that the fully frustrated Ising antiferromagnet on the $(2SG)_2$ never orders (not even at $T = 0$) similarly to the triangular lattice. We also obtained the P-AF phase boundaries and their respective critical exponent $\nu_T^{AF}(m)$ for the $q = 2, 3$ and 4−state Potts antiferromagnets on the $(mSG)_4$ family, which we expect to be the exact ones. In the cases where the ground state of this model is highly degenerated ($q = 2$ and 4; in the $q = 2$ case this degeneracy is due to frustration) it appears, above a certain critical fractal dimension $D_f^c(q, m)$, an unusual low temperature phase with an attractor at finite temperature. We have proved, for $q = 2$, that such a phase is characterised by a power law decay of correlations not only at the transition point, as usual, but throughout this entire phase. This result had already been suggested by Berker and Kadanoff [24] but, as far as we know, there has been no proof of this in the literature.

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Captions for Figures

Figure 1: (a) The first three stages (n) of construction of the $(2SG)_2$ fractal. The second sheet of the $n = 2$ stage connected to A,B and C is represented by just a single dashed line for visual purposes. (b) The generator ($n = 1$) stage of the $(2SG)_3$. (c) The generator of the $(2SG)_4$. The roots and internal sites are represented by empty and full points.

Figure 2: The RG transformation on the t−variable and the respective flow diagram for (a) the linear chain: renormalisation $b = 2 \rightarrow b' = 1$, (b) the diamond HL: renormalisation $b = 2 \rightarrow b' = 1$. • and ■ represent, respectively, the unstable and the fully stabel fixed points. The dooted line denotes the $t' = t$ curve and the arrows indicate the RG flow directions. By sucessive iterations, an initial value $t \neq 1$ (P-phase) in case (a) will converge to the paramagnetic (P) attractor $t_P^* = 0$. In (b), an initial value $t < t_0$ or $t > t_c^F$ will converge to the ferromagnetic attractor $t_F^* = 1$. Another initial value $t_0 < t < t_c^F$ (P-phase) will converge to the $t_P^* = 0$.

Figure 3: The RG transformation $t'(t)$ and the respective phase diagram for (a) the linear chain: renormalisation $b = 4 \rightarrow b' = 2$, (b) diamond HL: renormalisation $b = 4 \rightarrow b' = 2$. The solid and dash–dotted lines represent, respectively, the positive $(t'_+(t))$ and negative $(t'_-(t))$ solutions.

Figure 4: Typical plot of a RG transformation $t'(t)$ for negative values of t and the respective phase diagram for situation (3) ($\epsilon_F = \epsilon_I$). (a) the finite solution, $K'_2 < 0$, at $T = 0$ coverges after sucessive iterations to the paramagnetic

attractor (t_P^*), (b) when $|K_2'|$ is large enough it leads to a finite temperature antiferromagnetic attractor (AF).

Figure 5: The RG transformation (Eq. (22)) and the respective phase diagram for the Ising model on the $(mSG)_2$: renormalisation $b = 2 \rightarrow b' = 1$. (a) $m = 2$, (b) $m = 3$; the flow diagrams for $m > 3$ are qualitatively similar.

Figure 6: The exact P-F critical temperature as a function of m for the Ising model on the $(mSG)_2$: renormalisation $b = 2 \rightarrow b' = 1$.

Figure 7: Solutions of the RG transformation ($Eq.$ (25)) for the Ising model on the $(2SG)_2$: renormalisation $b = 4 \rightarrow b' = 2$. The dash–dotted lines denote, respectively, the positive and negative branches.

Figure 8: The RG transformation (Eq. (26)) and the respective phase diagram for $q = 2$ on the $(mSG)_4$ for different values of m: (a) $m = 50$ (typical of $m < m_c$), (b) $m = m_c \cong 115.57$, (c) $m = 140$ (typical of $m > m_c$). t_m is a marginal fixed point.

Figure 9: Critical temperatures of the para-ferromagnetic (T_c^F) and para-antiferromagnetic (T_c^{AF}) transitions of the Ising model on the $(mSG)_4$. Notice the jump discontinuity of $K_B\, T_c^{AF}\, /\, |J_2|$ at $D_f^c = 5.08$.

Figure 10: Critical exponent η vs. m along the whole unusual phase of the Ising model on the $(mSG)_4$ for $m > m_c \cong 115.57$.

Figure 11: The 3-state Potts model on the $(mSG)_4$. (a) Flow diagram for $m = 2$. The • and ■ points denote, respectively, the semi-stable and fully stable fixed points; the dashed and solid lines indicate the flows and the critical frontiers respectively. (b) $P - F$ and $P - AF$ critical frontiers for different values of m.

Figure 12: Flow diagrams for the 4-state Potts model on the $(mSG)_4$. (a) $m = 2$ case, which exihibts only the P and F phases. (b) $m = 20$ case, where it appears the AF phase with attractor at a finite temperature showed on the inset (c). The flow diagrams for other values of m with $m > m_c \cong 17.63$ are qualitatively similar to that of $m = 20$.

Figure 13: The respective correlation length critical exponents ν_T^F and ν_T^{AF} vs. $D_f(m)$ for para–ferromagnetic (a) and para–antiferromagnetic (b) phase transitions. The asymptots represent the lower critical dimensions for different values of q.

Captions for Tables

Table I: Real solutions of Eq. (17) in the $X = 0$ limit for different relations among ε_F, ε_I, ε_F', ε_I'. Whenever there is the possibility of having two solutions we indicate them by X_1' and X_2'.

Table II: Values of the semi-stable fixed points t_c^F and t_c^{AF} for $q = 3$ on the $(mSG)_4$ for different values of m.

Table III: Values of the semi-stable fixed points t_c^F, t_c^{AF} and the unusual $AF-$attractor (t_{AF}^*) for $q = 4$ on the $(mSG)_4$ for different values of m.

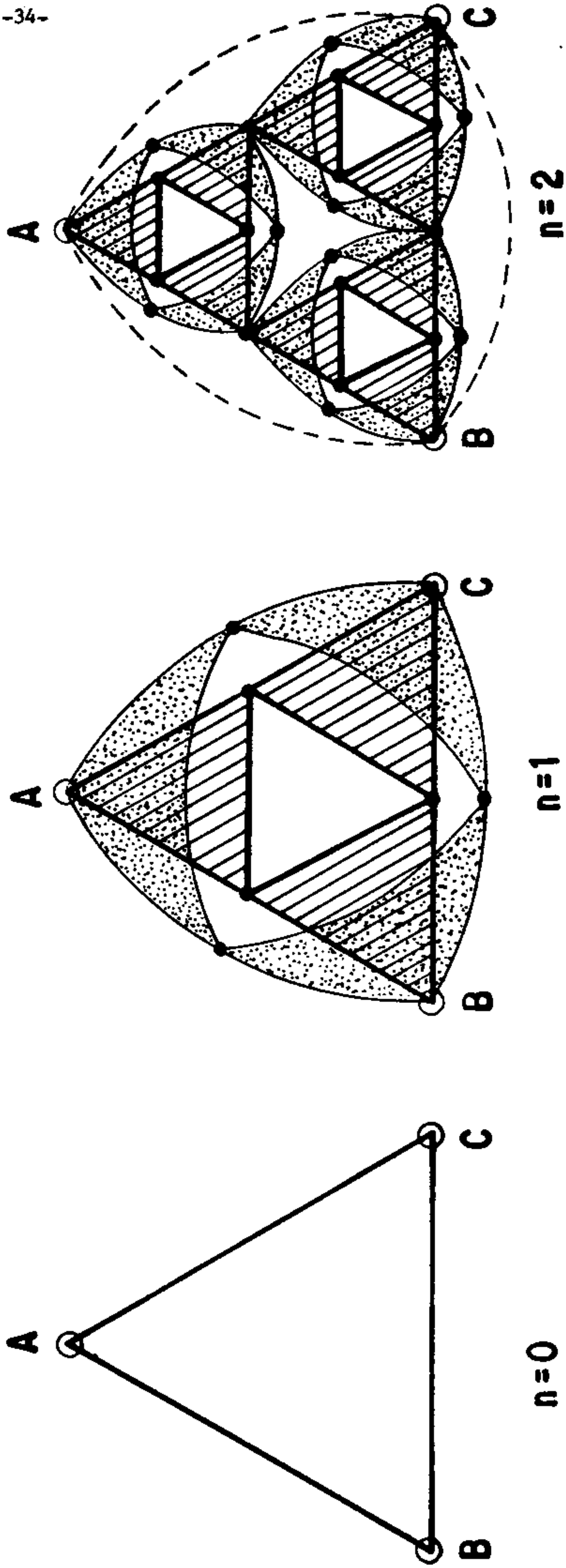


FIG.1a

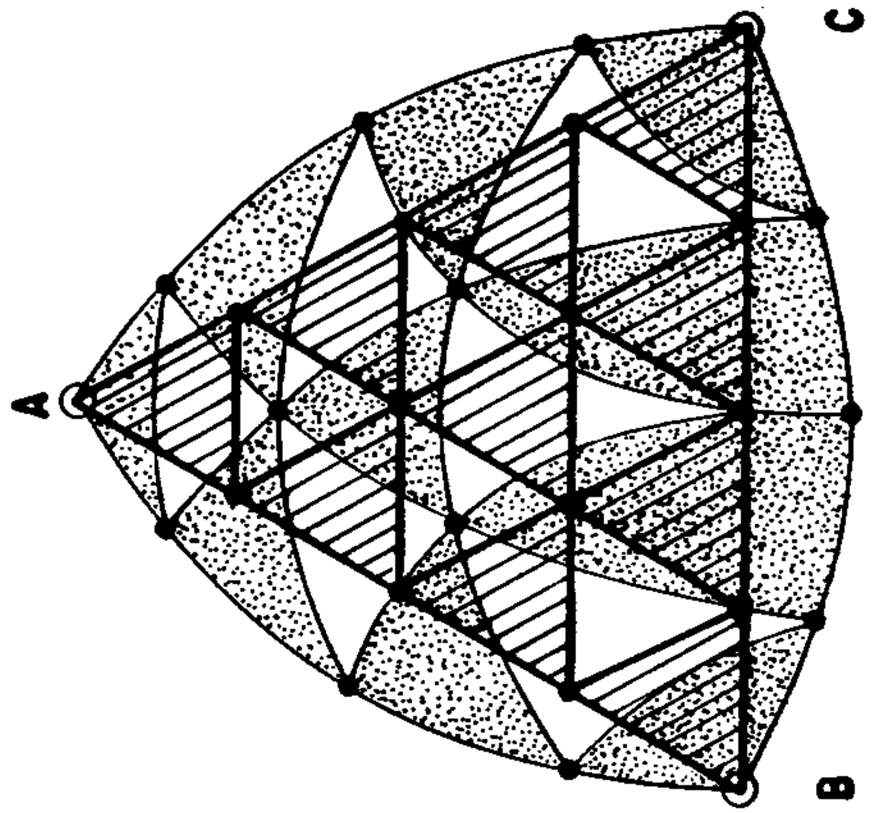


FIG.1c

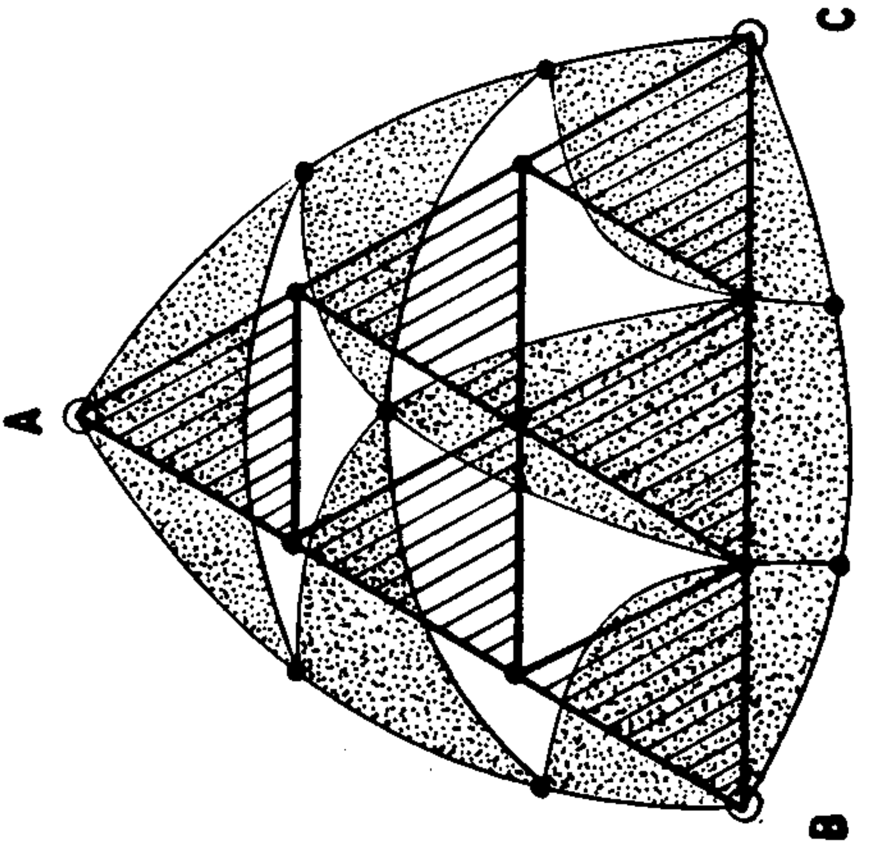


FIG.1b

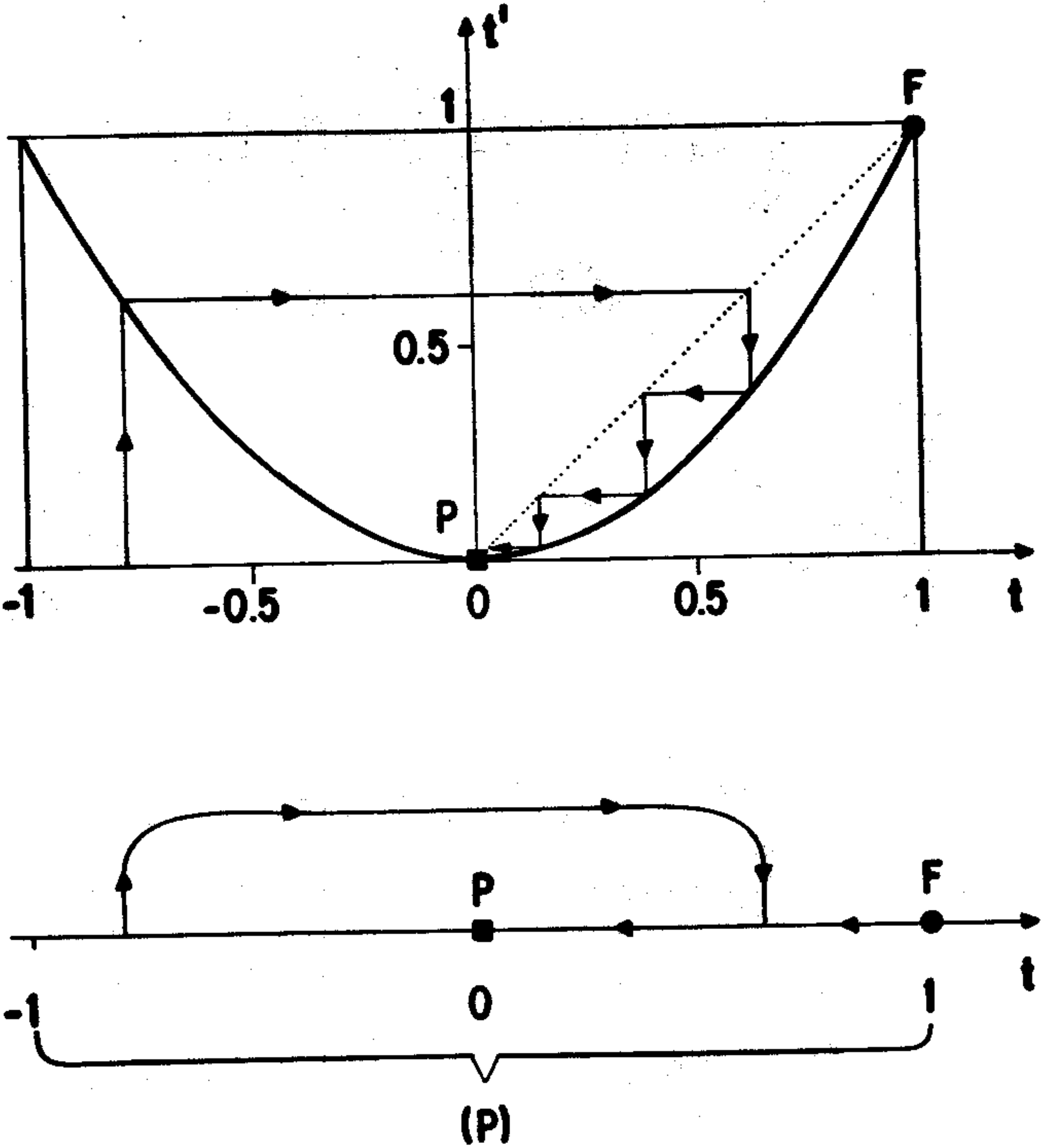


FIG. 2a

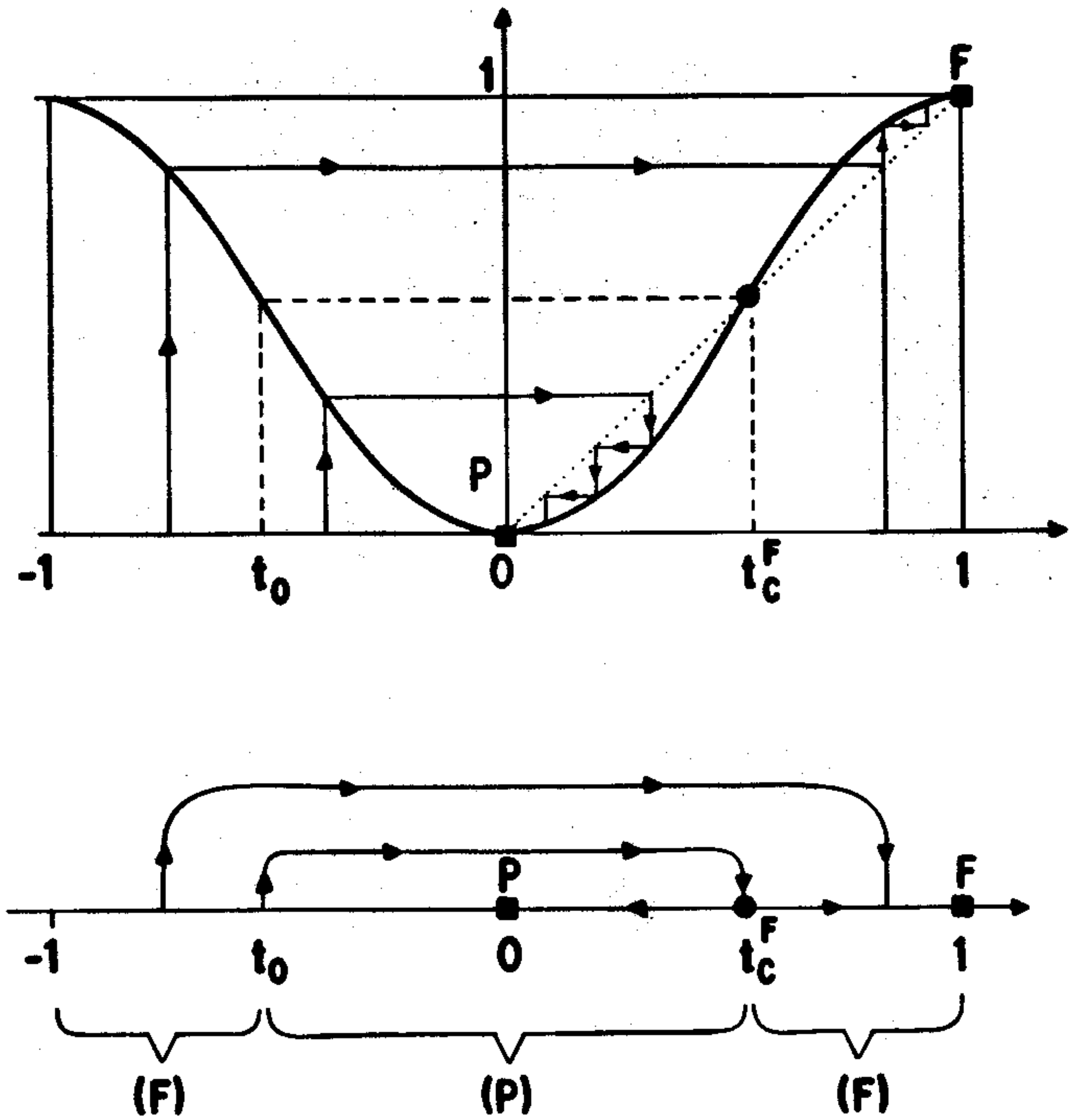


FIG.2b

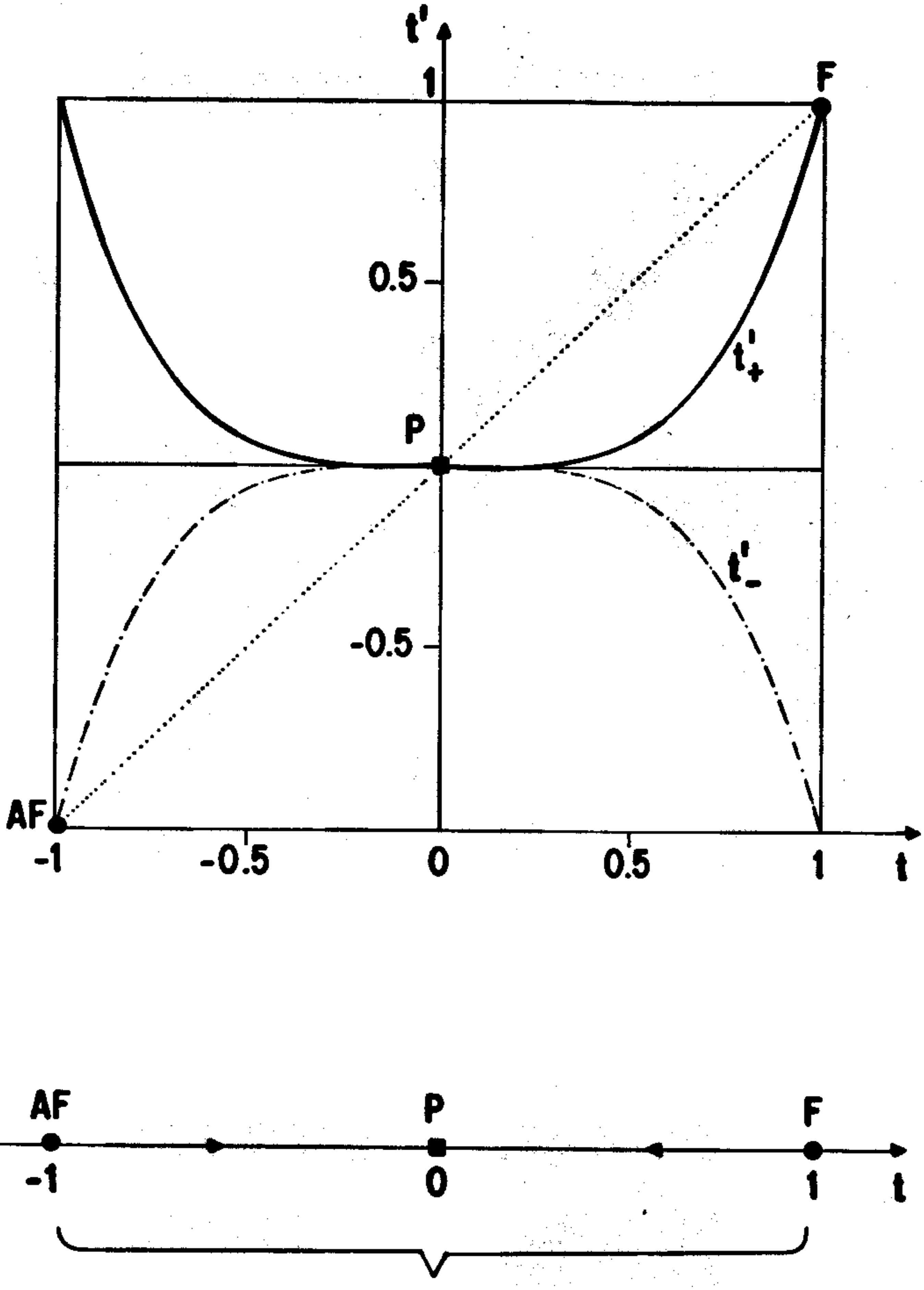
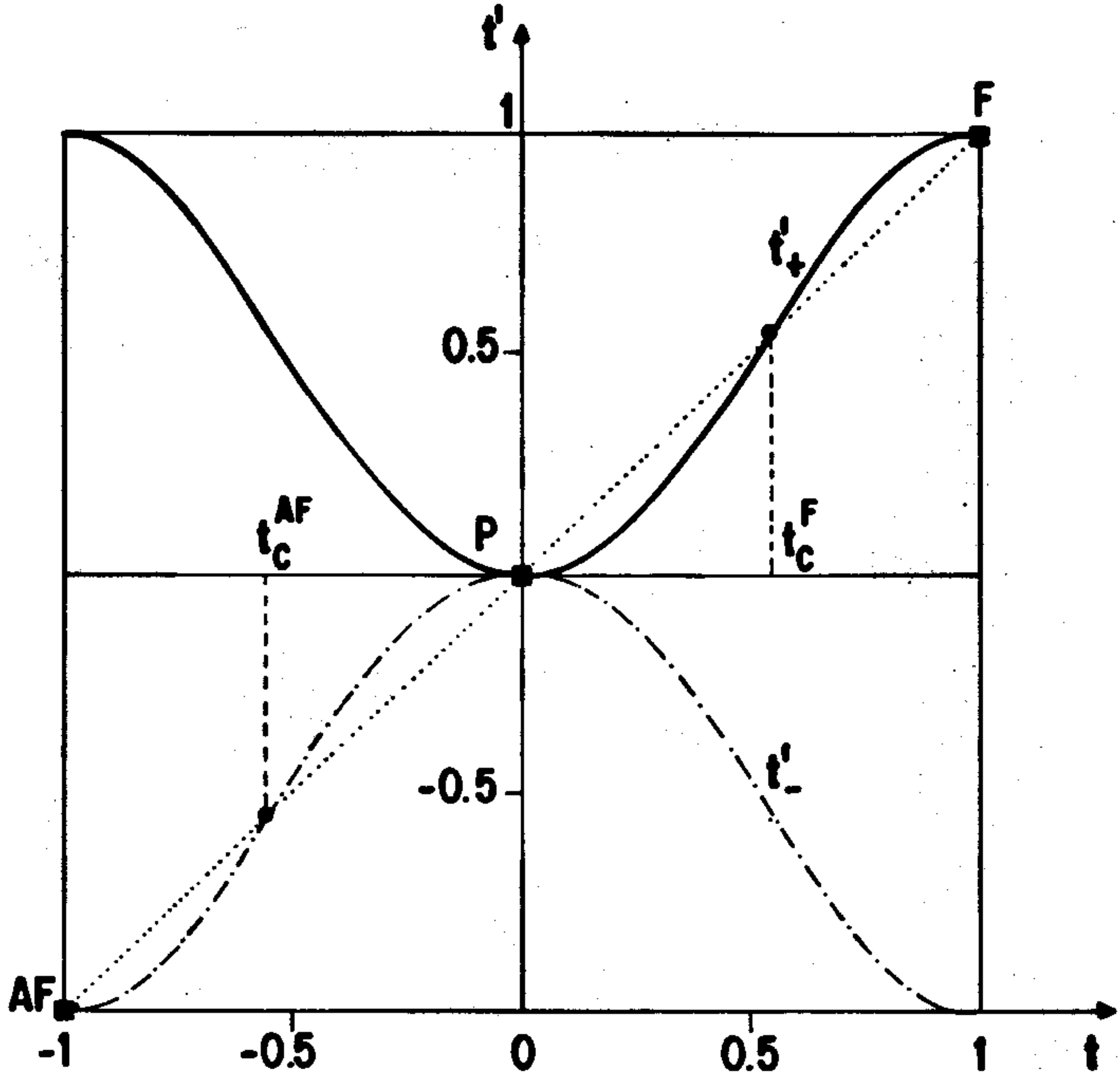


FIG.3a



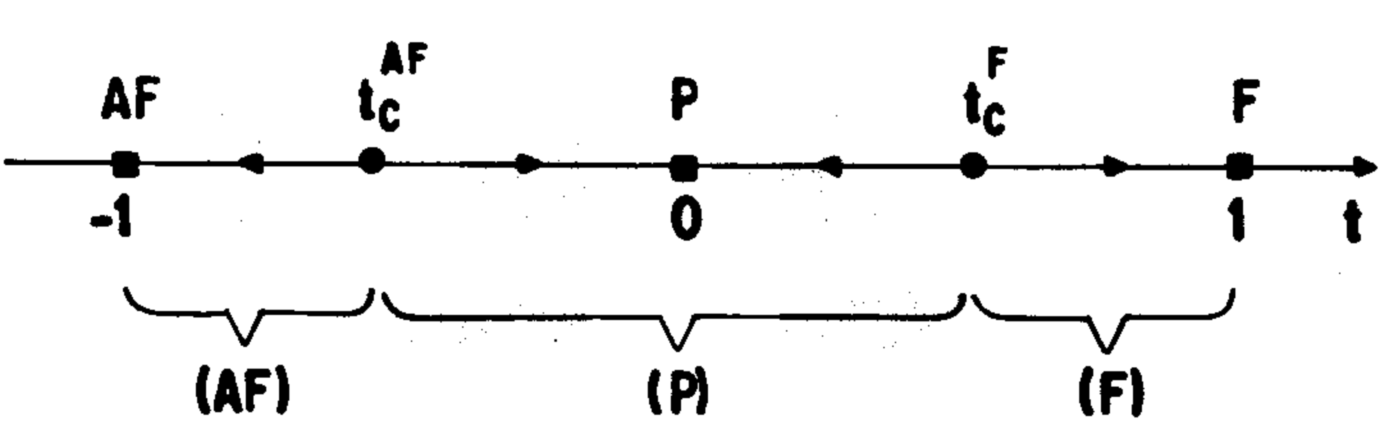


FIG.3b

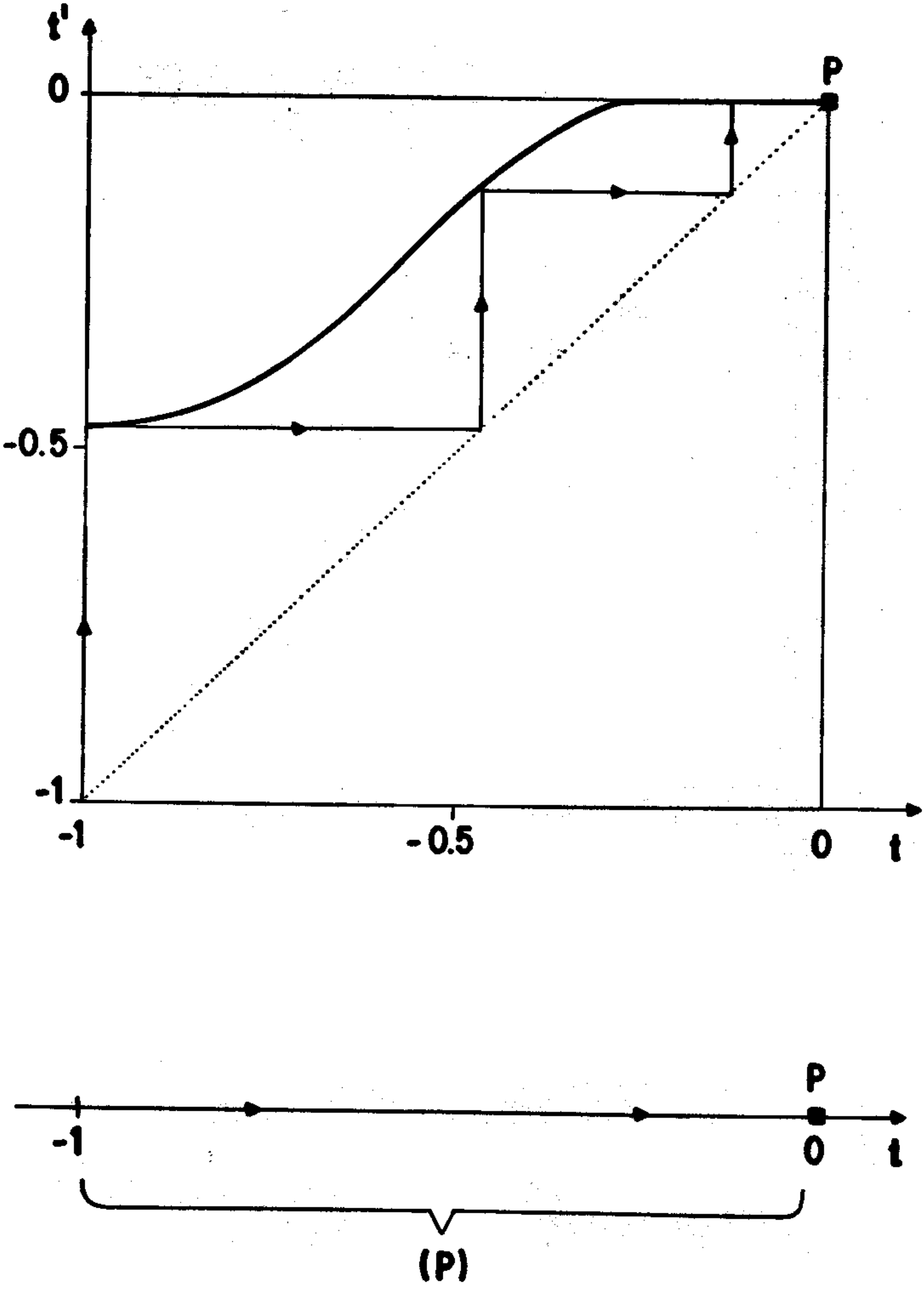


FIG.4a

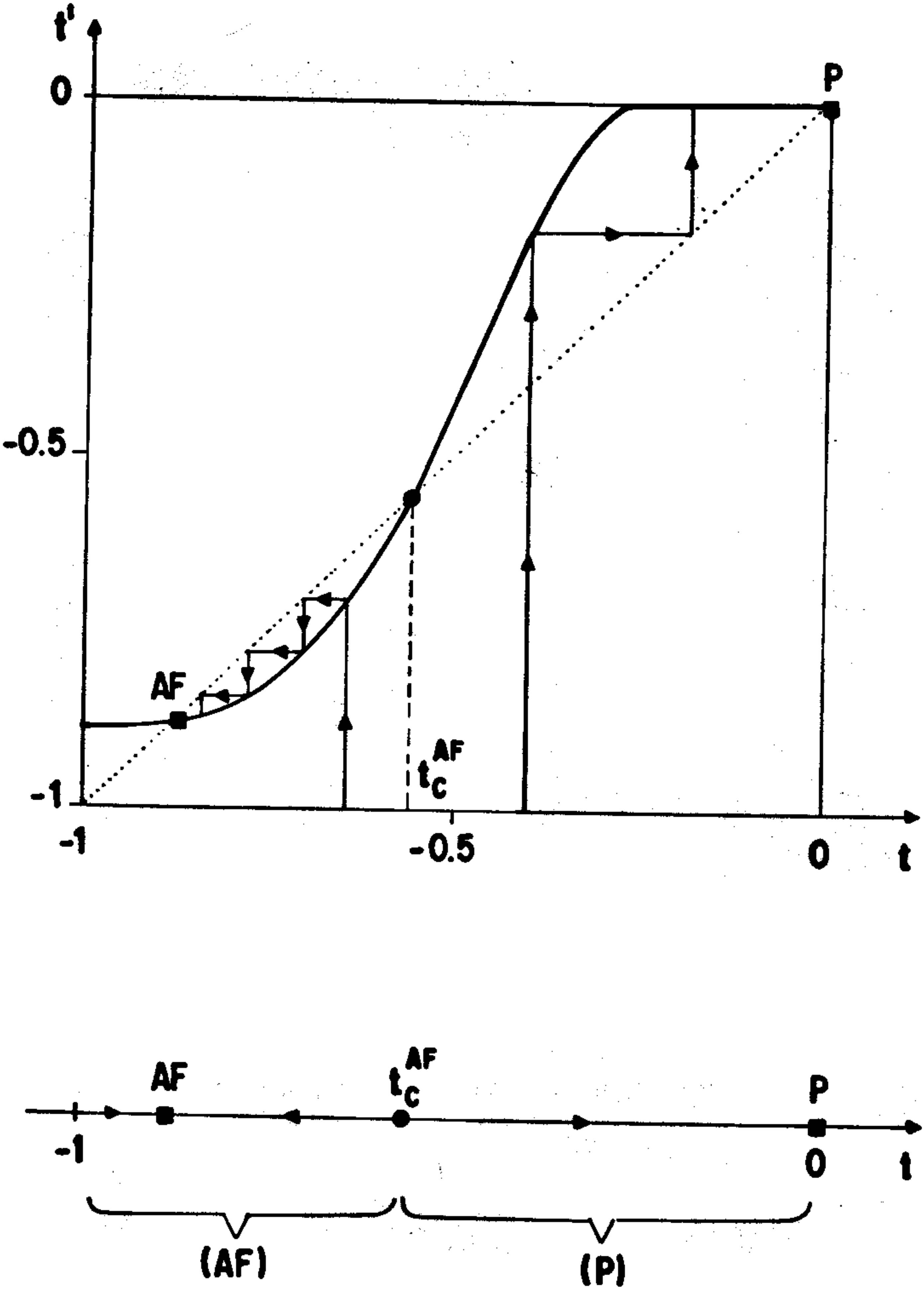


FIG. 4b

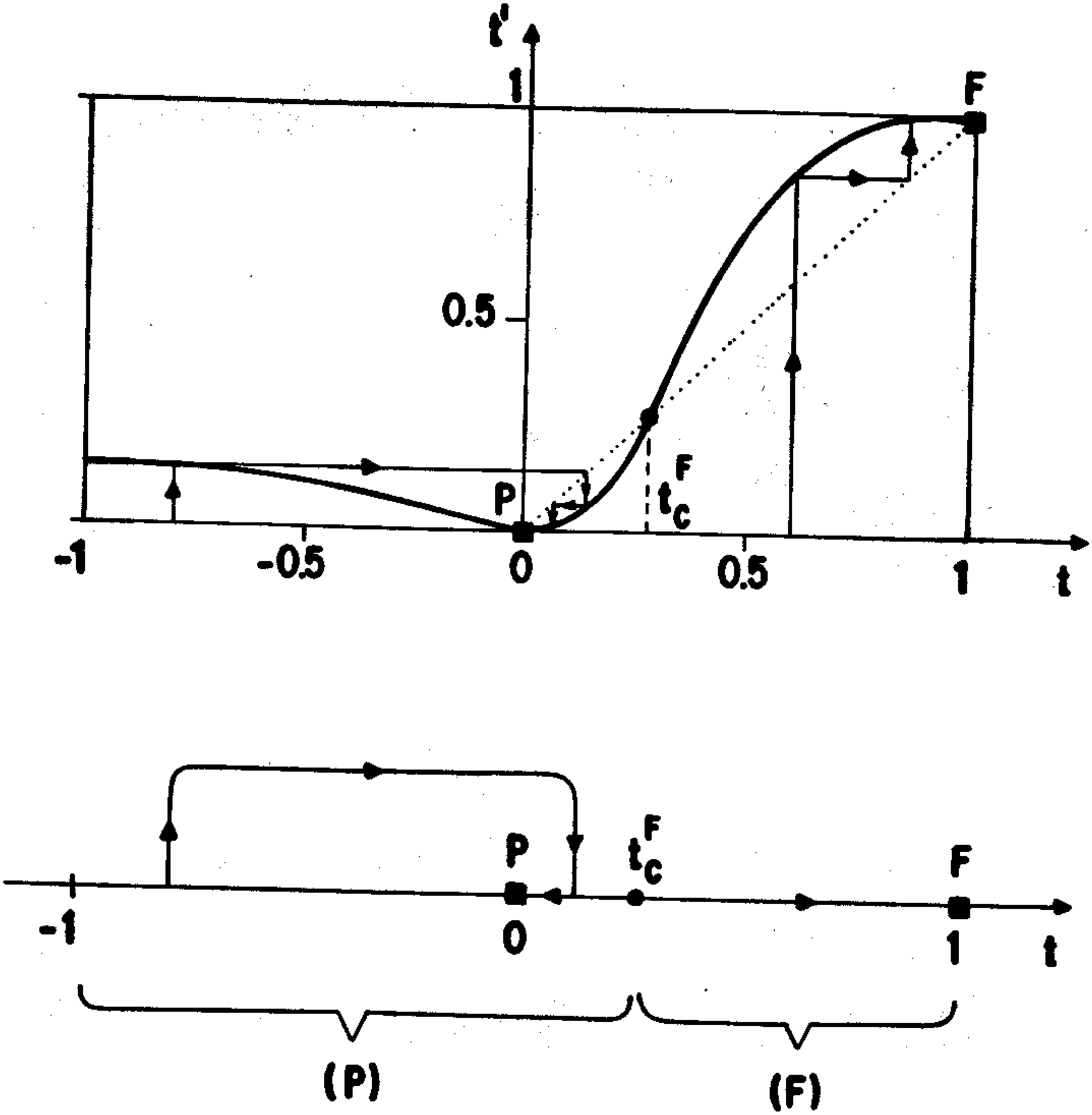


FIG.5a

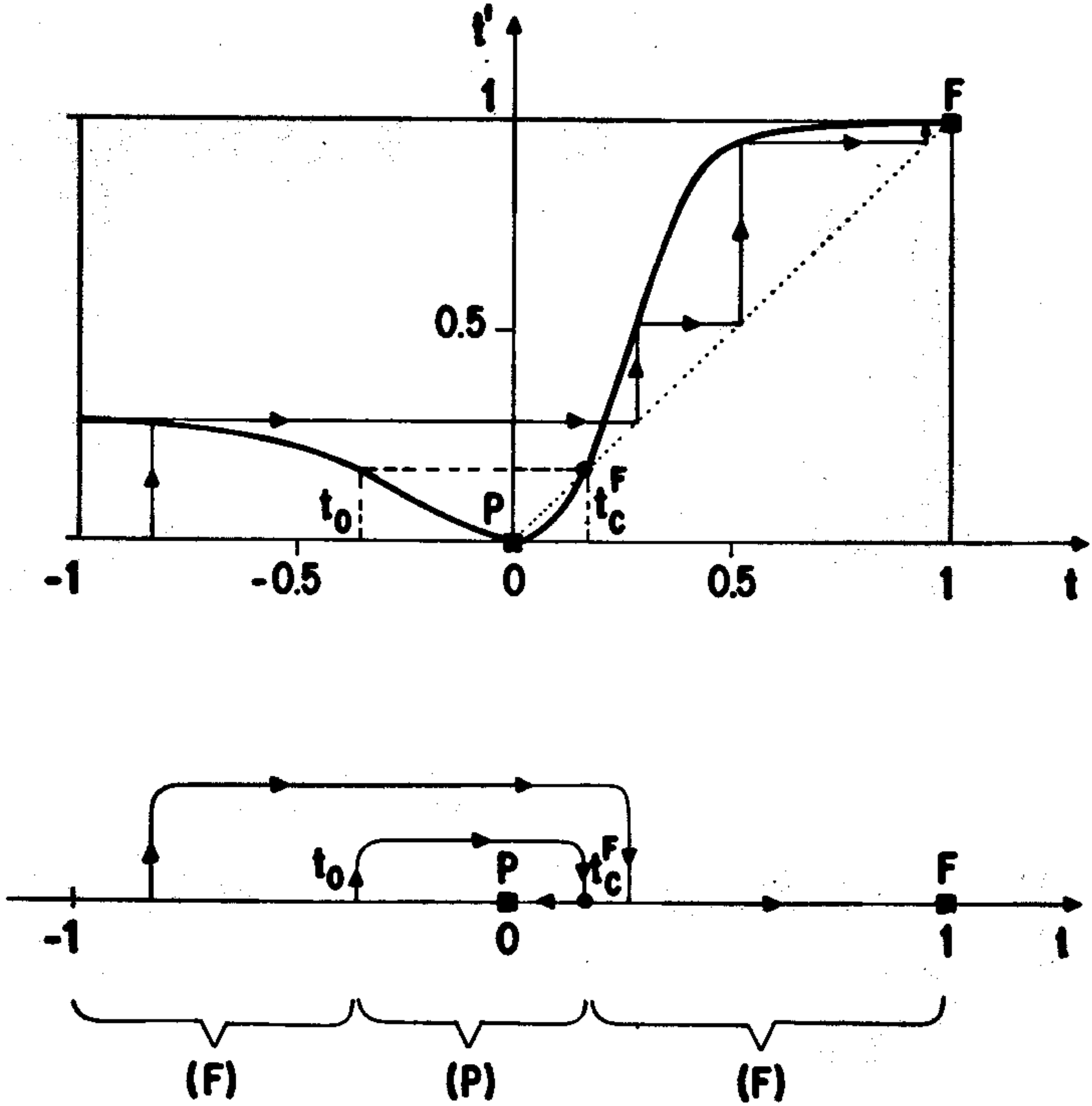


FIG.5b

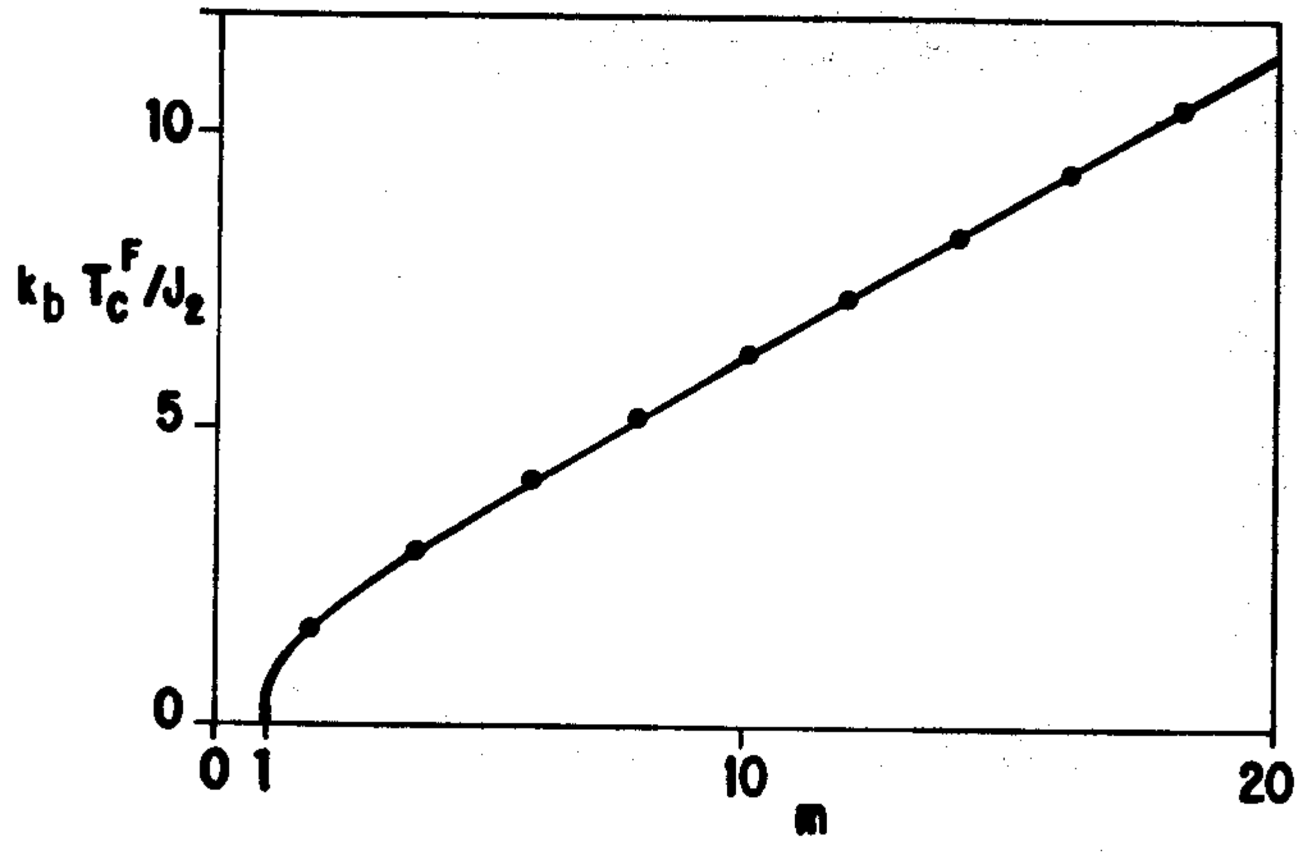


FIG.6

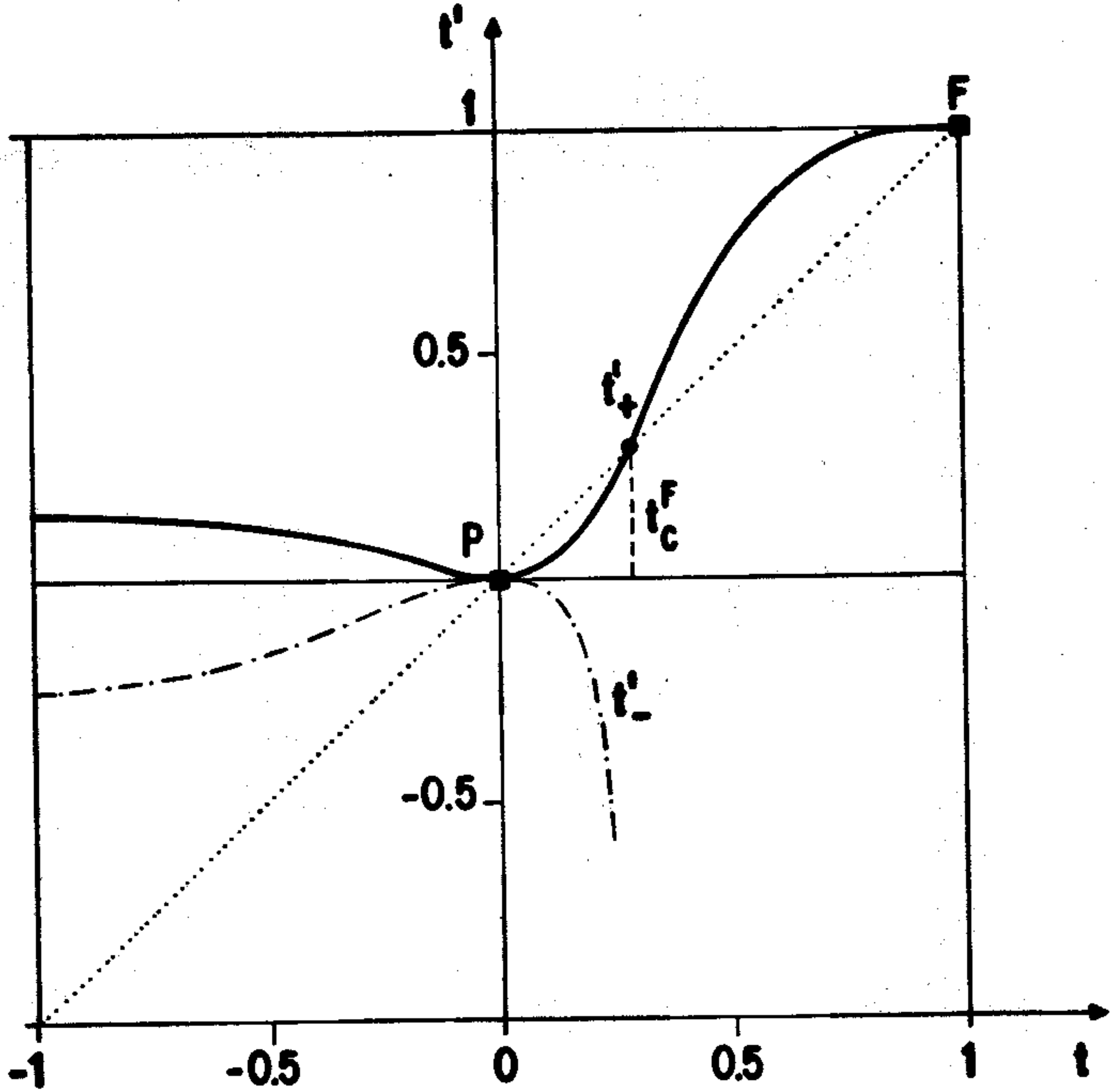


FIG. 7

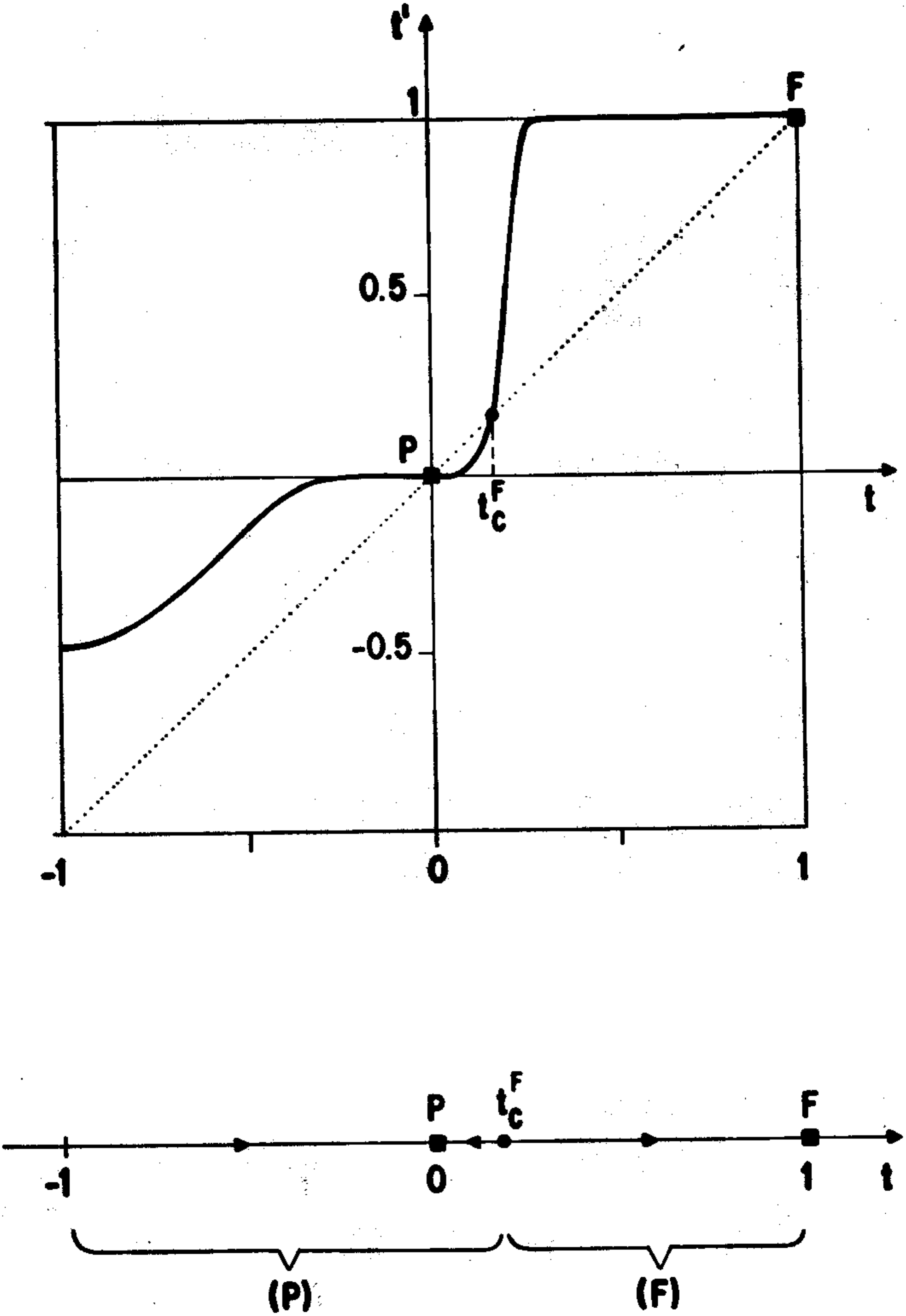


FIG.8a

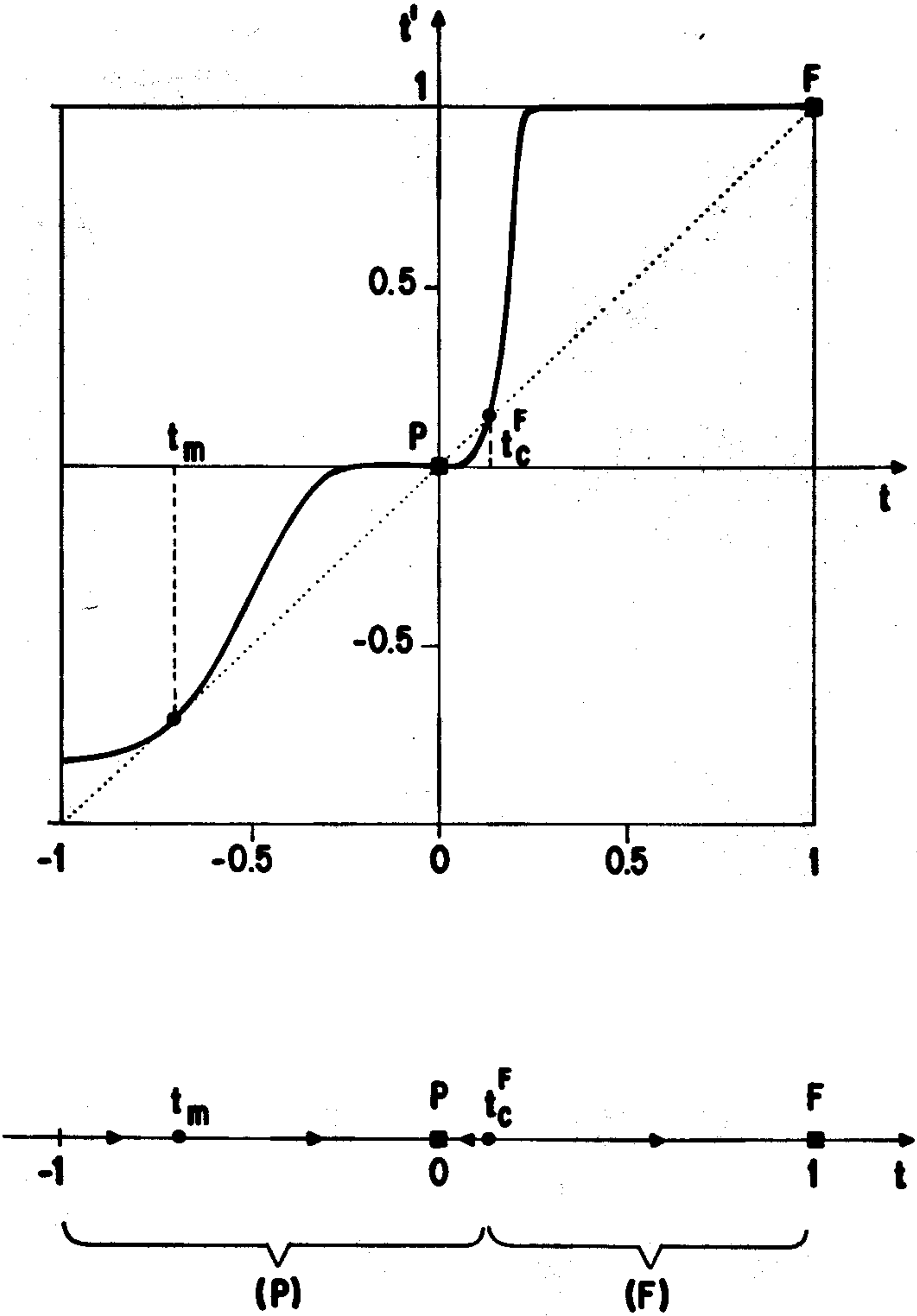


FIG.8b

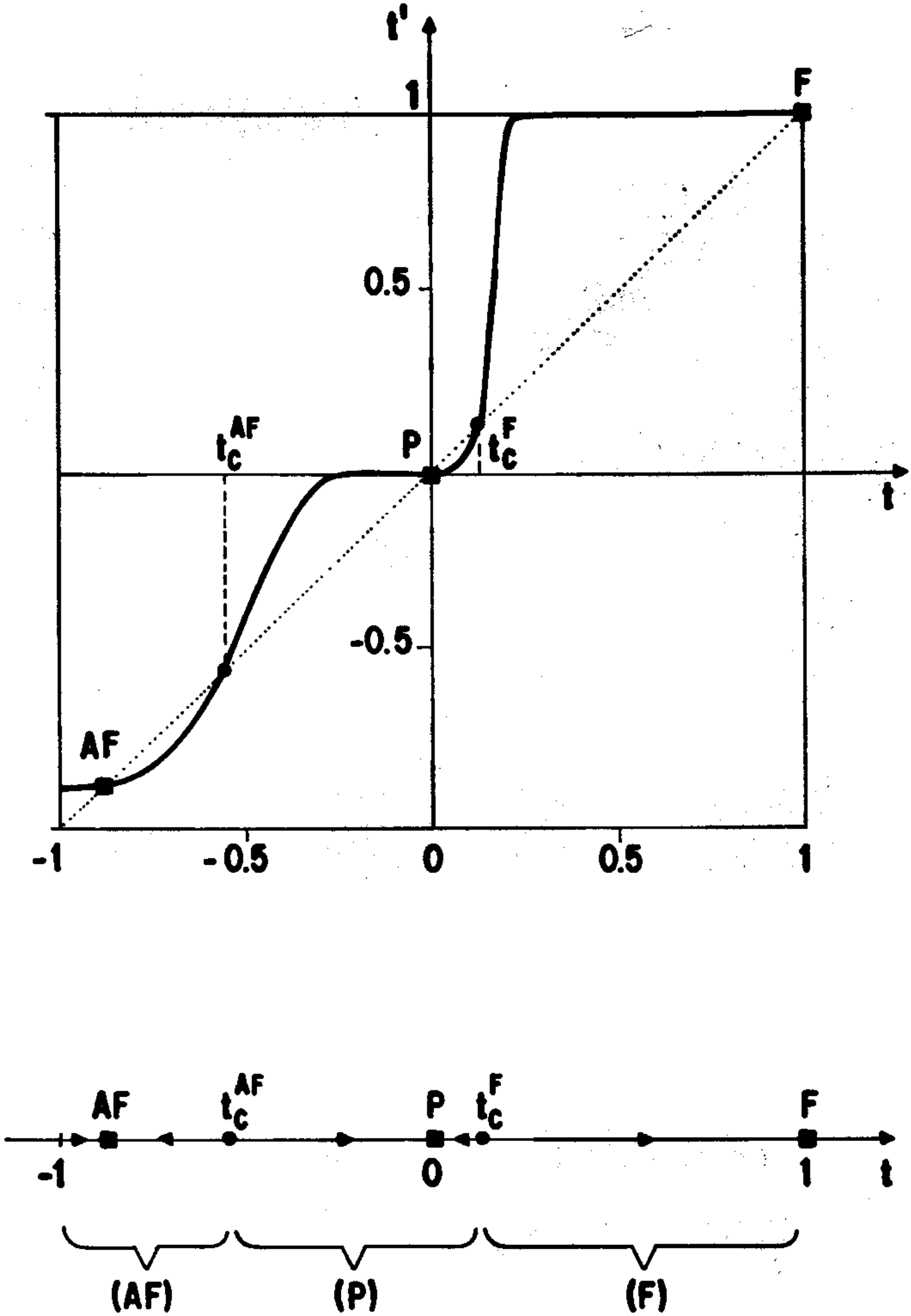


FIG.8c

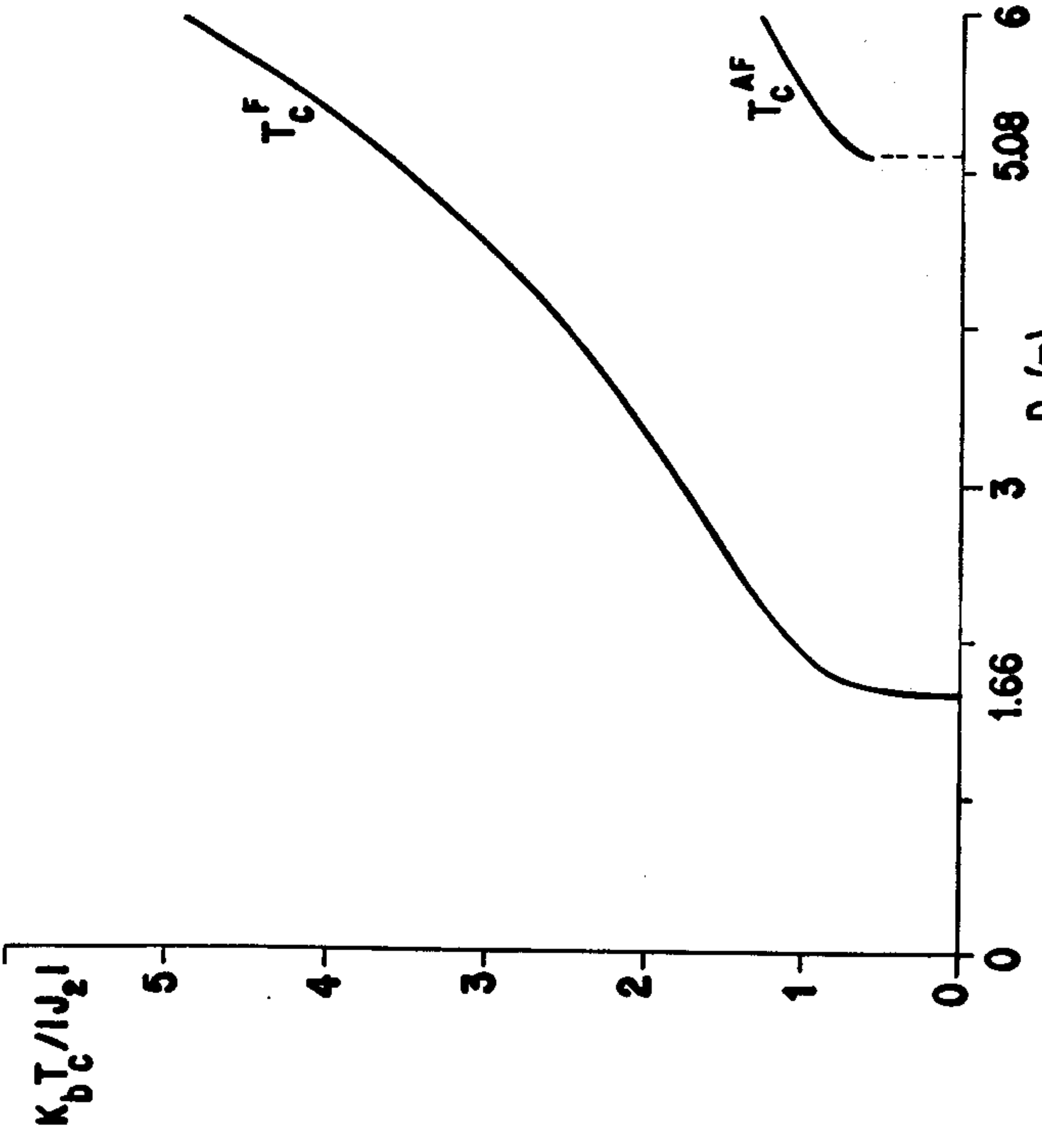


FIG.9

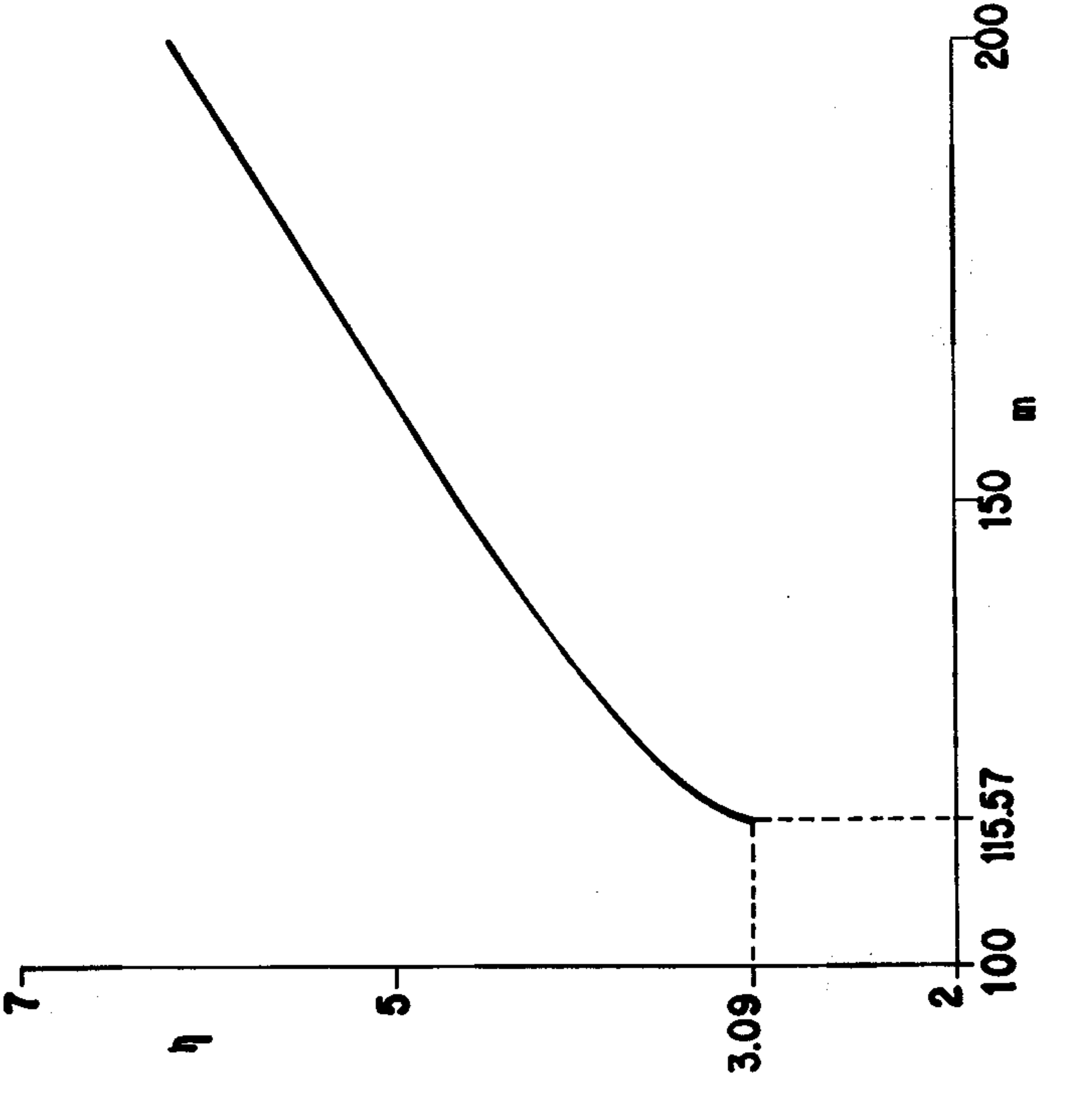


FIG.10

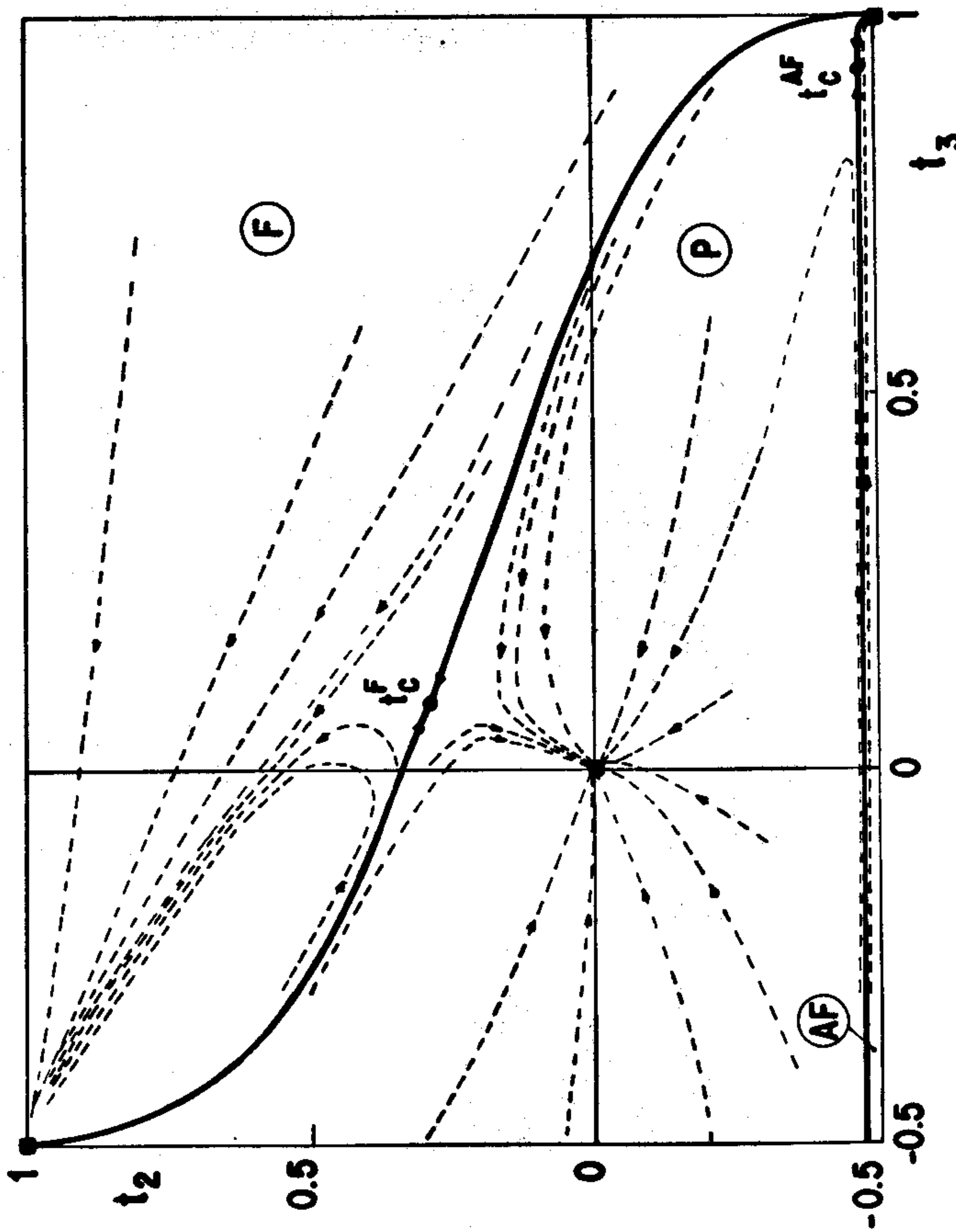


FIG.11a

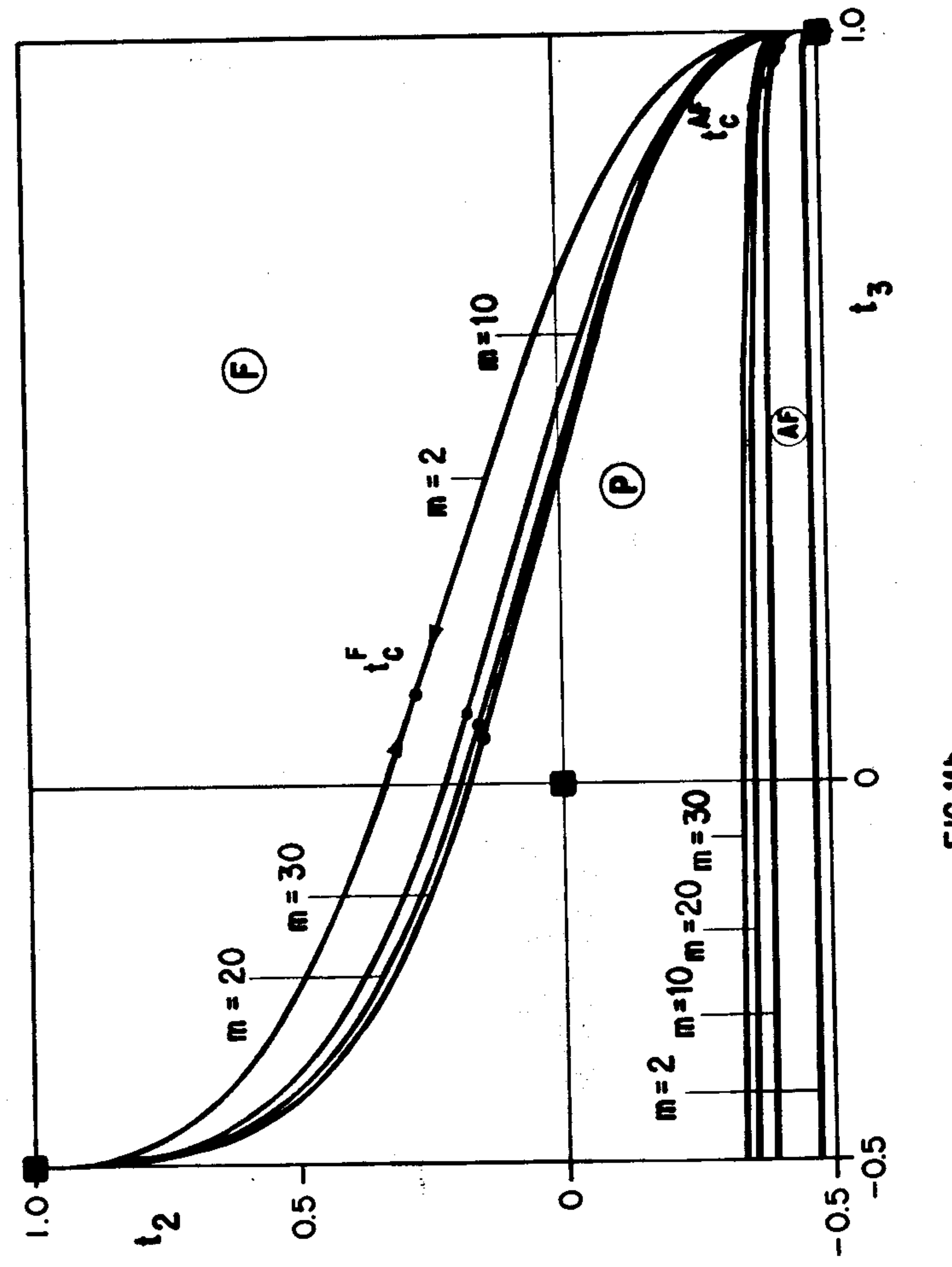


FIG.11b

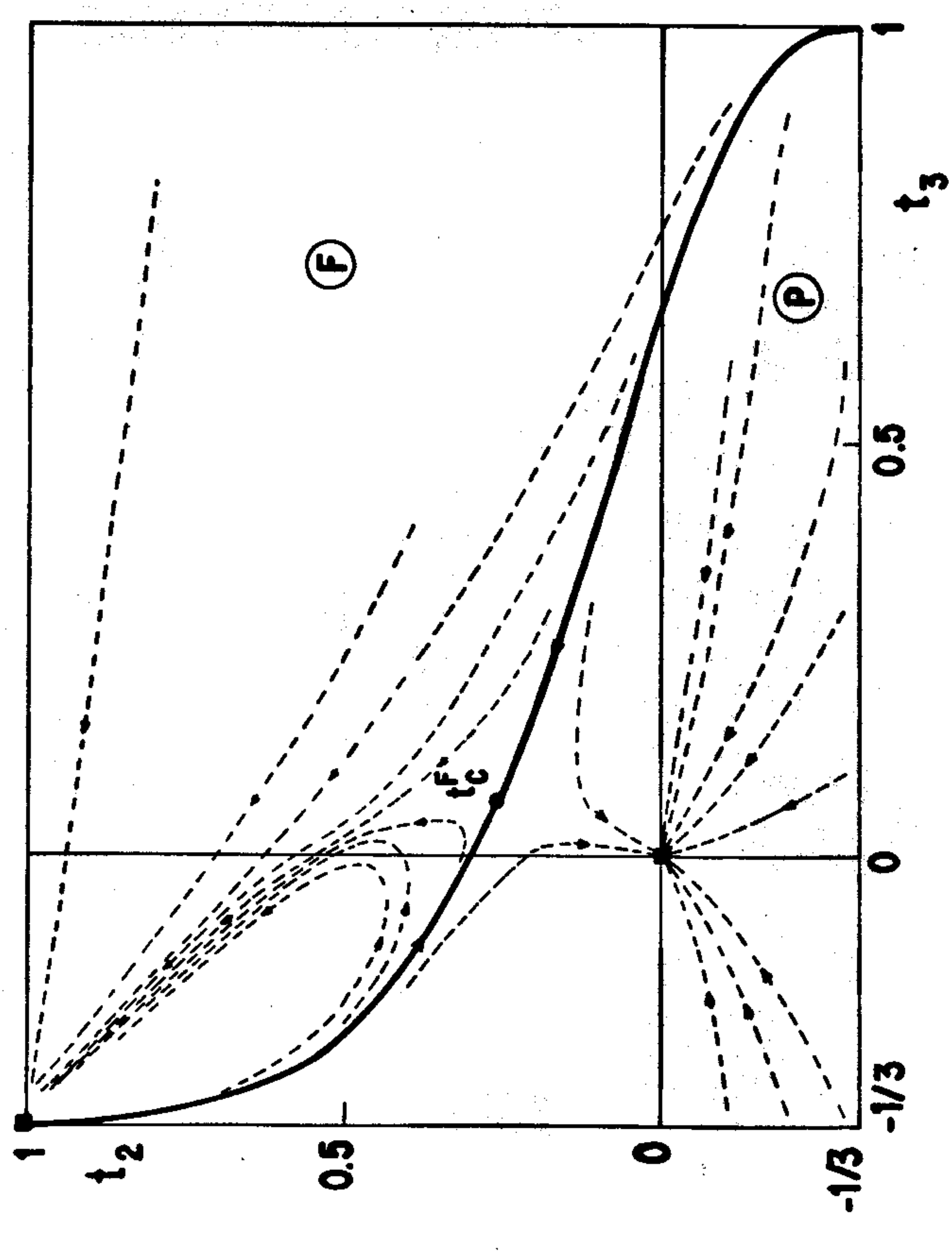


FIG.12a

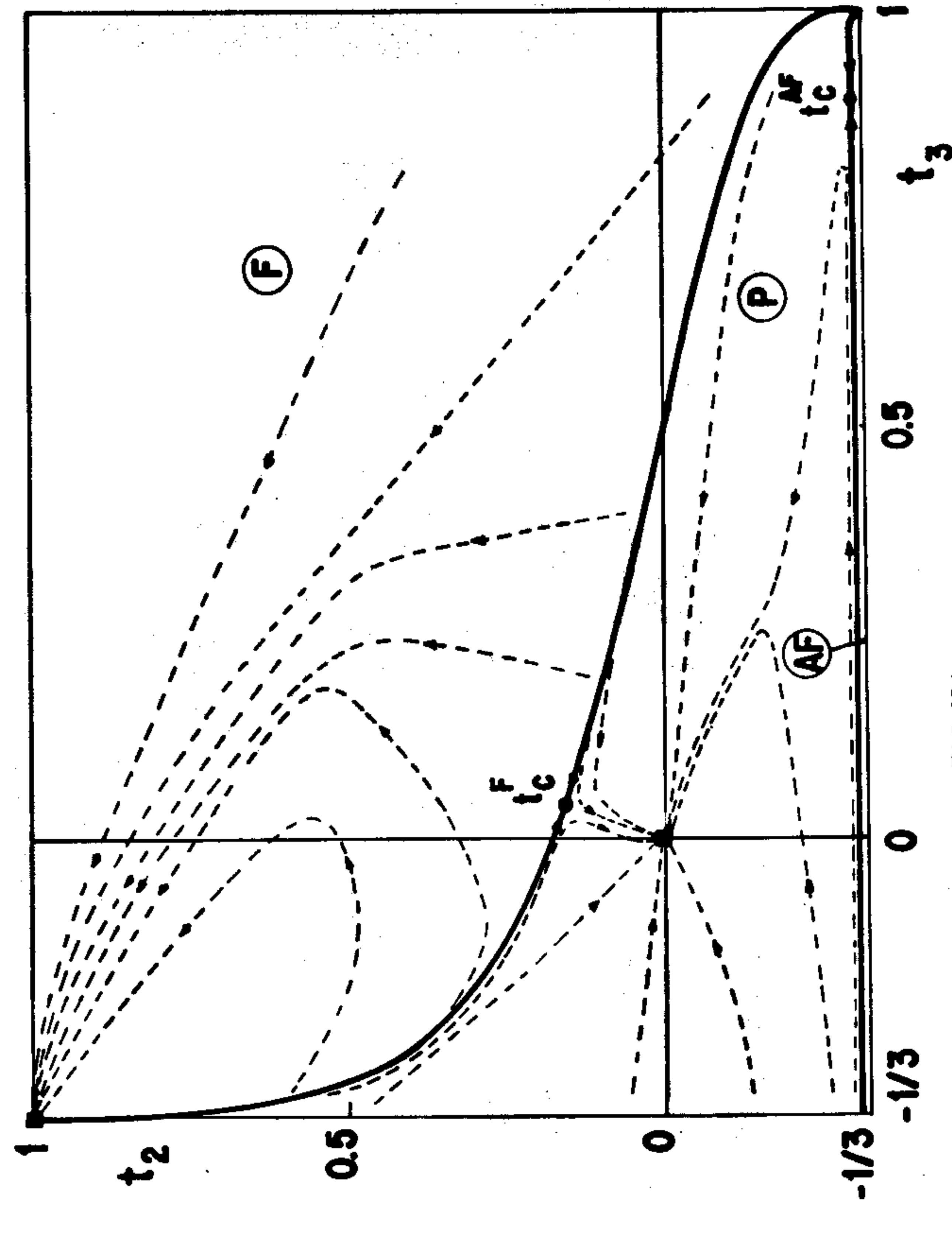


FIG.12b

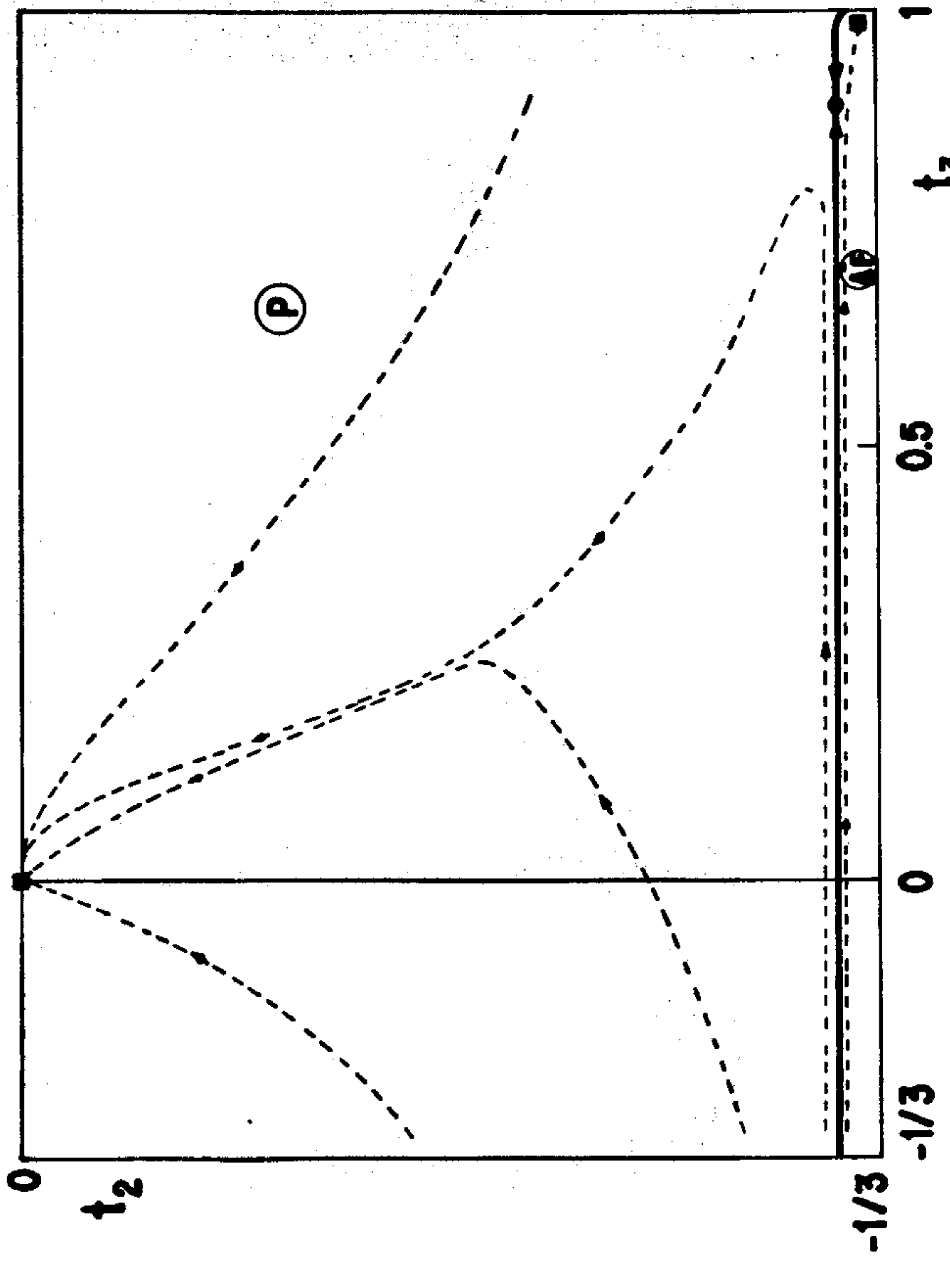
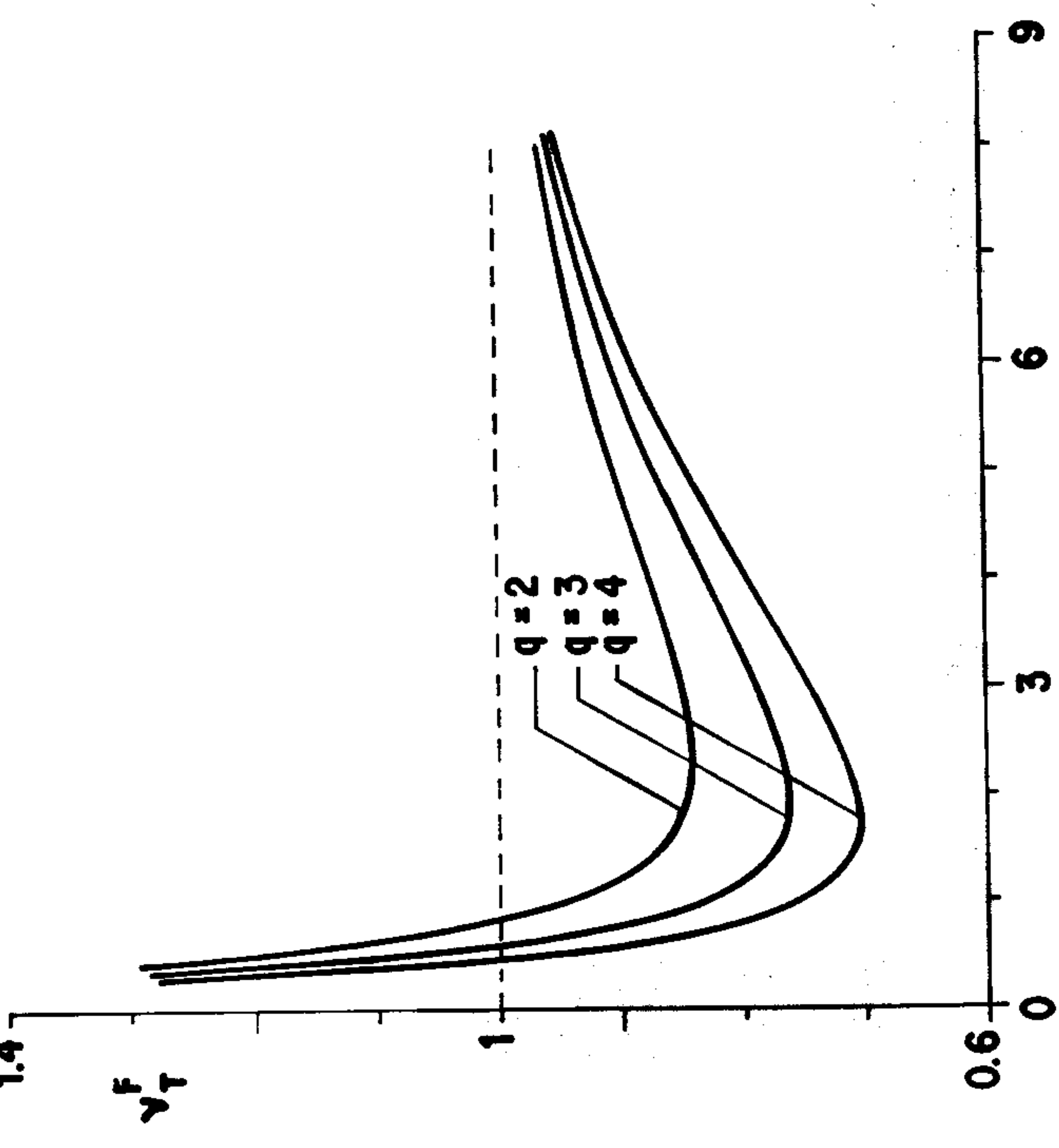


FIG.12c



$D_f(m) - D_f(1)$

FIG.13a

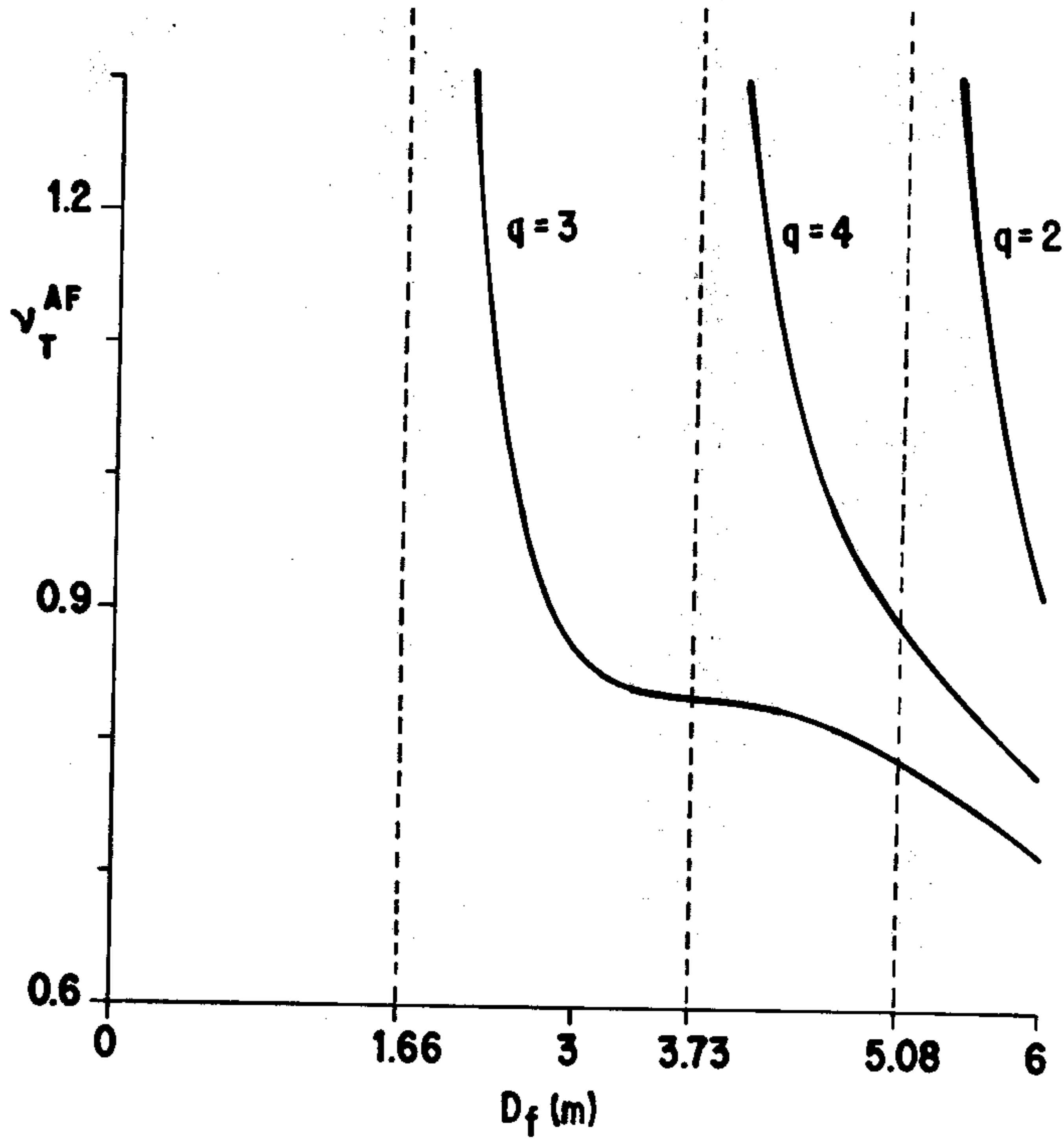


FIG.13b

Table I -

POSSIBLE CASES	SOLUTIONS OF EQ. (17) IN THE $X = 0$ LIMIT $\varepsilon'_F > \varepsilon'_I$	$\varepsilon'_F < \varepsilon'_I$	$\varepsilon'_F = \varepsilon'_I$
CASE (A) $\varepsilon_F > \varepsilon_I$	CASE $A1)$	CASE $A2)$	CASE $A3)$
$A(X') = 0$	$X' \to 0$	no real solution	no real solution
CASE (B) $\varepsilon_F < \varepsilon_I$	CASE $B1)$	CASE $B2)$	CASE $B3)$
$A(X') \to \infty$	$X' \to \infty$	$X'_1 \to 0$ and $X'_2 \to \infty$	$X' \to \infty$
CASE (C) $\varepsilon_F = \varepsilon_I$	CASE $C1)$	CASE $C2)$	CASE $C3)$
$A(X') = \frac{g_F}{g_I}$	$X' \neq 0$ and finite	$X' \neq 0$ and finite	$X'_1 \neq 0$ and finite $X'_2 \to 0$ if $\frac{g_F}{g_I} = \frac{g'_F}{g'_I}$

Table II -

m	$t_c^F \equiv (t_3, t_2)$	$t_c^{AF} \equiv (t_3, t_2)$
1	$(-1/2,\ 1)$	$(1,\ -1/2)$
2	(0.07007, 0.29470)	(0.93461, −0.47241)
3	(0.09403, 0.24317)	(0.86671, −0.45039)
4	(0.09306, 0.22328)	(0.81011, −0.43684)
5	(0.08780, 0.21234)	(0.76041, −0.42683)
10	(0.06297, 0.18952)	(0.58791, −0.39595)
30	(0.02823, 0.16199)	(0.37136, −0.34369)
50	(0.01794, 0.14894)	(0.31109, −0.31909)
70	(0.01301, 0.14016)	(0.28110, −0.30343)
100	(0.00909, 0.13081)	(0.25566, −0.28759)

Table III -

m	$t_c^F \equiv (t_3, t_2)$
1	(−1/3, 1)
2	(0.07306, 0.25262)
3	(0.09957, 0.20771)
4	(0.10037, 0.19144)
5	(0.09598, 0.18303)
10	(0.07143, 0.16725)
12	(0.06429, 0.16420)
14	(0.05841, 0.16172)
16	(0.05349, 0.15958)
18	(0.04932, 0.15769)
20	(0.04574, 0.15597)

m	$t_c^{AF} \equiv (t_3, t_2)$	$t^*_{AF} \equiv (t_3, t_2)$
$m_c = 17.63436$	(0.97021, −0.32647)	(0.97021, −0.32647)
18	(0.94670, −0.32270)	(0.98501, −0.32927)
20	(0.89129, −0.31522)	(0.99590, −0.33186)
25	(0.79587, −0.30394)	(0.99952, −0.33307)
30	(0.72491, −0.29580)	(0.99992, −0.33327)
35	(0.66927, −0.28925)	(0.99998, −0.33332)
40	(0.62444, −0.28374)	(0.99999, −0.33333)
45	(0.58755, −0.27898)	(0.99999, −0.33333)

References

[1] Y. Gefen, B.B. Mandelbrot and A. Aharony – Phys. Rev. Lett. 45 (1980) 855.

[2] Y. Gefen, A. Aharony and B.B. Mandelbrot – J. Phys. A16 (1983) 1267.

[3] Y. Gefen, A. Aharony, Y. Shapir and B.B. Mandelbrot – J. Phys. A17 (1984) 435.

[4] Y. Gefen, A. Aharony and B.B. Mandelbrot – J. Phys. A17 (1984) 1277.

[5] R. Riera and C.M. Chaves – Z. Phys. B. – Condensed Matter 62 (1986) 387.

[6] R. Riera – J. Phys. A19 (1986) 3395.

[7] W. Zidan, G. Changde and A. Holz – Phys. Rev. A34 (1986) 1531.

[8] U.M.S. Costa, I. Roditi and E.M.F. Curado – J. Phys. A20 (1987) 6001.

[9] M.P. Grillon and F.G. Brady Moreira – Phys. Lett. A142 (1989) 22.

[10] T. Abe – J. Phys. Soc. Japan 58 (1989) 1962.

[11] R.B. Stinchcombe – Phys. Rev. B41 (1990) 2510.

[12] R.B. Stinchcombe – Physica D38 (1989) 345.

[13] R.B. Griffiths and M. Kaufman – Phys. Rev. B26 (1982) 5022.

[14] M. Kaufman and R.B. Griffiths – Phys. Rev. B24 (1981) 496; Phys. Rev. B28 (1983) 3864.

[15] J.R. Melrose – J. Phys. A16 (1983) 1041; J. Phys. A16 (1983) 3077.

[16] Bambi Hu – Phys. Rev. Lett. 55 (1985) 2316.

[17] E.P.da Silva and C. Tsallis – Physica A167 (1990) 347.

[18] F.Y. Wu – Rev. Mod. Phys. 54 (1982) 235.

[19] C. Itzykson and J. M. Luck, in **Progress in Physics Vol. 11 – Critical Phenomena** (1983 Brasov School Conference), *edited by A. Jaffe, G. Parisi and D. Ruelle*, 1985 Birkhauser Boston, Inc. pag. 45.

[20] Y. Qin and Z.R Yang – Phys. Rev. B43 (1991) 8576.

[21] G.S. Grest – J. Phys. A14 (1981) L217.

[22] F.Y. Wu – J. Stat. Phys. 23 (1980) 773.

[23] G.H. Wannier – Phys. Rev. 79 (1950) 357.

[24] A.N. Berker and L.P. Kadanoff – J. Phys. A13 (1980) L259.

[25] J.R. Banavar, G.S. Grest and D. Jasnow – Phys. Rev. Lett. 45 (1980) 1424.

[26] G.S. Grest and J.R. Banavar – Phys. Rev. Lett. 46 (1981) 1458.

[27] W. Kinzel, W. Selke and F.Y. Wu – J. Phys. A14 (1981) L399.

[28] M.P. Ninghtingale and M. Shick – J. Phys. A15 (1982) L39.

[29] J.S Wang, R.H. Swendsen and R. Kotecky – Phys. Rev. Lett. 63 (1989) 109.

[30] R. Hilfer and A. Blumen – J. Phys. A17 (1984) 1537.

[31] Z. Borjan, S. Elezovic, M. Knezevic and S. Milosevic – J. Phys. A20 (1987) L715.

[32] P. Alstron, D. Stassinopoulos and H.E. Stanley, Physica A153 (1988) 20.

[33] S. Elezoviv, M. Knezevic and S. Milosevic – J. Phys. A20 (1987) 1215.

[34] T. Stovic, B. Stovic, S. Milosevic and H.E. Stanley – Phys. Rev. A37 (1988) 1747.

[35] B.B. Mandelbrot, "The Fractal Geometry of Nature", New York: W.H. Freeman, 1983.

[36] A. Bakchich, A. Benyoussef and N. Boccara – J. Phys. C3 (1991) 1727.

[37] C. Tsallis and S.V.F. Levy – Phys. Rev. Lett. 47 (1981) 950.

[38] P.M.C. de Oliveira – Le J. Physique 47 (1986) 1107.

[39] M.J. Stephen - Phys. Lett. A56 (1976) 149.

[40] L.R. da Silva and C. Tsallis – J. Phys. A20 (1987) 6013.

[41] K.G. Wilson and M.E. Fisher – Phys. Rev. Lett. 28 (1972) 24.