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Extensive Versus Nonextensive Physics

by

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Abstract

Within the framework of a recently generalized Statistical Mechanics and Thermodynamics, we establish the Fluctuation-dissipation theorem for magnetic systems, as well as an integral transformation (Hilhorst formula) which enables a quick derivation of the specific heat and the equation of states for the d-dimensional classical ideal gas. We finally discuss possible applications of this formalism to Condensed Matter Physics, Astrophysics and other areas.

Key-words: Generalized Statistical Mechanics; Nonextensive Thermodynamics; Magnetic fluctuations; Ideal gas.

I Introduction

During recent years, an interesting tendency is developing in Physics, namely towards *nonextensivity* (or *nonadditivity*, as sometimes referred to). These generalizations contain traditional extensive Physics as particular cases. They present a somehow holistic (context-dependent) nature, and mainly follow along two streams: Generalized Statistical Mechanics and Thermodynamics [1–19], and Quantum Groups [20–33]. Furthermore, a possible connection between those two areas has been very recently proposed [34]. Let us briefly review them.

Generalized statistical mechanics can be obtained through the following generalization of the entropy [1]:

$$S_q = k\frac{1-\sum_s p_s^q}{q-1} \quad (q \in \Re) \tag{1}$$

where p_s is the probability associated with the microscopic state s of the system, and k a conventionally chosen positive constant. The $q \to 1$ limit of S_q yields the well known Shannon expression $-k_B \sum_s p_s \ln p_s$ (where we have used $p_s^{q-1} \sim 1 + (q-1)\ln p_s$). S_q satisfies a great variety of properties and yields a great variety of results. Let us mention:

(i) $S_q \geq 0$, $\forall q$, $\forall \{p_s\}$;

(ii) S_q attains its extremum (maximum for $q > 0$ and minimum for $q < 0$) for equiprobability (i.e., $p_s = p_{s'}, \forall(s, s')$);

(iii) S_q is expansible for $q > 0$, i.e.,

$$S_q(p_1, p_2, \cdots, p_W) = S_q(p_1, p_2, \cdots, p_W, 0) \quad \forall \{p_s\} \tag{2}$$

(iv) S_q is concave (convex) for all $\{p_s\}$ if $q > 0$ ($q < 0$), a fact which guarantees thermodynamic stability for the system;

(v) H-theorem: under quite general conditions [5–7] dS_q/dt is nonnegative, vanishes and is nonpositive for $q > 0$, $q = 0$ and $q < 0$ respectively (t being the time);

(vi) If Σ and Σ' are two *independent* systems (i.e., $\hat{\rho}_{\Sigma\cup\Sigma'} = \hat{\rho}_{\Sigma}\hat{\rho}_{\Sigma'}$, where $\hat{\rho}$ denotes the density operator, whose eigenvalues are the $\{p_s\}$), S_q is pseudo-additive, i.e.,

$$(S_q^{\Sigma\cup\Sigma'}/k) = (S_q^{\Sigma}/k) + (S_q^{\Sigma'}/k) + (1-q)(S_q^{\Sigma}/k)(S_q^{\Sigma'}/k) \qquad (3)$$

Consequently, entropy is generically extensive for and only for $q = 1$; $S_q^{\Sigma\cup\Sigma'}$ is smaller (larger) than $S_q^{\Sigma} + S_q^{\Sigma'}$ if $q > 1 (q < 1)$;

If we define the *entropy operator* $\hat{S}_q \equiv k(\hat{1} - \hat{\rho}^{1-q})/(1-q)$ (so referred to because it satisfies $< \hat{S}_q >_q \equiv Tr\hat{\rho}^q \hat{S}_q = S_q$), Eq. (3) (which holds for arbitrary $(\hat{\rho}_{\Sigma}, \hat{\rho}_{\Sigma'})$) can be rewritten as follows:

$$(\hat{S}_q^{\Sigma\cup\Sigma'}/k) = (\hat{S}_q^{\Sigma}/k) + (\hat{S}_q^{\Sigma'}/k) + (q-1)(\hat{S}_q^{\Sigma}/k)(\hat{S}_q^{\Sigma'}/k). \qquad (4)$$

Notice the $(1-q) \to (q-1)$ changement from Eq. (3) to Eq. (4).

(vii) For generic and fixed $\{p_s\}$, S_q monotonically decreases for q increasing from $(-\infty)$ to $(+\infty)$; $\lim_{q\to-\infty} S_q(\{p_s\}) = \infty$ and $\lim_{q\to\infty} S_q(\{p_s\}) = 0$;

(viii) Canonical ensemble: The optimization of S_q under the constraints $Tr\hat{\rho} = 1$ and $Tr\hat{\rho}^q\hat{\mathcal{H}} \equiv < \hat{\mathcal{H}} >_q = U_q$ [3] (where $\hat{\mathcal{H}}$ is the Hamiltonian) yields the generalized equilibrium distribution [1, 3]

$$\hat{\rho} = \begin{cases} \frac{[1-\beta(1-q)\hat{\mathcal{H}}]^{\frac{1}{1-q}}}{Z_q} & \text{if } \hat{1} - \beta(1-q)\hat{\mathcal{H}} > 0 \\ 0 \text{ , otherwise} \end{cases} \qquad (5)$$

with

$$Z_q \equiv Tr[1 - \beta(1-q)\hat{\mathcal{H}}]^{\frac{1}{1-q}} \qquad (6)$$

where $\beta \equiv 1/kT$ is a Lagrange parameter. It can be shown [3] that $1/T = \partial S_q/\partial U_q, F_q \equiv U_q - TS_q = -\frac{1}{\beta}\frac{Z_q^{1-q}-1}{1-q}$ and $U_q = -\frac{\partial}{\partial\beta}\frac{Z_q^{1-q}-1}{1-q}$.

The diagonal form of (5) is given by

$$p_s = \begin{cases} \frac{[1-\beta(1-q)\varepsilon_s]^{\frac{1}{1-q}}}{Z_q} & \text{if } 1 - \beta(1-q)\varepsilon_s > 0 \\ 0 \text{ , otherwise} \end{cases} \qquad (7)$$

where $\{\varepsilon_s\}$ are the eigenvalues of $\hat{\mathcal{H}}$. In the $q \to 1$ limit, Eqs. (5) and (6) recover the well known Boltzmann-Gibbs distribution

$$\hat{\rho} = e^{-\beta\hat{\mathcal{H}}}/Z_1 \tag{8}$$

with

$$Z_1 = Tr\ e^{-\beta\hat{\mathcal{H}}} \tag{9}$$

Besides the above properties, the present generalized statistics has been shown to satisfy appropriate extensions of the Ehrenfest theorem [9], von Neumann equation [13], Jaynes Information Theory duality relations [9], fluctuation-dissipation theorem [16], Bogolyubov inequality [11], Langevin and Fokker-Planck equations [14], Callen's identity [15], among others. Last but not least, Plastino and Plastino [8] have pointed that $q \neq 1$ Thermodynamics overcomes the Boltzmann-Gibbs inability to provide *finite* mass for astrophysical systems within Chandrasekhar's polytropic model. We come back onto this point later on.

Let us now focus Quantum Groups (q_G-deformations, q_G-oscillators, q_G-calculus, where we use q_G, instead of the traditional notation q, in order to distinguish it from the present entropy parameter q). Quantum groups generalize standard Lie groups and algebras, which are recovered in the $q_G \to 1$ limit. The generalization occurs by appropriately modifying ("deforming") the commutator relations which determine the Lie groups and algebras (e.g., creation and annihilation bosonic operators might satisfy $\hat{A}\hat{A}^{+} - q_G^2\hat{A}^{+}\hat{A} = 1$ with $q_G \neq 1$). Nonextensivity appears because certain basic eigenvalues associated with $\Sigma \cup \Sigma'$ (Σ and Σ' being independent systems) differ from the sum of those associated with Σ and Σ' respectively. For example, the eigenvalues associated with the q_G-deformed bosonic number operator are given by ([33] and references therein)

$$[n] = \frac{q_G^{2n} - 1}{q_G^2 - 1} \quad (n = 0, 1, 2, ...) \tag{10}$$

If Σ and Σ' are independent bosonic systems (respectively characterized by $n_\Sigma = 0, 1, 2, \cdots$, and $n_{\Sigma'} = 0, 1, 2, \cdots$), we immediately verify

$$[n]_{\Sigma\cup\Sigma'} = [n]_\Sigma + [n]_{\Sigma'} + (q_G^2 - 1)[n]_\Sigma[n]_{\Sigma'} \tag{11}$$

Consequently, $[n]$ is generically extensive if and only if $q_G^2 = 1$. The analogy with Eq. (4) is, obviously, striking. Quantum groups have found applications in the inverse scattering method, vertex models, anisotropic spin chain

Hamiltonians, knot theory, conformed field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyons, the discussion of the existence of dark matter.

In a very recent work [34], we proposed (for the case where the q-source of nonextensitivity cancels the q_G-source of nonextensivity) a possible connection between Generalized Statistical Mechanics and Quantum Groups, q being a nonuniversal function of q_G, depending not only upon the system but also on its state. For example, in the $(q, q_G) \to (1,1)$ limit one expects, for a system in thermodynamic equilibrium, $q-1 \propto q_G - 1$ where the prefactor depends on the temperature.

In Section II we discuss fluctuations in a magnetic system for arbitrary q; in Section III we discuss an interesting integral transformation (Hilhorst formula), which we apply to the classical ideal gas in Section IV; in Section V we discuss the possible regions of physical relevance of the present nonextensive Thermodynamics; we finally conclude in Section VI.

II Fluctuations in a Magnetic System

We discuss, in the present section, energy fluctuations (specific heat) and magnetic dipolar moment fluctuations (isothermal susceptibility). We first review the fluctuation-dissipation form for the specific heat (established in [10]) and then establish the corresponding one for the susceptibility.

By using $U_q = Tr\hat{\rho}^q\hat{\mathcal{H}}$ with $\hat{\rho}$ given by (5) and (6) we straightforwardly obtain the specific heat $C_q \equiv T\partial S_q/\partial T = \partial U_q/\partial T$ [10]

$$\begin{aligned}\frac{C_q}{k} &= \frac{q}{(kT)^2}\left\{Tr\left[\hat{\rho}^q\frac{\hat{\mathcal{H}}^2}{1-\beta(1-q)\hat{\mathcal{H}}}\right]\right. \quad (12)\\ &- \left.[Tr\hat{\rho}^q\hat{\mathcal{H}}]\left[Tr\hat{\rho}\frac{\hat{\mathcal{H}}}{1-\beta(1-q)\hat{\mathcal{H}}}\right]\right\}\\ &\equiv \frac{q}{(kT)^2}\left\{\left\langle\frac{\hat{\mathcal{H}}^2}{1-\beta(1-q)\hat{\mathcal{H}}}\right\rangle_q - \left\langle\hat{\mathcal{H}}\right\rangle_q^2/Z_q^{1-q}\right\} \quad (12')\end{aligned}$$

This expression can be rewritten as follows:

$$\frac{C_q}{k} = \frac{qZ_q^{1-q}}{(kT)^2}\left\langle\left(\frac{\hat{\mathcal{H}}}{1-\beta(1-q)\hat{\mathcal{H}}} - \left\langle\frac{\hat{\mathcal{H}}}{1-\beta(1-q)\hat{\mathcal{H}}}\right\rangle_1\right)^2\right\rangle_1 \tag{13}$$

hence C_q/kq is, in all cases, a non-negative real number (thermodynamic stability with respect to energy fluctuations).

Let us focus now the magnetic susceptibility. We consider the Hamiltonian $\hat{\mathcal{H}}(\{\hat{S}_i\}) - \mu H\sum_i \hat{S}_i^z$, where μ is the elementary magneton, H is an uniform external magnetic field applied, along the z-axis, on the system of N interacting spins $\{\hat{S}_i\}$, and $\hat{\mathcal{H}}(\{\hat{S}_i\})$ is an arbitrary (H-independent) Hamiltonian. The total spin along z is given by

$$\hat{S}_z \equiv \sum_{j=1}^{N} \hat{S}_j^z \tag{14}$$

and the average total magnetic dipolar moment by

$$\begin{aligned} M_q &\equiv \mu < \hat{S}_z >_q \equiv \mu Tr\hat{\rho}^q\hat{S}_z \\ &= \mu\frac{Tr\left\{\left[1-\beta(1-q)(\hat{\mathcal{H}}(\{\hat{S}_i\}) - \mu H\sum_i \hat{S}_i^z)\right]^{\frac{q}{1-q}}(\sum_j \hat{S}_j^z)\right\}}{\left\{Tr\left[1-\beta(1-q)(\hat{\mathcal{H}}(\{\hat{S}_i\}) - \mu H\sum_i \hat{S}_i^z)\right]^{\frac{1}{1-q}}\right\}^q} \end{aligned} \tag{15}$$

It follows straightforwardly that the vanishing field isothermal magnetic susceptibility $\chi_q \equiv \lim_{H\to 0}(\partial M_q/\partial H)$ is given by

$$\begin{aligned} \chi_q &= \frac{q\mu^2}{kT}\left\{Tr\left[\hat{\rho}^q\frac{\hat{S}_z^2}{1-\beta(1-q)\hat{\mathcal{H}}}\right]\right. \\ &- \left.\left[Tr(\hat{\rho}^q\hat{S}_z)\right]\left[Tr\hat{\rho}\frac{\hat{S}_z}{1-\beta(1-q)\hat{\mathcal{H}}}\right]\right\} \end{aligned} \tag{16}$$

$$\equiv \frac{q\mu^2}{kT}\left\{\left\langle\frac{\hat{S}_z^2}{1-\beta(1-q)\hat{\mathcal{H}}}\right\rangle_q - \left\langle\hat{S}_z\right\rangle_q^2/Z_q^{1-q}\right\} \tag{16'}$$

This expression can be rewritten as follows

$$\chi_q = \frac{q\mu^2 Z_q^{1-q}}{kT} \left\langle \left(\frac{\hat{S}_z}{1-\beta(1-q)\hat{\mathcal{H}}} - \left\langle \frac{\hat{S}_z}{1-\beta(1-q)\hat{\mathcal{H}}} \right\rangle_1 \right)^2 \right\rangle_1 \tag{17}$$

hence χ_q/q is, in all cases, a non-negative real quantity (thermodynamic stability with respect to magnetic dipolar moment fluctuations). In the limit $q \to 1$, we recover the well known expression $\chi_1 = (\mu^2/k_B T) < (\hat{S}_z - < \hat{S}_z >_1)^2 >_1$. The general form for the fluctuation-dissipation theorem has just now been established [16], and recovers Eqs. (13) and (17) as particular cases.

III Hilhorst Formula

This Section is dedicated to an interesting integral transformation recently established by Hilhorst [35]. From the definition of the gamma function we have that

$$\mu^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx\; x^{\nu-1} e^{-\mu x} \quad (\mu > 0, \nu > 0) \tag{18}$$

If we identify now $\nu \equiv 1/(q-1)$ (hence $q > 1$) and $\mu \equiv 1 - \beta(1-q)\varepsilon_s > 0$ ($\{\varepsilon_s\}$ being the eigenvalues of an arbitrary Hamiltonian $\hat{\mathcal{H}}$) and use Eq. (18), Eq. (6) can be rewritten as follows:

$$Z_q(\beta) = \frac{1}{\Gamma(\frac{1}{q-1})} \sum_s \int_0^\infty dx\; x^{\frac{2-q}{q-1}} e^{-[1-\beta(1-q)\varepsilon_s]x} \tag{19}$$

Whenever $\sum_s \int = \int \sum_s$, this equation becomes

$$Z_q(\beta) = \frac{1}{\Gamma(\frac{1}{q-1})} \int_0^\infty dx\; x^{\frac{2-q}{q-1}} e^{-x} \sum_s e^{-\beta(q-1)\varepsilon_s\, x} \tag{20}$$

Finally, with $\alpha \equiv \beta(q-1)x$ we obtain

$$Z_q(\beta) = \frac{1}{\Gamma(\frac{1}{q-1})\,[\beta(q-1)]^{\frac{1}{q-1}}} \int_0^\infty d\alpha\; \alpha^{\frac{2-q}{q-1}} e^{-\frac{\alpha}{\beta(q-1)}} Z_1(\alpha) \quad (q > 1) \tag{21}$$

which is Hilhorst's formula. If we recall now that the Mellin transform $M(y; f(x))$ of a function $f(x)$ is defined by [36]

$$M(y; f(x)) \equiv \int_0^\infty dx\ f(x) x^{y-1} \tag{22}$$

we see that Hilhorst formula can be rewritten as follows

$$Z_q(\beta) = M\left(\frac{1}{q-1}; \frac{e^{-x} Z_1(\beta(q-1)x)}{\Gamma(\frac{1}{q-1})}\right) \tag{23}$$

Let us now verify the $q \to 1$ limit. Using Stirling formula

$$\Gamma(\frac{1}{q-1}) \sim \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} e^{\frac{1}{q-1}} / \sqrt{2\pi(q-1)} \tag{24}$$

we obtain

$$Z_q(\beta) \sim \frac{1}{\sqrt{2\pi(q-1)}\beta^{\frac{1}{q-1}}} \int_0^\infty d\alpha e^{\frac{1}{q-1} r(\alpha)} Z_1(\alpha) \tag{25}$$

where

$$r(\alpha) \equiv (2-q)\ln\alpha - \frac{\alpha}{\beta} + 1 \tag{26}$$

The derivative $r'(\alpha)$ vanishes at $\alpha = (2-q)\beta$, in the neighborhood of which we have

$$r(\alpha) \sim (2-q)\ln[(2-q)\beta] + (q-1) - \frac{1}{2(2-q)\beta^2}[\alpha - (2-q)\beta]^2 \tag{27}$$

Replacing this into integral (25) we obtain

$$Z_q(\beta) \sim \frac{e(2-q)^{\frac{2-q}{q-1}}}{\sqrt{2\pi(q-1)}\beta} \int_0^\infty d\alpha\ exp\left\{-\frac{[\alpha-(2-q)\beta]^2}{2(q-1)(2-q)\beta^2}\right\} Z_1(\alpha) \tag{28}$$

Introducing $v \equiv [\alpha/(2-q)\beta] - 1$ we obtain

$$Z_q(\beta) \sim \frac{Z_1(\beta)}{\sqrt{2\pi(q-1)}} \int_{-1}^\infty dv\ e^{-\frac{2-q}{2(q-1)}v^2} \tag{29}$$

$$\sim \frac{Z_1(\beta)}{\sqrt{2\pi(q-1)}} \int_{-\infty}^\infty dv\ e^{-\frac{2-q}{2(q-1)}v^2} \tag{29'}$$

$$\sim \frac{Z_1(\beta)}{\sqrt{2\pi(q-1)}} \sqrt{\frac{2\pi(q-1)}{2-q}} \sim Z_1(\beta) \tag{29''}$$

as expected. In other words, though in a nontrivial manner, Hilhorst formula reproduces identity in the $q \to 1$ limit.

Classical Ideal Gas

We discuss here a system constituted by N non-interacting non-relativistic particles of mass m free to move in a d-dimensional hypercubic box of volume $V \equiv L^d$ (with periodic boundary conditions). The classical partition function is given by

$$Z_1(\beta) = \frac{V^N}{N! h^{dN}} \left(\frac{2\pi m}{\beta} \right)^{dN/2} \equiv D\beta^{-dN/2} \tag{30}$$

where h is Planck constant. Replacing this into Eq. (21) we straightforwardly obtain

$$Z_q(\beta) = \frac{D\Gamma\left(\frac{1}{q-1} - \frac{dN}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)} \beta^{-dN/2} \tag{31}$$

for $1 < q < 1 + \frac{2}{dN}$. Consequently,

$$\begin{aligned} F_q &= -kT \frac{Z_q^{1-q} - 1}{1-q} \\ &= -\frac{kT}{1-q} \left[\frac{D\Gamma\left(\frac{1}{q-1} - \frac{dN}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)} \right]^{1-q} (kT)^{(1-q)\frac{dN}{2}} + \frac{kT}{1-q} \end{aligned} \tag{32}$$

hence, the specific heat $C_q = -T\partial^2 F_q / \partial T^2$ is given by

$$C_q = k\frac{dN}{2} \left[1 - (q-1)\frac{dN}{2} \right] \left[\frac{D\Gamma\left(\frac{1}{q-1} - \frac{dN}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)} \right]^{1-q} (kT)^{(1-q)\frac{dN}{2}} \tag{33}$$

We remark that: (i) in the $q \to 1$ limit, we recover the well known universal (mass independent) result $C_1 = k_B dN/2$; (ii) for $d = N = 1$, we obtain $C_q \propto T^{\frac{1-q}{2}}$, thus reproducing the result obtained in Ref.[10] (the prefactor depends on nonuniversal quantities such as m); (iii) nonuniform convergence aspects emerge (e.g., $\lim_{h \to 0} \lim_{q \to 1} C_q = kdN/2$, whereas $\lim_{q \to 1} \lim_{h \to 0} C_q = 0$), similar to those which occur in *quantum* Boltzmann-Gibbs statistics (e.g., $\lim_{T\to 0}\lim_{\hbar\to 0} C_1 = k_B dN/2$ whereas $\lim_{\hbar\to 0}\lim_{T\to 0} C_1 = 0$); (iv) were it not the very restrictive condition $1 \le q < 1 + \frac{2}{dN}$, the prefactor $\frac{dN}{2}\left[1-(q-1)\frac{dN}{2}\right]$ would make, for $q \ne 1$ and $N \to \infty$, a crossover from a N-behavior to a N^2-behavior (the crossover occuring at $N_{crossover} \equiv 2/d(q-1)$). In fact, Eq. (33) might be (in the present or in a similar form) correct under conditions larger than those which we have used to deduce it; for instance, it could be correct, as suggested by numerical results presented in Ref.[10] (see Fig. 9 therein), also for $1/2 \le q < 1$ (in which case, the prefactor $\frac{dN}{2}\left[1+(1-q)\frac{dN}{2}\right]$ *indeed* presents the above mentioned N to N^2 crossover in the $N \to \infty$ limit, the crossover occuring at $N_{crossover} = 2/d(1-q)$). The possibility of an enlarged validity (for instance, $1/2 \le q \le 1$ in the $dN \to \infty$ limit) for Eq. (33) is an interesting one. Indeed, Plastino and Plastino [8] used the $q = 1$ equation of states for the classical ideal gas as well as $U_q = Tr\hat{\rho}\hat{\mathcal{H}}$ and elegantly solved, for $q > 9/7$, the Chandrasekhar's polytropic model paradox. A fully consistent calculation would have to generalize *all* the steps of the calculation (and not only the entropy and its direct consequences). In particular, one should have to use $U_q = Tr\hat{\rho}^q\hat{\mathcal{H}}$ (as justified in [3, 9] and in many subsequent works) and the (still unknown for arbitrary q) generalized equation of states of the gas. Such calculation would certainly depend, for instance, on the energy spectrum being say entirely positive ($\varepsilon_n = An^2 + B$ with $A > 0$, $B \ge 0$ and $n = 0, \pm 1, \pm 2 \cdots$) or only partially positive ($B < 0$). So, certain conditions could exist such that the galaxy mass is finite for $q < q_c < 1$, diverges for $q \to q_c - 0$, and remains infinite for $q \ge q_c$ (in particular for $q = 1$, thus recovering the well known paradox). In such a situation ($q < 1$), maybe one could use Eq. (33). This possibility fits very nicely with a crossover to a N^2-behavior for a galaxy (made of a large number of stars which interact "all with all" because of the long-range gravitational forces)!

Let us now address the equation of states of the classical ideal gas. We have that

$$\frac{\bar{p}_q}{T} = \frac{\partial S_q}{\partial V} \tag{34}$$

where $\bar{p}_q$ is the pressure. But $S_q = (U_q - F_q)/T$, consequently

$$\frac{\bar{p}_q}{T} = \frac{1}{T}\frac{\partial}{\partial V}\left[-\frac{\partial}{\partial\beta}\left(\frac{Z_q^{1-q}-1}{1-q}\right) + \frac{1}{\beta}\frac{Z_q^{1-q}-1}{1-q}\right] \tag{35}$$

If we use Eq. (31) into Eq. (35) (D is defined in Eq. (30)) we easily obtain

$$\bar{p}_q V = N\left[1-(q-1)\frac{Nd}{2}\right]\left[\frac{D\Gamma\left(\frac{1}{q-1}-\frac{dN}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)}\right]^{1-q}(kT)^{1+\frac{dN}{2}(1-q)} \tag{36}$$

The same type of remarks we did for C_q are feasible here. In particular, in the $q \to 1$ limit, we recover the well known relation $\bar{p}_1 V = Nk_BT$. In addition to this, Eqs. (33) and (36) imply a remarkably simple relation, namely

$$\frac{dV\bar{p}_q}{2TC_q} = 1 \quad (\forall q) \tag{37}$$

hence

$$\frac{\bar{p}_q}{\bar{p}_1} = \frac{C_q}{C_1} \tag{37'}$$

In addition to the above discussion of the polytropic model paradox, let us mention that, at fixed (N,T), Eq. (36) implies $\bar{p}_q/\bar{p}_1 \propto (h^d/V)^{(q-1)N}$ (where we have used the definition of D). Consequently, if $q > 1$ (as suggested in [8]), the pressure abnormally *increases (decreases)* if the volume *decreases (increases)* and/or if h increases (decreases), i.e., if quantum effects become stronger (weaker). In other words, for fixed number of stars and in a roughly isothermal situation, this effect on one hand helps, for small values of V, compensating the gravitational tendency towards collapse of the galaxy. On the other hand, for large values of V, the pressure is abnormally small, hence the gas remains relatively confined, thus behaving as a physical cut-off which (presumably) impeaches the total mass integral to diverge (as occurs for $q = 1$).

IV Increasing N

We want to focus here the role of the number N of particles of a system, and its possible relevance in the extensive vs. nonextensive discussion. To simplify the discussion, let us assume N classical rigid spheres with radius b contained in a volume $V \equiv L^d$.

We further assume that the spheres are homogeneously distributed and interact through two-body interactions characterized by a potential energy

$\varphi(\vec{r}_{ij})$ $(\vec{r}_{ij} \equiv \vec{r}_j - \vec{r}_i)$ such that

$$\varphi(\vec{r}_{ij}) = \begin{cases} +\infty & \text{if } r_{ij} < b \\ \varphi_0 < 0 & \text{if } b \leq r_{ij} \leq \xi \\ 0 & \text{if } r_{ij} > \xi \end{cases} \quad (38)$$

where $\xi > b$ is the range of the forces. We assume, for simplicity once more, that ξ is a multiple of b (i.e., $\xi/b \in \mathcal{N}$). Let us discuss the ground state of the $d = 1$ *compact* system (i.e., $V \equiv L^d = Nb^d$). The total energy E is given by

$$E(N) = \begin{cases} \frac{N(N-1)}{2}\varphi_0 & \text{if } 1 \leq N \leq N^* \equiv (\xi/b) + 1 \\ \left[\frac{N(N-1)}{2} + (N - N^*)\frac{\xi}{b}\right]\varphi_0 & \text{if } N \geq N^* \end{cases} \quad (39)$$

If the system is *half-compact*, Eq. (39) still holds but ξ/b is replaced by $\xi/2b$. If $L/N = \xi$, then $E(N) = (N-1)\varphi_0$, and, if $L/N > \xi$, then $E(N) = 0$, $\forall N$. Typical cases are represented in Fig. 1. When $E(N)$ exhibits a N^2-type growth, we refer to the system as *nonextensive* (NEXT); when it exhibits a N-type growth, we refer to the system as *extensive* (EXT). The situations that might occur are depicted in Fig. 2. Along the line $V = b^d(N-1)$ of case (a) we have a typical compact clustering situation of Condensed Matter Physics, and nonextensive behavior is expected for *small* clusters. Case (b) corresponds to relatively diluted long-ranged-force systems, such as domain walls in some $d = 2$ systems or grain elastic attraction (or repulsion) in $d = 2$ or $d = 3$ nucleation in alloys (or similar situations in Nuclear or Elementary Particles Physics, in cases where there are "droplets" whose size is smaller than the range of the forces. Case (d) (or case (c)) corresponds to situations such as a galaxy, where the system is essentially nonextensive (as strongly suggested by the result of Ref. [8]).

Let us now focus the N-dependence of a typical physical quantity P_q for various values of q. Suppose we are increasing N in case (a) of Fig. 2 along a line roughly parallel to the forbidden-nonforbidden frontier (i.e., $\partial V/\partial N \simeq b^d$) and slightly *above* it. A typical behavior one expects for P_q is depicted in Fig. 3. Nonextensivity should become relevant only for $N_{min} < N < N_{max}$ (*intermediate* size). Otherwise, all values of q should merge into the $q = 1$ behavior. The situation mostly encountered corresponds to $N_{min} \simeq N_{max}$, hence no $q \neq 1$ behavior appears. However, depending on b,

ξ and the path we are following (in the (N,V) space), it might happen that $N_{min} \simeq 0$ (hence nonextensivity is expected to appear at *small* systems), or $N_{max} \to \infty$ (hence nonextensivity is expected to appear at *large* systems; see [32]), or both $N_{min} \simeq 0$ and $N_{max} \to \infty$ (hence nonextensivity should appear at *all* sizes, as seems to be the case in gravitational astrophysical systems).

If, instead of Eq. (38), we have

$$\varphi(\vec{r}_{ij}) = \begin{cases} +\infty \text{ if } r_{ij} < b \\ \frac{\o}{r_{ij}^{\alpha}} e^{-r_{ij}/\xi} \text{ , otherwise} \\ \qquad\qquad \text{(with } \xi > 0; \o < 0) \end{cases} \tag{40}$$

the situation is much richer since it can be $b \geq 0$, $\xi^{-1} \geq 0$ and $\alpha \gtreqless \alpha_c(d)$ where $\alpha_c(d)$ is a crossover value which monotonically increases with d; there are consequently $2 \times 2 \times 3 = 12$ different cases. We may have singularities *only at r=0* if $b = 0$ and $\alpha \geq \alpha_c(d)$ and $\xi^{-1} > 0$ or if $b = 0$ and $\alpha > \alpha_c(d)$ and $\xi^{-1} = 0$ (analogous to the case (b) of Fig. 2), or singularities *only at* $r = \infty$ if $b > 0$ and $\alpha \leq \alpha_c(d)$ and $\xi^{-1} = 0$ or if $b = 0$ and $\alpha < \alpha_c(d)$ and $\xi^{-1} = 0$ (analogous to the case (c) of Fig. 2), or *both* singularities if $b = 0$ and $\xi^{-1} = 0$ and $\alpha = \alpha_c(d)$, (analogous to case (d) of Fig. 2), or *no singularities* if $b > 0$ and $\xi^{-1} > 0$ and $\alpha \gtreqless \alpha_c(d)$ or if $b = 0$ and $\xi^{-1} > 0$ and $\alpha < \alpha_c(d)$ or if $b > 0$ and $\xi^{-1} = 0$ and $\alpha > \alpha_c(d)$ (analogous to case (a) of Fig. 2). The N-dependence of P_q/P_1 should closely follow the discussion presented for Eq. (38). The $b = \xi^{-1} = 0$ (i.e., $\varphi(r_{ij}) \propto 1/r_{ij}^{\alpha}$) *diluted* (i.e., N/V small enough) case deserves some more words. If $\alpha > \alpha_c(d)$ an essentially extensive behavior is expected for all values of N/V small enough (hence the $q = 1$ description should be satisfactory). If $\alpha \leq \alpha_c(d)$, nonextensive behavior is expected to emerge in an increasingly stronger manner for increasingly larger (though still small) values of N/V (see, also, [32]). Then a $q \neq 1$ description is expected to be unavoidable. In particular, unphysical singularities (such as the mass divergence in Chandrasekhar's polytropic model for stellar matter) are expected to be present for $q \geq q_c(d,\alpha)$ if $q_c(d,\alpha) < 1$ or for $q \leq q_c(d,\alpha)$ if $q_c(d,\alpha) > 1$ (this is the situation found in [8] by Plastino and Plastino, whose calculation yields $q_c(3,1) = 9/7$). Also, it seems reasonable to expect $q_c(d,\alpha_c(d)) = 1$, $\forall d$.

V Conclusion

In the present work we have briefly reviewed the main aspects of the Generalized Statistical Mechanics and Quantum Groups, have presented, for the first time, Hilhorst formula, and have established: (i) the striking analogy between Eqs. (4) and (11); (ii) the fluctuation-dissipation theorem for magnetic systems (Eq. (17)); (iii) the specific heat (Eq. (33)) and equation of states (Eq. (36)) which imply a simple relation, namely Eq. (37)) for the classical ideal gas; (iv) the expected extensive and nonextensive physical regions (Fig. 2), in terms of b (rigid sphere radius or, more generally speaking, minimal allowed distance), ξ (cut-off length of the forces) and α (characterizing the range of the forces if $b = \xi^{-1} = 0$), as well as the expected N-dependence of arbitrary physical properties for arbitrary values of q (Fig. 3). As a final remark, let us recall that, within the possible connection between Generalized Statistical Mechanics (characterized by q) and Quantum Groups (characterized by q_G) [34], $q-1$ is roughly proportional (strictly proportional in the $q_G \to 1$ limit) to $q_G - 1$, the proportionality coefficient being a nonuniversal function of the thermodynamic state of the system, in particular of the temperature (if the system is at thermodynamic equilibrium). Consequently, various calculations that have been done here at fixed q, could be redone at fixed q_G; in general, they differ, excepting of course for the particular case $q = q_G = 1$ (extensive physics).

Naturally, experimental and further theoretical work in Astrophysics and Gravity (galaxies, massive stars, black-body radiation), Condensed Matter Physics (domain walls, grain interaction in nucleation in alloys, cluster physics), Nuclear and Elementary Particle Physics ("droplets" whose linear size is smaller than the range of the interactions) as well as in Human Sciences (learning curves, Neural networks, Economics) are strongly needed (in order to clearly establish the conditions under which nonextensive physics is unavoidable) and extremely welcome.

I am deeply indebted to H.J. Hilhorst for communicating to me his interesting integral transform, to A.J.R. da Silva with whom I longly discussed the content of Section V, to T. Kodama who first suggested the "droplets" as possible candidates for nonextensivity, and to M.R.-Monteiro and I. Roditi for permanently illuminating my knowledge of Quantum Groups. Useful remarks from L.M. Falicov, M. Berry, R. Maynard, J.M. Sanchez, B. Koiller, F.C. de Sá Barreto, M. Saraceno, C.A. Balseiro, A. Craievich, P.A. Lingård, I.K. Schuller, E.M.F. Curado and W. de Heer are also acknowledged with pleasure. Last but not least, I am grateful to J.L. Morán-López who made possible my participation at the interesting II Latin-American Workshop on Magnetism, Magnetic Materials and their Applications, where I found great inspiration.

Caption for Figures

Fig. 1 - N-dependence of the ground state energy $E(N)$ ($\varphi_0 < 0$) for typical situations associated with a simple $d = 1$ classical system (Eq. (38)); $\xi/b = 8$; L is the accessible length. •(∘) denotes nonextensive (extensive) behavior; the last • on each curve is located at N^*.

Fig. 2 - Physical regions of extensive (EXT) and nonextensive (NEXT) behaviors related to the model characterized by Eq. (38) ($V \equiv d$-dimensional accessible volume, $b \equiv$ rigid sphere radius, $\xi \equiv$ cut-off length of the two-body forces): (a) $b > 0$ and $\xi^{-1} > 0$; (b) $b = 0$ and $\xi^{-1} > 0$; (c) $b > 0$ and $\xi^{-1} = 0$; (d) $b = \xi^{-1} = 0$.

Fig. 3 - Expected N-dependence of an arbitrary physical property P_q for typical fixed values of q (the choice $P_q(N) > P_1(N)$ in the intermediate region $N \in [N_{min}, N_{max}]$ is a conventional one). N_{min} and N_{max} depend on the physical path followed in Fig. 2. (see the text).

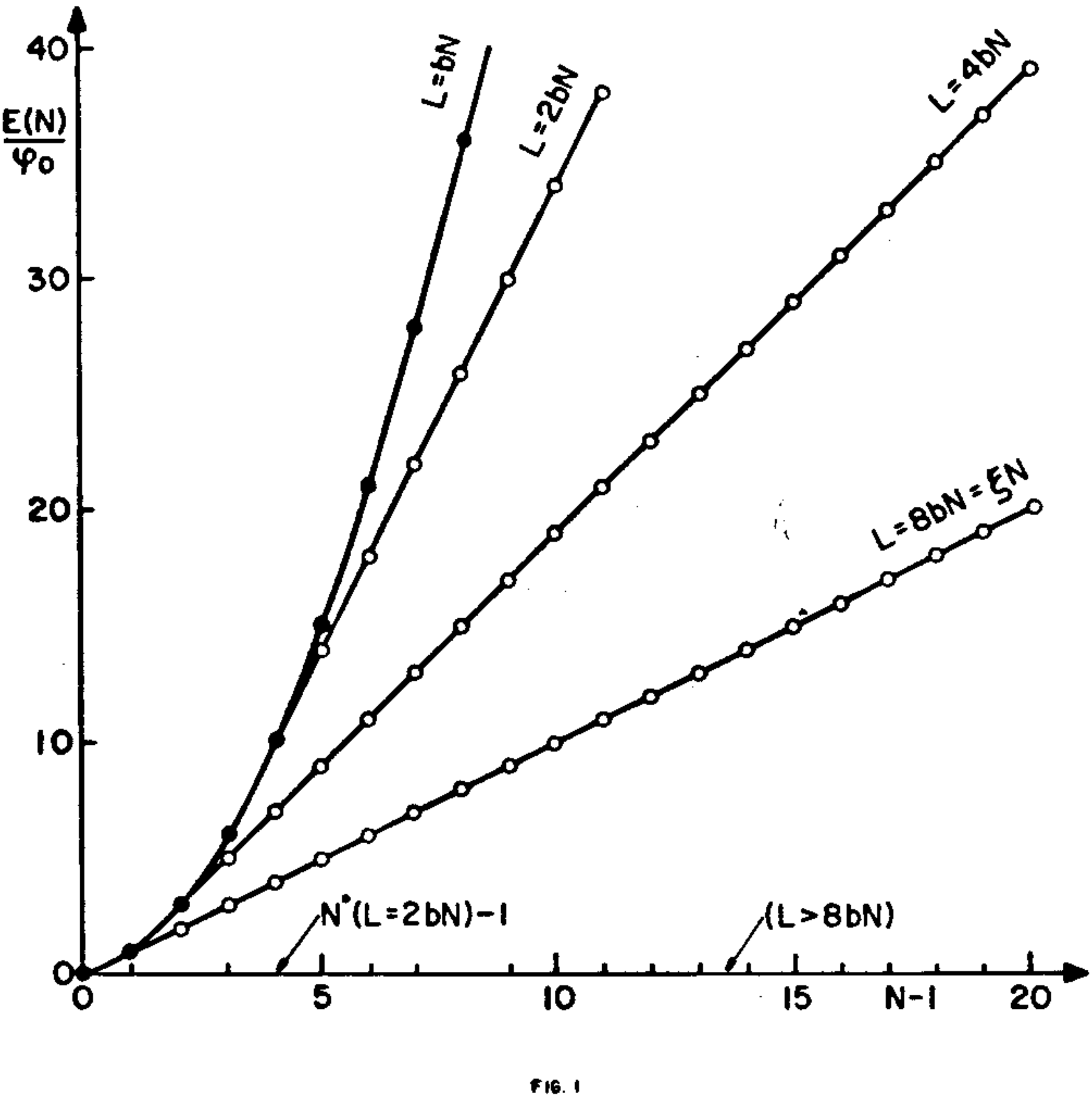
40
30
20
10
0
E(N)/φo
L=bN
L=2bN
L=4bN
L=8bN=ξN
N*(L=2bN)-1
(L>8bN)
0
5
10
15
N-1
20

FIG. 1

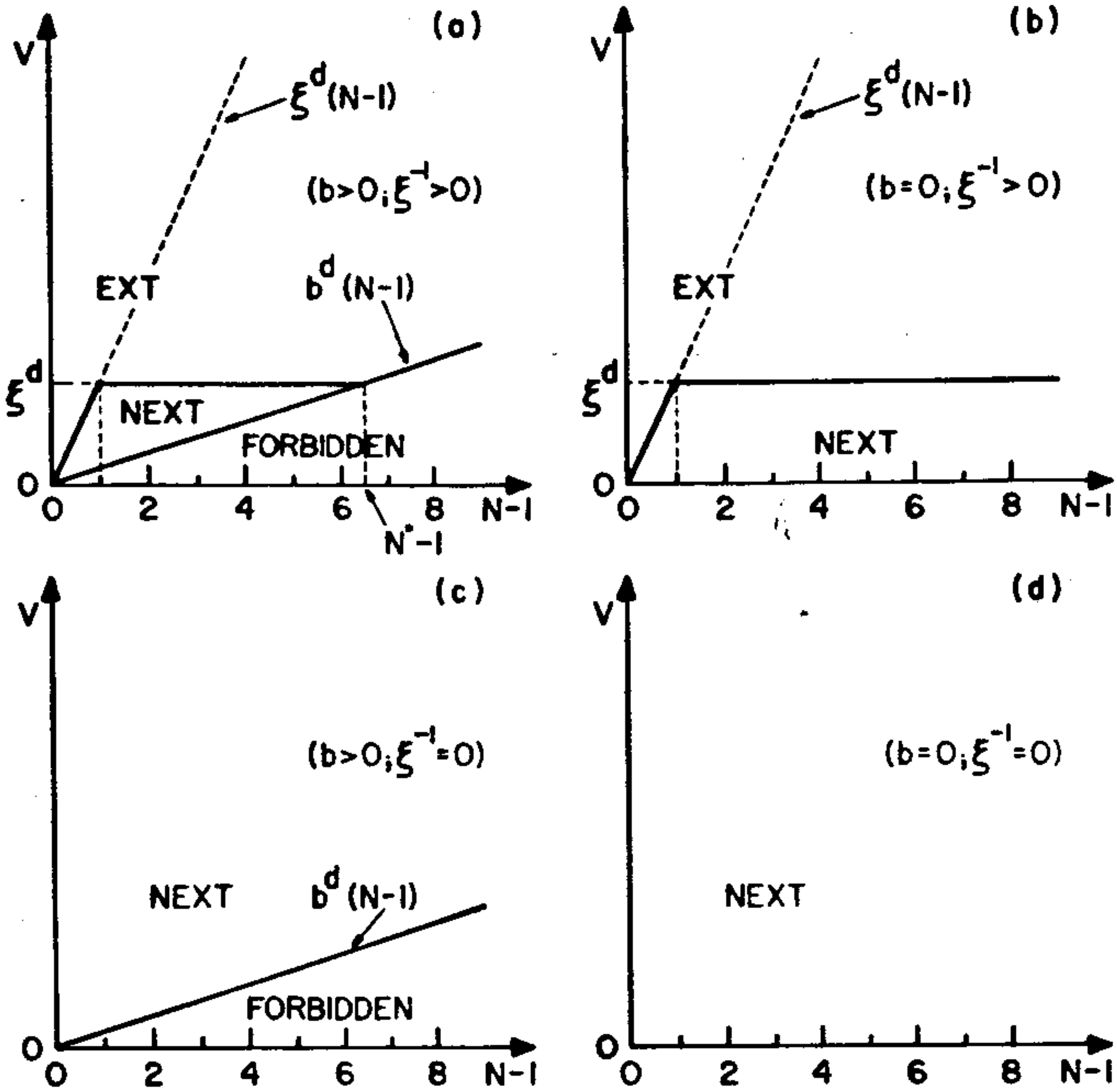
(a)
V
$\xi^d(N-1)$
$(b>0;\xi^{-1}>0)$
EXT
$b^d(N-1)$
ξ^d
NEXT
FORBIDDEN
0
0 2 4 6 8 N-1
N^*-1
(b)
V
$\xi^d(N-1)$
$(b=0;\xi^{-1}>0)$
EXT
ξ^d
NEXT
0
0 2 4 6 8 N-1
(c)
V
$(b>0;\xi^{-1}=0)$
NEXT
$b^d(N-1)$
FORBIDDEN
0
0 2 4 6 8 N-1
(d)
V
$(b=0;\xi^{-1}=0)$
NEXT
0
0 2 4 6 8 N-1

FIG. 2

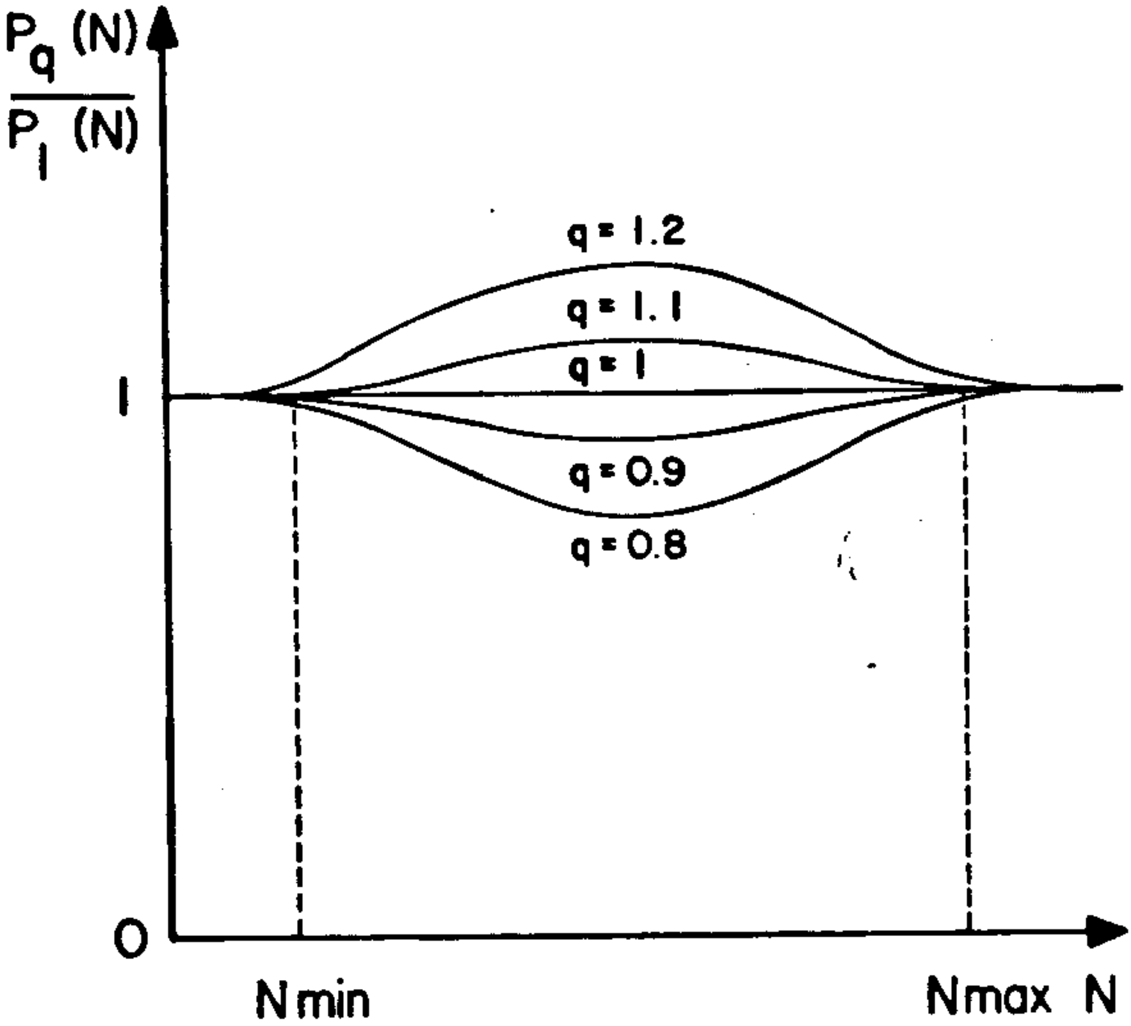

FIG. 3

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