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Nonlinear Theories of Spin-2 Field in Terms of Fierz Variables

by

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ABSTRACT

Exceptional non linear Lagrangeans for a spin-two field in terms of the Fierz variables are obtained using the method proposed by Lax. They are equivalent to the Born-Infeld Lagrangean for a spin-one field theory and are free from unbounded growth of wave velocities.

I INTRODUCTION

The aim of this paper is to propose a new method to arrive at non linear equations of motion for spin two-fields. This method has been successfully applied by Boillat to generate a special case of non linear Eletrodynamics. Traditionally, from the theory of representation of the Lorentz-Poincaré group, a spin-two field is described in terms of a symmetric second order tensor $\phi_{\mu\nu}$. Besides such a standard formulation it is possible to use a third order tensor $A_{\alpha\beta\mu}$ to represent such a field. This alternative description, proposed some 50 years ago by Fierz, has been almost forgotten, until very recently when it the enterprise of reviving it was undertaken.. In ref. (1) a complete self-consistent theory of linear spin-two field using the $A_{\mu\nu\lambda}$ variable was presented.

This Fierz representation deals with a third-order tensor $A_{\alpha\beta\mu}$ whose properties will now be synthesized.

The tensor $A_{\alpha\beta\mu}$ is anti-symmetric in the first pair of indices, that is

$$A_{\alpha\beta\mu} + A_{\beta\alpha\mu} = 0$$

and it is pseudo-trace free, that is $\overset{*}{A}{}^{\alpha\beta}{}_{\beta} = 0$. The dual $A^{*}_{\alpha\beta\nu}$ is defined as usual by

$$A^{*}_{\alpha\beta\nu} = \frac{1}{2}\, \eta_{\alpha\beta}{}^{\rho\delta} A_{\rho\delta\nu}$$

in which $\eta_{\alpha\beta\rho\delta} = \sqrt{-g}\ \varepsilon_{\alpha\beta\rho\delta}$ and $\varepsilon_{\alpha\beta\rho\delta}$ is the completely anti-symmetric Levi-Civita symbol. The determinant of the background metric $g_{\mu\nu}$ is represented by g. This is equivalent to the cyclic condition

$$A_{\alpha\beta\mu} + A_{\mu\alpha\beta} + A_{\beta\mu\alpha} = 0$$

These conditions reduce the number of independent components of $A_{\alpha\beta\mu}$ from 64 to 20, which is still a very large number, far beyond the minimum required to describe a spin-two field. It is worth to comment that the same difficulty occurs with standard variables. Indeed $\phi_{\mu\nu}$ has too many unnecessary variables to describe the true physical degrees of freedom associated to a spin two field. The reason for the use of any variable that contains unobservable quantities lies at the kernel of modern Physics which has at its foundations the existence of general symmetry principles.

Until now the study of the spin-two field equations in terms of Fierz variables had been limited to the linear case. Two main goals were then achieved: the complete description of a Lagrangian and Hamiltonian formulation of the theory and its quantum version[2]. We can sum up the achievements of such an enterprise by just noticing that, by using Fierz variables, the theory of spin two field shows up in a complete analogy to Electrodynamics.

On the other hand, we have learned, from the analysis made by Thirring, Feynman and many others, that any theory which is a candidate to describe gravitational processes must be non-linear. In this vein, it is natural to examine non-linear theories for the $A_{\alpha\beta\mu}$ field.

We then face the following question: how to obtain a non-linear equation of motion for $A_{\alpha\beta\mu}$? To be more specific, what should the criteria to which one should adhere to guide us towards this goal be? The simplest way should be to proceed by analogy to the case of the standard second-order tensor variable $\phi_{\mu\nu}$ by using an interative process[3]. However, attempts to provide an equivalent prescription for $A_{\alpha\beta\mu}$ along this line were not successful. This motivated us to look into other possible methods.

It turned out that a very practical and efficient method to solve this question was the one suggested some years ago by Lax[4], in another context, for the arbitrary field theory. In order to obtain at a well defined set of equations of motion for a given field, a theory must satisfy three preliminary conditions:

a) The equations of motion must be of hyperbolic type;

b) A well posed standard Cauchy problem must be defined;

c) Stability property of arbitrary disturbances through any characteristic surface must be satisfied.d.

Condition "a" guarantees that the velocity of the disturbances is finite and real; condition "b" is the basic requirement of any classical field theory; and finally "c" is the true novelty of the Lax method. Let us review briefly this method. The main idea rests on an analysis of the behaviour of the propagation of successive wave fronts. In order to describe this method, let us introduce some definitions. We call exceptional waves some small disturbances such that its associated velocity of propagation remains finite. A perturbation which satisfies such a requirement does not contain accelerating disturbances which could, at least in principle, be responsible for the generation of undesirable shock waves. Hence, the difficulties associated to unbounded growth of perturbed velocities are automatically excluded. Indeed, if such limitations on the perturbations are not imposed, there appears a difficulty to treat the Cauchy problem in a standard way, since it is no more possible to propagate initial data arbitrarily. In the cases in which all the possible wave disturbances are exceptional, the theory and its associated Lagrangean are called exceptional.

In ref. (5), Boillat exhibits an example of the use of Lax criteria which yields a model for Electrodynamics. Astonishingly enough, it turned out that the resultant equation of motion for this case is nothing but the one proposed in the early 30's by Born and Infeld[6]. It seems worth to remark that the "leitmotiv" of Born-Infeld, which guided them to suggest a non linear Electrodynamics, was very distinct from the above considerations. Indeed, these authors were looking for a regular theory which did not contain singularities of any sort. In particular they argued that there should be a maximal value for any electromagnetic field. This was achieved by the hypothesis, made by Born and Infeld, that the non-linearities of Electrodynamics should be manifested through the imposition of a particular Lagrangean from which the bondedness of any electromagnetic field would follow. It is indeed an intriguing property of Lax's method that the Born-Infeld proposal can be derived from an exceptional Lagrangean.

We shall see that the treatment used by Boillat in Electrodynamics can be employed in the spin two case with almost trivial modifications. The new aspects of the theory can be atributed to the existence of a higher number of degrees of freedom which do not have a correspondence to the Electromagnetic case.

We then obtain at a very appealing result. The dynamics obtained by using Lax suggestion in the Fierz variables is nothing but a similar reproduction, for the spin two-field case, of Born-Infeld theory for spin one.

In the next section we review the classical dynamics of the Fierz variables in the linear case. Then we analyse some aspects of wave dynamics and hyperbolic systems. Finally, in the last section, we construct two exceptional Lagrangeans in terms of the Fierz variables which are analogous to the Born-Infeld theory.

II THE LINEAR THEORY

Recently, the spin-2 linear theory in terms of Fierz variables has been thoroughly examined[1,2].

The theory is constructed in terms of a traceless field $C_{\alpha\beta\mu\nu}$ defined by:

$$C_{\alpha\beta\mu\nu} \equiv A_{\alpha\beta[\mu;\nu]} + A_{\mu\nu[\alpha;\beta]} + \frac{1}{2} A_{(\alpha\nu)}\gamma_{\beta\mu} + \frac{1}{2} A_{(\beta\mu)}\gamma_{\alpha\nu} - \frac{1}{2} A_{(\alpha\mu)}\gamma_{\beta\nu} +$$

$$- \frac{1}{2} A_{(\beta\nu)}\gamma_{\alpha\mu} + \frac{1}{6} A^{\lambda\sigma}{}_{\lambda;\sigma}(\gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\mu}\gamma_{\beta\nu}) \tag{2.1}$$

where $\gamma_{\mu\nu}$ is the Minkowski metric written in an arbitrary coordinate system. The quantity $A_{\alpha\mu}$ is defined by: $A_{\alpha\mu} := A_{\alpha}{}^{\varepsilon}{}_{\mu;\varepsilon} - A_{\alpha}{}^{\varepsilon}{}_{\varepsilon;\mu}$.

In order to construct a dynamics by analogy with Maxwell's Electrodynamics one is tempted to set the Lagrangean

$$L = - \frac{1}{8} C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} \tag{2.2}$$

Variation of $A_{\alpha\beta\mu}$ in (2.2) yields the equation of motion

$$C^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \tag{2.3}$$

From Fierz additional requirement, which is nothing but a gauge fixing procedure, we set (see[1] for a complete review of the properties):

$$A^{\alpha\beta}{}_{\beta} = 0 \tag{2.4a}$$

and

$$A^{\alpha\beta\mu}{}_{;\mu} = 0 \tag{2.4b}$$

Then, using these conditions into (2.3), we obtain the wave equation

$$\Box\, A_{\alpha\beta\mu} = 0 \tag{2.5}$$

In this linear theory there are some simple relations between the Fierz and the standard variable, which can be written in a self-consistent way:

$$A_{\mu\lambda\nu} = \phi_{\nu[\mu,\lambda]} + B\phi_{,[\mu}\gamma_{\lambda]\nu} - B\gamma_{\nu[\lambda}\phi_{\mu]}{}^{\beta}{}_{,\beta} \quad . \tag{2.6}$$

$$\phi_{\mu\nu} = -\frac{1}{2m} A_{(\mu}{}^{\lambda}{}_{\nu),\lambda} + \frac{Q}{2m}\left(A_{\mu}{}^{\lambda}{}_{\lambda,\nu} + A_{\nu}{}^{\lambda}{}_{\lambda,\mu}\right) - \frac{1}{2m} A^{\alpha\beta}{}_{\beta,\alpha}\gamma_{\mu\nu} \tag{2.7}$$

where

$Q = \frac{1 - B}{1 - 3B}$ and B is an arbitrary constant.

In a sense these relations can be interpreted as follows: the standard variable ($\phi_{\mu\nu}$) has a potential ($A_{\mu\nu\lambda}$) in such a way that it becomes the potential of its potential. That is, eq. (2.7) shows that $A_{\mu\nu\lambda}$ is the potential of $\phi_{\mu\nu}$; and eq. (2.6) shows that $\phi_{\mu\nu}$ is the potential of $A_{\mu\nu\lambda}$. The combination of these two eqs. yields the consistency represented by equation (2.5).

The condition (2.6) can be covariantly written as:

$$\overset{*}{A}{}^{\alpha\beta\mu}{}_{;\beta} = 0 \qquad (2.8)$$

This is just Fierz's condition which allows the variable $A_{\alpha\beta\mu}$ to represent just a single spin-2 field. We remark that in the absence of this kind of condition, one deals whith two independent spin-2 parts of the field $A_{\sigma\beta\mu}$.

This linear theory was quantized in two equivalent formulations: (i) Using the "Fermi-Gupta-Bleuler-method" and (ii) using Dirac's proposal[(2)]. We do not extend the analysis of this linear theory in this paper, since it has been lengthily presented elsewhere[(2)] and we go imediately into our main subject, namely the construction of a non linear model.

Before this, let us review briefly some results of Lax's method.

III GENERAL FORMALISM

Let $\vec{u}(x^\beta)$ be an arbitrary tensor field the evolution of which is governed by a set of quasi-linear differential equations represented by:

$$A^\alpha(\vec{u},x^\beta)\partial_\alpha\vec{u} = \vec{f}(\vec{u},x^\beta) \tag{3.1}$$

in which A is a $n \times n$ matrix. Let S be a hypersurface defined by the equation $\phi(x) = 0$.

We define the discontinuity, denoted by the symbol [ ], through the hypersurface S in the standard way

$$\left[\frac{\partial\vec{u}}{\partial\phi}\right] := \frac{\partial\vec{u}}{\partial\phi}\bigg|_{0+} - \frac{\partial\vec{u}}{\partial\phi}\bigg|_{0-} \tag{3.2}$$

Discontinuity of equation (3.1) through this hypersurface implies that S is a characteristic surface if the determinantal condition

$$\left|A^\alpha\partial_\alpha\phi\right| = 0 \tag{3.3}$$

is satisfied or, otherwise, if

$$\left|A_n - \lambda I\right| = 0 \tag{3.4}$$

where,

$$A_n = (A^0)^{-1} A^i n_i \tag{3.5}$$

$$\lambda = \frac{\partial_0 \phi}{|\nabla\phi|} \quad \text{and} \quad \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} \tag{3.6}$$

since A^0 is supposed to be a regular matrix.

The solutions of (3.4) will have the form $\lambda = \lambda(\vec{n})$.

When the equations of motion are non-linear a suitable choice of the initial disturbances can produce an accelerated wave. In this case the disturbances grow without limit generating a shock wave.

However, there is a class of waves for which this phenomenon does not arise, that is the disturbances do not cease to be finite. This fact, which seems to have been firstly noticed by Lax[4], requires that the velocity λ must be continuous through the chracteristic surface, i.e.

$$\left[\frac{\partial\lambda}{\partial\phi}\right] = \nabla_u \lambda . \left[\frac{\partial\vec{u}}{\partial\phi}\right] = 0 \tag{3.7}$$

If the field equation is covariant, the caracteristic equation (3.3) will be given by:

$$\psi = G^{\alpha\beta\mu\ldots\nu} \phi_\alpha \phi_\beta \phi_{\mu\ldots} \phi_\nu = 0 \tag{3.8}$$

where, $\phi_\alpha := \partial_\alpha \phi := k_\alpha$ and $\alpha = 1,2,\ldots n$

The condition (3.7) can be written as:

$$\nabla_u\psi \cdot \left[\frac{\partial\vec{u}}{\partial\phi}\right] = |\nabla\phi| \frac{\partial\psi}{\partial\phi_0} \nabla_\mu\lambda \cdot \left[\frac{\partial\vec{u}}{\partial\phi}\right] = \nabla G^{\alpha\beta\mu\ldots\nu} \cdot \left[\frac{\partial\vec{u}}{\partial\phi}\right] \phi_\alpha\phi_\beta\phi_\mu\ldots\ldots\phi_\nu = 0$$

that is,

$$\left[G^{\alpha\beta\mu\ldots\nu}\right]\phi_\alpha\phi_\beta\phi_{\mu\ldots}\phi_\nu = 0 \tag{3.9}$$

This is the essence of Lax's method. We will apply this method to spin-2 field equations in terms of the Fierz variable. Before, as an example, we will show how this method works in the well known case of Electrodynamics.

IV NON-LINEAR ELECTRODYNAMICS

In this section we will make a brief summary of Boillat's paper(5) on non-linear Electrodynamics.

The electromagnetic field.

$$F_{\mu\nu} = A_{[\mu,\nu]} \tag{4.1}$$

allows the construction of two invariants

$$F = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{4.2a}$$

$$G = \frac{1}{4} F_{\mu\nu}\overset{*}{F}{}^{\mu\nu} \tag{4.2b}$$

The most general Lagrangean that may describe non-linear Electrodynamics is an arbitrary function of these two invariants L(F,G) The action is given by:

$$S = \int L(F,G)d^4x \tag{4.3}$$

Variation of S with respect to the vector potential A_μ yields the following equations of motion:

$$L_{F,\nu}F^{\mu\nu} + L_F F^{\mu\nu}{}_{,\nu} + L_{G,\nu}\overset{*}{F}{}^{\mu\nu} = 0 \tag{4.4.}$$

where, $L_F \equiv \frac{\partial L}{\partial F}$ and $L_G \equiv \frac{\partial L}{\partial G}$

In order to apply Lax's method to Electrodynamics, we must start to examine the characteristic equation corresponding to (4.4).

Following Hadamard's[7] discontinuity condition we set:

$$[A_{\alpha,\beta}] = \ell_\alpha k_\beta \tag{4.5}$$

where ℓ_α is the vector that characterizes the discontinuity and k_β is the gradient vector orthogonal to the discontinuity surface.

Thus, from (4.4), and nothing that the discontinuity operator [ ] acts as a differentiation operator, it follows that

$$[L_F]F^{\mu\nu}k_\nu + L_F[F^{\mu\nu}]k_\nu + [L_G]\overset{*}{F}{}^{\mu\nu}k_\nu = 0 \tag{4.6}$$

Defining the vectors

$$V^{\mu} \equiv F^{\mu\nu}k_{\nu} \tag{4.7a}$$

$$U^{\mu} \equiv \overset{\bullet}{F}{}^{\mu\nu}k_{\nu} \tag{4.7b}$$

and using (4.5), equation (4.6) may be rewritten in the following way:

$$\ell^{\mu} = \alpha V^{\mu} + \beta U^{\mu} \tag{4.8}$$

where,

$$a k^{2} L_{F} + [L_{F}] = 0 \tag{4.9a}$$

$$\beta k^{2} L_{F} + [L_{G}] = 0 \tag{4.9b}$$

Thus, the problem of finding the vector ℓ^{μ} diferent from zero is reduced to the problem of finding non vanishing values for α or β.

Using the Lorentz condition

$$A^{\mu}{}_{,\mu} = 0 \quad . \tag{4.10}$$

and Leibniz rule we can write:

$$[L_F] = L_{FF}[F] + L_{FG}[G] \tag{4.11a}$$

$$[L_G] = L_{GF}[F] + L_{GG}[G] \tag{4.11b}$$

The system (4.9) then becomes:

$$\alpha\left\{k^2(L_F + FL_{FF} + GL_{FG}) - \tau L_{FF}\right\} + \beta\left\{k^2(GL_{FF} - FL_{FG}) - \tau L_{FG}\right\} = 0 \tag{4.12a}$$

$$\alpha\left\{k^2(FL_{FG} + GL_{GG}) - \tau L_{FG}\right\} + \beta\left\{k^2(L_F + GL_{FG} - FL_{GG}) - \tau L_{GG}\right\} = 0 \tag{4.12b}$$

where,

$$\tau := \tau^{\mu\nu}k_\mu k_\nu := (Fg^{\mu\nu} - F^{\mu\rho}F^{\nu}{}_{\rho})k_\mu k_\nu \tag{4.13}$$

The system of equations (4.12) has non trivial solutions if

$$a\mu^2 + b\mu + c = 0 \tag{4.14}$$

and, in which a,b and c are given, respectively, by:

$$a := L_{FF}L_{GG} - L_{FG}^2 \tag{4.14a}$$

$$b := -L_F^2(L_{GG} + L_{FF}) \tag{4.14b}$$

$$c := L_F^2 + 2L_F L_{FG} - a(F^2 + G^2) + FL_F(L_{FF} - L_{GG}) \tag{4.14c}$$

where $\mu := -\frac{\tau}{k^2}$ (4.15)

Boillat shows that, in the case in which equation (4.13) has a unique solution, i.e., when

$$\Delta = b^2 - 4ac = 0 \tag{4.16}$$

the Lagrangean L(F,G) satisfying (4.16) has the form:

$$L(F,G) = \sqrt{-G^2+2nF+n^2} \tag{4.17}$$

where n is a constant with the same dimensions of F. The Lagrangean (4.17) is nothing but the Born-Infeld Lagrangean. If we insert the solution μ of (4.13) with L(F,G) given by (4.17) into the characteristic equation (4.16),which is the particular form that equation (3.8) takes for this theory, it can be verified that this theory is exceptional in the sense of section III.This procedure will be developed in detail in the next section.

Let us point out once more that Born and Infeld obtained precisely this Lagrangean (4.17) by a very distinct motivation, e.g., to have a well behaved theory of the electron and its corresponding electromagnetic field. In particular, they searched a model in which a spherically symmetric field which does not diverge at the origin can be generated by the electron.

In the next section we will apply the same method developed in this example to construct a non linear theory for the Fierz field $A_{\alpha\beta\mu}$.

V A NOW LINEAR THEORY FOR FIERZ VARIABLE: THE RIEMANN CASE

In this section we show an example of exceptional Lagrangean which depend only on two invariants A and B. These invariants are constructed with the tensor $S_{\alpha\beta\mu\nu}$, which has the same symmetries of Riemann's tensor and is defined by (compare with (2.1) above)

$$S_{\alpha\beta\mu\nu} := A_{\alpha\beta[\mu;\nu]} + A_{\mu\nu[\alpha;\beta]} \tag{5.1}$$

where (;) is the covariant derivative taken in the flat metric $\gamma_{\mu\nu}$ and $A_{\alpha\beta\nu}$ is the Fierz variable.

We can define two invariants of second order A and B given by:

$$A = S_{\alpha\beta\mu\nu} S^{\alpha\beta\mu\nu} \tag{5.2a}$$

$$B = S_{\alpha\beta\mu\nu} \overset{*}{S}{}^{\overset{*}{\alpha\beta}\mu\nu} \tag{5.ab}$$

The action S is given in terms of the lagrangean L(A,B) which is an arbitrary function constructed in terms of the invariants (5.2).

$$S = \int d^4x \sqrt{-\gamma}\, L(A,B) \tag{5.3}$$

and γ is the determinant of $\gamma_{\mu\nu}$.

Then, the equation of motion is given by:

$$(L_A S^{\alpha\beta\mu\nu} + L_B \overset{*}{S}{}^{\overset{*}{\alpha\beta}\mu\nu})_{;\nu} = 0 \tag{5.4}$$

in which,

$$L_A = \frac{\delta L}{\delta A} \quad , \quad L_B = \frac{\delta L}{\delta B} \ .$$

In order to apply the criteria described in section III to obtain an exceptional Lagrangean, we will examine the behaviour of the evolution of the disturbances of the field $A_{\alpha\beta\mu}$. Let us consider the case in which, across the surface S defined by

$$\phi(x^{\mu}) = 0 \tag{5.5}$$

the field has a discontinuity, which obeys Hadamard's condition given by:

$$\left[A_{\alpha\beta\mu,\nu}\right] = A_{\alpha\beta\mu,\nu(0+)} - A_{\alpha\beta\mu,\nu(0-)} = \phi_{\alpha\beta\mu} k_{\nu} \tag{5.6}$$

Thence,

$$\left[S_{\alpha\beta\mu\nu}\right] = \left[A_{\alpha\beta[\mu;\nu]}\right] + \left[A_{\mu\nu[\alpha;\beta]}\right] = \phi_{\alpha\beta[\mu} k_{\nu]} + \phi_{\mu\nu[\alpha} k_{\beta]} \tag{5.7}$$

in which we assumed that the metric tensor $\gamma^{\mu\nu}$ is continuous through the surface S, as in the previous case of Electrodynamics.

The Fierz's condition, which eliminates one of the two spin-2 fields represented by $A_{\alpha\beta\mu}$, is given by:

$$\overset{*}{A}{}^{\alpha\beta\mu}{}_{;\beta} = 0 \tag{5.8}$$

According to the discontinuity (5.6), this condition (5.8) turns into:

$$\phi^{\alpha\beta\mu}k^{\nu} + \phi^{\nu\alpha\beta}k^{\mu} + \phi^{\mu\nu\alpha}k^{\beta} = 0 \tag{5.9}$$

Consequently,

$$\phi^{\alpha\beta\mu}k_{\mu} = 0 \tag{5.10}$$

Combining the conditions (5.10) and (5.7) it follows that

$$\left[S^{\alpha\beta\mu\nu}{}_{;\nu}\right] = \left[S^{\alpha\beta\mu\nu}\right]k_{\nu} = 2\phi^{\alpha\beta\mu}k^2 \tag{5.11}$$

where

$$k^2 = k_{\nu}k^{\nu} \tag{5.12}$$

Besides this, condition (5.9) implies that the double dual $\overset{*}{S}{}^{\alpha\beta\mu\nu}{}^{*}$ is divergence-free:

$$\overset{*\ *}{S}{}^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \tag{5.13}$$

Using (5.13) into the equation of motion, it follows that

$$2L_A\phi^{\alpha\beta\mu}k^2 + [L_A]S^{\alpha\beta\mu\nu}k_\nu + [L_B]\overset{*\ *}{S}{}^{\alpha\beta\mu\nu}k_\nu = 0 \tag{5.14}$$

in which we used the relations

$$[L_{A;\nu}] = [L_A]k_\nu \tag{5.15a}$$

$$[L_{B;\nu}] = [L_B]k_\nu \tag{5.15b}$$

We define the quantities $U^{\alpha\beta\mu}$ and $V^{\alpha\beta\mu}$, which have the same algebraic symmetries as $A_{\alpha\beta\mu}$ by the expressions (compare with (4.7)).

$$U^{\alpha\beta\mu} = S^{\alpha\beta\mu\nu}k_\nu \tag{5.16a}$$

$$V^{\alpha\beta\mu} = \overset{*\ *}{S}{}^{\alpha\beta\mu\nu}k_\nu \tag{5.16b}$$

Then we can rewrite (5.14) as:

$$\phi^{\alpha\beta\mu} = pU^{\alpha\beta\mu} + qV^{\alpha\beta\mu} \tag{5.17}$$

where p and q are given by

$$2L_A k^2 p + [L_A] = 0 \qquad (5.18a)$$

$$2L_A k^2 q + [L_B] = 0 \qquad (5.18b)$$

In order to obtain the caracteristic equation, let us evaluate the discontinuities $[L_A]$ and $[L_B]$.

From Leibniz rule we have

$$[L_A] = L_{AA}[A] + L_{AB}[B] \qquad (5.19a)$$

$$[L_A] = L_{BA}[A] + L_{BB}[B] \qquad (5.19b)$$

The discontinuities, [A] and [B] can be evaluated by the formula:

$$[A] = 2S_{\alpha\beta\mu\nu}\left[S^{\alpha\beta\mu\nu}\right] = 8S_{\alpha\beta\mu\nu}\phi^{\alpha\beta\mu}k^{\nu} \qquad (5.20)$$

Thus,

$$[A] = 8U_{\alpha\beta\mu}\phi^{\alpha\beta\mu} \qquad (5.21a)$$

Using the same procedure, we obtain for the discontinuity [B]

$$[B] = 8V_{\alpha\beta\mu}\phi^{\alpha\beta\mu} \qquad (5.21b)$$

Inserting this result into equation (5.17) and taking account of (T.1), (T.2) and (T.3) (see Appendix), we obtain:

$$[L_A] = p\left\{2k^2\left[(A - \tau)L_{AA} - (A - \tau)L_{AB}\right] + 8QL_{AA}\right\} +$$

$$+ q\left\{2k^2\left[(A + \tau)L_{BA} - (A - \tau)L_{AA}\right] - 8QL_{AB}\right\} \qquad (5.22a)$$

$$[L_B] = p\left\{2k^2\left[(A - \tau)L_{BA} - (A - \tau)L_{BB}\right] + 8QL_{BA}\right\} +$$

$$+ q\left\{2k^2\left[(A + \tau)L_{BB} - (A - \tau)L_{AB}\right] - 8QL_{BB}\right\} \qquad (5.22b)$$

where $Q = 4\tau^{\mu\nu}k_\mu k_\nu$ and $\tau^{\mu\nu} = S^{\alpha\beta\lambda\mu}S_{\alpha\beta\lambda}{}^{\nu} - \frac{1}{4}(2A + B)g^{\mu\nu}$

We can thus rewrite equations (5.18) under the form:

$$p\left\{L_A + (A-\tau)L_{AA} - (A-\tau)L_{AB} + \mu L_{AA}\right\} + q\left\{(A+\tau)L_{AB} - (A-\tau)L_{AA} - \mu L_{AB}\right\} = 0 \qquad (5.23a)$$

$$p\left\{(A-\tau)L_{BA} - (A-\tau)L_{BB} + \mu L_{BA}\right\} + q\left\{L_A + (A+\tau)L_{BB} - (A-\tau)L_{BA} - \mu L_{BB}\right\} = 0 \qquad (5.23b)$$

where

$$4Q - \mu k^2 = 0 \qquad (5.24)$$

The determinantal equation reads:

$$\left[L_A + (A-\tau)L_{AA} - (A-\tau)L_{AB} + \mu L_{AA}\right]\left[L_A + (A+\tau)L_{BB} - (A-\tau)L_{AB} - \mu L_{BB}\right] +$$

$$- \left[(A+\tau)L_{AB} - (A-\tau)L_{AA} - \mu L_{AB}\right]\left[(A-\tau)L_{BA} - (A-\tau)L_{BB} + \mu L_{BA}\right] = 0 \qquad (5.25)$$

This equation is a second degree equation for μ:

$$a\mu^2 + b\mu + c = 0 \tag{5.26}$$

where

$$a = L_{AB}^2 - L_{AA}L_{AB} \tag{5.27a}$$

$$b = L_A(L_{AA} - L_{BB}) - 2a\tau \tag{5.27b}$$

$$c = L_A^2 + (A-\tau)(L_{BB}+L_{AA})L_A + (A-\tau)(L_{AA}-2L_{BA})L_A - 2a\tau(A-\tau) \tag{5.27c}$$

Equation (5.26) has a unique solution if its discriminant vanishes,

$$\Delta = b^2 - 4ac = 0 \tag{5.28}$$

A rather long but straightforward calculation shows that a function L which is a solution of equation (5.28) is given by:

$$L(A,B) = \sqrt{B^2 - 2nA + n^2} \tag{5.29}$$

in which n is an arbitrary constant.

The caracteristic of this Lagrangean is provided by inserting the unique solution μ of (5.26) into (5.24) and using (5.29):

$$\left\{\tau_{\mu\nu} - (2A + B - n)\gamma_{\mu\nu}\right\}k^{\mu}k^{\nu} = 0 \tag{5.30}$$

which corresponds to (3.8).

According to Lax's characterization (see Boillat, op. cit.), the above Lagrangean (5.29) is exceptional if the discontinuities of (5.30) through the hypersurface S vanish identically. We have (compare (5.30) with (3.8) in order to identify $G_{\mu\nu} \equiv \tau_{\mu\nu} - (2A + B - n)\gamma_{\mu\nu}$):

$$[G_{\mu\nu}]k^{\mu}k^{\nu} = 4[A_{\mu\nu}]k^{\mu}k^{\nu} - 2[A]k^{2} \tag{5.31}$$

in which,

$$A_{\mu\nu} \equiv S_{\mu\alpha\beta\lambda}S_{\nu}{}^{\alpha\beta\lambda}$$

From equation (T.1)

$$[A_{\mu\nu}]k^{\mu}k^{\nu} = 4U_{\alpha\beta\lambda}\phi^{\alpha\beta\lambda}k^{2}$$

and from equation (5.21a) it follows that

$$[G_{\mu\nu}]k^{\mu}k^{\nu} = 0 \tag{5.32}$$

identically.

Eq. (5.32) implies that the exceptional waves of this theory are not restricted to the null case $k^2 = 0$.

VI THE WEYL-LIKE CASE

Let us examine Lagrangeans constructed with the tensor $C_{\alpha\beta\mu\nu}$ defined in (2.1).

In this case the possible invariants, D and E, are defined by:

$$D = C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} \tag{6.1a}$$

$$E = \overset{*}{C}_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} \tag{6.1b}$$

In the same way as in the last section, we can construct an arbitrary Lagrangean L(D,E), which yields the following equations of motion:

$$(L_D C^{\alpha\beta\mu\nu} + L_E \overset{*}{C}{}^{\alpha\beta\mu\nu})_{,\nu} = 0 \tag{6.2}$$

We apply condition (5.8) to obtain:

$$L_{D;\nu}C^{\alpha\beta\mu\nu} + L_D C^{\alpha\beta\mu\nu}{}_{;\nu} + L_{E;\nu}\overset{*}{C}{}^{\alpha\beta\mu\nu} + L_E \overset{*}{C}{}^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \tag{6.3}$$

The discontinuity equation is:

$$[L_D]k_\nu C^{\alpha\beta\mu\nu} + L_D[C^{\alpha\beta\mu\nu}]k_\nu + [L_E]k_\nu \overset{*}{C}{}^{\alpha\beta\mu\nu} + L_E[\overset{*}{C}{}^{\alpha\beta\mu\nu}]k_\nu = 0 \tag{6.4}$$

But,

$$[C^{\alpha\beta\mu\nu}]k_\nu = 3\phi^{\alpha\beta\mu}k^2 \tag{6.5}$$

Defining,

$$Y^{\alpha\beta\mu} = C^{\alpha\beta\mu\nu}k_{\nu} \tag{6.6a}$$

$$Z^{\alpha\beta\mu} = \overset{*}{C}{}^{\alpha\beta\mu\nu}k_{\nu} \tag{6.6b}$$

we can rewrite equation (6.3) as:

$$\phi^{\alpha\beta\mu} = aY^{\alpha\beta\mu} + bZ^{\alpha\beta\mu} \tag{6.7}$$

where,

$$3k^2(aL_D - bL_E) + [L_D] = 0 \tag{6.8a}$$

and

$$3k^2[bL_D + aL_E] + [L_E] = 0 \tag{6.8b}$$

Using Leibniz rule, we have:

$$[L_D] = L_{DD}[D] + L_{DE}[E] \tag{6.9a}$$

$$[L_E] = L_{ED}[D] + L_{EE}[E] \tag{6.9b}$$

After a straightforward calculation using the identities (T.4) and (T.5), we obtain:

$$[E] = 2(aE + bD)k^2 \tag{6.10a}$$

$$[D] = 2(aD + bE)k^2 \tag{6.10b}$$

Thus,

$$[L_D] = 2\left\{a(EL_{DE} + DL_{DD}) + b(EL_{DD} - DL_{ED})\right\}k^2 \tag{6.11a}$$

$$[L_E] = 2\left\{a(EL_{EE} + DL_E) + b(EL_{DE} - DL_{EE})\right\}k^2 \tag{6.11b}$$

Using equations (6.12) and (6.9) we obtain:

$$\left\{(3L_D + 2EL_{DE} + 2DL_{DD})a + (2EL_{DD} - 2DL_{DE} - 3L_E)b\right\}k^2 = 0 \tag{6.12a}$$

$$\left\{(2EL_{EE} + 2DL_{ED} + 3L_E)a + (3L_D + 2EL_{DE} - 2DL_{EE})b\right\}k^2 = 0 \tag{6.12b}$$

This system admits linearly independent solutions if the determinantal equation is satisfied:

$$k^4\left\{9(L_D^2+L_E^2)+6(L_{DD}-L_{EE})(DL_D-EL_E)+12L_{DE}(EL_D+DL_E)+4(D^2+E^2)(L_{DE}^2-L_{DD}L_{EE})\right\}=0 \tag{6.13}$$

There are two classes of solutions for this equation

i)

$$9(L_D^2+L_E^2)+ 6(DL_D-EL_E)(L_{DD}-L_{EE}) + 12L_{DE}(EL_D+DL_E) + 4(D^2+E^2)(L_{DE}^2-L_{DD}L_{EE})\neq 0 \tag{6.14}$$

and

$$k^2 = \gamma_{\mu\nu} k^\mu k^\nu = 0 \tag{6.15}$$

ii)

$$9(L_D^2+L_E^2) + 6(DL_D-EL_E)(L_{DD}-L_{EE}) + 12L_{DE}(EL_D+DL_E) + 4(D^2+E^2)(L_{DE}^2-L_{DD}L_{EE})=0 \tag{6.16}$$

In the first case, there is a set of exceptional Lagrangeans because the equation

$$[\gamma_{\mu\nu}]k^\mu k^\nu = 0 \tag{6.17}$$

is identically satisfied ($\gamma_{\mu\nu}$ is continuous).

In this case, one of the possible solutions is provided precisely by the Born-Infeld type Lagrangean

$$L = \sqrt{E^2 - 2qD + q^2} - 1 \tag{6.18}$$

This Lagrangean has as first approximation

$$L = qD = qC^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu}$$

which is nothing but the Lagrangean (2.2) used in the linear theory.

From (6.15), it can be seen that the exceptional waves coming from Lagrangean (6.18) have $k^2 = 0$.

VII CONCLUSION

Using the Lax method described in section II we have obtained non-linear Lagrangeans for the Fierz field which are exceptional. They are similar to the Born-Infeld Lagrangean for Electrodynamics and may describe spin-2 dynamics with self-interaction. The Lagrangean constructed with the invariants $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ and $C_{\alpha\beta\mu\nu}\overset{*}{C}^{\alpha\beta\mu\nu}$ given by equation (6.18) has their exceptional waves with $k^2 = 0$ and its first approximation corresponds to the Lagrangean that describes the theory of spin-2 fields in terms of the Fierz variable, which is equivalent to the linear theory in terms of the standard variable $\phi^{(1)}_{\mu\nu}$. Whether this theory may describe real Gravity is an open question. We know that the Fierz variable is the linear approximation[1] of the Lanczos potential[10] of the Weyl tensor which can be used to describe Gravity via Jordan-Lichnerowicz equations[8,9]. However, we do not know the explicit relation of the Lanczos potential with the metric tensor of the curved space and, as a consequence, we do not know how to express Lagrangean (6.19) in terms of geometrical quantities (we cannot compare, e.g., the series expansion of (6.19) with the expansion of the Lagrangean $-\sqrt{-g}\, W_{\alpha\beta\mu\nu}W^{\alpha\beta\mu\nu}$, where $W_{\alpha\beta\mu\nu}$ is the Weyl tensor, because it depends on the metric tensor $g_{\mu\nu}$). We could also fix $A_{\alpha\beta\mu}$ as a fundamental variable and obtain the tensor $\phi_{\mu\nu}$ by means of a non linear relation involving covariant derivatives of the Fierz field which satisfy equation (2.7) in the linear approximation. Using this relation together with the equations of motion for $A_{\alpha\beta\mu}$, the goal would be to obtain equations of motion for $\phi_{\mu\nu}$ which would have as first integrals the GR equations of motion in the form presented in reference (10). However, this is not a straightforward task. We leave this for a future work. Moreover we could identify the geometry in terms of Fierz variables trough the interaction between the spin-2 field and matter[11].

These are open questions.

TABLE I

Properties of Riemann's and Weyl's Tensors

$$R_{\mu\alpha\beta\sigma}R_{\nu}^{\ \alpha\beta\sigma} = \frac{1}{4}\,(A - \tau)g_{\mu\nu} + \tau_{\mu\nu} \qquad \text{(T.1)}$$

$$R_{\mu\alpha\beta\sigma}\overset{**}{R}_{\nu}^{\ \alpha\beta\sigma} = -\frac{1}{4}\,(A - \tau)g_{\mu\nu} \qquad \text{(T.2)}$$

$$R^{*}_{\ \mu\alpha\beta\sigma}\overset{*}{R}_{\nu}^{\ \alpha\beta\sigma} = -\frac{1}{4}\,(A - \tau)g_{\mu\nu} + \tau_{\mu\nu} \qquad \text{(T.3)}$$

$$R_{\mu\alpha\beta\sigma}R_{\nu}^{\ *\alpha\beta\sigma} = \frac{1}{4}\,Cg_{\mu\nu} + 2R_{\mu\alpha\nu\beta}R^{\alpha\beta} \qquad \text{(T.4)}$$

where, $C = R_{\mu\alpha\beta\sigma}\overset{*}{R}^{\mu\alpha\beta\sigma}$

$$W_{\mu\alpha\beta\sigma}W_{\nu}^{\ \alpha\beta\sigma} = \frac{1}{4}\,Dg_{\mu\nu} \qquad \text{(T.5)}$$

$$W_{\mu\alpha\beta\sigma}\overset{*}{W}_{\nu}^{\ \alpha\beta\sigma} = \frac{1}{4}\,Eg_{\mu\nu} \qquad \text{(T.6)}$$

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