ISSN 0029-3865

CBPF-NF-054/89

POLYCHROMATIC MAJORITY MODEL: CRITICALITY AND REAL SPACE RENORMALIZATION GROUP

by

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ABSTRACT

A generalization of a simple-majority rule model is presented. The system, say a d-dimensional hypercubic checkerboard, whose elements are coloured with one out of q colours with probabilities $p_1, p_2, \ldots, p_q$, presents a continuous phase transition. Using a real space renormalization group (RG) approach, we establish the phase diagram as well as the correlation length critical exponent ν. The various types of convergence of the RG numerical values for ν towards the (presumably) exact answer are analysed in connection with finite size scalings.

Key-words: Majority model; Renormalization group; Phase diagram; Finite size scaling.

In this work we present a generalization of a majority-rule model developed by one of us [1]. We consider a majority model on say a d-dimensional hypercubic checkerboard (simply "checkerboard" from now on) whose elements are coloured with one out of q colours with probabilities p_1, p_2,......, p_q respectively ($\sum_{i=1}^{q} p_i = 1$). We arbitrarily choose a colour which will de referred as the first one; we then consider a cluster of elements of the checkerboard and check if there is a simple majority of the first colour. If this is the case we choose, at random, a larger cluster: if the majority of the first colour is still preserved we keep increasing the cluster size until a different colour achieves the majority. The mean size of the cluster size ξ at which the majority was shifted to other colours is then calculated. For small values of p_1, finite values of $\xi(\{p_i\})$ are expected and a divergence will show up for increasing values of p_1. This threshold defines the critical frontier (critical point if $q = 2$, critical line if $q = 3$, etc). The divergence of ξ at the critical frontier is presumably given by $(p^* - p_1)^{-\nu}$ and we want to calculate the critical exponent ν. More specifically, if $q = 2$, then $p_1^* = 1/2$ and $\nu = 2/d$ [1]. For $q = 3$, the critical frontier is given by $p_1 + p_2/2 = 1/2$ if $p_2 \leq p_1$, and by $p_1/2 + p_2 = 1/2$ if $p_1 \leq p_2$. It can be shown within the RG framework that along these lines the critical exponent is the same as the one obtained for the $q = 2$ case, excepting for the particular point $p_1 = p_2 = p_3 = 1/3$, where a possibly new value of ν is expected. In fact it will be shown that the numerical results suggest $\nu = 2/d$ for any point on the critical frontier for any q.

Notice that, for a given value of q, taking the probability of one of the colours p_j equal to zero leads to the (q-1) model.

We will use the real space RG technique to calculate the critical point $P^* = (p_1^*, p_2^*, \ldots, p_q^*)$ (the point which is common to all the branches of the critical frontier) and the corresponding critical exponent ν. Based on these calculations we will discuss the convergence of ν to the exact value on the frame of finite scaling hypothesis.

In order to define the RG transformation we take a cluster of side length b (in elementary cell units) so the cluster has $b^d = a$ unitary cells, and consider the probability of having a majority of cells of the first colour. The probabilities of the configurations for which the first colour shares the majority with r other colours are weighted with a factor $(r+1)^{-1}$. In this way we have an unambiguous assignation for p_1' (probability associated with the cluster with first colour majority)

$$p_1' = R_1^a(\{p_i\}) \equiv \sum \frac{a!\, p_1^{n_1} p_2^{n_2} \ldots p_q^{n_q}}{n_1!\, n_2! \ldots n_q!} W(n_1, n_2, \ldots, n_q) \qquad (1)$$

where the sum is over $n_1, n_2, \ldots, n_q$ running over non-negative integers with the constraints

$$n_1 + n_2 + \ldots + n_q = b^d = a \qquad (2.a)$$

$$n_1 \geq n_j \,, \quad (j=2,3, \ \ ,q) \qquad (2.b)$$

and $W(n_1, n_2, \ldots, n_q)$ is defined as follows

$$W(n_1,n_2,\ldots n_q) = \begin{cases} 1 & \text{if } n_1 > n_j \quad (j=2,3,\ldots,q) \\ (r+1)^{-1} & \text{if the first colour shares} \\ & \text{majority with } r \text{ other colours} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Notice the symmetry of $W(n_1,n_2,\ldots,n_q)$ under permutations among the last $(q-1)$ variables $n_2, n_3,\ldots,n_q$.

In analogous form the renormalized probabilities for the other colours are

$$p_j' = R_j^a(\{p_i\}) \equiv \sum \frac{a!\, p_1^{n_1} p_2^{n_2} \ldots p_q^{n_q}}{n_1!\, n_2! \ldots n_q!} W(n_j,n_1,\ldots) \quad (j \neq 1) \tag{4}$$

also with the constraint (2.a).

Before we go on we consider, as an example, the case $q = 3$. Let us refer to the colours black, white and red and their probabilities $p_1 = p_B$, $p_2 = p_W$ and $p_3 = p_R$. For a given value of a, the renormalized probability p_B' is given by

$$p_B' = \sum \frac{a!\, p_B^{n_1}\, p_W^{n_2}\, p_R^{n_3}}{n_1!\, n_2!\, n_3!} W(n_1,n_2,n_3) \tag{5}$$

with

$$W(n_1,n_2,n_3) = \begin{cases} 1 & \text{if } n_1 > n_2,\ n_3 \\ 1/2 & \text{if } n_1 = n_2 > n_3 \text{ or } n_1 = n_3 > n_2 \\ 1/3 & \text{if } n_1 = n_2 = n_3 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

and the constraint $n_1 + n_2 + n_3 = a$.

In the particular case $a = 3$, Eq.(5) becomes

$$p'_B = p_B^3 + 3\, p_B^2\,(1-p_W) + 3\, p_B^2\,(1-p_W) + 2\, p_B\, p_W\, p_R \qquad (7)$$

The p'_W and p'_R are obtained anologously. The RG transformation presents a fully unstable fixed point $P^* = (1/3,1/3,1/3)$ and three semi-stable fixed points (1/2,1/2,0), (1/2,0,1/2), and (0,1/2,1/2). The phase diagram and the RG critical frontier is shown in Fig (1). The frontier separates the black, white and red prevalent colour regions.

We return now to the general case of q colours. Eqs (1) and (4) define the RG transformation from probabilities of elementary cells to a cell $b^d = a$ times larger. There are $2^q-(q+1)$ non-trivial fixed points, one of which, $P^* = (1/q,1/q, \ldots,1/q)$ is fully unstable, the rest of them being semi-stable. The fixed point P^* is at the geometrical center of a hypertetrahedron in a (q-1)-dimensional space and the semi-stable ones are located at the centers of the faces and of the edges.

We will study the linearized RG recursion relations in the neighborhood of P^*. At the fixed point Eq.(1) yields

$$\left(\frac{1}{q}\right)^{a-1} = \sum \frac{a!\, W(n_1,n_2 \ldots, n_q)}{n_1!\; n_2! \ldots\; n_q!} \qquad (8)$$

The derivative of p'_i with respect to p_j is

$$\frac{\partial p_1'}{\partial p_j} = \sum \frac{a!\; p_1^{n_1}\cdots p_j^{n_j-1}\cdots p_q^{n_q-1}}{n_1!\; n_2!\cdots n_q!} (n_j p_q - n_q p_j)\; W(n_1, n_2, \ldots, n_q)$$

$$\xrightarrow[p \to p^*]{} \sum \frac{a!\; W(n_1, n_2 \ldots, n_q)}{n_1!\; n_2!\cdots n_q!} \left[\frac{1}{q}\right]^{a-1} (n_j - n_q) = 0 \quad (j = 2, \ldots q) \tag{9}$$

This expession vanishes due to the symmetry of $W(n_1, n_2, \ldots, n_q)$, therefore the linearized transformation at the critical point P^* is diagonal. Since all colours play the same role, the diagonal elements coincide and are given by

$$\left.\frac{\partial p_1'}{\partial p_1}\right|_{P^*} = \frac{a!\; (n_1, n_2, \ldots, n_q)}{n_1\; n_2!\cdots n_q!} \left[\frac{1}{q}\right]^{a-1} (n_1 - n_q)$$

$$= \frac{q}{(q-1)} \sum \frac{a!\; W(n_1, n_2, \ldots, n_q) n_1}{n_1!\; n_2!\cdots n_q!} \left[\frac{1}{q}\right]^{a-1} - \frac{a}{(q-1)} \equiv \lambda_a \tag{10}$$

From this eigenvalue we obtain the critical exponent

$$\nu_{a,1} = \frac{\ln(b)}{\ln(\lambda_a)} = \frac{1}{d}\,\frac{\ln(a)}{\ln(\lambda_a)} \tag{11}$$

The RG transformation, Eqs. (1) and (4), define a change of scale with a factor b. Through this transformation we may define a recursive relation between the probabilities assigned to a cluster of side length b' ($a' = b'^d$) and that assigned to the cluster of side length b, namely

$$p_j^{a'} = R_j^{a'}(P) = R_j^{a'}(R^{a-1}(p^a)) \qquad (12)$$

It follows the critical enponent $\nu_{a,a'}$ given by

$$\nu_{a,a'} = \frac{1}{d}\frac{\ln(a/a')}{\ln(\lambda_a/\lambda_{a'})} \qquad (13)$$

We now pass to discuss the RG results we have obtained for the critical exponent ν.

First of all we should notice (see Eq.(11)), that the dimensionality d of the problem always enters in the same way in the expression for ν. We can as well think that theclusters of elements of the checkerboard are arranged in a one dimensional array of length a (d = 1). In Ref[1] is proved that $\nu d = 2$ is the exact answer. Besides this, the RG scheme shows that the $a \to \infty$ asymptotic behaviour of $(\nu_{a,1} - \nu_{exact})$ is of the form $1/\ln(a)$, whereas the behaviour of $(\nu^{-1}_{exact} - \nu^{-1}_{a,a-2})$ is of the form $1/a$.

Our calculations for $q > 2$ strongly suggest that the exact value is $\nu d = 2$ for any q. The asymptotic behaviour of $\nu_{a,1} - \nu_{exact}$ is also of the form $1/\ln(a)$ as can be seen in Fig.(3).

The finite size scaling hypothesis [2,3] would indicate that faster convergence is obtained when we consider the asymptotic sequence $\nu_{a,a'}$ with a' as close to a as possible. Due to the way through which renormalization procedure treats the case in which the majority is shared by two or more colours, an oscillation of period q is shown in $\nu_{a,1}$ as a function of a. These oscillations become less important as a goes to infinity: see Fig.(2). This effect is rather dramatic in the case q = 2 since an infinite value of $\nu_{a,a-1}$ is obtained if a equals an even integer, though a good convergent sequence is obtained if we consider $\nu_{a,a-2}$ for any parity of a. In general, we have almost always considered sequences of the form $\nu_{a,a-q}$ since they exhibit smooth convergences to the exact value of ν.

Fig.(3) shows, for different values of q, the numerical results of $\nu_{a,1}$ as functions of 1/ln(a). The plot of $(\nu^{-1}_{exact} - \nu^{-1}_{a,a-q})$ vs a in log-log scale is shown in Fig.(4). As we have already mentioned, the case q = 2 is well fitted with a slope (-1) straight line, whereas for all the other cases (q = 3, 4, 5, 6) the slopes are in the interval (-0.52,-0.44). This sudden change in the asymptotic behaviour of $\nu^{-1}_{a,a-q}$ with q > 2 still requires further study. It might be related to the fact that, whenever q > 2 , an important degeneracy is present. For example, for q = 3 , a 55% presence of black cells can be obtained in many manners by varying the frequency of white and red in the 45% of non-black cells. Analytical solutions of the present mathematically simple model would be very welcome. Also, a further generalization of the present model which would use higher than simple majority (and would consequently present hysteresis) would be interesting to consider.

One of us (D.P.) has been partially supported by CONICET (Argentina) and has benefitted from warm hospitality at the

Centro Brasileiro de Pesquisas Fisicas as well as from discussions with F.A. Tamarit and S. Cannas. Another one (C.T.) acknowledges useful discussion with D. Stauffer. Computational assistance by G. Schwachheim and M.C.S. Izzo is gratefully acknowledged.

Captions for figures

Fig. 1 : Phase diagram for the case q = 3. B, W and R respectively refer to the black, white and red colours. ■ , ● and ○ denote fully stable, semi-stable and fully unstable fixed points. The arrows indicate the RG flow.

Fig. 2 : The critical exponents $\nu_{a,1}$ as a function of a for the case q = 3. Notice the oscillations with period 3.

Fig. 3 : The critical exponents $\nu_{a,1}$ as a function of 1/ln(a) for typical values of q.

Fig. 4 : RG numerical results for ($\nu^{-1}_{exact} - \nu^{-1}_{a,a-q}$) vs a for typical values of q. The straight lines are the fitted $a \to \infty$ asymptotic behaviour.

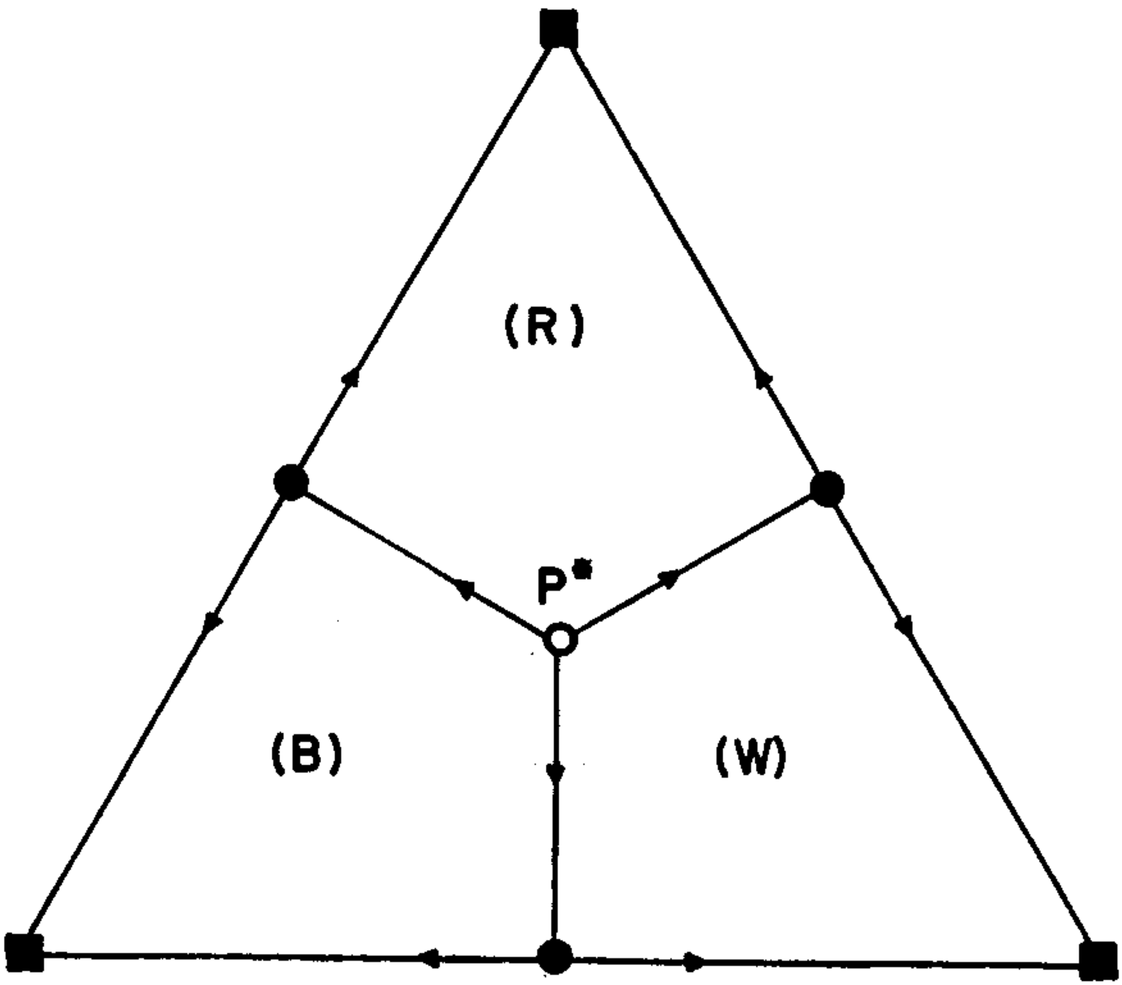

FIG.1

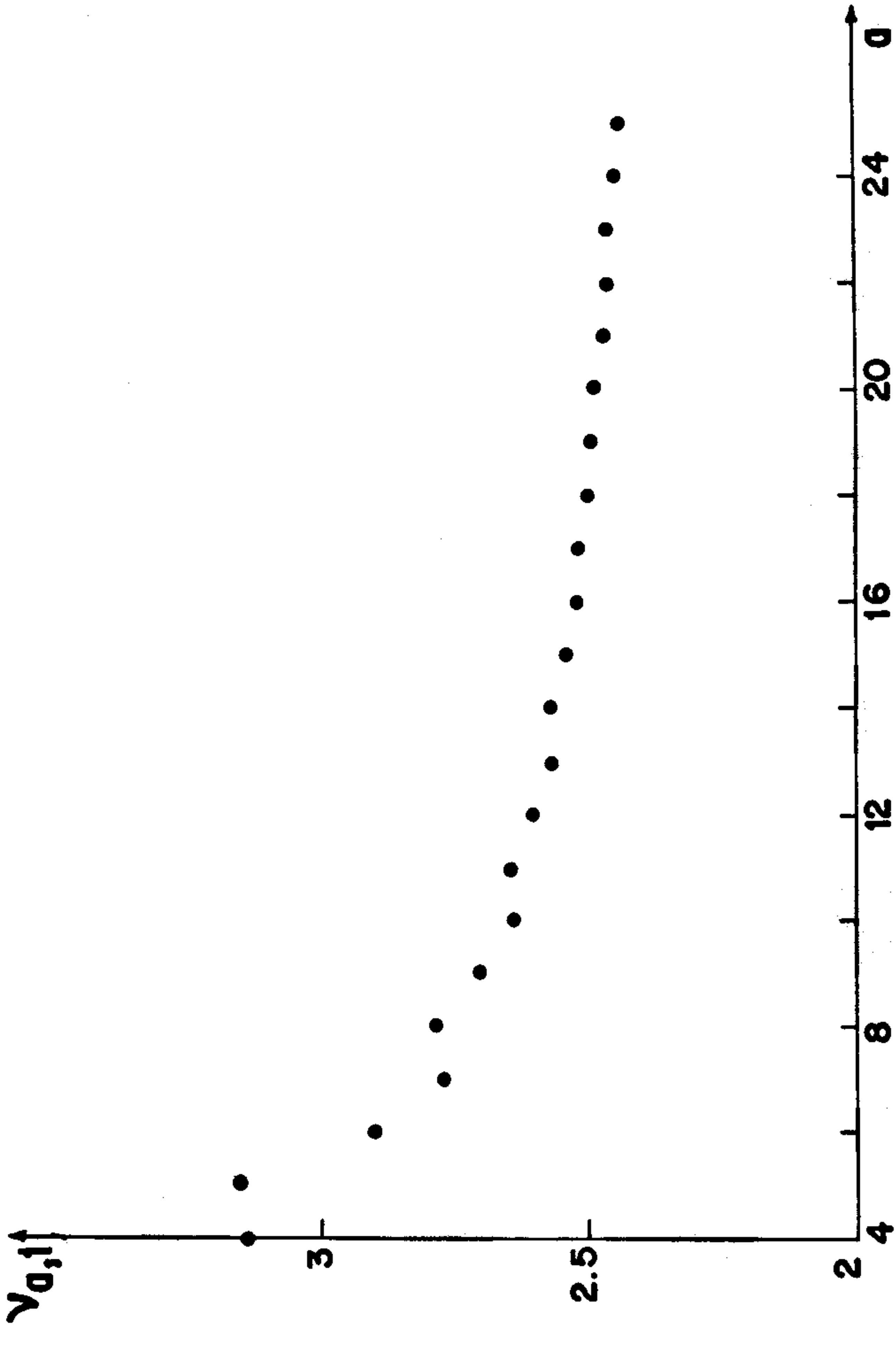

FIG.2

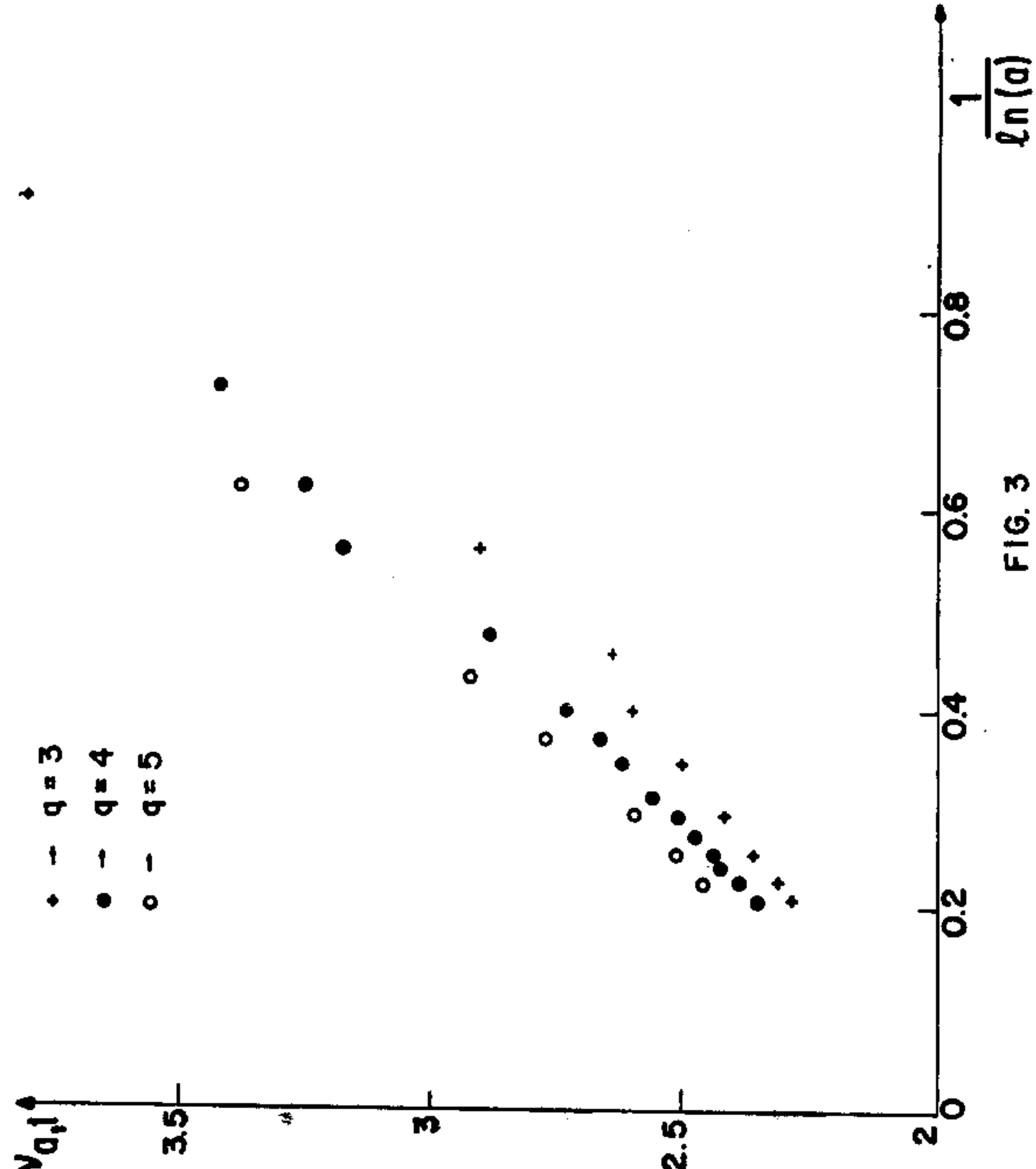

FIG. 3

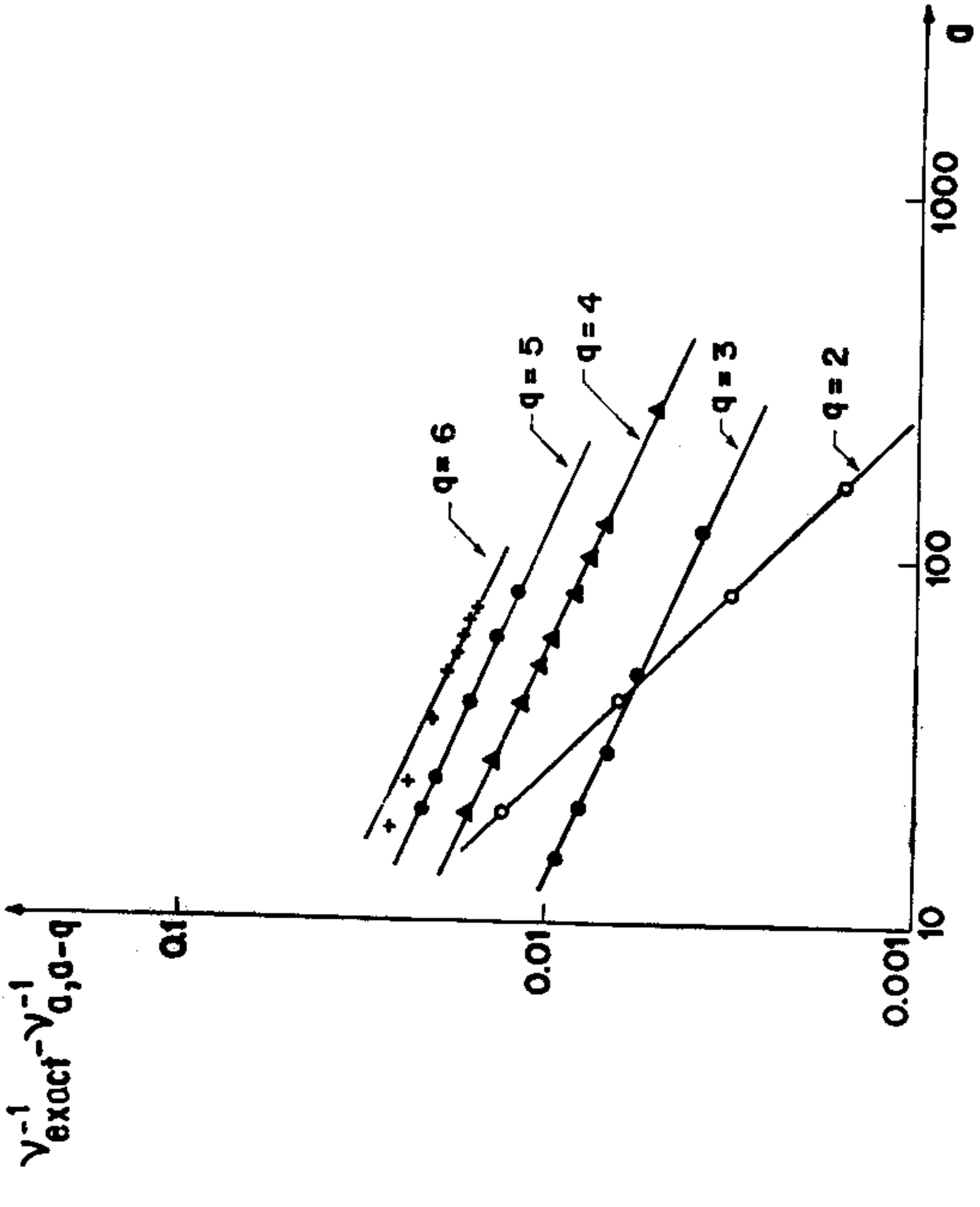

FIG.4

REFERENCES

[1]. Tsallis C., J. Physique Lettres 43 L-471 (1982).

[2]. Fisher M. E. in Proceedings of the International Summer School Enrico Fermi 1970, Course 51, Varenna, Italy, ed. by M. S. Green (Academic, New York) 1971.

[3]. Derrida, B, de Seze L. and Vannimenus, J., Disordered Systems and Localization (Conference Rome 1981), Lecture Notes in Physics 149, 46 by C. Castellani, C. Di Castro and L. Peliti (Berlin, Springer-Verlag).