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SUSCEPTIBILITY OF THE POTTS MODEL IN AN HIERARCHICAL LATTICE: RENORMALISATION GROUP APPROACH

by

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ABSTRACT

Using a real space renormalisation group method, we calculate the thermal dependence of the susceptibility of the q-state Potts model (ferro-and antiferromagnet) on self-dual Wheatstone-bridge-like hierarchical lattices. The influence of external fields on the antiferromagnetic phase diagram is discussed as well.

Key-words: Potts model; Hierarchical lattices; Renormalisation group; Thermal susceptibility.

I INTRODUCTION

Real space renormalisation group (RG) methods have been widely used to study critical properties (phase diagrams and critical exponents among others) of various systems. In particular, some RG formalisms [1,2] enable the calculation of the relevant thermodynamic energy for arbitrary values of external parameters such as temperature and magnetic field. Through appropriate derivatives of this energy we can obtain the equation of states, specific heat, susceptibility, etc. It has been recently introduced [3,4] a new RG procedure which enables the *direct* calculation of the equation of states in the absence of external fields.

Following along the lines of Caride and Tsallis [4], we extend here the above direct procedure in order to include the case where external fields are present. We apply this formalism to the study of the q-state Potts model in (self-dual Wheatstone--bridge-like) hierarchical lattices. In particular, we study the susceptibility (and its related critical exponent γ) associated with both ferro (arbitrary values of q) and antiferromagnetic (q=2) cases, as well as the phase diagram corresponding to the $q = 2$ antiferromagnetic case, with special emphasis given to the role played by the weights through which the external field is taken into account.

In Section II we present the RG formalism and its application to the Potts model; in Section III, we present the results, and finally conclude in Section IV.

II MODEL AND FORMALISM

Let us first consider a d-dimensional hypercubic lattice of linear size L, whose first-neighbouring sites interact (say ferromagnetically) through a dimensionless coupling constant $K \equiv J/k_BT$. The order parameter M can be defined as $M = N_L(K)/L^d$ in the thermodynamical limit $L \to \infty$, where $N_L(K)$ is the thermal canonical average number of sites whose spin is pointing in the easy magnetisation direction (e.g.,one of the q states of the Potts model, say the state $\sigma_i = 0$) minus those whose spin is pointing in any other possible direction.

Following Kadanoff we divide the system of L^d sites into a system of L'^d cells of linear size $B = L/L' > 1$. Through rescaling, the total magnetic momentum of the system must be preserved since it is an extensive quantity. Then, by associating an elementary dimensionless magneton μ with each site of the lattice, we have

$$N_{L'}(K')\mu' = N_L(K)\mu \tag{1}$$

where K' and μ' denote the renormalised variables. Dividing now both sides of Eq. (1) by L^d and performing n iterations we obtain, in the limit $n \to \infty$,

$$M(K^{(o)}) = \lim_{n\to\infty} \frac{M(K^{(\infty)})\mu^{(n)}}{B^{nd}} \tag{2}$$

where we have arbitrarily chosen $\mu^{(o)} = 1$. This formula has to be used together with the (standard) RG recurrence for the coupling constant, namely

$$K' = f(K) \tag{3}$$

This recursive relation normally admits three fixed points, namely
(i) $K = 0$, which characterizes the paramagnetic phase;
(ii) $K = \infty$, which characterizes the ferromagnetic phase;
(iii) $K = K_c$, unstable fixed point indicating the critical point.

In general, when $K < K_c$, $K^{(\infty)}$ vanishes hence $M(K^{(\infty)}) = 0$. This leads, through Eq. (2), to $M(K^{(o)}) = 0$ as desired. On the other hand, when $K > K_c$, $K^{(\infty)}$ diverges hence $M(K^{(\infty)}) = 1$ (conventional value for $T = 0$), and consequently

$$M(K^{(o)}) = \lim_{n\to\infty} \frac{\mu^{(n)}}{B^{nd}} \tag{4}$$

This formula provides the thermal dependence of the order parameter in the non-trivial region (i.e., $T < T_c$) as soon as we have established a recursive relation for μ. In what follows we describe how to establish such relation by using, as an illustration, the transformation indicated in Fig. 1(a) (which generates the Wheatstone-bridge hierarchical lattice).

The procedure follows along two steps, namely

(i) In order to establish the equation for the order parameter, we shall break the symmetry. To do so, we impose that the spin of say terminal 1 (of both small and large graphs of Fig. 1(a)) be along the easy magnetisation direction, while the rest of the spins (all the internal spins as well as that of terminal 2) are left free to take all possible orientations (q states for the Potts model);

(ii) Since an hierarchical lattice is not translationally invariant an inhomogeneity must be present in the Gibbs probability distribution [5]. It seems very reasonable to us that the order parameter is inhomogeneous too [4]. In fact these inhomogeneities have been recently exhibited [6] on this type of hierarchical lattices. Here we assume the order parameter to be proportional to the coordination number at any given site. Although this is not strictly true [6], it seems to be a good approximation (in fact, it might be strictly true in average). Also, we shall consistently assume that the relevant (inhomogeneous) external field is proportional to the coordination number as well.

To implement the calculation each cluster configuration is weighted by the corresponding Boltzmann factor and is associated with a value for the cluster magnetic momentum m (each spin contributes, to m, proportionally to its coordination number). We then impose (analogously to what we did in Eq. (1))

$$\langle m \rangle_{\substack{\text{small}\\ \text{cluster}}} = \langle m \rangle_{\substack{\text{large}\\ \text{cluster}}} \tag{5}$$

where $\langle ... \rangle$ denotes thermal canonical average. This equation has the form

$$u(K')\mu' = v(K)\mu \tag{6}$$

where u(K') and v(K) are the explicit functions that appear when we impose Eq. (5).

The recurrence relation between K and K' is established in a standard way namely by preserving the correlation function between the roots (terminals 1 and 2 of Fig. 1) of both graphs. In other words, we impose (say for Fig. 1(a))

$$e^{-\beta \mathcal{H}'_{12}} = \operatorname*{Tr}_{\{\sigma_3,\sigma_4\}} e^{-\beta \mathcal{H}_{1234}} \tag{7}$$

where $\mathcal{H}'_{12}$ and $\mathcal{H}_{1234}$ are the Hamiltonians respectively associated with the small and large clusters. This equation yields the recurrence we were looking for, namely Eq. (3).

Summing up, Eqs. (6) and (3) give the renormalisation in the (K,μ) space, which in turn enables the calculation of the magnetisation through Eq. (4).

For clarification, let us illustrate on Fig. 1(a) various relevant quantities appearing in the calculation. The chemical distances between the terminals of the large and small graphs are respectively b = 2 and b' = 1, hence the scaling factor is given by B = b/b' = 2. The intrinsic fractal dimensionality of the hierarchical lattice is given (see [7] and references therein) by $d_f = \ell n\ 5/\ell n\ 2$, consequently the denominator appearing in Eq. (4) is given by $B^{d_f} = 5$.

The extension of the above procedure to non vanishing external field is straightforward. Eq. (6) is generalised into

$$u(K',H')\mu' = v(K,H)\mu \tag{8}$$

and Eq. (3) is generalised into

$$K' = f(K,H) \tag{9}$$

$$H' = g(K,H) \tag{10}$$

where the dimensionless field H is given by $H \equiv \beta h$ ($h \equiv$ external field). We can verify of course that $u(K',0) = u(K')$, $v(K,0) = v(K)$, $f(K,0) = f(K)$ and $g(K,0) = 0$. Eqs. (8), (9) and (10) together with Eq. (4) yield the equation of states $M(K,H)$. Through derivation with respect to H we straightforwardly obtain the isothermal susceptibility χ.

Let us now return to Fig. 1(a). The dimensionless Hamiltonians are given by

$$-\beta\mathcal{H}'_{12} = qK'\delta_{\sigma_1,\sigma_2} + H' \sum_{i=1}^{2} \frac{q\delta_{\sigma_i,0}-1}{q-1} + K'_0 \tag{11}$$

$$-\beta\mathcal{H}_{1234} = qK \sum_{<i,j>=1}^{5} \delta_{\sigma_i,\sigma_j} + H \sum_{i=1}^{4} C_i \frac{q\delta_{\sigma_i,0}-1}{q-1} \tag{12}$$

where $\sigma_i = 0,1,\ldots,q-1$, K'_0 is an additive constant (necessary to satisfy Eq. (7)) and $\{C_i\}$ are the coordination numbers ($C_1 = C_2 = 2$, $C_3 = C_4 = 3$). We have introduced, in Eqs. (11) and (12), the random variables which lead to the Potts model order parameter $(q <\delta_{\sigma,0}>-1)/(q-1)$. The functions $u(K',H')$ and $v(K,H)$ are determined in Table 1. Eqs. (9) and (10) become

$$K' = \frac{1}{q}\left(\ell n \frac{\sqrt{R_{00}R_{11}}}{R_{01}}\right) \tag{13}$$

and

$$H' = \frac{q-1}{2q}\ \ell n \frac{R_{00}}{R_{11}} \tag{14}$$

where

$$R_{00} \equiv e^{5qK+2(C_1+C_3)H} + 2(q-1)e^{2qK+(2C_1+C_3-\frac{C_3}{q-1})H} + \\ + (q-1)\left[e^{qK}+q-2\right]e^{2(C_1-\frac{C_3}{q-1})H} \tag{15.a}$$

$$R_{01} \equiv e^{3qK+(C_1+2C_3-\frac{C_1}{q-1})H} + 2\left[e^{2qK}+(q-2)e^{qK}\right]e^{(C_1+C_3-\frac{C_1}{q-1}-\frac{C_3}{q-1})H} \\ +\left[e^{3qK}+3(q-2)e^{qK}+(q-2)(q-3)\right]e^{H(C_1-\frac{C_1}{q-1}-\frac{2C_3}{q-1})} \tag{15.b}$$

$$R_{11} \equiv e^{qK+H(-\frac{2C_1}{q-1}+2C_3)} + 2\left[e^{2qK} + q - 2\right] e^{H(-\frac{2C_1}{q-1} - \frac{C_3}{q-1} + C_3)}$$
$$+ \left[e^{5qK} + 2(q-2)e^{2qK} + (q-2)e^{qK} + (q-2)(q-3)\right] e^{\frac{-2H}{q-1}(C_1+C_3)} \tag{15.c}$$

Before closing this section, let us go back to the general case and establish an useful formula for the exponent γ. By using Eqs. (4) and (8) we obtain

$$\frac{M(K',H')}{M(K,H)} = \frac{B^d}{\ell(K,H)} \tag{16}$$

where

$$\ell(K,H) \equiv \frac{v(K,H)}{u(f(K,H),g(K,H))} \tag{17}$$

Differentiating Eq. (16) with respect to H and then taking $H = 0$ we obtain

$$\frac{\chi(K')}{\chi(K)} = \frac{B^d}{\ell(K,0)\left.\frac{\partial g}{\partial H}\right|_{H=0}} . \tag{18}$$

In the neighbourhood of the critical point K_c we have

$$\frac{\chi(K')}{\chi(K)} \sim \left(\frac{K' - K_c}{K - K_c}\right)^{-\gamma} \tag{19}$$

Since $(K'-K_c)/(K-K_c) = \left[\partial f(K,0)/\partial K\right]_{K_c}$ we have

$$\left[\frac{\partial f(K,0)}{\partial K}\right]_{K_c}^{-\gamma} = \frac{B^d}{\ell(K_c,0)\left.\frac{\partial g}{\partial H}\right|_{\substack{K=K_c\\H=0}}} \tag{20}$$

hence

$$\gamma = \frac{\ell n\left\{\frac{\ell(K_c,0)\left.\frac{\partial g}{\partial H}\right|_{\substack{K=K_c\\H=0}}}{B^d}\right\}}{\ell n\left\{\left.\frac{\partial f(K,0)}{\partial K}\right|_{K=K_c}\right\}} \tag{21}$$

III RESULTS

III.1 Potts ferromagnet

For calculating the susceptibility of the q-state Potts ferromagnet we have used the RG transformation indicated in Fig. 1(a), and have assumed that the external fields are proportional to the coordination numbers [2,8,9]. We present in Fig. 2(a), the $q = 2$ results (both ordered and disordered phases) for typical values of H, and, in Fig. 2(b), the $H = 0$ results for typical values of q. We present, in Fig. 3, the q-dependence of γ. In particular, for $q = 2$, we obtain $\gamma = 2.31$ which reproduces the value associated by Melrose [8] with the Wheatstone-bridge hierarchical lattice.

Let us now consider the Rushbrooke scaling law, namely $\alpha + 2\beta + \gamma = 2$, where α and β are the specific heat and magnetisation critical exponents. We want to check whether the present calcu lation satisfies this scaling relation. We obtain $\alpha(q)$ from the relation $2 - \alpha(q) = d_f \nu(q)$ where $d_f = \ell n5/\ell n2$ and $\nu(q)$ has been calculated in [4] and references therein. We used $\beta(q)$ obtained in [4], and $\gamma(q)$ from the present calculation. The results are presented in Fig. 4. We note, for values of $q \neq 2$, a slight violation of Rushbrooke equality. The reason for this is not clear but it could be related to the (strictly not true) assumption we have done that the local magnetisations are proportional to the coordination numbers. In any case, the present result cannot be considered as a proof that the Rushbrooke equality is indeed violated for hierarchical lattices.

III.2 Ising antiferromagnet

To discuss the $q = 2$ antiferromagnet we have used the RG transformation indicated in Fig. 1(b) [10]. Indeed, Fig. 1(a) does not preserve the ground state of an antiferromagnet (the chemical distance between the terminals of Fig. 1(a) is an even number, whereas that of Fig. 1(b) is an odd one). We present in Fig. 5 the susceptibility in the paramagnetic phase for typical values of H. To perform the calculations we have used the T vs. H phase diagram indicated in Fig. 6 [11]. To calculate the susceptibility in the ordered phase one should know the associated equation of states, which has not been calculated herein. If we denote by T_m the temperature at which a maximum occurs for the zero field susceptibility, our results yield $T_m/T_c \simeq 1.5$ which approximately coincides with the square lattice series result by Fisher and Sykes [12].

We want now to exhibit the influence of the local weights within the present RG procedure. To do this we shall study the T vs. H phase diagram. We shall attribute the weights (C_1, C_2, C_3) to the sites of the cluster as indicated in the Fig. 1(b). The phase diagram corresponding to the assumption of proportionality of the local external fields with the coordination numbers (i.e., $(C_1, C_2, C_3) = (3,2,4)$) is shown in Fig. 6, and the phase diagrams corresponding to other typical choices are shown in Fig. 7 together with their RG flows.

The T = 0 point of the critical line in the T vs. H plane can be obtained either through a $T \to 0$ (numerically rough) extrapolation of the $T \neq 0$ line, or through direct evaluation by considering the energies of the possible ground states. These two procedures should yield the same value. They do so within the present RG framework, only for $(C_1, C_2, C_3) = (3,2,4)$, which is consistent with the analogous proportionality assumption we have done for the local magnetisations.

IV CONCLUSION

A real space renormalisation group scheme has been formulated which, for the first time avoiding the calculation of the thermodynamical energy, enables a simple study of the magnetic susceptibility (and its critical exponent γ). The method is based on the inspection of the spin configurations of small clusters. Opera tionally speaking, the calculation is as simple as a mean field one while providing, for arbitrary temperatures and fields, non trivial results which can be systematically improved (by enlarging the cluster).

We have applied the procedure to the calculation of the susceptibility and γ of the Wheatstone-bridge hierarchical lattice by assuming ferromagnetic q-state Potts interactions. Differently from the Diamond hierarchical lattice case [13], in this lattice the divergence of the susceptibility occurs only at the critical temperature. The validity of the Rushbrooke equality has been focused. For $q = 2$ we recover a value for γ available in the literature, which is considered to be exact for the hierarchical lattice.

In addition to that, we have calculated the susceptibility associated with the paramagnetic phase of the $q = 2$ antiferromagnet as well as the corresponding critical line in the (T,H) plane. The influence of the weights of the local fields has been exhibited as well.

We acknowledge fruitful discussions with A.O. Caride as well as interesting remarks from E.M.F. Curado, A.M.N. Chame, M. L. Martins, S.A. Cannas and H.F.V. Resende. Furthermore, one of us (EPS) has benefitted from computational advice from A.O. Caride as well as from a CAPES/Brazil Fellowship.

CAPTION FOR FIGURES AND TABLE

Figure 1 - Renormalisation group transformation used to calculate:
(a) the susceptibility of the Potts ferromagnet;
(b) the susceptibility for the paramagnetic phase of the Ising antiferromagnet as well as phase diagrams.

Figure 2 - (a) Behavior of the susceptibility as a function of the temperature, for decreasing values of field;
(b) The logarithm of the zero field susceptibility as a function of the temperature.

Figure 3 - The critical exponent γ as a function of the number of states of of the Potts model. In the limit $q \to \infty, \gamma$ tends to $\gamma_\infty \simeq 0.4$ (dashed line).

Figure 4 - The test of the Rushbrooke scaling relation for various values of the q-state Potts model.

Figure 5 - The susceptibility in the paramagnetic phase for typical values of field, for an Ising antiferromagnet.

Figure 6 - Phase diagram of an Ising antiferromagnet [11], corresponding to the assumption of proportionality of the local external fields with the coordination numbers (i.e., $(C_1,C_2,C_3) = (3,2,4)$)

Figure 7 - Phase diagrams corresponding to other typical choices together with their RG flows. The diagram corresponding to the weights (3,2,3) presents, besides the trivial fixed point (2.26,0), the new fixed point (1.33,0.68).

TABLE 1 - Establishment of Eq. (5) for the Potts ferromagnet

$$q_i = q - i \quad ; \quad i = 1,2,3$$

(a) $$\langle m \rangle_{\substack{\text{small}\\ \text{cluster}}} = \left(2\mu' e^{qK'+2H'} + \mu'\left(1 - \frac{1}{q-1}\right)(q-1)\, e^{H'\left(1-\frac{1}{q-1}\right)} \right) \Bigg/ \left(e^{qK'+2H'} + (q-1)\, e^{H'\left(1-\frac{1}{q-1}\right)} \right)$$

(b) $$\langle m \rangle_{\substack{\text{large}\\ \text{cluster}}} = \left(2(c_1+c_3)\mu e^{5qK+2(C_1+C_3)H} + \left(2c_1 + c_3 - \frac{c_3}{q-1}\right)\mu 2(q-1) e^{2qK+\left(2C_1+C_3-\frac{C_3}{q-1}\right)H} + \ldots \right) \Bigg/$$

$$\left(e^{5qK+2(C_1+C_3)H} + 2(q-1)e^{2qK+\left(2C_1+C_3-\frac{C_3}{q-1}\right)H} + \ldots \right) .$$

The symbol * means all the (q-2) possible states different from "↑" and "↓".

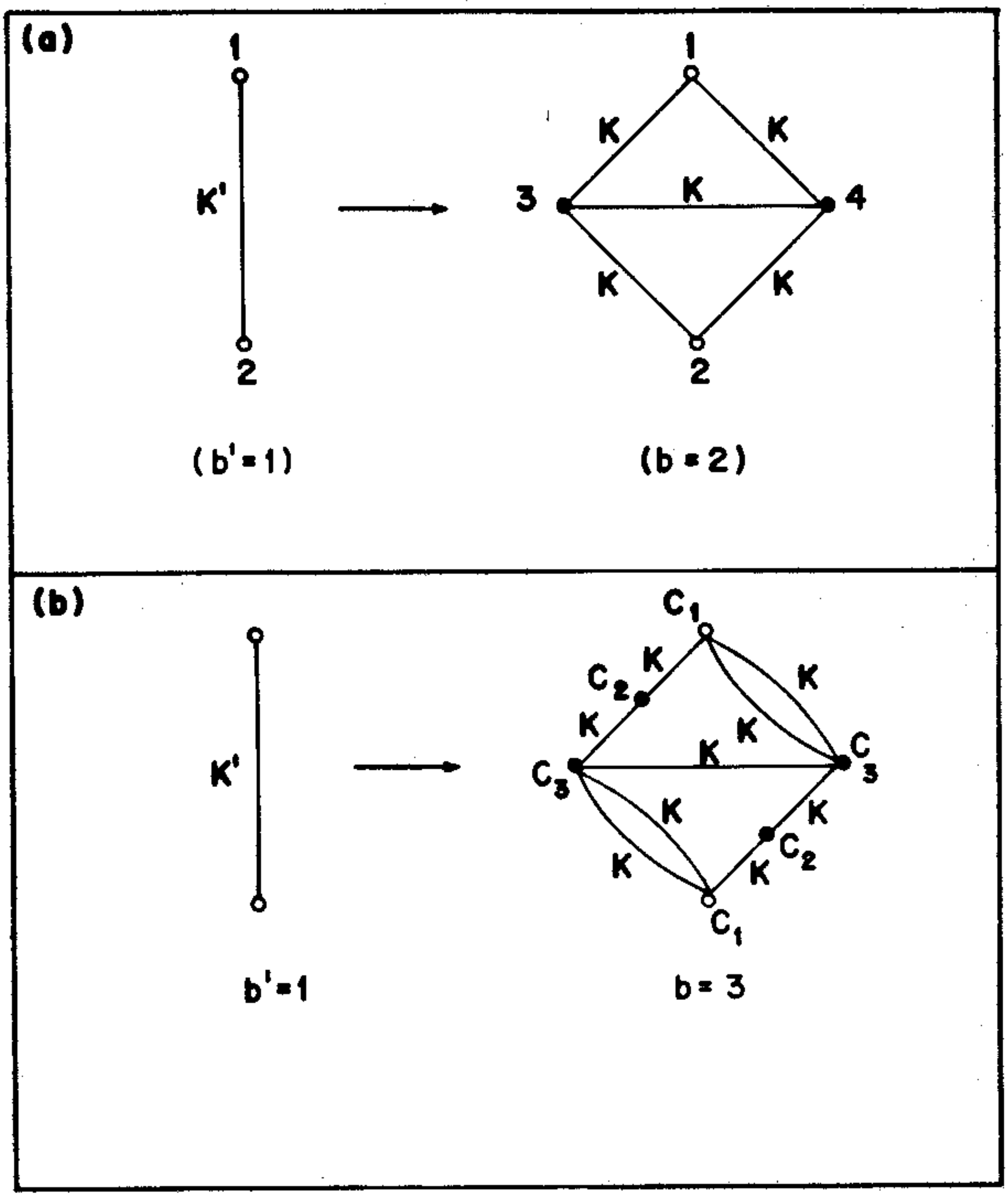


FIG.1

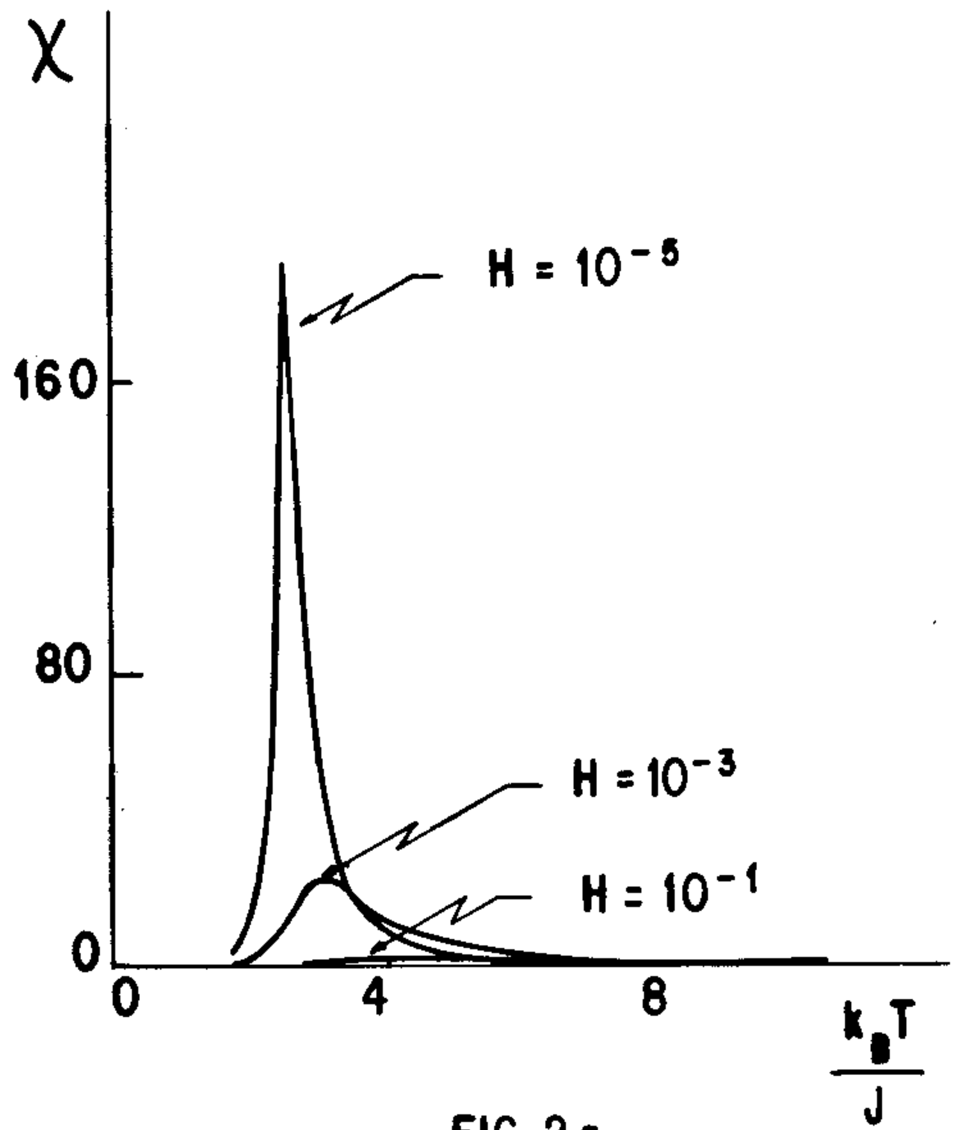


FIG. 2a

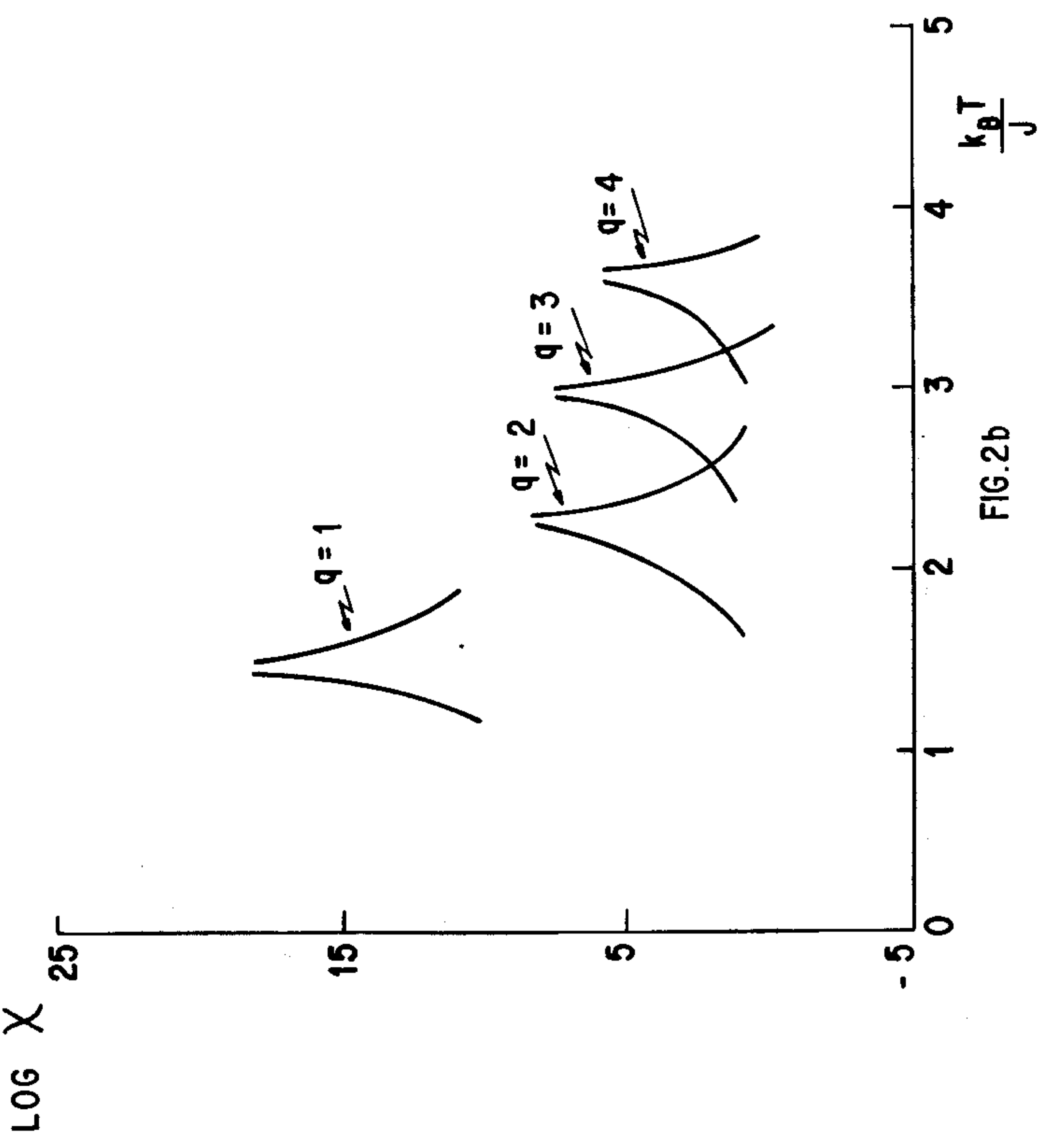


FIG.2b

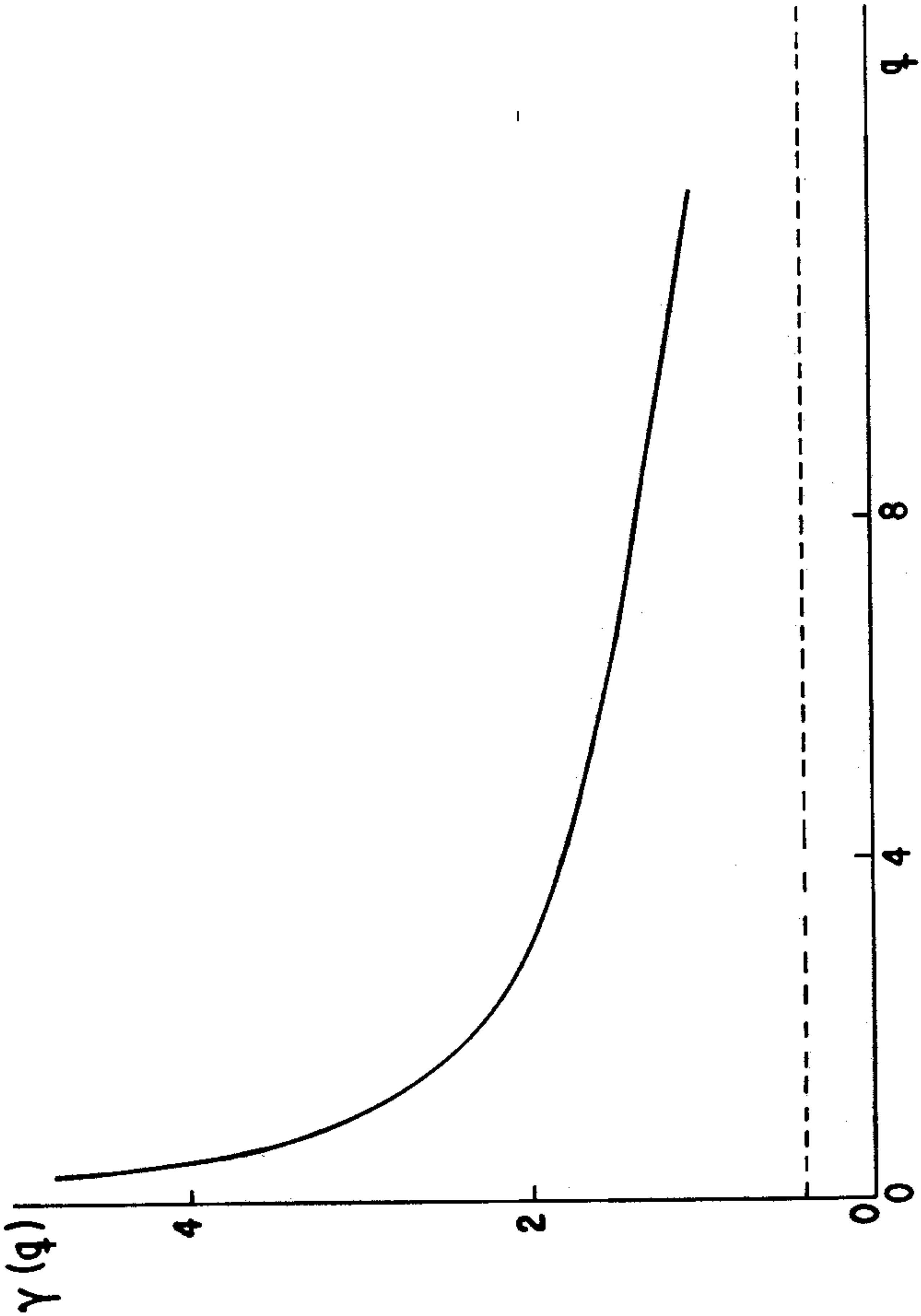


FIG.3

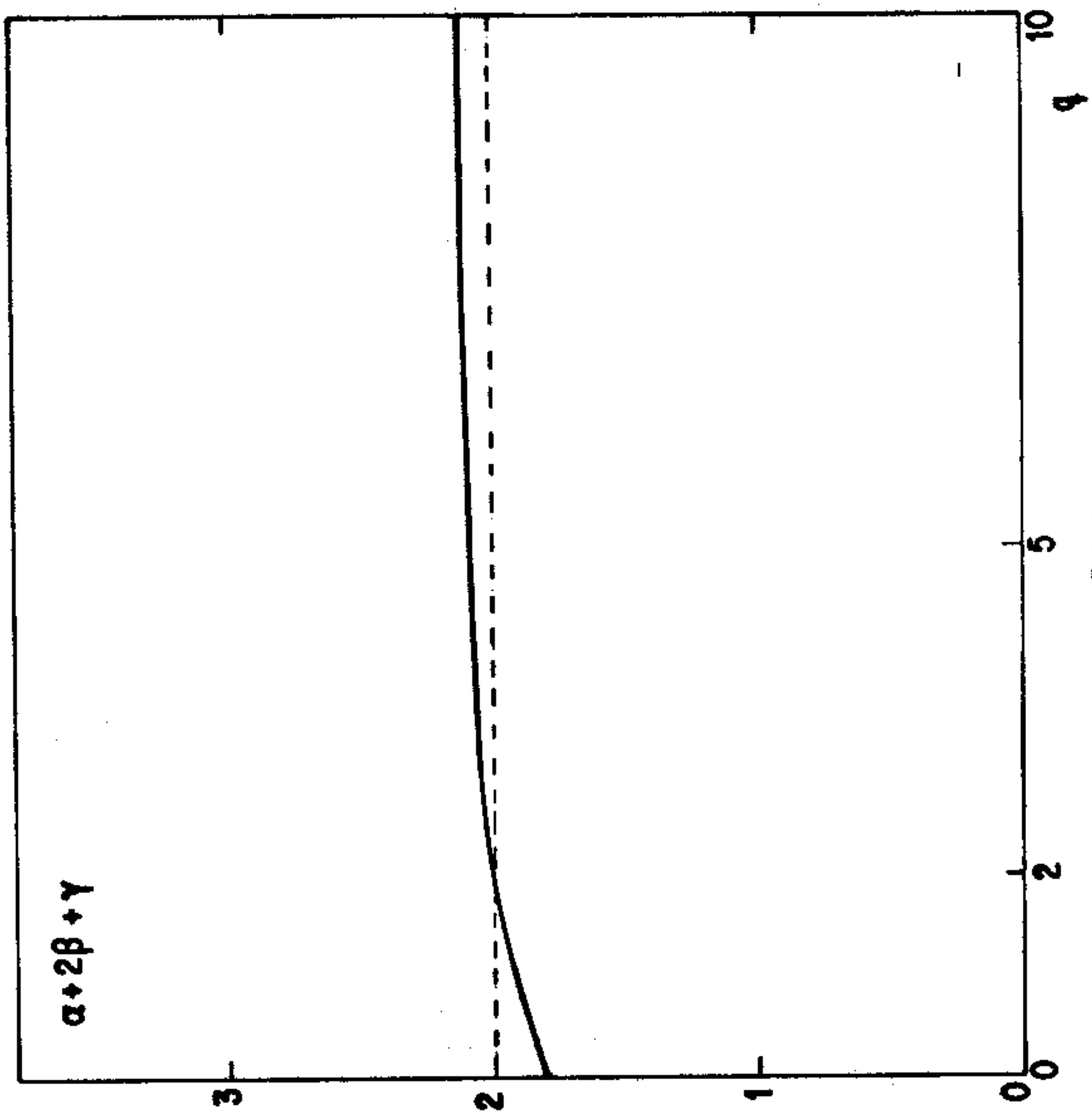


FIG.4

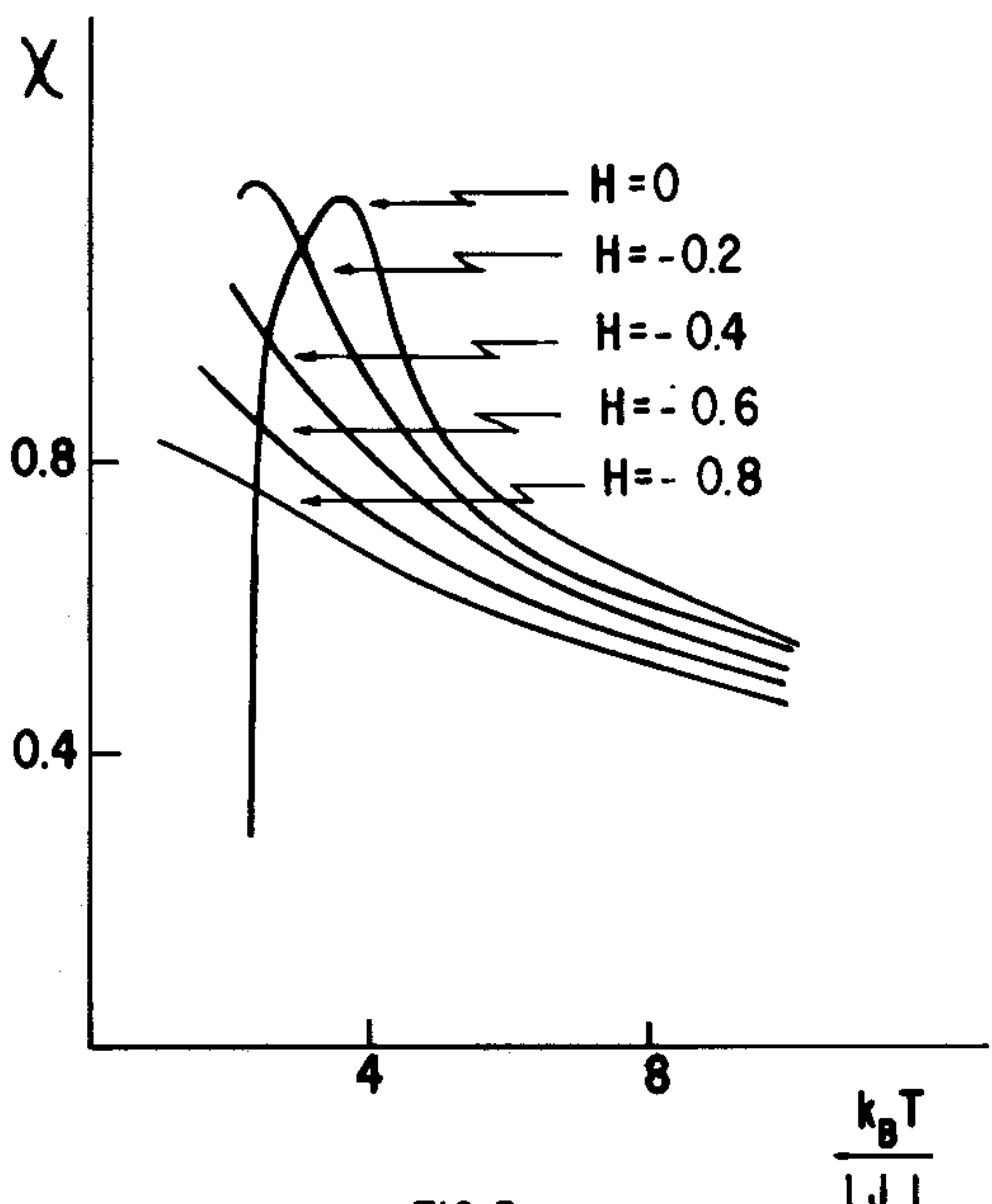


FIG.5

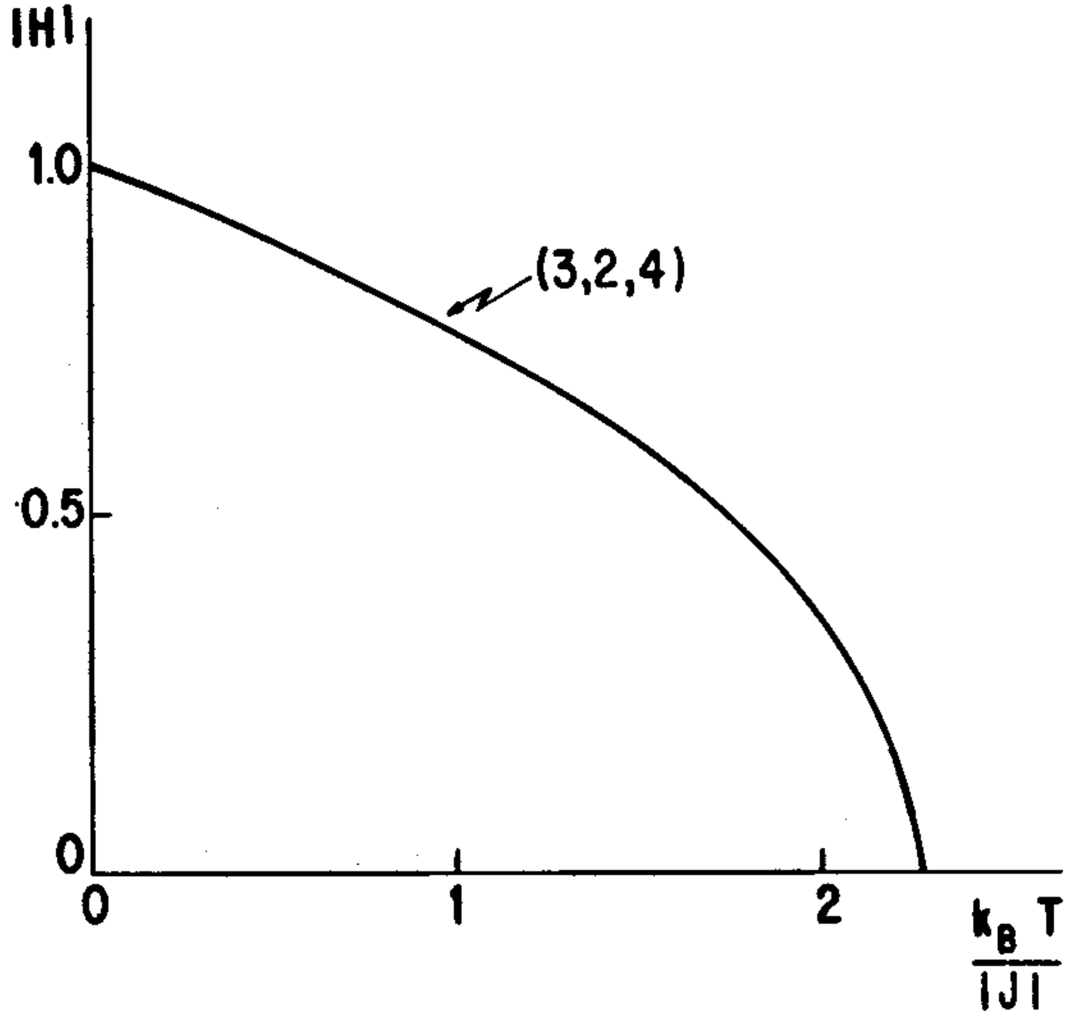


FIG.6

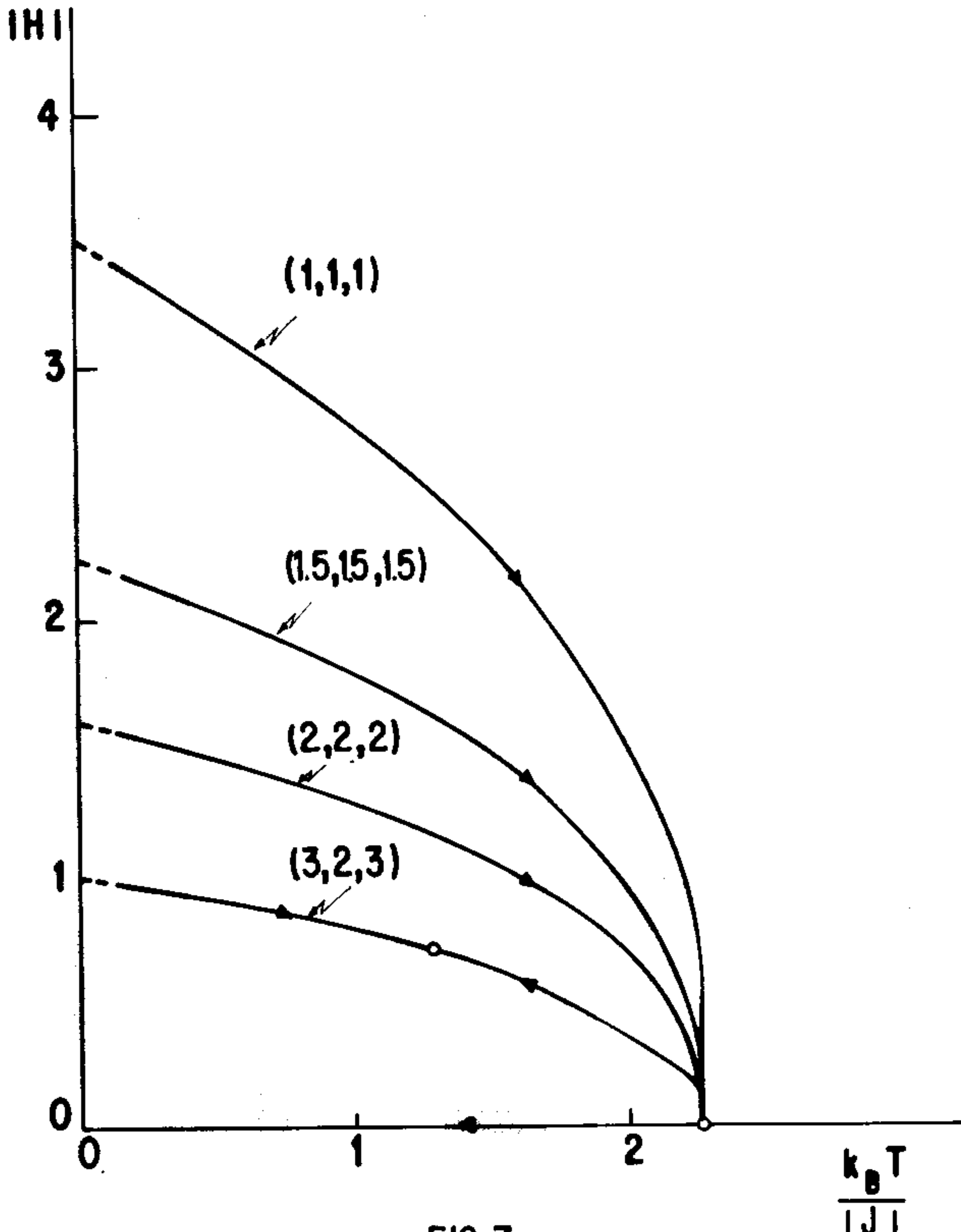
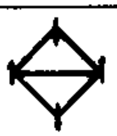
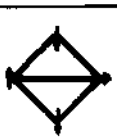


FIG. 7

TABLE 1

(a) small cluster configurations	weight	m
	$e^{qK'+2H'}$	$2\mu'$
	$q_1 e^{H'-\frac{H'}{q_1}}$	$\mu' - \frac{\mu'}{q_1}$

TABLE 1 (continued)

(b) large cluster configuration	weight	m
	$e^{5qK+2(C_1+C_3)H}$	$2(C_1+C_3)\mu$
	$2q_1e^{2qK+(2C_1+C_3-\frac{C_3}{q_1})H}$	$(2C_1+C_3-\frac{C_3}{q_1})\mu$
	$(q_1e^{qK}+q_1q_2)e^{2(C_1-\frac{C_3}{q_1})H}$	$2(C_1-\frac{C_3}{q_1})\mu$
	$e^{3qK+(C_1-\frac{C_1}{q_1}+2C_3)H}$	$(C_1-\frac{C_1}{q_1}+2C_3)\mu$
	$2\left(e^{2qK}+q_2e^{qK}\right).\,e^{(C_1+C_3-(C_1+C_3)/q_1)H}$	$(C_1+C_3-\frac{(C_1+C_3)}{q_1})\mu$
	$(e^{3qK}+3q_2e^{qK}+q_2q_3).\,e^{(C_1-\frac{C_1}{q_1}-\frac{2C_3}{q_1})H}$	$(C_1-\frac{C_1}{q_1}-\frac{2C_3}{q_1})\mu$
	$e^{qK+(-\frac{2C_1}{q_1}+2C_3)H}$	$(-\frac{2C_1}{q_1}+2C_3)\mu$
	$2(e^{2qK}+q_2).\,e^{(-\frac{2C_1}{q_1}-\frac{C_3}{q_1}+C_3)H}$	$(-\frac{2C_1}{q_1}-\frac{C_3}{q_1}+C_3)\mu$
	$\left(e^{5qK}+2q_2e^{2qK}+q_2e^{qK}+q_2q_3\right)e^{-2(C_1+C_3)H/q_1}$	$-\frac{2}{q_1}(C_1+C_3)\mu$

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