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CASIMIR EFFECT IN A D-DIMENSIONAL FLAT SPACETIME AND THE CUT-OFF METHOD

by

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ABSTRACT

The Casimir effect in a D-dimensional spacetime produced by a Hermitian massless scalar field in the presence of a pair of perfectly reflecting parallel flat plates is discussed. The exponential cut-off regularization method is employed. The regularized vacuum energy and the Casimir energy of this field are evaluated and a detailed analysis of the divergent terms in the regularized vacuum energy is carried out. The two-dimensional version of the Casimir effect is discussed by means of the same cut-off method. A comparison between the above method and the zeta function regularization procedure is presented in a way which gives us the unification between these two methods in the present case.

Key-words: Quantum Field Theory; Vacuum energy; Renormalization.

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I - INTRODUCTION

In recent years the problem of the infinite of the vacuum energy of quantized fields has been extensively studied. One of the first successful approaches concerns Casimir's[1] work. Since than, this problem has been studied for a variety of geometries[2,3] and topologies[4], and in different spacetime dimensions[5].

Following this trend we study the vacuum fluctuation of a Hermitian massless scalar field defined in an arbitrary higher dimensional ($D \geq 4$) Minkowskian spacetime. We calculate the Casimir energy in this higher dimensional spaces between parallel ($D-2$)-dimensional perfectly reflecting flat plates. Our approach is similar to those used by Casimir[1], Fierz[6] and Boyer[7].

Another method to obtain the Casimir energy deals with the analytic regularization — the so called "zeta function regularization method". Although apparently simpler, the physical meaning of this approach was left somewhat obscure in the past. In this paper we present a connection between the zeta function method and the cut-off method which intends to shed some light on this subject. We will show that the use of the zeta function can be interpreted as a cut-off method and we unificate this methods in $D = 2$.

The organization of this paper is as follows:

In Section II we obtain the expression of the Casimir energy in a D-dimensional flat spacetime. In the first step we evaluate the energy per unit area of the vacuum state of the field inside a perfectly reflecting box (Dirichlet's boundary conditions) which depends on the cut-off function employed. We employ the standard definition of the Casimir energy, i.e., the energy per unit area of the configuration, which depends on the finite distance between one of the pairs of plates, is subtracted from the energy per unit area associated to a configuration which does not depend on this finite distance. The divergences that appear are canceled out using this approach of Casimir generalized for the D-dimensional case.

In Section III we show that the zeta function regularization can be understood in terms of the cut-off method. Conclusions are given in Section IV. The evaluation of a certain integral in the complex plane (used in Section II) is carried out in the Appendix.

II - QUANTIZATION OF A REAL MASSLESS SCALAR FIELD IN A D-DIMENSIONAL FLAT SPACETIME CONFINED IN A (D-1)-DIMENSIONAL BOX

A free Hermitian massless scalar field $\phi(x^0,x^1,\ldots,x^{D-1})$ defined in a D-dimensional Minkowski spacetime must satisfy the generalized Klein-Gordon equation ($\hbar = c = 1$)

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2 - \sum_{j=1}^{D-1}\left(\frac{\partial}{\partial x^j}\right)^2\right]\phi(x) = 0 \quad . \tag{2.1}$$

If we restrict the field to the interior of a (D-1)-dimensional box with lenghts $(L_1 \times L_2 \times \ldots L_{D-2} \times A)$, the field should be expanded as

$$\Phi(x) = \sum_{n_1,n_2,\ldots,n_{D-1}} \phi_{n_1 n_2 \ldots n_{D-1}} \, a_{n_1 n_2 \ldots n_{D-1}} + $$

$$+ \phi^*_{n_1 n_2 \ldots n_{D-1}} \, a^\dagger_{n_1 n_2 \ldots n_{D-1}} \quad , \qquad (2.2)$$

where

$$\phi_{n_1 n_2 \ldots n_{D-1}} = f_{n_1 n_2 \ldots n_{D-1}}(x^0) \left[\sin(\frac{n_1 \pi}{L_1}) \, \sin(\frac{n_2 \pi}{L_2} x_2) \ldots \right.$$

$$\left. \ldots \sin(\frac{n_{D-2}\pi}{L_{D-2}} x_{D-2}) \, \sin(\frac{n_{D-1}\pi}{A} x_{D-1})\right] \quad , \qquad (2.3)$$

the n_j's are positive integers once we impose Dirichlet boundary conditions.

The field modes (2.3) and their respective complex conjugates form a complete orthonormal basis and $a_{n_1 n_2 \ldots n_{D-1}}$, $a^\dagger_{n_1 n_2 \ldots n_{D-1}}$ are interpreted in the standard way, i.e., as destruction and creation operators of quanta of the field $\Phi(x)$ with energy $\omega_{n_1 n_2 \ldots n_{D-1}}$, given by

$$\omega_{n_1 n_2 \ldots n_{D-1}} = \left[(\frac{n_1 \pi}{L_1})^2 + (\frac{n_2 \pi}{L_2})^2 + \ldots (\frac{n_{D-2}\pi}{L_{D-2}})^2 + (\frac{n_{D-1}\pi}{A})^2\right]^{1/2} . \qquad (2.4)$$

In this Fock representation, there must exist a particular vector $|0\rangle$, called the vacuum or no-particle state. The energy of this vacuum state is

$$E_D(L_1, L_2, \ldots L_{D-2}, A) = \frac{1}{2} \sum_{n_1,n_2,\ldots,n_{D-1}} \left[(\frac{n_1 \pi}{L_1})^2 + (\frac{n_2 \pi}{L_2})^2 + \ldots \right.$$

$$\left. \ldots (\frac{n_{D-2}\pi}{L_{D-2}})^2 + (\frac{n_{D-1}\pi}{A})^2\right]^{1/2} . \qquad (2.5)$$

Using the condition $A \ll L_i$ $(i = 1,2,\ldots,D-2)$ the expression (2.5) becomes

$$E_D(L_1,L_2,\ldots,L_{D-2},A) = \frac{1}{2}\left(\prod_{i=1}^{D-2} L_i\right)\frac{1}{\pi^{D-2}}\int_0^\infty dk_1\ldots\int_0^\infty dk_{D-2}\sum_{n=1}^{\infty}\left[k_1^2 + k_2^2 + \ldots k_{D-2}^2 + \left(\frac{n\pi}{A}\right)^2\right]^{1/2} . \quad (2.6)$$

The summation starts at $n = 1$ because the scalar field does not include the modes for which one of the integers $n_1,n_2,\ldots,n_{D-1}$ vanishes.

The angular part of the integral over (D-2)-dimensional k space is trivial, once we have:

$$\int_0^\infty d^{D-2}k = \int_0^\infty dr\, r^{D-3}\int_0^{2\pi} d\theta_1\int_0^\pi d\theta_2 \sin\theta_2\ldots\int_0^\pi \sin^{D-4}\theta_{D-3}\, d\theta_{D-3} = $$

$$= \frac{2(\pi)^{\frac{D-2}{2}}}{\Gamma\left(\frac{D-2}{2}\right)}\int_0^\infty dr\, r^{D-3} \quad . \quad (2.7)$$

Using (2.7) we obtain

$$E_D(L_1,L_2,\ldots,L_{D-2},A) = \left(\prod_{i=1}^{D-2} L_i\right)\frac{1}{2^{D-2}}\frac{1}{\Gamma\left(\frac{D-2}{2}\right)}\frac{1}{\pi^{(D-2)/2}}\sum_{n=1}^{\infty}\int_0^\infty dr\, r^{D-3}\left[r^2 + \left(\frac{n\pi}{A}\right)^2\right]^{1/2} . \quad (2.8)$$

Defining a function F(D) by setting

$$F(D) = \frac{1}{2^{D-2}}\frac{1}{\Gamma\left(\frac{D-2}{2}\right)}\frac{1}{\pi^{(D-2)/2}} \quad , \quad (2.9)$$

the energy per unit area can be written as

$$\frac{E_D(L_1,L_2\ldots L_{D-2},A)}{\Pi\, L_i} = \varepsilon_D(A) = F(D)\sum_{n=1}^{\infty}\int_0^{\infty} dr\; r^{D-3}\left(r^2+(\frac{n\pi}{A})^2\right)^{1/2} . \tag{2.10}$$

The expression (2.10) clearly diverges. It is possible to regularize such expression introducing a convergent factor, a ultraviolet regulator function, e.g.,

$$\exp\left[-\lambda\left(r^2+(\frac{n\pi}{A})^2\right)^{1/2}\right] , \tag{2.11}$$

$Re\,\lambda > 0$.

The regularized energy per unit area will be denoted by $\varepsilon_D\,(\lambda,A)$, thus

$$\varepsilon_D(\lambda,A)=F(D)\sum_{n=1}^{\infty}\int_0^{\infty} dr\; r^{D-3}\left(r^2+(\frac{n\pi}{A})^2\right)^{1/2}\exp\left[-\lambda\left(r^2+(\frac{n\pi}{A})^2\right)^{1/2}\right] . \tag{2.12}$$

This expression is convergent provided $Re\lambda > 0$. The Casimir energy $U_D(A)$ (the vacuum energy per unit area) is defined by

$$U_D(A) = \lim_{\substack{\lambda\to 0\\ R\to\infty}}\left[\varepsilon_D(\lambda,A) + \varepsilon_D(\lambda,R-A) - \varepsilon_D(\lambda,\eta R)+\varepsilon_D(\lambda,(1-\eta)R)\right] , \tag{2.13}$$

with

$$0 < \eta < 1 .$$

In the end of this Section it will become clear why this definition is adopted.

Let us evaluate $\varepsilon_D(\lambda,A)$. We define a new variable X by:

$$X = \frac{\lambda\pi}{A}\,(1 + \frac{A^2r^2}{n^2\pi^2})^{1/2} . \tag{2.14}$$

It is straightforward then to show that in terms of the new variable

$$\varepsilon_D(\lambda,A) = (-1)^{D-1}F(D)\,\frac{d^{D-1}}{d\lambda^{D-1}}\int_{\lambda\pi/A}^{\infty}\frac{dX}{X}\left[1-\left(\frac{\lambda\pi}{AX}\right)^2\right]^{\frac{D-4}{2}}\frac{1}{e^X-1}\ . \tag{2.15}$$

The expression (2.15) contains a power of the binomial $(1-(\frac{\lambda\pi}{AX})^2)$. When D is even the power is an integer and the simple use of the Newton's binomial theorem will give a very direct way of evaluating $\varepsilon_D(\lambda,A)$. When D is odd the expansion of $\left[1-(\frac{\lambda\pi}{AX})^2\right]^{\frac{D-4}{2}}$ yields an infinite power series. Verscheld et al.[8] elude this problem by using a Laplace transformation and integral representations of the cylindrical Bessel functions. However, the use of the power series (finite or infinite) does allow us to evaluate $\varepsilon_D(\lambda,A)$, with greater physical insight concerning the nature of the infinities which appear in the Casimir effect as we will show now.

Using the notation

$$\begin{aligned} C_p^0 &= 1\ , \\ C_p^1 &= \frac{p}{1!} \\ C_p^2 &= \frac{p(p-1)}{2!} \\ &\vdots \\ C_p^k &= \frac{p(p-1)\ldots(p-k+1)}{k!}\ , \end{aligned} \tag{2.16}$$

and setting $p = \frac{D-4}{2}$, the expression (2.15) may be written as

$$\varepsilon_D(\lambda,A) = (-1)^{D-1}F(D)\sum_{k=0}^{\infty}(-1)^k C_p^k\,\frac{d^{D-1}}{d\lambda^{D-1}}\left(\frac{\lambda\pi}{A}\right)^{2k}\int_{\pi\lambda/A}^{\infty}\frac{dX}{X^{2k+1}}\,\frac{1}{e^X-1}\ . \tag{2.17}$$

Substituting $\frac{X}{e^X-1}$ by its power serie expansion around $X = 0$ in the expression (2.17) we get

$$\int_u^\infty \frac{dX}{X^{2k+1}} \frac{1}{e^X-1} = - \sum_{\substack{q=0 \\ q\neq 2k+1}}^{\infty} \frac{B_q}{q!} \frac{u^{q-2k+1}}{q-2k-1} - \frac{1}{(2k+1)!} B_{2k+1} \ln u + \gamma_k \, , \tag{2.18}$$

where

$$\mathrm{Re}\; u > 0$$

$$2\pi > |u| > 0 \quad ,$$

γ_k is a constant and B_q are the Bernoulli numbers.

Substituting the identity (2.18) in the equation (2.17) we get:

$$\varepsilon_D(\lambda,A) = -(-1)^{D-1} F(D) \sum_{k=0}^{\infty} (-1)^k C_p^k \frac{B_{2k+1}}{(2k+1)!} \frac{d^{D-1}}{d\lambda^{D-1}} \left(\frac{\lambda\pi}{A}\right)^{2k} \ln \frac{\lambda\pi}{A} +$$
$$- (-1)^{D-1} F(D) \sum_{k=0}^{\infty} \sum_{\substack{q=0 \\ q\neq 2k+1}}^{\infty} (-1)^k C_p^k \frac{B_q}{q!} \frac{1}{q-2k-1} \frac{d^{D-1}}{d\lambda^{D-1}} \left(\frac{\lambda\pi}{A}\right)^{q-1} +$$
$$+ (-1)^{D-1} F(D) \sum_{k=0}^{\infty} (-1)^k C_p^k \gamma_k \frac{d^{D-1}}{d\lambda^{D-1}} \left(\frac{\lambda\pi}{A}\right)^{2k} \quad . \tag{2.19}$$

Our discussion will be greatly simplified if we study separatedly the even and odd-dimensional cases:

A) - The even case

The first term in the expression (2.19) yields

$$- \frac{1}{2^{D-2}} \frac{1}{\pi^{\frac{D-2}{2}}} \frac{(D-2)!}{\Gamma(\frac{D-2}{2})} \frac{1}{\lambda^{D-1}} \quad . \tag{2.20}$$

The second term in (2.19) yields

$$\frac{1}{4}\,\frac{1}{\pi^{D/2}}\,(D-1)(D-2)\,\Gamma\left(\frac{D-2}{2}\right)\frac{A}{\lambda^{D}} + \left(-\,\frac{\pi}{2}\right)^{D/2}\frac{B_D}{D}\,\frac{1}{(D-1)!!}\,\frac{1}{A^{D-1}} +$$

$$+ \text{ terms containing positive powers in } \lambda\;. \tag{2.21}$$

The third term vanishes because the sum in k is finite $(0 \leq k \leq \frac{D-4}{2})$.

B) - The odd case

The first term in the expression (2.19) has the same functional form as (2.20).

The second term in the expression (2.19) give

$$\frac{1}{4}\,\frac{1}{\pi^{D/2}}\,(D-1)(D-2)\,\Gamma\left(\frac{D-2}{2}\right)\,\frac{A}{\lambda^{D}} + \text{positive powers in } \lambda\;. \tag{2.22}$$

The contribution of the third term in this case does not vanish and becomes (after the evaluation of the constant $\gamma_{\frac{D-1}{2}}$) equal to

$$\frac{1}{2}\,(-1)^{D-1}F(D)\;C_p^{\frac{D-1}{2}}\;\zeta(D)\,(D-1)!\,\frac{1}{(2A)^{D-1}} + \text{positive powers in } \lambda\;. \tag{2.23}$$

Simple algebraic manipulation provides an unique expression for the energy per unit area, valid for both — even and odd dimensional cases.

$$\varepsilon_D(\lambda,A) = \frac{1}{4}\,\frac{\Gamma(\frac{D-2}{2})}{\pi^{D/2}}\,(D-1)(D-2)\,\frac{A}{\lambda^D} - \frac{1}{2^{D-1}}\,\frac{1}{\pi^{(\frac{D-2}{2})}}\,\frac{(D-2)!}{\Gamma(\frac{D-2}{2})}\,\frac{1}{\lambda^{D-1}} +$$

$$- (4\pi)^{-D/2}\Gamma(D/2)\zeta(D)\,\frac{1}{A^{D-1}} + g(D,\lambda) \quad , \qquad (2.24)$$

where $\lim_{\lambda \to 0} g(D,\lambda) = 0$.

The first term in the expression (2.24) dominates the behavior of $\varepsilon_D(\lambda,A)$ when $\lambda \to 0$, and gives rise to a total divergent energy proportional to the volume between the plates. So in order to obtain the value of the Casimir energy we have to subtract the energy of isovolumetric configurations as it was done previously in expression (2.13). Although the second term does not dominate $\varepsilon_D(\lambda,A)$ when $\lambda \to 0$, it is a negative divergence and gives rise to an infinite total energy proportional to the surface of the plates. So, in order to eliminate this divergence it is enough to subtract configurations with an equal number of pairs of plates. The third term (which is the Casimir energy) vanishes when $A \to \infty$. In this way the auxiliary plates must be infinitely separated in order that this auxiliary plates do not give any measurable contribution to the renormalized energy.

Now, it's clear the convenience of the definition used by Casimir for the renormalized vacuum energy presented in expression (2.13). It's important to stress that since we supposed that $L_i \gg A$ $(i = 1,2,\ldots,D-2)$ the geometric parameters of the configuration are the distance between the pair of plates not infinitedly appart and the area of this pair of plates (which is much greater than the total area of the remaining plates).

So we have two physical parameters for the configuration and they appear in the divergent terms in expression (2.24).

The use of the cut-off method in a two-dimensional (D=3) box ($L_1 \approx A$) will give us divergent terms proportional to the "volume", $L_1 A$ and to the "surface" L_1+A of the box. In the three dimensional box ($L_1 \approx L_2 \approx A$) we will obtain three divergent terms proportional to the volume, area and perimeter ($L_1 + L_2 + A$) of the box. And so forth for higher dimensional boxes. In the (D-1) dimensional box ($L_1 \approx L_2 \approx \ldots L_{D-2} \approx A$) the divergent term of higher order must be proportional to the volume of the box. Although the divergent term of lower order in equation (2.24) does not give any contribution to the force between the plates (which leads some authors to disregard it), in the finite box the divergent terms of lower order do contribute to the force between any parallel pair of plates of the box. So these terms cannot be a priori disregarded.

Using the definition (2.13), the Casimir energy $U_D(A)$ is given by:

$$U_D(A) = -(4\pi)^{-D/2}\,\Gamma(D/2)\zeta(D)\,\frac{1}{A^{D-1}} \quad . \tag{2.25}$$

This energy per unit area is always negative, then the force per unit area is attractive and given by

$$-\frac{\partial}{\partial A}U_D(A) = -(D-1)(4\pi)^{-D/2}\Gamma(D/2)\zeta(D)\,\frac{1}{A^D} \quad . \tag{2.26}$$

The result (2.25) can be obtained using two different methods. One of them consists in the analysis of the behavior of the vacuum stress-tensor. This technique is referred to as the Green function method[9]. The other one is based on the direct evaluation of an infinite series with an appropriate regularization procedure. This second method can be implemented in three differents ways. The first one is the cut-off method, the second one is based on the analytic continuation in the number D of dimensions(10). The third one is based on a sugestion given by Hawking(7) and uses the analytic extension of a generalized zeta-function(12,13). In the next section we will discuss the zeta regularization method and its connection with the exponential cut-off method.

III - THE ZETA FUNCTION REGULARIZATION AS A CUT-OFF METHOD

The zeta function regularization method provides the value of the Casimir energy almost automatically and with no "aparent" need of infinite subtractions.

Ruggiero et al.(12) claim that the cut-off method can be interpreted as a tool to obtain an analytic regularization of the function $\sum_{n=1}^{\infty} e^{-\lambda\omega_n}$. The poles (at $\lambda = 0$) are subtracted and taking the $\partial/\partial\lambda$ of this function at $\lambda = 0$ we get the Casimir energy. So it seems at a first sight that the cut-off method is a particular form of the analytic regularization.

However, we will now show in this Section that, in a flat spacetime the zeta function regularization method (an analytic regularization method "per excellentia") can be interpreted as a cut-off method. This approach is valuable because it unifies these two methods and reveals, in a simple way, the ultimate physical reason for the result to be the same.

The zeta function method evaluate the vacuum energy inside a finite (D-1)-dimensional box by performing summation in (D-1) indices. The exponential cut-off method employed in Section II using the approximation $A \ll L_i$ performs integration over (D-2) indices and summation over only in one index. In order to compare these two methods we will study the D = 2 case, when for both methods we perform just only one summation. Note that the expression (2.24) can not be applied in D = 2. The vacuum energy in D = 2 is

$$E_2(A) = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n\pi}{A} \quad . \tag{3.1}$$

A direct calculation using the exponential cut-off gives

$$E_2(\lambda,A) = \frac{A}{2\pi}\frac{B_0}{\lambda^2} - \frac{\pi}{4}\frac{B_2}{A} + O(\lambda^2) \quad . \tag{3.2}$$

Now we have only one divergence proportional to the "volume" of the box. Subtracting isovolumetric configurations we eliminate this divergence. So we can define

$$U_2(A) = \lim_{\substack{\lambda\to 0\\ R\to\infty}}\left[E_2(\lambda,A) + E_2(\lambda,R-A) - E_2(\lambda,R)\right] . \tag{3.3}$$

To study the zeta function regularization method it is important to point out a very simple fact. Every analytic extension is done in connected domains. So it must be possible to travel from one point (when the function is well behaved) to another one (where we shall find our physical results, but in which our formula does not work) along a smooth path.

Now we inquire: Is the exponential cut-off used too much strong ? Let us try a weaker cut-off function $\omega_n^{-\sigma}$. The regularized energy becomes

$$E_2(\sigma,A) = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n^{(1-\sigma)} = \frac{1}{2} \left(\frac{A}{\pi}\right)^{\sigma-1} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-1}} \quad . \qquad (3.4)$$

This cut-off works only if Re $\sigma > 2$. After some manipulations we get

$$E_2(\sigma,A) = \frac{1}{2} \left(\frac{A}{\pi}\right)^{\sigma-1} \frac{1}{\Gamma(\sigma-1)} \left[\int_0^1 x^{\sigma-2} Z_1(x)\,dx + \int_1^{\infty} x^{\sigma-2}\, Z_1(x)\,dx \right] \quad , \qquad (3.5)$$

where $Z_1(x) = \frac{1}{e^x-1}$.

So, when $\sigma \to 2^+$, $\sigma \in \mathbb{R}$ our cut-off choice fails.

Let us study the divergence in the "path". If in the first integral in (3.5) we make $Z_1(x) = (Z_1(x) - \frac{B_0}{x}) + \frac{B_0}{x}$ we get:

$$E_2(\sigma,A) = \frac{1}{2} \left(\frac{A}{\pi}\right)^{\sigma-1} \frac{1}{\Gamma(\sigma-1)} \left[\int_0^1 x^{\sigma-2} (Z_1(x) - \frac{B_0}{x})\,dx + \int_1^{\infty} x^{\sigma-2}\, Z_1(x)\,dx \right] + \frac{1}{2} \left(\frac{A}{\pi}\right)^{\sigma-1} \frac{1}{\Gamma(\sigma-1)} \frac{B_0}{\sigma-2} \; . \qquad (3.6)$$

The last term in (3.6) has an isolated singularity which is a first order pole. The divergence part (for $\sigma = 2$) of (3.6) is given by $\frac{1}{2}\frac{A}{\pi}\frac{B_0}{\sigma-2}$. This divergence is proportional to the "volume" of the box and vanishes if we make the Casimir-like regularization, subtracting isovolumetric configurations. So let us define

$$Y_2(\sigma,A,R) = E_2(\sigma,A) + E_2(\sigma,R-A) - E_2(\sigma,R) \quad , \qquad (3.7)$$

We note that $Y_2(\sigma,A,R)$ so defined, as a function of σ can be analyticaly extended to the whole complex plane, including $\sigma = 2$

$$Y_2(\sigma,A,R) = \frac{1}{2}\left[\left(\frac{A}{\pi}\right)^{\sigma-1} + \left(\frac{R-A}{\pi}\right)^{\sigma-1} - \left(\frac{R}{\pi}\right)^{\sigma-1}\right] \times$$

$$\times \frac{1}{\Gamma(\sigma-1)}\left[\int_0^1 x^{\sigma-2}\left(Z_1(x) - \frac{B_0}{x}\right)dx + \int_1^\infty x^{\sigma-2} Z_1(x)\,dx\right] +$$

$$+ \frac{1}{2}\left[\left(\frac{A}{\pi}\right)^{\sigma-1} + \left(\frac{R-A}{\pi}\right)^{\sigma-1} - \left(\frac{R}{\pi}\right)^{\sigma-1}\right] \frac{1}{\Gamma(\sigma-1)}\,\frac{B_0}{\sigma-2} \quad . \qquad (3.8)$$

Going back (using (3.4) and (3.7)) we get

$$Y_2(\sigma,A,R) = \frac{1}{2}\left[\left(\frac{A}{\pi}\right)^{\sigma-1} + \left(\frac{R-A}{\pi}\right)^{\sigma-1} - \left(\frac{R}{\pi}\right)^{\sigma-1}\right]\zeta(\sigma-1) \qquad (3.9)$$

The Casimir energy is given by

$$U_2(A) = \lim_{\substack{R\to\infty \\ \sigma\to 0}} Y_2(\sigma,A,R) = -\frac{\pi}{4}\,\frac{B_2}{A} \quad . \qquad (3.10)$$

Now we can unify both methods in order to exhibit the link between them. Let us define a mixed cut-off regularized energy:

$$E_2(\sigma,\lambda,A) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n\, \omega_n^{-\sigma}\, e^{-\lambda\omega_n} \quad , \qquad (3.11)$$

with

$$\mathrm{Re}\lambda > 0$$
$$\mathrm{Re}\sigma > 2 \quad .$$

Then:

$$Y_2(\sigma,\lambda,A,R) = E_2(\sigma,\lambda,A) + E_2(\sigma,\lambda,R-A) - E_2(\sigma,\lambda,R) \ . \tag{3.12}$$

Y_2 is defined in the region $\lambda \geq 0$, $\sigma \geq 0$. The fact that the results obtained by the traditional cut-off method and the zeta function regularization method yields the same result follows from a very simple fact: Y_2 is continuous in the domain and then the limits

$$\lim_{\lambda\to 0} Y_2(0,\lambda,A,R) = \lim_{\sigma\to 0} Y_2(\sigma,0,A,R) \quad , \tag{3.13}$$

must be the same.

IV - CONCLUSIONS

In this paper we derived two interesting results. In Section II we extended in a straightforward way the approach used by Casimir, Fierz and Boyer to calculate the vacuum renormalized energy to a higher dimensional flat spacetime. We have stressed that, for the success of this method, we need to subtract configurations not only isovolumetric but also configurations with the same area of plates.

In the Section III, using the insight achieved by the above observation, we proved that the zeta function regularized method in D = 2 can be interpreted as a cut-off method, that needs the same artificie: isovolumetric configurations subtraction. Based in this fact we proved that the exponential cut-off and the zeta function methods are totally equivalent in D = 2.

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APPENDIX

The expression (2.19) is

$$\int_u^\infty \frac{dX}{X^{2k+1}} \frac{1}{e^X-1} = -\sum_{\substack{q=0 \\ q\neq 2k+1}}^{\infty} \frac{B_q}{q!} \frac{u^{q-2k-1}}{q-2k-1} - \frac{1}{(2k+1)!} B_{2k+1} \ln u + \gamma_k , \qquad (A.1)$$

where

Re. $u > 0$

$2\pi > |u| > 0$.

We will define:

$$f_1(X) = \frac{1}{X^{2k+1}} \frac{1}{e^X-1} - \sum_{q=0}^{2k+1} \frac{B_q}{q!} X^{q-2k-2} , \qquad |X| < 2\pi. \qquad (A.2)$$

The expression (A.2) is the integrand without its polar parts around zero. Using (A.2) in (A.1) we get, if $k > 0$

$$\int_u^\infty \frac{dX}{X^{2k+1}} \frac{1}{e^X-1} = \int_u^\infty f_1(X)dX - \sum_{q=0}^{2k} \frac{B_q}{q!} \frac{u^{q-2k-1}}{q-2k-1} . \qquad (A.3)$$

Using (A.3) in (A.1), $k > 0$ we get:

$$\gamma_k = \int_u^\infty f_1(X)dX + \sum_{q=2k+2}^{\infty} \frac{B_q}{q!} \frac{u^{q-2k-1}}{q-2k-1} . \qquad (A.4)$$

When $u \to 0$ the summation in (A.4) vanishes, but because $f_1(X)$ is even we get

$$\gamma_k = \frac{1}{2} \int_{-\infty}^{\infty} f_1(X)dX . \qquad (A.5)$$

In the complex plane the function $f_1(X)$ have poles in $X = 2n\pi$ ($n \neq 0$, $n \in \mathbb{N}$). Using the Cauchy theorem and the path of integration - a semi circle with center in the origin in the uper half plane when $R = (n + \frac{1}{2})$ we have:

$$\gamma_k = \pi i \sum_{n=1}^{\infty} \text{Res}\ (f_1(X),\ 2\pi ni) =$$

$$= \frac{1}{2} \frac{1}{(2\pi i)^{2k}} \zeta(2k+1) \quad . \tag{A.6}$$

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