ISSN 0029 - 3865

CBPF-NF-048/91

ARBITRARY POWERS OF D'ALEMBERTIANS AND THE HUYGENS' PRINCIPLE

by

C.G. BOLLINI[1,2] and J.J. GIAMBIAGI[1,3]

[1]Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

[2]Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de La Plata
Argentina and Comisión de Investigaciones Cientificas de la Provincia de Buenos Aires - Argentina

[3]Centro Latino Americano de Fisica - CLAF
Av. Wenceslau Braz, 71 - fundos
22290 - Rio de Janeiro, RJ - Brasil

ABSTRACT

By means of some reasonable rules we define the operators that can represent arbitrary powers of the D'Alembertian and their corresponding Green functions. We find which powers lead to the validity of Huygens' Principle.

We discuss the specially interesting case of powers that are half an odd integer in spaces of odd dimensionality, they obey Huygens' Principle and can be expressed as iterated D'Alembertians of the retarded potential.

We also discuss arbitrary powers of the Laplacian operator as well as their corresponding Green functions.

Key-words: Field theory; Pseudo differential operators; Bosonization; Wave equations. Distributions.

§1 Introduction

The ordinary wave equation, as well as its relation to the Huygens' principle (HP), has received considerable attention and has also been the object of some beautiful works. We would like to mention the classical book on the subject by B.B. Baker and E.T. Copson[1], and the elegant analytic continuation method of M. Riesz[2]. It is well known that HP is valid for the usual wave equation when the number n of space-time dimensions is even, but not when it is odd.

Nowadays some physicists are not happy living in a world of only four dimensions. Furthermore, second order wave equations are no longer mandatory for the description of the evolution of physical particles or fields. For example, in gravitational theories, terms quadratic in the curvature tensor are some times introduced in the lagrangian. Then, in some approximation the iterated D'Alembertian ($\Box^2$) is found to operate on the field[3]. There are also examples, in particular for the bosonization in 2 + 1[4], in which the equation of motion involves the square root of the D. Alembertian ($\Box^{1/2}$).

The observations lead us to consider the general problem of constructing arbitrary powers of the Lorentz invariant differential operator $\Box$, and then of finding, in any number of dimensions, their relation to a general HP which we are going to specify later.

In §2, with the aid of some reasonable rules, we find the general form of $\Box^\alpha$ which although dependent somewhat on the boundary conditions, it is almost completely specified.

In §3 we define the Green functions $G^{(\alpha)}$ and find some of their properties.

In §4 we introduce the Huygens' principle. In §5 we study the analytic distribution Q_+^λ. In §6, the relations of $G^{(\alpha)}$ with HP are expressed in terms of the properties found in previous paragraphs. In §7 we study in particular the interesting and less known case of space-time with odd dimensionality (n = odd). Finally, in §8, we introduce and discuss arbitrary powers of the laplacian operator and their Green functions.

In an appendix we show how to evaluate the Fourier tranform of Riesz's classical retarded Green-function.

§2. Definition of $\Box^\alpha$

We suppose that space-time has d + 1 = n dimensions, d being the number of euclidean space dimensions.

The D'Alembertian operator is:

$$\Box = \partial_0^2 - \sum_{i=1}^{d} \partial_i^2 = \partial_0^2 - \Delta \qquad (1)$$

For the operator $\Box^s$ (s = positive integer) the Fourier transform will be:

$$\tilde{\Box}^s = F\{\Box^s\} = (-1)^s K_+^s + K_-^0 \qquad (2)$$

where in general:

$$K_{+}^{\alpha} = \left(k_0^2 - k^2\right)^{\alpha} \quad \text{if} \quad k_0^2 > k^2, \text{ zero otherwise} \tag{3}$$

$$K_{-}^{\alpha} = \left(k^2 - k_0^2\right)^{2} \quad \text{if} \quad k_0^2 < k^2, \text{ zero otherwise} \tag{4}$$

we now define $\Box^{\alpha}$ (any α) to be such that

$$\tilde{\Box}^{\alpha} = F\{\Box^{\alpha}\} = f(\alpha)K_{+}^{\alpha} + K_{-}^{\alpha} \quad , \quad \text{with } f(s) = (-1)^{s}, \tag{5}$$

and impose the condition:

$$\Box^{\alpha} * \Box^{\beta} = \Box^{\alpha+\beta} \tag{6}$$

which is equivalent to:

$$\tilde{\Box}^{\alpha} . \tilde{\Box}^{\beta} = \tilde{\Box}^{\alpha+\beta} \tag{6'}$$

But: $K_{+}^{\alpha}K_{+}^{\beta} = K_{+}^{\alpha+\beta}$; $K_{-}^{\alpha}K_{-}^{\beta} = K_{-}^{\alpha+\beta}$; $K_{+}^{\alpha}K_{-}^{\beta} = 0$.

So that, for (6') to hold we must impose:

$$f(\alpha)f(\beta) = f(\alpha+\beta) \therefore f(\alpha) = e^{\lambda\alpha}$$

And, due to (5), we must have:

$$f(\alpha) = e^{i\pi\varepsilon\alpha}$$

where ε is + 1 or -1.

It is now easy to see that there are essentially four Lorentz-invariant solutions for $\square^{\alpha}$, namely:

$$\tilde{\square}^{\alpha}_{\pm} = e^{\pm i\pi\alpha}K^{\alpha}_{+} + K^{\alpha}_{-} \tag{7}$$

$$\tilde{\square}^{\alpha}_{\substack{R\\A}} = e^{\pm i\pi\alpha . sgk_0}K^{\alpha}_{+} + K^{\alpha}_{-} \tag{8}$$

Where in (8) sgk_0 is Lorenrz-invariant as K^{α}_{+} is zero outside the light-one (cf. (3)).

If we compare (7) with the definition for $(K + io)^{\alpha}$ given in ref. [5] we find that

$$\tilde{\square}^{\alpha}_{\pm} = e^{\pm i\pi\alpha}\left(K^{\alpha}_{+} + e^{\mp i\pi\alpha}K^{\alpha}_{-}\right) = e^{\pm i\pi\alpha}(K \mp io)^{\alpha} \tag{9}$$

So that $\tilde{\square}^{\alpha}_{\pm}$ is the causal D'Alembertian already discussed in ref. [6].

From (7) and (8) is easy to see that we have the relations.

$$\tilde{\square}^{\alpha}_{+} = \theta(k_0)\tilde{\square}^{\alpha}_{R} + \theta(-k_0)\tilde{\square}^{\alpha}_{A} \tag{10}$$

$$\tilde{\square}^{\alpha}_{-} = \theta(k_0)\tilde{\square}^{\alpha}_{A} + \theta(-k_0)\tilde{\square}^{\alpha}_{R} \tag{11}$$

$\theta(x)$ being Heaviside's step function.

The operators $\Box_{\pm}^{\alpha}$ are then not independent of $\Box_{\substack{R\\A}}^{\alpha}$. They can be contructed by taking the positive frequency part of $\Box_{R}^{\alpha}$ $\left(\text{Resp } \Box_{A}^{\alpha}\right)$ and the negative frequency part of $\Box_{A}^{\alpha}$ $\left(\text{Resp } \Box_{A}^{\alpha}\right)$.

For the explicit form of $\Box_{\pm}^{\alpha}$ we take the anti-Fourier transform of (7) or (9), by using the results of ref. [5].

$$\Box_{\pm}^{\alpha} = \pm\, i e^{\pm i\pi\left(\alpha + \frac{n}{2}\right)} 4^{\alpha} (4\pi)^{\frac{n}{2}} \frac{\Gamma\left(\alpha + \frac{n}{2}\right)}{\Gamma(-\alpha)} (Q \pm io)^{-\alpha-\frac{n}{2}} \tag{12}$$

where Q is the quadratic form

$$Q = x_0^2 - \sum_{i=1}^{d} x_i^2 = t^2 - r^2$$

In the Appendix we show how to evaluate the anti-Fourier transform of (8). The result is:

$$\Box_{\substack{R\\A}}^{\alpha} = \frac{2.4^{\alpha} Q_{+}^{-\alpha-\frac{n}{2}}\, \theta(\mp t)}{\pi^{\frac{n}{2}-1}\, \Gamma(1-\alpha-\frac{n}{2})\Gamma(-\alpha)} \tag{13}$$

This is the operator found by M. Riesz by a generalization of the Rieman-Liouville complex integral (cf. ref. [2]).

Note that the by taking half the sum of the retarded plus the advanced solutions (13), we obtain an operator whose Fourier transform is

$$\frac{1}{2}\,\tilde{\Box}^{\alpha}_{R} + \frac{1}{2}\,\tilde{\Box}^{\alpha}_{A} = \frac{1}{2}\,\tilde{\Box}^{\alpha}_{+} + \frac{1}{2}\,\tilde{\Box}^{\alpha}_{-} = \cos\pi\alpha K^{\alpha}_{+} + K^{\alpha}_{-} \tag{14}$$

which for $\alpha = s =$ integer coincides with (2) but does not satisfy (6).

We will show below that for $\alpha = s =$ positive integer (12) and (13) reduce to

$$\Box^{s}_{\pm} = \Box^{s}_{\substack{R\\A}} = \Box^{s}\delta(x) \tag{15}$$

So that in a convolution $\Box^{\alpha}$ acts effectively as a differential operator when $\alpha = s$:

$$\Box^{\alpha}*f\big|_{\alpha=s} = \Box^{s}\delta*f(x) = \Box^{s}f(x) \qquad s = \text{positive integer.}$$

§3 The Green Function $G^{(\alpha)}$

The Green function for the operator $\Box^{\alpha}$ is the fundamental solution of the equation:

$$\Box^{\alpha}*f = g \tag{16}$$

$$\Box^{\alpha} * G^{(\alpha)} = \delta(x) \tag{17}$$

By taking Fourier transform we find:

$$\tilde{\Box}^{\alpha} . \tilde{G}^{(\alpha)} = 1$$

So that

$$\tilde{G}^{(\alpha)} = \tilde{\Box}^{-\alpha} \tag{18}$$

we have then (cf. (7), (8))

$$\tilde{G}^{\alpha}_{\pm} = e^{\mp i\pi\alpha} K_{+}^{-\alpha} + K_{-}^{-\alpha} \tag{19}$$

$$\tilde{G}^{\alpha}_{\substack{R\\A}} = e^{\mp i\pi\alpha\, \mathrm{sgk}_0} K_{+}^{-\alpha} + K_{-}^{-\alpha} \tag{20}$$

And of course (cf. (12) and (13)):

$$G_{\pm}^{(\alpha)} = \pm\, i e^{\pm i\pi\left(\frac{n}{2}-\alpha\right)} \frac{(4\pi)^{\frac{n}{2}}}{4^{\alpha}} \frac{\Gamma\left(\frac{n}{2}-\alpha\right)}{\Gamma(\alpha)} (Q \pm io)^{\alpha-\frac{n}{2}} \tag{21}$$

$$G^{(\alpha)}_{\substack{R\\A}} = \frac{2.4^{-\alpha} Q_{+}^{\alpha-\frac{n}{2}}\,\theta(\mp t)}{\pi^{\frac{n}{2}-1}\,\Gamma\left(1+\alpha-\frac{n}{2}\right)\Gamma(\alpha)} \tag{22}$$

If we take into account (10) and (11), we can write $G^{(\alpha)}_{\pm}$ in terms of $G^{(\alpha)}_{\substack{A\\R}}$ as:

$$G^{(\alpha)}_{\pm} = \delta^{\pm} * G^{(\alpha)}_{R} + \delta^{\mp} * G^{(\alpha)}_{A} \tag{23}$$

where

$$\delta^{\pm} = F\{\theta(\pm\, k_0)\} \tag{24}$$

So that the $G^{(\alpha)}_{\pm}$ Green functions propagate the positive (negative) frenquencies with the retarded $G^{(\alpha)}_{R}$ Green function and the negative (positive) frequencies with the advanced one.

For the ordinary wave equation ($\alpha = 1$) in four dimensions ($n = 4$), eq. (21) gives the massless Feynman propagator in coordinate space.

$$G^{(1)}_{\pm} = \pm\,\frac{4\pi^2 i}{Q \pm io} \qquad (n = 4) \tag{25}$$

while eq. (22) gives the usual retarded (advanced) potential (see below §5)

$$G^{(1)}_{\substack{A\\R}} = \frac{1}{2\pi}\,\delta(Q)\theta(\mp t) = \frac{\delta(r \pm t)}{4\pi r} \quad , \quad (n = 4) \tag{26}$$

The Fourier transform of (25) is given by (9) with $\alpha = -1$.

$$\tilde{G}^{(1)}_{\pm} = -\frac{1}{K \mp io} \qquad (n = 4) \tag{27}$$

The Fourier transform of (26) can be found from (8) if care is taken with the poles of $K^{\alpha}_{\pm}$ at $\alpha = -1$ (see below). The result is:

$$\tilde{G}^{(1)}_{\substack{A\\R}} = -\frac{1}{K \mp sgk_0 io} \qquad (n = 4) \tag{28}$$

§4 The Huygens' Principle

The equation corresponding to the pseudo-differential operators introduced in paragraph 2 are of the form

$$\Box^{\alpha} * f = g \tag{29}$$

The solution f can be found by using the Green function $G^{(\alpha)}$ defined by (17): see also [7]

$$f = G^{(\alpha)} * g \tag{30}$$

Note that (29) and (30) are dual of each other as $G^{(\alpha)}$ is the operator $\Box^{-\alpha}$, and (30) can be considered to be an equation for the determination of g, if f is given.

There are several statements which can be considered to represent the principle that Huygens introduced to describe the propagation of light waves (See ref. [1] for a discussion of this point). We are going to adopt the following statement:

-. The solution (30) of eq. (29) is said to obey Huygens' Principle (HP) if the Green function $G^{(\alpha)}$ has its support on the surface of the light - cone:-

This HP implies that the signals generated by the source propagate with one sharp velocity, that of the light.

Due to eq. (23), we see that the properties of $G_{\pm}^{(\alpha)}$ can be deduced from those of $G_{\substack{R\\A}}^{(\alpha)}$. In fact $G_{+}^{(\alpha)}$ propagates the positive frenquencies of the source by means of $G_{R}^{(\alpha)}$ and the negative frequencies by means of $G_{A}^{(\alpha)}$. In this sense we can say that $G_{\pm}^{(\alpha)}$ obeys HP if $G_{R}^{(\alpha)}$ and $G_{A}^{(\alpha)}$ do so. It is then enough to examine $G_{R}^{(\alpha)}$ ($G_{A}^{(\alpha)}$ is similar) to find out when Hp is satisfied.

From (26) we see immediately that $G_{R}^{(\alpha)}$ obey HP in n = 4, as $\delta(Q)$ has its support on the light-cone Q = 0. For n = odd number, it follows from (22) for d = 1 that

$$G_{R}^{(1)} \cong Q_{+}^{1-\frac{n}{2}}$$

which is well defined and zero outside the light-cone (cf. (3)) but it is different from zero everywhere *inside* the light-cone and so as is well-known, the solutions of the ordinary wave equation obey HP for $n = 4$ (n = even) but do not obey HP for n = odd.

In the general case we have to examine the singularities of the functions on which $G_R^{(\alpha)}$ depends (cf. (22)). The positions and residues of the poles, of Eulers Γ functions are well known. For Q_+^λ, as analytic function of λ, we transcribe the results found in ref. [5].

5§ The analytic distribution Q_+^λ

In the next paragraph it will be evident that the structure of the singularities of Q_+^λ determine the relation of the Green function $G_R^{(\alpha)}$ with HP.

We are considering only one time and d space coordinates in the quadratic form $Q = t^2 - r^2$, then according to ref. [5], the distribution Q_+^λ is an analytic function of λ which:

a) *-for n = odd* has simple poles at $\lambda = -1, -2, \ldots, -k, \ldots$ and at $\lambda = -\frac{n}{2}$, $-\frac{n}{2} - 1$, -1, $-\frac{n}{2} - 2$, $\ldots$, $-\frac{n}{2} - k$, $\ldots$

The residues are:

$$\operatorname*{Res}_{\lambda=-k} Q_+^\lambda = \frac{(-1)^{k-1}\delta^{(k-1)}}{\Gamma(k)}(Q) \qquad (k = 1,2,\ldots) \tag{31}$$

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} Q_+^\lambda = \frac{(-1)^{\frac{d}{2}}\pi^{\frac{n}{2}}\Box^k\delta(x)}{4^k\Gamma(k+1)\Gamma\left(\frac{n}{2}+k\right)} \qquad (k = 0,1,2,\ldots) \tag{32}$$

b) *for n = even* has simple poles at $\lambda = -1, -2, \ldots, 2 - \frac{n}{2}, 1 - \frac{n}{2}$ and double poles at $\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, \ldots, -\frac{n}{2} - k, \ldots$

$$\operatorname*{Res}_{\lambda=-k} Q_+^\lambda = \frac{(-1)^{k-1}}{\Gamma(k)} \delta^{(k-1)}(Q) \qquad \left(k = 1, 2, \ldots, \frac{n}{2} - 1\right) \tag{33}$$

Near $\lambda = -\frac{n}{2} - k$, the double poles have the form

$$Q_+^\lambda = \frac{(-1)^{\frac{n}{2}} \pi^{\frac{n}{2}-1}}{4^k \Gamma(k+1) \Gamma\left(\frac{n}{2}+k\right)} \frac{\square^k \delta(x)}{\left(\lambda + \frac{n}{2} + k\right)^2} + \ldots \qquad (k = 0, 1, 2, \ldots) \tag{34}$$

We now observe that Q_+^λ has the types of singularities that present the product $\Gamma(1+\lambda)\Gamma\left(\lambda + \frac{n}{2}\right)$. In fact, when n = odd, this product has simple poles at $\lambda = -k$ (k = positive integer), and at $\lambda = -\frac{n}{2} - k$ (k = positive integer or zero), just as in a). Further, when n = even (as in b), the product presents simple poles for $\lambda = -k$ $\left(0 < k < \frac{n}{2}\right)$, and double poles for $\lambda = -k$, if $k \geq \frac{n}{2}$.

For these reasons, if we divide Q_+^λ by that product, we obtain

$$Q'(\lambda) = \frac{Q_+^\lambda}{\Gamma(1+\lambda)\Gamma\left(\frac{n}{2} + \lambda\right)} \tag{35}$$

And $Q'(\lambda)$ is an entire analytic function of λ.

Furthermore, $Q'(\lambda)$ has the following properties:

a') For n = odd and $\lambda = -k$ (k = positive integer)

$$Q'(-k) = \frac{1}{\Gamma\left(\frac{n}{2} - k\right)} \delta^{(k-1)}(Q) \tag{36}$$

For n = odd and $\lambda = -\frac{n}{2} - k$ $(k = 0,1,2,\ldots)$

$$Q'\left(-\frac{n}{2} - k\right) = \frac{\Pi^{\frac{n}{2}-1}}{4^k} \Box^k \delta(x) \tag{37}$$

b') For n= even and $\lambda = -k$ $\left(k = 1,2,\ldots,\frac{n}{2} - 1\right)$

$$Q'(-k) = \frac{1}{\Gamma\left(\frac{n}{2} - k\right)} \delta^{(k-1)}(Q) \tag{38}$$

For n = even and $\lambda = -\frac{n}{2} - k$ $(k = 0,1,\ldots)$

$$Q'\left(-\frac{n}{2} - k\right) = \frac{\Pi^{\frac{n}{2}-1}}{4^k} \Box^k \delta(x) \tag{39}$$

§6 The $G_R^{(\alpha)}$ that obey HP

We first observe that $\Box_R^\alpha$ (eq. (13)) and $G_R^{(\alpha)}$ (eq. (22)) can be expressed in terms of $Q'(\lambda)$ (eq. (35)) as:

$$\Box_R^\alpha = \frac{2.4^\alpha}{\Pi^{\frac{n}{2}-1}} \frac{Q_+^{-\alpha-\frac{n}{2}} \theta(-t)}{\Gamma(-\alpha)\Gamma\left(1 - \alpha - \frac{n}{2}\right)} = \frac{2.4^\alpha}{\Pi^{\frac{n}{2}-1}} Q'\left(-\alpha - \frac{n}{2}\right)\theta(-t) \tag{40}$$

and

$$G_R^{(\alpha)} = \Box_R^{-\alpha} = \frac{2.4^{-\alpha}}{\pi^{\frac{n}{2}-1}} Q'\left(\alpha - \frac{n}{2}\right)\theta(-t) \tag{41}$$

As a consequence of the properties of $Q'(\lambda)$ pointed out in §5, we have that $\Box_R^{\alpha}$ and of course $G_R^{(\alpha)}$ are entire analytic functions of α. See also [7].

For any n and $\alpha = -k$ $(k = 0,1,2...)$, it follows from (37), (39) and (41) that

$$G_R^{(-k)} = \Box_R^{k} = \Box^k\delta(x) \qquad (k = 0,1,2...) \tag{42}$$

Where, due to the presence of the factor $\theta(-t)$, the contribution of the retarted cone is only a half of the quoted value in (37) and (39).

It is now easy to see when the Green function $G_R^{(\alpha)}$ obeys HP. The only cases for which $Q'(\lambda)$ has its support on the light-cone are those for which (36) and (38) are valid, i.e. for $\alpha = \frac{n}{2} - k$.

From (41) we then obtain:

$$G_R^{(\frac{n}{2}-k)} = \frac{2.4^{k-\frac{n}{2}}}{\pi^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2} - k\right)} \theta(-t)\delta^{(k-1)}(Q) \quad (k = \text{positive integer}) \tag{43}$$

when n is even the values of k are restricted to be less than $\frac{n}{2}$ $\left(k < \frac{n}{2}\right)$, but for n = odd, k is an unrestricted positive integer.

For $n = 4$ we have the usual retarded potential (26). Further, this kind of potential holds in any number of dimensions for $k = 1$:

$$G_R^{(\frac{n}{2}-1)} = \frac{2\theta(-t)\delta(Q)}{(4\pi)^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}-1\right)} = \frac{\delta(r+t)}{(4\pi)^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}-1\right).r} \tag{44}$$

$$\Box^{\frac{n}{2}-1} * G_R^{(\frac{n}{2}-1)} = \delta(x) \tag{45}$$

Eqs. (44) and (45) are true for any n (even or odd).

The usual wave equation $\Box f = g$ is the only one whose solution obeys HP in any even number of dimensions ($n > 2$).

The once iterated D'Alembertian equation:

$$\Box\Box f = \Box^2 f = g$$

does not obey HP in four dimensions, but it does satisfy that principle for $n = 6,8,10,\dots$.

In general for $\Box^s f = g$ to obey HP it is necessary that $n = 2(s+k) \geq 2(s+1)$ ($k = 1,2,\dots$). See §5 and 41.

§7 The case n = odd

The results found in §6, eq. (43) do not seem to be well known for n = odd, and they are interesting enough to deserve explicit mention, at least for low values of n. See also [8]. For any odd n there are an infinite number of convolution operators whose Green function obeys HP. They are

$$\Box_R^{\frac{n}{2}-1}\ ,\quad \Box_R^{\frac{n}{2}-2}\ ,\ \ldots,\ \Box^{\frac{n}{2}-k}\ ,\ \ldots \tag{46}$$

From (36) and (40) we get:

$$\Box_R^{\frac{n}{2}-k} = \frac{2.4^{\frac{n}{2}-k}}{\pi^{\frac{n}{2}-1}\Gamma\left(k-\frac{n}{2}\right)}\,\theta(-t)\delta^{(n-k-1)}(Q)\quad ,\ k < n \tag{47}$$

$$\Box_R^{\frac{n}{2}-k} = \frac{2.4^{\frac{n}{2}-k}}{\pi^{\frac{n}{2}-1}\Gamma\left(k-\frac{n}{2}\right)}\,\frac{\theta(-t)}{\Gamma(1+k-n)}\,Q_+^{k-n}\quad ,\quad k \geq n \tag{48}$$

For k < n, we can also write:

$$\Box_R^{\frac{n}{2}-k} = \Box^{n-k-1+1-\frac{n}{2}} = \Box^{n-k-1}\Box_R^{1-\frac{n}{2}} = \Box^{n-k-1}G_R^{(\frac{n}{2}-1)} \tag{49}$$

where $G_R^{(\frac{n}{2}-1)}$ is the usual restarded potential given by (44). With the aid of (49) we can compute the action of the operator $\Box_R^{\frac{n}{2}-k}$ on a function f, as:

$$\Box_R^{\frac{n}{2}-k} * f = \Box^{n-k-1} G_R^{(\frac{n}{2}-1)} * f = G_R^{(\frac{n}{2}-1)} * \Box^{n-k-1} f \quad , \quad (k < n) \tag{50}$$

In this way the action of $\Box_R^{\frac{n}{2}-k}$ on f is represented by the retarded potential produced by $\Box^{n-k-1}f$.

For example in n = 3, we have

$$\Box^{\frac{1}{2}} = -\frac{2}{\pi}\,\theta(-t)\delta'(Q) = \frac{\Box}{\pi}\,\theta(-t)\delta(Q) \tag{51}$$

$$\Box_R^{-\frac{1}{2}} = \frac{1}{\pi}\,\theta(-t)\delta(Q) \tag{52}$$

$$\Box_R^{-\frac{3}{2}} = \frac{1}{2\pi}\,\theta(-t)\theta(Q) \tag{53}$$

$$\Box_R^{-\frac{5}{2}} = \frac{1}{12\pi}\,\theta(-t)Q_+ \tag{54}$$

Note also that the operator $\Box_R^{+\frac{1}{2}}$ depends on n:

$$\Box_R^{\frac{1}{2}} = \Box_R^{\frac{n}{2}-\frac{n-1}{2}} = \Box_R^{\frac{n-1}{2}} G_R^{(\frac{n}{2}-1)} \tag{55}$$

where use has been made of (49) and $G_R^{(\frac{n}{2}-1)}$ is proportional to $\frac{1}{r}\delta(r+t)$ for any n.

The Green functions corresponding to the operators (46) can also be expressed in terms of the retarded potential (44):

$$G_R^{(\frac{n}{2}-k)} = \Box_R^{k-\frac{n}{2}} = \Box^{k-1}\Box_R^{1-\frac{n}{2}} = \Box^{k-1}G_R^{(\frac{n}{2}-1)} \tag{56}$$

(compare with (49)).

So that the causal solution of

$$\Box_R^{\frac{n}{2}-k} * f = g$$

is

$$f = G_R^{(\frac{n}{2}-k)} * g = G_R^{(\frac{n}{2}-1)} * \Box^{k-1} g \tag{57}$$

(Compare with (50)).

§8 Arbitrary powers of the Laplacian

Just as for the D'Alembertian, we can define arbitrary powers of the Laplacian operator Δ. See also [7].

For the d-dimensional euclidean space we define:

$$R = r^2 = \sum_1^d x_i^2 \tag{58}$$

$$\kappa = k^2 = \sum_1^d k_i^2 \tag{59}$$

For s = positive integer:

$$F\{\Delta^s\} = \tilde{\Delta}^s = (-1)^s K^s \tag{60}$$

We generalize this formula to:

$$F\{\Delta^\alpha\} = \tilde{\Delta}^\alpha = e^{i\pi\alpha}\kappa^\alpha \tag{61}$$

This definition satisfies:

$$\Delta^\alpha * \Delta^\beta = \Delta^{\alpha+\beta} \quad ; \quad \Delta^0 = \delta(x) \tag{62}$$

and gives for Δ^α the expression (see ref. [5])

$$\Delta^\alpha = \frac{e^{i\pi\alpha} 4^\alpha \Gamma\left(\alpha + \frac{d}{2}\right)}{\Pi^{\frac{d}{2}} \Gamma(-\alpha)} R^{-\alpha-\frac{d}{2}} \tag{63}$$

The Green function corresponding or Δ^α is

$$\Delta^{\alpha} * G^{(\alpha)} = \delta(x)$$

$$\therefore G^{(\alpha)} = \Delta^{-\alpha} \tag{64}$$

$$G^{(\alpha)} = \frac{e^{-i\pi\alpha}\Gamma\left(\frac{d}{2} - \alpha\right)}{4^{\alpha}\Pi^{\frac{d}{2}}\Gamma(\alpha)} R^{\alpha-\frac{d}{2}} \tag{65}$$

According to ref. [5], the distribution R^{λ} has simple poles for $\lambda = - s - \frac{d}{2}$ $(s = 0,1,2,\dots)$ with residues

$$\operatorname*{Res}_{\lambda=-s-\frac{d}{2}} R^{\lambda} = \frac{\Pi^{\frac{d}{2}}\Delta^{s}\delta(x)}{4^{s}\Gamma(s+1)\Gamma\left(s + \frac{d}{2}\right)} \tag{66}$$

So that, from (63) and (66) we obtain:

$$\Delta^{s} * f = \Delta^{s}\delta(x) * f = \Delta^{s} f \qquad (s = 0,1,2\dots) \tag{67}$$

In the expression (65), the poles of $R^{\alpha-\frac{d}{2}}$ are compensated or neutralized by the poles of $\Gamma(\alpha)$.

However, the Green function $G^{(\alpha)}$ has simple poles for $\alpha = \frac{d}{2} + s$ $(s = 0,1,2,\dots)$, which are due to the presence of $\Gamma\left(\frac{d}{2} - \alpha\right)$. The residues of $G^{(\alpha)}$ at these poles are proportional to R^{s} (a polynomial in x_1^2) and they are solutions of the homogeneous equation:

$$\Delta^{\frac{d}{2}+s} * R^{s} = 0 \tag{68}$$

This is trivial for d = even, but it is also true for d = odd, as can be proved by computing $R^{-s-d} * R^{s}$. See [5] p. 361.

For this reason we can drop the poles of $G^{(\alpha)}$ and define, for $\alpha = \frac{d}{2} + s$.

$$G^{(\frac{d}{2}+s)} = \mathrm{Pf}G^{(\alpha)}\Big|_{\alpha=\frac{d}{s}+s} = \frac{d}{d\alpha}\left\{\left(\alpha - \frac{d}{2} - s\right)G^{(\alpha)}\right\}_{\alpha=\frac{d}{2}+s} \tag{69}$$

$$G^{(\frac{d}{2}+s)} = \frac{e^{-i\pi\frac{d}{2}} R^{s} \ell n R}{4^{\frac{d}{2}+s}\,\Pi^{\frac{d}{2}}\,\Gamma(s+1)\Gamma\left(s + \frac{d}{s}\right)} \qquad (s = 0,1,2,\dots) \tag{70}$$

where we have droped terms proportional to R^{s}. (Residues)

In particular for d = 2, and s = 0 we have the well-known logarithmic potential:

$$\Delta G^{(1)} = \delta$$

$$G^{(1)} = - \frac{\ell n\ R}{4\pi} \tag{71}$$

As a matter of facts, the logarithmic potential is the Green function corresponding to the operator $\Delta^{\frac{d}{2}}$ in any number of dimensions:

$$\Delta^{\frac{d}{2}} * \ell n R \approx \delta(x)$$

In four dimensions for example (d = 4), the iterated Laplacian has a logarithmic potential as a fundamental solution

$$\Delta\Delta G^{(2)} = \delta(x)$$

$$G^{(2)} = \frac{\ell n R}{16\pi^2} \qquad (d = 4) \tag{72}$$

We may ask in general, which is the operator which has a potential of the form R^{β} in a d-dimensional euclidean space. The answer is given by (63) and (65). See ref. [7].

For the Green function to be proportional to R^{β}, we must have $\alpha - \frac{d}{2} = \beta$, so that the operator is

$$\Delta^{\beta+\frac{d}{2}} = \frac{e^{i\pi\left(\beta+\frac{d}{2}\right)} 4^{\beta+\frac{d}{2}} \Gamma(\beta+d)}{\pi^{\frac{d}{2}} \Gamma\left(-\beta - \frac{d}{2}\right)} R^{-\beta-d} \tag{73}$$

and

$$G^{(\beta+\frac{d}{2})} = \frac{e^{-i\pi(\beta+\frac{d}{2})}\,\Gamma(-\beta)}{4^{\beta+\frac{d}{2}}\pi^{\frac{d}{2}}\Gamma\left(\beta+\frac{d}{2}\right)}\,R^{\beta} \tag{74}$$

The logarithmic potential corresponds to $\beta = 0$.

For the newtonian potential $r^{-1} = R^{-\frac{1}{2}}$, $\beta = -\frac{1}{2}$ and (73), (74) give:

$$\Delta^{\frac{d-1}{2}} = \frac{e^{i\pi\frac{d-1}{2}}\,4^{\frac{d-1}{2}}\,\Gamma\left(d-\frac{1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{1-d}{2}\right)}\,R^{\frac{1}{2}-d} \tag{75}$$

$$G^{(\frac{d-1}{2})} = \frac{e^{-i\pi\frac{d-1}{2}}}{4^{\frac{d-1}{2}}\,\pi^{\frac{d-1}{2}}\,\Gamma\left(\frac{d-1}{2}\right)}\,\frac{1}{r} \tag{76}$$

For odd dimensional spaces, (75) is just the laplacian iterated $\frac{d-1}{2}$ times. In $d = 3$ it is the usual laplacian Δ. In $d = 5$ it is $\Delta^2 = \Delta\Delta$, etc.

For even dimensional spaces (70) gives an exponent which is half an odd integer.

Appendix

To evaluate the Fourier transform of $G_R^{(\alpha)}$ (eq. (22)) we start with:

$$F\left\{Q_+^\lambda \theta(-t)\right\} = \int d^{n-1}x\; e^{i\vec{k}.\vec{r}} \int_{-\infty}^{-r} dt\,(t^2-r^2)^\lambda e^{-ik_0 t} \tag{A.1}$$

and use the table of Integral Transforms (Bateman project Vol. 1, p.11 and p. 69) to write:

$$\int_{-\infty}^{-r} dt\,(t^2-r^2)^\lambda e^{-ik_0 t} = \frac{\Pi^{\frac{1}{2}} 2^{\lambda-\frac{1}{2}} \Gamma(1+\lambda) r^{\lambda+\frac{1}{2}}}{\operatorname{sen}\pi\left(\lambda + \frac{1}{2}\right) |k_0|^{\lambda+\frac{1}{2}}} \left\{ e^{i\operatorname{sg}k_0 (\lambda+\frac{1}{2})\pi} J_{-\lambda-\frac{1}{2}}(|k_0| r) - J_{\lambda+\frac{1}{2}}(|k_0| r) \right\} \tag{A.2}$$

We must also take into account that the angular integral in (A.1) gives:

$$\int d\Omega e^{ikr\cos\theta} = \frac{2\Pi^{\frac{n-1}{2}}}{\left(\frac{rk}{2}\right)^{\frac{n-3}{2}}} J_{\frac{n-3}{2}}(kr) \tag{A.3}$$

We now need integrals of the form

$$\int_0^\infty dr\ r^{\frac{n}{2}+\lambda} J_{\frac{n-3}{2}}(kr) J_{\pm(\lambda+\frac{1}{2})}(k_0 r) \tag{A.4}$$

which are found in a table of integrals (Ref. [9] p. 692).

Replacing now in (A.1) we obtain

$$F\left\{Q_+^\lambda \theta(-t)\right\} = \frac{2^{2\lambda+n-1}\Gamma\left(\frac{n}{2}+\lambda\right)\Gamma(\lambda+1)\Pi^{\frac{n}{2}-1}}{\sin\pi\left(\lambda+\frac{1}{2}\right)}$$

$$\left\{e^{i\pi\left(\lambda+\frac{1}{2}\right)\mathrm{sgk}_0} K_+^{-\lambda-\frac{n}{2}} \sin\left(\frac{n}{2}+\lambda\right)\pi - K_+^{-\lambda-\frac{n}{2}} \sin\Pi^{\frac{n-1}{2}} + K_-^{-\lambda-\frac{n}{2}} \sin\Pi\left(\lambda+\frac{1}{2}\right)\right\} \tag{A.5}$$

Now we wright

$$\sin\left(\lambda+\frac{n}{2}\right)\pi = \mathrm{sen}\Pi\left(\frac{n-1}{2}\right)\cos\Pi\left(\lambda+\frac{1}{2}\right) + \sin\Pi\left(\lambda+\frac{1}{2}\right)\cos\Pi\left(\frac{n-1}{2}\right)$$

$$\cos\left(\lambda+\frac{1}{2}\right)\pi = e^{-i(\lambda+\frac{1}{2})\mathrm{sgk}_0} + i\,\mathrm{sgk}_0 \sin\left(\lambda+\frac{1}{2}\right)\pi$$

and using these equalities in (A.5):

$$F\left\{Q_+^\lambda \theta(-t)\right\} = 2^{2\lambda+n-1}\Pi^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}+\lambda\right)\Gamma(\lambda+1)\left\{e^{i\pi\left(\lambda+\frac{n}{2}\right)\mathrm{sgk}_0} K_+^{-\lambda-\frac{n}{2}} + K_-^{-\lambda-\frac{n}{2}}\right\} \tag{A.6}$$

So we have for M. Rierz Green function (eq. (22)):

$$F\left\{G^{\alpha}_{R}\right\} = e^{i\pi\alpha\,\mathrm{sg}k^{0}} K_{+}^{-\alpha} + K_{-}^{-\alpha} \tag{A.7}$$

Bibliography

[1] The Mathematical Theory of Huygen's Principle. B.B. Baker & E.T. Copson Oxford at Clarendon Press 1939.

[2] M. Riesz Acta Math. 81 (1948).

[3] T.V. Ruzmaikina & A.A. Ruzmaikin Soviet Physics JETP 30 (1970) 372
A.J. Accioly Rev. Bras. Fis. 18 (1988) 593.

[4] E.C. Marino Nota Cientifica Preprint 16/90. Pontificia Universidade Católica do Rio de Janeiro.

[5] I.M. Gelfand & G.E. Shilov. Les distributions Dumod. Paris 1962.

[6] C.G. Bollini, J.J. Giambiagi, A. González Domingues, Il Nuovo Cimento 31 (1964) 550.

[7] C.G. Bollini, J.J. Giambiagi, 80. Obregón Preprint Universidad Autónoma Metropolitana. P.O. Box 55-534-México D.F. C.P. 09340.

[8] J.J. Giambiagi, Preprint Notas de Física CBPF-NF-025/91.

[9] Table of Integrals Series and Products, I.S. Gradshteyn, I.M. Ryzhir. Academic Press 1965.