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DIRAC'S EQUATION AS A GEODESIC IN RIEMANN SPACE

by

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Dirac's equation is derived from a geometrical variational principle which involves a line-element matrix in flat space (for a free fermion and for a fermion in interaction with a gauge field) or a in Riemann space (for a fermion in interaction with the spinor connection, the gauge field of the local Lorentz group).

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relativistic wave equation;

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In the relativistic theory of gravitation[1], as is well known, the line element in Riemann space is expressed in terms of the metric tensor $g_{\mu\nu}(x)$:

$$ds^2 = g_{\mu\nu}(x)\ dx^\mu\ dx^\nu \qquad (1)$$

which is postulated to describe the gravitational potential. The classical equation of motion of a particle in a gravitational field follows in the familiar way from the variational principle

$$\delta \int_A^B ds = 0 \qquad (2)$$

and equivalently describes a geodesic in Riemann space.

In this letter we wish to point out that Dirac's equation for a free fermion can be deduced from a variational principle similar to the principle:

$$\delta \int_A^B d\sigma = 0 \qquad (3)$$

where

$$d\sigma^2 = \eta_{\mu\nu}\ dx^\mu\ dx^\nu \qquad (4)$$

which gives the classical equations of motion of a free particle.

In the case of a free fermion as well as for a fermion in a gravitational field a line element matrix is defined in a flat or in Riemann space where spinors and Dirac matrices are defined in the usual way[2]. The variational principle involves a line-element spinor and yields Dirac's equation which can thus be regarded as a spinor geodesic in this space and contains the spin connection when gravity is taken into account.

The procedure is generalized to the gauge theories where the line element is an element of a Lie group. The variational principle gives Dirac's equation for a fermion in interaction with the gauge field.

The line element (4) may be linearized by the following matrix:

$$d\,\Sigma_0 = \gamma_a dx^a \qquad (5)$$

sum over the space-time indices a=0,1,2,3. In flat space, the γ_a's obey the usual anticommutation rule:

$$\frac{1}{2}\ \{\gamma_a, \gamma_b\} = \eta_{ab} \tag{5a}$$

from which the line element follows with the identity matrix in spinor space.

Let us now define a spinor field ψ (x) and a line-element spinor associated to it:

$$d\Sigma = \gamma_a\, dx^a\, \psi\ (x) \tag{6}$$

The variational principle

$$\delta \int_A^B d\Sigma = 0 \tag{7}$$

will give us the condition, for an arbitrary variation δx^α of the spinor world line between points A and B:

$$\partial_a\ (\gamma_b\, \psi) - \partial_b(\gamma_a\ \psi) = 0$$

Therefore $\gamma_a\, \psi$ must be the gradient of a spinor which we economically take as ψ(x) itself. We may therefore write:

$$\gamma_a\, \psi = 4i\, \frac{\hbar}{mc}\quad \partial_a\psi \tag{8}$$

which is actually Dirac's equation for a free particle of mass m

$$i\hbar\, \gamma^a\, \partial_a\, \psi - mc\, \psi = 0 \tag{8a}$$

The vierbeins[3] e^μ_a (x) allow us to define the x-dependent γ-matrices.

$$\gamma^\mu(x) = e^\mu_a\ (x)\ \gamma^a$$

which obey the commutation rule:

$$\frac{1}{2}\ \{\gamma_\mu\ (x),\ \gamma_\nu(x)\} = g_{\mu\nu}\ (x)$$

and the relationship of the vierbeins:

$$e^\mu_a\ (x)\ g_{\mu\nu}\ (x)\ e^\nu_b\ (x) = \eta_{ab}$$

The spin connection[3] ω^{ab}_μ (x) is the gauge field of the local Lorentz group and describes the interaction of a spinor with a gravitational field as given by Dirac's equation:

$$\left[i\hbar\ \gamma^\mu\ (x)\ \left(\partial_\mu + \Gamma_\mu\ (x)\right)\ -mc\right]\ \psi(x) = 0 \tag{9}$$

where, the generators of the local Lorentz group being $\frac{\sigma_{ab}}{2}$:

$$\Gamma_\mu\ (x) = -\frac{i}{2}\ \omega^{ab}_\mu\ (x)\ \frac{\sigma_{ab}}{2} \tag{10}$$

In terms of the Christoffel symbols the spin connection is expressed in the following way:

$$\omega^{ab}_\mu = \Gamma^\lambda_{\mu\nu}\ e^a{}_\lambda\ \ e^{\nu b} + e^a_\nu\ (\partial_\mu e^{\nu b})$$

but the second term gives no contribution to the commutation of the derivatives on a spinor, which defines the usual field tensor, the Riemann curvature tensor $R_{\mu\nu}{}^{ab}$:

$$[\nabla_\mu,\ \nabla_\nu]\ \psi = -\frac{i}{2}\ R^{ab}_{\mu\nu}\ \frac{\sigma_{ab}}{2},$$
$$\nabla_\mu = \partial_\mu + \Gamma_\mu\ .$$

In order to derive equation (9) from a geometrical variational principle we have to extend the line-element matrix (5) by replacing γ_a by $\gamma_\mu(x)$ and by adding a new term with the gauge field of the local Lorentz group (10). We set:

$$d\ \sigma_g = (\gamma_\mu\ (x)\ -\ 4i\ \frac{\hbar}{mc}\ \Gamma_\mu(x)\ )\ dx^\mu$$

and postulate the variational principle

$$\delta \int d\Sigma_g = 0 \tag{11}$$

where $d\,\Sigma_g$ is the line-element spinor:

$$d\Sigma_g = \left(\gamma_\mu(x) - 4i\,\frac{\hbar}{mc}\,\Gamma_\mu(x)\right)\,\psi\,(x)\quad dx^\mu \tag{12}$$

The equation (11) will imply, for arbitrary variations δx^α :

$$\partial_\alpha\,\left[\left(\gamma_\mu\,(x) - 4i\,\frac{\hbar}{mc}\,\Gamma_\mu\,(x)\right)\,\psi\,(x)\right] =$$

$$\partial_\mu\,\left[\left(\gamma_\alpha(x) - 4i\,\frac{\hbar}{mc}\,\Gamma_\alpha\,(x)\right)\,\psi\,(x)\right]$$

and this relation leads to Dirac's equation

$$\left(\gamma_\mu\,(x) - 4i\,\frac{\hbar}{mc}\,\Gamma_\mu\,(x)\right)\,\psi\,(x) = 4i\,\frac{\hbar}{mc}\,\partial_\mu\psi$$

which will have the form (9).

As this equation corresponds to the classical equation of motion, it might well be called a spinor geodesic in Riemann space. We note that only for integer spin fields in interaction with gravity and for the continuous group of transformations is the line element (1) sufficient. For a spin 1/2 field the line-element spinor (6) is needed to derive Dirac's equation for a free particle. And for spin $\frac{1}{2}$ fields in interaction with gravitation a line-element spinor which contains an element of the local Lorentz group - its gauge field - is necessary to obtain the equation (9).

This procedure may be extended to gauge fields (abelian and non-abelian) in flat space. For a SU(n) group with n^2-1 generators Γ_k a line-element spinor may be defined in the following form:

$$d\,\Sigma_A = \left(\gamma_\mu\,(x) + \frac{4g}{mc}\,\tilde{A}_\mu\,(x)\right)\psi(x). \tag{13}$$

where:

$$\tilde{A}_\mu\,(x) = \Sigma_k\,A_{\mu k}\,\Gamma_k$$

The variational principle leads to the equation:

$$(\dot{\gamma}^\mu\,(i\hbar\,\partial_\mu - g\,\tilde{A}_\mu) - mc)\,\psi(x) = 0 \tag{13a}$$

where the γ_g^μ, satisfy the rule (5a) in flat space.

In his 1916 paper, §14, Einstein[1] made a distinction between gravitational field and matter and denoted as matter everything but the gravitational field - a distinction justified by the fact that it is this everything else that contributes to the (matter) energy - momentum tensor $\Gamma_{\mu\nu}$, the source of the gravitational field in his equations.

Nowadays, we may rather make a distinction between fundamental matter - quarks and leptons - the multiplets which span the gauge group space, and the gauge fields - gravity, electromagnetic, Yang-Mills fields - which are associated to the group generators and describe the basic interactions. We recuperate part of Einstein's ideal of geometrization of physics, by showing that the fundamental matter spinor equations follow from a variational principle involving a convenient line-element in Riemann or in flat space.

We note that a scalar line element may be obtained from the line-element matrix (6), (12) or (13) by considering a constant spinor χ and the scalar product $\bar{\chi}\,d\,\Sigma$ with its adjoint.

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1. A. Einstein, Die Grandlage der Allgemeinen Relativitätstheorie,Johann Ambrosais Barth, Leipzig, 1916.

2. Cf. J.Leite Lopes, Gauge Field Theories: an introduction pg. 182, Pergamon Press, Oxford 1981.

3. Cf. M.G. Green, J.H.Schwartz and E. Witten, Superstring theory, 2nd. vol. pg. 271, Cambridge University Press, 1987.